Measure-valued limits of interacting particle systems with k-nary interactions I. One-dimensional limits.

Vassili N. Kolokoltsov

Abstract. Results on existence, uniqueness, non-explosion and stochastic monotonicity are obtained for one-dimensional Markov processes having non-local pseudo-differential generators with symbols of polynomial growth. It is proven that the processes of this kind can be obtained as the limits of random evolutions of systems of identical indistinguishable particles with k-nary interaction.

Key words. Interacting particles, k-nary interaction, measure-valued processes, one-dimensional Feller processes with polynomially growing symbols, duality, stochastic monotonicity, heat kernel.

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Content. 1. Introduction. 2. Preliminaries: analytic treatment of one-dimensional diffusions. 3. Estimates for the resolvent of one-dimensional diffusion operators. 4. One-dimensional Feller processes with polynomial generators. 5. K-nary interaction of indistinguishable particles. 6. Limits of k-nary interacting particle systems and duality. Appendix. Heat kernel estimates.

1. Introduction.

1. *Aims of the paper.* Suppose the state of a system is characterized by a number $n \in \mathbf{Z}_+ = \{0, 1, 2, ...\}$ of identical indistinguishable particles. If any single particle, independently of others, can die after a random life-time producing a random number $l \in \mathbf{Z}_+$ of offspring, the generator of such a process will clearly have the form

$$(G_1 f)(n) = n \sum_{m=-1}^{\infty} g_m^1 (f(n+m) - f(n))$$

with some non-negative constants g_m^1. More generally, if any group of k particles randomly chosen (with uniform distribution, say) from a given group of n particles can be transformed (at some random time) into a group of $l \in \mathbf{Z}_+$ particles (due to some process of birth, death, coagulation, etc), the generator of such a process will have the form

$$(G_k f)(n) = C_n^k \sum_{m=-k}^{\infty} g_m^k (f(n+m) - f(n)) \qquad (1.1)$$

with some non-negative constants g_m^k, where C_n^k denote the usual binomial coefficients and where it is understood that these coefficients vanish whenever $n < k$. Finally, a spontaneous birth (or input) of a random number of particles (if allowed to occur) will contribute a term of type $\sum_{m=0}^{\infty} g_m^0 (f(n+m) - f(n))$ to the generator of our process. The generator of the type $\sum_{k=0}^{K} G_k$ describes all k-nary interaction with $k \le K$. The usual scaling of the state space $n \mapsto nh$, h being a positive parameter, combined with the scaling of the interaction $G_k \mapsto h^k G_k$ leads to the Markov chain on $h\mathbf{Z}_+$ with the generator

$$G^h = \sum_{k=0}^{K} G_k^h, \quad (G_k^h f)(hn) = h^k C_n^k \sum_{m=-k}^{\infty} g_m^k (f(hn+hm) - f(hn)). \tag{1.2}$$

We are interested in the limit $n \to \infty$, $h \to 0$ with $nh \to x \in \mathbf{R}_+$, and where $g_m^k = g_m^k(h)$ may also depend on h. To analyze this limiting procedure we shall consider operator (1.2) as a restriction on $h\mathbf{Z}_+$ of the operator (which we shall again denote by G^h with some abuse of notations) defined on functions on $(0, \infty)$ by $G^h = \sum_{k=0}^K G_k^h$, where

$$(G_k^h f)(x) = \frac{x(x-h)...(x-(k-1)h)}{k!} \sum_{m=\max(-k,-x/h)}^{\infty} g_m^k(h)(f(x+hm) - f(x)) \tag{1.3}$$

for $x \geq h(k-1)$ and vanishes otherwise. Clearly $x(x-h)...(x-(k-1)h)$ tends to x^k as $h \to 0$ and one can expect that (with an appropriate choice of $g_m^k(h)$), the sum of the k-nary interaction generators (1.3) will tend to the generator of a stochastic process on $\mathbf{R}_+$ with generator of the form $\sum_{k=0}^K x^k N_k$, where each N_k is the generator of a spatially homogeneous process with i.i.d. increments (i.e. a Lévy process) on $\mathbf{R}_+$, which is given therefore by the Lévy-Khintchine formula with the Lévy measure having support in $\mathbf{R}_+$. We conclude that the study of measure-valued limits of process with k-nary interaction lead us to the study of Feller process having non-local pseudo-differential generators with increasing (at least polynomially, if K is finite) coefficients (more precisely, with polynomially increasing symbols). At the moment, it seems that there are almost no rigorous general results on processes with generators of this kind, even in case of finite-dimensional process in Euclidean spaces. The known results are devoted essentially to the bounded symbols (see e.g. review [JS]).

This paper is devoted to (i) the problems of existence, uniqueness, non-explosion, stochastic monotonicity and heat kernel estimations of one-dimensional Markov processes having non-local generators with coefficients of polynomial growth, (ii) to the proof of rigorous results on convergence of random evolution of systems of identical particles with k-nary interaction (given by generators of type (1.2)) to the processes of this kind.

2. Formulation of the main result. We shall denote by $C[0, \infty]$ the Banach space of continuous bounded functions on $(0, \infty)$ having limits as $x \to 0$ and as $x \to \infty$ (with the usual sup-norm). We shall also use the closed subspaces $C_0[0, \infty]$ or $C_\infty[0, \infty]$ of $C[0, \infty]$ consisting of functions such that $f(0) = 0$ or $f(\infty) = 0$ respectively, and a dense subspace $\tilde{C}[0, \infty]$ that is a linear span of constant functions and the set of smooth functions on $[0, \infty)$ with a compact support.

Consider an operator L in $C[0, \infty]$ given by the formula

$$(Lf)(x) = \sum_{k=1}^{K} x^k \left(a_k f''(x) - b_k f'(x) + \int_0^\infty (f(x+y) - f(x) - f'(x)y)\nu_k(dy) \right), \tag{1.4}$$

where K is a natural number, a_k and b_k are real constants, $k = 1, ..., K$, all a_k are non-negative, and all ν_k are Borel measures on $(0, \infty)$ satisfying

$$\int \min(\xi, \xi^2)\nu_k(d\xi) < \infty. \tag{1.5}$$

As a natural domain $D(L)$ of L we take the space of twice continuously differentiable functions $f \in C[0, \infty]$ such that $Lf \in C[0, \infty]$.

To formulate our main result, let us introduce the following notations. Let $k_1 \leq k_2$ (respectively, $l_1 \leq l_2$) denote the bounds of those indexes k where a_k (respectively, b_k) do not vanish, i.e. $a_{k_1} > 0$, $a_{k_2} > 0$ and $a_k = 0$ for $k > k_2$ and $k < k_1$ (respectively, $b_{l_1} \neq 0$, $b_{l_2} \neq 0$ and $b_k = 0$ for $k > l_2$ and $k < l_1$).

Theorem 1.1. *Suppose that*
(i) ν_k vanish for $k < k_1$ and $k > k_2$,
(ii) if $l_2 < k_2$, then $\nu_{k_2} = 0$,
(iii) if $l_1 = k_1 - 1$ and $b_{l_1} = -l_1 a_{l_1}$, then $\nu_{k_1} = 0$,
(iv) $b_{l_2} > 0$ whenever $l_2 \geq k_2 - 1$,
(v) if $l_2 = k_2$, then there exists $\delta > 0$ such that

$$\frac{1}{a_{l_2}} \int_0^\delta \xi^2 \nu_{l_2}(d\xi) + \frac{1}{|b_{l_2}|} \int_\delta^\infty \xi \nu_{l_2}(d\xi) < \frac{1}{4}.$$

Then

(i) if $k_1 > 1$ (respectively $k_1 = 1$), L generates a strongly continuous conservative semigroup on $C[0, \infty]$ (respectively not conservative semigroup on $C_0[0, \infty]$));

(ii) the corresponding process $X_x(t)$ (x denotes a starting point) is stochastically monotone: $P(X_x(t) \geq y)$ is a non-decreasing function of x for any y (where $P(E)$ denotes, as usual, the probability of the event E);

(iii) there exists a dual process $\tilde{X}(t)$ (with a generator given explicitly in Section 6, see (6.9)), whose distribution is connected with the distribution of $X(t)$ by the formula

$$P(\tilde{X}_x(t) \leq y) = P(X_y(t) \geq x). \tag{1.6}$$

Remark. The long list of conditions (i)-(v) in the Theorem is made in order to cover the most reasonable situations where either the diffusion (second order) term or the drift (first order) term of L dominates the jumps and where consequently the perturbation theory can be used for the analysis of L with the jump part considered as a perturbation. As a simple example with all conditions satisfied one can choose an operator (1.4) with $a_1 > 0$, $a_K > 0$, $b_K > 0$ and with $\nu_1 = \nu_K = 0$.

We shall describe now the simplest natural approximation of the Markov process $X(t)$ by systems of interacting particles with k-nary interactions, i.e. by Markov chains with generators of type (1.2), (1.3).

Let the finite measures $\tilde{\nu}_k$ be defined by $\tilde{\nu}_k(dy) = \min(y, y^2)\nu_k(dy)$. Let β_k^1, β_k^2 be arbitrary positive numbers such that $\beta_k^1 - \beta_k^2 = b_k$ and let ω be an arbitrary constant in $(0, 1)$. Consider the operator $G^h = \sum_{k=1}^K G_k^h$ with

$$(G_k^h f)(hn) = h^k C_n^k \big[\frac{a_k}{h^2}(f(hn + h) + f(hn - h) - 2f(hn))$$

$$+\frac{\beta_k^1}{h}(f(hn+h)-f(hn))+\frac{\beta_k^2}{h}(f(hn-h)-f(hn))$$

$$+\sum_{l=[h^{-\omega}]}^{\infty}\big(f(nh+lh)-f(nh)+lh\frac{f(nh-h)-f(nh)}{h}\big)v_k(l,h)\big],$$

where $[h^{-\omega}]$ denotes the integer part of $h^{-\omega}$ and where

$$v_k(l,h)=\max\left(\frac{1}{hl},\frac{1}{h^2l^2}\right)\tilde{\nu}_k[lh,lh+h).$$

Theorem 1.2. *For any $h>0$, under the assumptions of Theorem 1.1, there exists a unique (and hence non-explosive) Markov chain $X^h(t)$ on $h\mathbf{Z}_+$ with the generator G^h given above. If the initial point nh of this chain tends to a point $x\in\mathbf{R}_+$ as $h\to 0$, then the process $X^h_{nh}(t)$ converges, as $h\to 0$, in the sense of distributions, to the process $X_x(t)$ from Theorem 1.1. The convergence also holds in the sense of the strong convergence of the corresponding semigroups (i.e. in the sense of Theorem 6.1, from Ch. 1 of [EK]).*

Remarks. 1. An approximating interacting particle system for a given process on $\mathbf{R}_+$ is by no means unique. The essential features of the approximations are the following: (1) k-nary interaction corresponds to pseudodifferential generators $L(x,\frac{\partial}{\partial x})$ being polynomials of degree k in x, and requires the common scaling of order h^k, (2) acceleration of small jumps ($g^k_m(h)$ in (1.3) of order h^{-2} for small $|m|$) gives rise to a diffusion term, (3) slowing down of large jumps gives rise to non-local terms of the limiting generator. 2. We have taken all $k\geq 1$ in (1.4) thus excluding from the corresponding approximating particle systems the processes with spontaneous (or external) inputs that are described by a term $k=0$ in (1.1). This is done just in order to simplify the formulation of our theorems.

Theorems 1.1 and 1.2 are proved in Section 6. The methods developed in the paper allow to obtain other similar results with various assumptions on the coefficients of operator (1.4) and the moments of the measures ν_k. For example, for the case of Lévy measures having a finite first moment, the corresponding conditions on the coefficients can be taken from Theorem 4.1 of Section 4.

3. Content of the paper. The preliminary Section 2 is devoted to a short exposition of some known facts from the analytic theory of one-dimensional diffusions that we shall need. Namely, we shall consider the operator

$$A=a(x)\frac{d^2}{dx^2}+b(x)\frac{d}{dx}\tag{1.7}$$

on $(0,\infty)$, where a and b are smooth functions on $(0,\infty)$ (a is everywhere strictly positive) and describe an explicit construction of its resolvent, which depends crucially on the asymptotic behavior of a and b near 0 and ∞.

In Section 3 we start with some general estimates on the resolvents of operators (1.7) and then apply these estimates to a particular case of (1.7), where $a(x)$ and $b(x)$ behave like some powers of x as $x\to\infty$ or $x\to 0$. In particular, we give a complete classification of the boundary points and rather precise estimates for the resolvents of such operators.

The main example which is related to the interacting particle systems is given by the case of a and b being polynomials:

$$a(x) = \sum_{j=k_1}^{k_2} a_j x^j, \quad b(x) = -\sum_{j=l_1}^{l_2} b_j x^j, \tag{1.8}$$

where $1 \le k_1 \le k_2$, $1 \le l_1 \le l_2$, $a_{k_1} > 0$, $a_{k_2} > 0$, $b_{l_1} \neq 0$, $b_{l_2} \neq 0$, and where all a_j are non-negative.

Section 4 is devoted to a study of Markov processes on $(0, \infty)$ with polynomial pseudodifferential generators, i.e. the generators of the form $L = A + N$, where A is given by (1.7), (1.8) and where

$$Nf = \sum_{k=l}^{K} x^k N_k f \tag{1.9}$$

with non-local (Lévy type) operators

$$N_k f(x) = \int (f(x+\xi) - f(x) - f'(x)\xi)\nu_k(d\xi), \tag{1.10}$$

and where all ν_k are Borel measures on $(0, \infty)$ satisfying (1.5). If ν_k satisfy stronger assumptions $\int \xi \nu_k(d\xi) < \infty$, one considers usually simpler nonlocal operators N_k of the form

$$N_k f(x) = \int (f(x+\xi) - f(x))\nu_k(d\xi). \tag{1.11}$$

We give criteria on existence, uniqueness and non-explosion for processes with such generators.

In Section 5, we study Markov chains with generators of type (1.1) discussing the questions of uniqueness, monotonicity and of the construction of the dual chain.

In Section 6 we combine the results on one-dimensional Feller processes obtained in Sections 2 - 4 with simple results on interacting particle systems obtained in Section 5 and prove Theorems 1.1 and 1.2. In particular, we shall use the monotonicity and duality results for interacting particle systems to prove the corresponding results on monotonicity and to construct duals for the processes on $\mathbf{R}_+$ with generators having coefficients of polynomial growth. The results on duality give a rigorous meaning (in the framework considered) of the well known idea of duality between fragmentation and coagulation processes as discussed e.g. in [Al].

We have studied the semigroups of our processes by their resolvents. An important problem is to estimate their transition probability densities. As the processes with nonlocal generators were studied by perturbation arguments from diffusions it is natural to start with the analysis of transition probability densities (also called heat kernels or Green functions) of diffusion operators having polynomial coefficients. This seems to be a difficult problem, which is of independent interest. In Appendix we put forward an approach to this problem based on the direct construction of the asymptotic probability density using semiclassical approximation (see [Kol2], [Kol3] and references therein for systematic exposition of this method for Feller processes in case of local and non-local generators with bounded coefficients). Notice that a familiar method of frozen coefficients widely used for asymptotic construction of the Green function of differential equations seems to be not very promising for rapidly growing coefficients.

We consider only the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} a x^2 \frac{\partial^2 u}{\partial x^2} - \frac{1}{2}(bx + cx^2)\frac{\partial u}{\partial x} \tag{1.12}$$

with $b > 0$ and $c > 0$ (in case $c = 0$, this is an exactly solvable model, so called Black-Scholes diffusion, in case $c < 0$, the corresponding process explodes in finite time almost surely). This equation describes a diffusion approximation to particle systems with pairwise interaction, which is the simplest and the mostly used type of interaction. As a corollary of the general results of Section 6 it follows that the process $X(t)$ on $(0, \infty)$ corresponding to (1.12) is stochastically monotone and has a dual process defined by the evolution equation

$$\frac{\partial \tilde{u}}{\partial t} = \frac{1}{2} a x^2 \frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{1}{2}(bx + 2ax + cx^2)\frac{\partial \tilde{u}}{\partial x}.$$

We shall construct an asymptotic Green function (or a heat kernel) for equation (1.12).

This asymptotics can be used to construct the processes with non-local generators (and their transition probabilities) using perturbation techniques directly in terms of semigroups and not in terms of resolvents as in Section 4. One can hope that this method can be generalized to at least some multi-dimensional generators.

It seems worth noticing here that in order to see how the heat kernel for a diffusion with polynomial coefficients must look like, one can observe that the equation

$$\frac{\partial u}{\partial t} = \frac{h}{2} x^4 \frac{\partial^2 u}{\partial x^2}, \quad h > 0,$$

(describing a diffusion on $(0, \infty)$ having an inaccessible entrance boundary at infinity (the classification of boundary points is recalled in Section 2)) has an exact heat kernel of the form

$$\frac{1}{\sqrt{2\pi h t}} \frac{x}{x_0^3} \exp\left\{ -\frac{1}{2th} \left(\frac{1}{x} - \frac{1}{x_0} \right)^2 \right\}.$$

4. Other related models. (1) Systems with cancellation (particle-antiparticle systems) and Feller processes on $\mathbf{R}$. So far we have considered processes on $\mathbf{Z}_+$ and their limiting processes on $\mathbf{R}_+$. Similarly one can obtain processes on $\mathbf{R}$ as limits of particle-antiparticle interacting systems on $\mathbf{Z}$. Namely, let us consider a model with two type of particles, a particle and an antiparticle, that can not coexist (they annihilate simultaneously). Then a state of the system is described by an integer number $n \in \mathbf{Z}$, where $|n|$ is interpreted as the number of particles if $n > 0$ and as the number of antiparticles if $n < 0$. The process of k-nary interaction is described as follows. Any group of k particles from a given family of $|n|$ particles or antiparticles can produce a random number $m \in \mathbf{Z}$ of offspring (m particles if $m > 0$ or $|m|$ antiparticles if $m < 0$) thus changing the the state from n to

$n+m$ (cancellation is taken into account). Considering then the same limiting procedure as above one arrives at Feller processes on the whole line with polynomial generators.

(2) Putting $K = \infty$ in (1.4) one can get similar results for some pseudodifferential generators of Markov processes with analytic symbols.

(3) If the measures ν_k in (1.4) do not satisfy (1.8), but the weaker condition

$$\int \min(1, \xi^2) \nu_k(d\xi) < \infty,$$

the existence of the process and of the approximating particle system can still be obtained. However, it requires an additional correcting term in the approximating generators G_k^h in case of ∞ being an entrance boundary.

5. Conclusions and future work.

This paper describes a $\mathbf{R}_+$-valued limit of a re-scaled number of particles under k-nary interaction generalizing the famous continuous time Galton-Watson model (in particular, Feller diffusion) that corresponds to the case of $K = 1$ in (1.4). Notice that $\mathbf{R}_+$ can be considered as the space of measures on a one-point set. In this sense, the Galton-Watson process is considered as a simplest one-dimensional (measure-valued) superprocess (see [Dyn]), the general (infinite-dimensional) superprocess being obtained formally by the same limiting procedure as the Galton-Watson process. Similarly, the generalization of the Galton-Watson model considered in this paper is important for the author not only in its own right, but as a simplest (one-dimensional) toy model of a quite general limiting procedure that leads to bona fide (infinite dimensional) measure-valued processes which constitute a far reaching generalization of superprocesses. These more general processes will be studied in the next publication of this series (see [Kol4]). The deterministic versions of these infinite dimensional processes have been obtained formally in [Be], [BK].

The one-dimensional case considered here deserves a special treatment, because, on the one hand, this is a natural first step in the study of general models, and on the other hand, a beautiful analytic theory of one-dimensional diffusions is available that gives an explicit construction of their resolvents, which in turn allows for the construction of processes with non-local generators by means of perturbation theory giving a full description of these processes.

2. Preliminaries: analytic treatment of one-dimensional diffusions.

We recall here some known facts from analytic theory of one-dimensional diffusion equations developed essentially in [Fe] and [Hi]. Our exposition is based on the Chapter 2 of book [Man], where one can find proofs of all the results presented here.

We shall study the properties of the operator (1.2) considered as an unbounded operator in the Banach space $C[0,\infty]$ with the domain $D(A)$ being the set of twice continuously differentiable functions $f \in C[0,\infty]$ such that $Af \in C[0,\infty]$.

Let a and b be infinitely smooth functions on $(0,\infty)$ such that a is everywhere strictly positive. Let

$$B(x) = \int_1^x b(y)(a(y))^{-1}\, dy$$

and let functions m and p on $(0,\infty)$ be defined by the requirements that $p(1)=m(1)=0$ and that

$$m'(x)=(a(x))^{-1}e^{B(x)}, \quad p'(x)=e^{-B(x)}. \tag{2.1}$$

Then the diffusion operator (1.7) can be written in the form

$$A=a(x)\frac{d^2}{dx^2}+b(x)\frac{d}{dx}=a(x)e^{-B(x)}\frac{d}{dx}\left(e^{B(x)}\frac{d}{dx}\right)=D_mD_p,$$

where

$$D_m=(m'(x))^{-1}\frac{d}{dx}, \quad D_p=(p'(x))^{-1}\frac{d}{dx}.$$

The classification of the boundary points is given in terms of the following non-negative functions

$$u^1(x)=\int_1^x m(s)\,dp(s), \quad v^1(x)=\int_1^x p(s)\,dm(s).$$

Definition. *The boundary point* 0 *(respectively* ∞*) is called accessible boundary of* D_mD_p *if* $u^1(0)<\infty$ *(respectively* $u^1(\infty)<\infty$*), and inaccessible otherwise. An inaccessible boundary, say* ∞*, is called an entrance boundary if* $v^1(\infty)<\infty$*, and a natural boundary if* $v^1(\infty)=\infty$*. An accessible boundary, say* ∞*, is called regular, if* $v^1(\infty)<\infty$*, and it is called an exit boundary if* $v^1(\infty)=\infty$*.*

Notice that a boundary is regular if both p and m are finite in a neighbourhood of the boundary.

Example. If $b(x)$ vanishes in (1.2), then $p(x)=x-1$, $m'(x)=1/a(x)$, and ∞ is always inaccessible. It is a natural or an entrance boundary if the integral $\int_1^\infty (x/a(x))\,dx$ is infinite or finite, respectively.

Let a sequence of functions $u^n(x)$ on $(0,\infty)$ be defined inductively by

$$u^0(x)=1, \quad u^{n+1}(x)=\int_1^x\int_1^y u^n(s)\,dm(s)\,dp(y), \quad n=1,2,...,$$

Proposition 2.1.
(i) the series

$$u(x,\lambda)=\sum_{n=0}^\infty \lambda^n u^n(x)$$

is convergent for all λ*;*

(ii) for $\lambda>0$ *the functions*

$$u_+(x,\lambda)=u(x,\lambda)\int_x^\infty u(y,\lambda)^{-2}\,dp(y), \quad u_-(x,\lambda)=u(x,\lambda)\int_0^x u(y,\lambda)^{-2}\,dp(y)$$

are well defined for $x\in(0,\infty)$*, and* $u(x,\lambda),u_+(x,\lambda),u_-(x,\lambda)$ *are non-negative solutions of the homogeneous equation*

$$\lambda w=\left(a(x)\frac{d^2}{dx^2}+b(x)\frac{d}{dx}\right)w=D_mD_pw;$$

(iii) for $\lambda > 0$, $D_p u_+$ and $D_p u_-$ are increasing functions of x and moreover

$$D_p u_+(x,\lambda) = \lambda \int_1^x u_+(y,\lambda)\,dm(y) - 1 \leq 0,$$

$$D_p u_-(x,\lambda) = \lambda \int_1^x u_-(y,\lambda)\,dm(y) + 1 \geq 0$$

for all x; this implies, in particular, that u_+ (respectively u_-) is a decreasing (respectively, increasing) function of x, and that the finite limits $D_p u_+(\infty,\lambda)$ and $D_p u_-(0,\lambda)$ exist;

(iv) if 0 (respectively, ∞) is inaccessible, then $D_p u_-(0,\lambda) = 0$ ($D_p u_+(\infty,\lambda) = 0$, respectively).

Let us introduce the Wronskian

$$W = W_\lambda[u_-,u_+] = (D_p u_-(x,\lambda))u_+(x,\lambda) - (D_p u_+(x,\lambda))u_-(x,\lambda). \tag{2.2}$$

By direct differentiation one shows that $W = W_\lambda[u_-,u_+]$ does not depend on x. From Proposition 2.1 (iii) it follows that $W > 0$. Let us now define an operator $\mathcal{L}_\lambda$ in $C[0,\infty]$ by the formula

$$\mathcal{L}_\lambda f(x) = W^{-1}u_+(x) \int_0^x u_-(y)f(y)\,dm(y) + W^{-1}u_-(x) \int_x^\infty u_+(y)f(y)\,dm(y). \tag{2.3}$$

Proposition 2.2. *For an arbitrary $f \in C[0,\infty]$, the function $F = \mathcal{L}_\lambda f$ is twice continuously differentiable and satisfies the non-homogeneous equation*

$$\lambda F - D_m D_p F = f. \tag{2.4}$$

Moreover, if 0 (respectively, ∞) is accessible, then $F(0) = 0$ (respectively, $F(\infty) = 0$). If 0 (respectively, ∞) is a natural boundary, then $F(0) = \lambda^{-1}f(0)$ (respectively, $F(\infty) = \lambda^{-1}f(\infty)$).

Proposition 2.3. *(i) $D_m D_p$ is the generator of a strongly continuous semigroup on $C[0,\infty]$ and hence of Markov process on $(0,\infty)$ if and only if both 0 and ∞ are inaccessible boundaries. In this case, formula (2.3) defines the resolvent of $D_m D_p$, i.e. $\mathcal{L}_\lambda = (\lambda - D_m D_p)^{-1}$ and $F = \mathcal{L}_\lambda f$ defines the unique bounded solution of equation (2.4). In particular, $\mathcal{L}_\lambda \mathbf{1} = 1/\lambda$, where $\mathbf{1}$ denotes the function that equals 1 identically.*

(ii) If 0 is accessible and ∞ is inaccessible (respectively 0 is inaccessible and ∞ is accessible), then $D_m D_p$ generates a strongly continuous contraction semigroup in the space $C_0[0,\infty]$ (respectively, $C_\infty[0,\infty]$). If both 0 and ∞ are accessible, then $D_m D_p$ generates a contraction semigroup in $C_\infty[0,\infty] \cap C_0[0,\infty]$. In all these cases, formula (2.3) still defines the resolvent, but the corresponding process is only sub-Markovian (i.e. non-conservative). In particular, $\lambda \mathcal{L}_\lambda \mathbf{1} \leq 1$, but does not equal 1 identically.

3. Estimates for the resolvent of one-dimensional diffusion operators.

The following is the key estimate for the derivatives of the resolvent of a one-dimensional diffusion operator that we shall need.

Proposition 3.1. *For an arbitrary $f \in C[0,\infty]$*

$$|\frac{d}{dx}\mathcal{L}_\lambda f(x)| \le 2\|f\|p'(x)(|m(x)| + \frac{C}{\sqrt{\lambda}}), \tag{3.1}$$

$$|\frac{d^2}{dx^2}\mathcal{L}_\lambda f(x)| \le \|f\|(2(a(x))^{-1} + |p''(x)|D_p(\mathcal{L}_\lambda f)(x)) \tag{3.2}$$

hold with some constant C and $\lambda > 1$. Moreover, if ∞ or 0 is inaccessible, then also

$$|\frac{d}{dx}\mathcal{L}_\lambda f(x)| \le 2\|f\|p'(x)\int_x^\infty m'(y)\,dy, \tag{3.3}$$

or respectively

$$|\frac{d}{dx}\mathcal{L}_\lambda f(x)| \le 2\|f\|p'(x)\int_0^x m'(y)\,dy. \tag{3.4}$$

Proof. From (2.4) on gets by integration that for an arbitrary positive x_0

$$F'(x) = \frac{d}{dx}\mathcal{L}_\lambda f(x) = p'(x)\left(\int_{x_0}^x m'(y)(\lambda F(y) - f(y))\,dy + (D_pF)(x_0)\right).$$

This implies

$$|\frac{d}{dx}\mathcal{L}_\lambda f(x)| \le p'(x)\left(2\|f\|\int_{x_0}^x |m'(y)|\,dy + |(D_pF)(x_0)|\right) \tag{3.5}$$

for $\lambda\mathcal{L}_\lambda \mathbf{1} \le 1$. If ∞ or 0 is inaccessible one takes x_0 to be ∞ or 0 respectively in (3.5). Using Proposition 2.1 (iv), one obtains (3.3) or (3.4) respectively. To get (3.1), one takes $x_0 = 1$ in (3.5) and one needs to prove only that

$$(D_pF)(1) \le C\|f\|/\sqrt{\lambda}.$$

To do this let us differentiate (2.3) to find that

$$(\mathcal{L}_\lambda f)'(x) = W^{-1}u_+'(x)\int_0^x u_-(y)f(y)\,dm(y) + W^{-1}u_-'(x)\int_x^\infty u_+(y)f(y)\,dm(y).$$

Using the estimates

$$\int_x^\infty u_+(y,\lambda)\,dm(y) \le \frac{W}{\lambda}(u_-(x,\lambda))^{-1}, \quad \int_0^x u_-(y,\lambda)\,dm(y) \le \frac{W}{\lambda}(u_+(x,\lambda))^{-1}.$$

(which are due to $\mathcal{L}_\lambda \mathbf{1} \le 1/\lambda$ and (2.3)) and the fact that $|D_pu_+(1)| = |D_pu_-(1)| = 1$ (see Proposition 2.1 (iii)), yields

$$(D_pF)(1) \le \|f\|\frac{1}{\lambda}\left(\frac{1}{u_+(1,\lambda)} + \frac{1}{u_-(1,\lambda)}\right). \tag{3.6}$$

Next, from the definition of $u^n(x)$ and $u(x,\lambda)$ it follows that there are constants C_1, C_2 such that

$$\frac{1}{(2n)!}C_1^{2n}(x-1)^{2n} \le u^n(x) \le \frac{1}{(2n)!}C_2^{2n}(x-1)^{2n}$$

and hence

$$C_1 \exp\{C_1\sqrt{\lambda}(x-1)\} \le u(x,\lambda) \le C_2 \exp\{C_2\sqrt{\lambda}(x-1)\}$$

for $x \in (1/2, 2)$ (in particular, $u(1,\lambda) = 1$). Consequently

$$u_+(1,\lambda) \ge C\int_1^2 (u(y,\lambda))^{-2}\, dy \ge \frac{C}{C_2^2}\int_0^1 e^{-C_2\sqrt{\lambda}x}\, dx \ge \tilde{C}/\sqrt{\lambda}$$

for $\lambda > 1$ with some $\tilde{C}$, and similar estimate holds for $u_-(1,\lambda)$. These estimates and (3.6) imply the required estimate for $(D_pF)(1)$ thus completing the proof of (3.1). At last, the identity $d^2/dx^2 = m'p'D_mD_p + p''D_p$ and equation (2.4) imply

$$(\mathcal{L}_\lambda f)''(x) = m'p'(\lambda\mathcal{L}_\lambda f - f) + p''D_p(\mathcal{L}_\lambda f),$$

which implies (3.2).

Proposition 3.2. *Let A be given by (1.7) and suppose that real numbers k, l, α, β are given with $\alpha > 0$, $\beta \ne 0$ such that as $x \to \infty$*

$$a(x) = \alpha x^k(1 + O(x^{-1})), \quad b(x) = -\beta x^l(1 + O(x^{-1})),$$

and these estimates can be differentiated, i.e. $a'(x) = O(x^{k-1})$ and $b'(x) = O(x^{l-1})$ as $x \to \infty$. Then

(i) ∞ is inaccessible natural boundary if and only if $\max(l, k-1) \le 1$;

(ii) suppose $\max(l, k-1) > 1$; (1) if $l < k-1$, then ∞ is inaccessible entrance boundary, (2) if $l > k-1$ and $\beta > 0$ (respectively $\beta < 0$) then ∞ is inaccessible entrance (accessible exit, respectively), (3) if $l = k-1$ (it implies, in particular, that $k > 2$) and $1-k < \beta/\alpha < -1$, then ∞ is accessible regular, (4) if $l = k-1$ and $\beta/\alpha \le 1-k$, then ∞ is accessible exit, (5) if $l = k-1$ and $\beta/\alpha \ge -1$, then ∞ is inaccessible entrance boundary;

(iii) if $l < k-1$, then as $x \to \infty$, $\lambda \to \infty$,

$$|\frac{d}{dx}\mathcal{L}_\lambda f(x)| \le \|f\|\frac{2}{\alpha}(1 + O(x^{-1}) + O(\lambda^{-1/2})) \times \begin{cases} x^{1-k}/|k-1|, & k \ne 1, \\ \ln x, & k = 1 \end{cases}, \tag{3.7}$$

and

$$|\frac{d^2}{dx^2}\mathcal{L}_\lambda f(x)| \le \frac{2\rho}{\alpha}\|f\|x^{-k}(1 + o(1))); \tag{3.8}$$

with $\rho = 1$;

(iv) if $l > k-1$, then as $x \to \infty$, $\lambda \to \infty$, one has (3.8) with $\rho = 2$ and

$$|\frac{d}{dx}\mathcal{L}_\lambda f(x)| \le \frac{2}{|\beta|}x^{-l}\|f\|(1 + O(x^{-1}) + O(\lambda^{-1/2})), \tag{3.9}$$

(term $O(\lambda^{-1/2})$ can be omitted for $\beta > 0$);

(v) if $l = k-1$ and $1-k < \beta/\alpha < -1$, then (3.8) holds with $\rho = 1$ and

$$|\frac{d}{dx}\mathcal{L}_\lambda f(x)| \leq 2\|f\|x^{\beta/\alpha}m(\infty)(1 + O(x^{-1}) + O(\lambda^{-1/2})); \tag{3.10}$$

(vi) if $l = k-1$, but conditions of (v) do not hold, then

$$|\frac{d}{dx}\mathcal{L}_\lambda f(x)| \leq \|f\|\frac{2}{\alpha}x^{1-k}(1 + O(x^{-1}) + O(\lambda^{-1/2})) \times \begin{cases} |1-k-\beta/\alpha|^{-1}, & 1-k \neq \beta/\alpha, \\ \ln x, & 1-k = \beta/\alpha \end{cases} \tag{3.11}$$

and (3.8) holds with $\rho = 1 + |1-k-\beta/\alpha|^{-1}$ (respectively with $\rho = 1 + \ln x$) if $1-k \neq \beta/\alpha$ (respectively otherwise).

Proof. This consists of tedious but explicit calculations of all asymptotics first classifying the boundary point ∞ and then estimating the derivatives of the resolvent by means of (3.2),(3.3) if ∞ is inaccessible and $m(\infty) < \infty$, and (3.2),(3.1) otherwise.

(i) Suppose $l < k-1$. Then B is finite at infinity and $B(x) = B(\infty) + O(x^{l-k+1})$ as $x \to \infty$. Next, $p'(x) \to e^{-B(\infty)}$ and $m'(x) \sim \alpha^{-1}x^{-k}e^{B(\infty)}$, which implies that $u^1(\infty) = \infty$ and thus ∞ is inaccessible. Since $p \sim xe^{-B(\infty)}$, v^1 behaves like $\int x^{1-k}dx$ and is finite if and only if $k > 2$. Hence ∞ is natural for $k \leq 2$ and entrance for $k > 2$. At last, m is finite at infinity if and only if $k > 1$. Hence one uses (3.1) for $k \leq 1$ and (3.3) for $k > 1$ and obtains (3.7). Estimate (3.8) is obtained from (3.2), because the first term on the r.h.s. of (3.2) gives precisely the estimate (3.8) and the second term is of the lower order.

(ii) Suppose $l > k-1$ and $\beta > 0$. Then

$$B(x) = -\frac{\beta}{\alpha(l-k+1)}x^{l-k+1}(1 + O(\frac{1}{x})),$$

as $x \to \infty$. Hence $m(\infty) < \infty$, $p'(\infty) = p(\infty) = \infty$ and $u^1(\infty) = \infty$, and therefore ∞ is inaccessible and one can use (3.3) to estimate the resolvent and to get (3.9), because

$$\int_x^\infty m'(y)\,dy = -\frac{1}{\alpha}x^{-k}(B'(x))^{-1}e^{B(x)}(1 + O(x^{-1})),$$

$$p'(x)\int_x^\infty m'(y)\,dy = \frac{1}{\beta}x^{-l}(1 + O(x^{-1})).$$

At last, v^1 behaves like $\int x^{-l}\,dx$ and is finite if and only if $l > 1$ and consequently ∞ is a natural or an entrance boundary for $l \leq 1$ or $l > 1$ respectively. On the r.h.s. of (3.2) both terms have now the same estimate, which gives (3.8) with $\rho = 2$.

(iii) Suppose $l > k-1$ and $\beta < 0$. The same formula for B as above in (ii) holds, but now $p'(\infty) = 0$, $m(\infty) = m'(\infty) = \infty$ and one uses (3.1), (3.2) to get (3.9). Since p tends to a finite constant as $x \to \infty$, v^1 behaves like m at infinity and thus $v^1(\infty) = \infty$. At last, u^1 behaves like $\int x^{-l}\,dx$ and is finite if and only if $l > 1$. Hence ∞ is an inaccessible natural boundary for $l \leq 1$ and an accessible exit boundary for $l > 1$.

(iv) Suppose $l = k - 1$. Then

$$B(x) = -\frac{\beta}{\alpha}\ln x + O(x^{-1}), \quad m'(x) = \frac{1}{\alpha}x^{-k-\beta/\alpha}(1+O(x^{-1})), \quad p'(x) = x^{\beta/\alpha}(1+O(x^{-1}))$$

as $x \to \infty$. (1). If $1 - k < \beta/\alpha < -1$ (this can happen only for $k > 2$), then both p and m are finite at infinity. Hence u^1 and v^1 behaves like p and m respectively and thus are both finite at infinity. Hence ∞ is an accessible regular boundary. Using (3.1) yields (3.10). Again (3.8) holds with $\rho = 1$, because only the first term on the r.h.s. of (3.2) plays the role. (2). If $-1 \le \beta/\alpha \le 1 - k$ (this can happen only for $k \le 2$), then $p(\infty) = m(\infty) = \infty$ and $v^1(\infty) = u^1(\infty) = \infty$. Hence ∞ is inaccessible natural boundary and using (3.1) yields (3.11). (3). If $\beta/\alpha < -1$ and $\beta/\alpha \le 1 - k$, then $p(\infty) < \infty$, $m(\infty) = \infty$ and again using (3.1) yields (3.11). Next v^1 behaves like m and tends to infinity as $x \to \infty$. At last u^1 behaves like $\int x^{1-k}\,dx$ (may be with an additional multiplier $\ln x$) and is finite if and only if $k > 2$. Hence ∞ is an inaccessible natural for $k \le 2$ and an accessible exit for $k > 2$. (4). If $\beta/\alpha \ge -1$ and $\beta/\alpha > 1 - k$, then $p(\infty) = \infty$, $m(\infty) < \infty$, u^1 behaves like p and thus tends to infinity as $x \to \infty$. Hence ∞ is inaccessible and one gets (3.11) using (3.3). At last, v^1 behaves like $\int x^{1-k}\,dx$ and is finite for $k > 2$ (implying ∞ is an entrance boundary) and tends to infinity as $x \to \infty$ for $k \le 2$ (implying that ∞ is a natural boundary).

Proposition 3.3. *Let A be given by (1.7) and suppose that real numbers k, l, α, β are given with $\alpha > 0$, $\beta \ne 0$ such that as $x \to 0$*

$$a(x) = \alpha x^k(1 + O(x)), \quad b(x) = -\beta x^l(1 + O(x)).$$

(i) 0 is inaccessible natural boundary if and only if $\min(l, k-1) \ge 1$*;*

(ii) suppose $\min(l, k-1) < 1$*; (1) if $l < k - 1$ and $\beta < 0$ (respectively, $\beta > 0$), then 0 is an inaccessible entrance boundary (respectively, accessible exit); (2) if $l > k - 1$, then 0 is accessible, regular for $k < 1$ and exit for $1 \le k < 2$; (3) if $l = k - 1$ (and hence $k < 2$) and $-1 < \beta/\alpha < 1 - k$, then 0 is accessible regular, (4) if $l = k - 1$ and $\beta/\alpha \ge 1 - k$, then 0 is accessible exit, (5) if $l = k - 1$ and $\beta/\alpha \le -1$, then 0 is inaccessible entrance boundary;*

(iii) if $l > k - 1$ and $k \ge 1$, one has (3.7), (3.8) with $\rho = 1$ as $x \to 0$ with $O(x^{-1})$ replaced by $O(x)$; if $l > k-1$ and $k < 1$, one has the estimate $(\mathcal{L}_\lambda f)'(x) = O(1)+O(\lambda^{-1/2})$;

(iv) if $l < k - 1$, then (3.8) with $\rho = 2$ and (3.9) hold as $x \to 0$ with $O(x^{-1})$ replaced by $O(x)$;

(v) if $l = k - 1$ and $-1 < \beta/\alpha < 1 - k$, then (3.10) holds as $x \to 0$ with $O(x^{-1})$ replaced by $O(x)$;

(vi) if $l = k - 1$ and conditions of (v) do not hold, then, as $x \to 0$, (3.11) holds and (3.8) holds with ρ being the same as in Proposition 3.2 (vi) (again $O(x^{-1})$ is replaced by $O(x)$ everywhere).

Proof. Is the same as that of Proposition 3.2. Let us consider only the case $l > k - 1$. Then B is finite near the origin and

$$B(x) = B(0) + O(x), \quad p'(x) = e^{-B(0)}(1 + O(x)), \quad m'(x) = \frac{1}{\alpha}x^{-k}e^{B(0)}(1 + O(x))$$

as $x \to 0$. Therefore $m(0) < \infty$ if and only if $k < 1$, v^1 behaves like m and hence $v^1(0) < \infty$ if and only if $k < 1$. Next, u^1 behaves like the integral of m and is finite if and only if $k < 2$. We conclude that 0 is inaccessible natural for $k \geq 2$, accessible exit for $1 \leq k < 2$ and accessible regular for $k < 1$. Moreover, if $k \geq 1$, we get (3.7) (with $O(x^{-1})$ replaced by $O(x)$), using (3.4) for $k \geq 2$ and using (3.1) for $1 \leq k < 2$. If $k < 1$, we use (3.1) to get the boundedness of $(\mathcal{L}_\lambda f)'(x)$.

4. One-dimensional Feller processes with non-local generators.

When considering a non-local term of a generator as a perturbation of a local operator one finds that the allowed growth of the symbol of a non-local term with respect to the space variable depends on the moments of the corresponding Lévy measures. We demonstrate this connection by proving here a pair of results on the existence and uniqueness of one-dimensional Feller processes with non-local generators using perturbation theory and the estimates from the previous section. One result is valid for Lévy measures with a finite first moment and the other for general Lévy measures. The conditions of Theorems 4.1 and 4.2 below are designed in the attempt to present reasonably general but still not too overcomplicated assumptions that ensure that respectively the drift or the diffusion term in L dominates the jumps. As we mentioned in the introduction, the simplest example when all conditions of Theorem 4.2 hold is given by an operator (1.4) with $a_1 > 0$, $a_K > 0$, $b_K > 0$ and with $\nu_1 = \nu_K = 0$. Other similar results can be obtained under various assumptions on k_j, l_j and the moments of the Lévy measures ν_j.

First let us give a classification of boundary points for operators of type $A + N$ with A given by (1.7) and N being a non-local operator of the Lévy type, for example of form (1.4).

Definition. *Let $L = A + N$, where A is given by (1.7) and N is a non-local operator of type (1.9) (or with a more general dependence on x). Let us say that both ∞ and 0 are inaccessible boundary points for L if (the closure of) L (defined on twice continuously differentiable functions $f \in C[0,\infty]$ such that $Lf \in C[0,\infty]$) generates a conservative strongly continuous semigroup on $C[0,\infty]$. If this is the case, we call ∞ (0, respectively) a natural boundary, if the subspace $C_\infty[0,\infty]$ (respectively, $C_0[0,\infty]$) is invariant under this semigroup, and an entrance boundary otherwise. We say that 0 is accessible and ∞ is inaccessible (respectively ∞ is accessible and 0 is inaccessible) if L generates a non-conservative strongly continuous semigroup on $C_0[0,\infty]$ (respectively, on $C_\infty[0,\infty]$). We say that both 0 and ∞ are accessible if L does not generate a semigroup neither on $C_0[0,\infty]$, nor on $C_\infty[0,\infty]$, but only on their intersection.*

Using the theory of one-dimensional diffusion sketched in Section 2 one can show that this definition is consistent with the classification of boundary points of one-dimensional diffusions given above.

Theorem 4.1. *Let a differential operator A be given by (1.7), (1.8), where $b_{l_2} > 0$ whenever $l_2 = k_2 - 1$. Let N be given by (1.9), (1.11), where Borel measures ν_j do not vanish only for $j = \min(l_1, k_1 - 1), ..., \max(l_2, k_2 - 1)$ and are such that $\int_0^\infty \xi \nu_j(d\xi) < \infty$ for all j and moreover,*

$$\frac{1}{|b_{l_2}|} \int_0^\infty \xi \nu_{l_2}(d\xi) < \frac{1}{2} \tag{4.1}$$

in case $l_2 > k_2 - 1$, or

$$\frac{1}{|a_{l_2}|(k_2-1)} \int_0^\infty \xi \nu_{k_2-1}(d\xi) < \frac{1}{2} \tag{4.2}$$

in case $l_2 \le k_2 - 1$. Moreover, suppose that $\nu_{l_1} = 0$ if $l_1 = k_1 - 1$ and $b_{l_1} = -l_1 a_{l_1}$. Then the operator $A+N$ (defined on the same domain $D(A)$ as A), generates a strongly continuous contraction semigroup on the same space $C[0,\infty]$ (or its subspaces $C_\infty[0,\infty]$, $C_0[0,\infty)$ or $C_\infty[0,\infty] \cap C_0[0,\infty)$) and with the same type of boundary points as the semigroup defined by A, and hence a Markov (or sub-Markov) process on $(0,\infty)$.

Remarks. 1. The assumption $b_{l_2} > 0$ for $l_2 = k_2 - 1$ is made above only to simplify the formulation. The same remark concerns Theorem 4.2 below. 2. Recall (see (1.8) and the description of the coefficients afterwards) that we consider only the case $k_1 \ge 1$ and $l_1 \ge 1$ (which is of major interest from the point of view of interacting particle systems); however, similarly, one can analyze non-positive k_1 and l_1. 3. It is plausible that the constant 1/2 on the r.h.s. of (4.1), (4.2) can be improved to become 1 (using more refined methods, see [Kol4]). However, it seemingly can not be allowed to be more than one, because then the integral term N could dominate the local operator A, which could lead to the explosion.

The proof of the theorem is based on the following technical statement.

Proposition 4.1. *The assumptions of Theorem 4.1 imply the estimate*

$$\|N\mathcal{L}_\lambda\| < 1, \quad \lambda > \Lambda, \tag{4.3}$$

for some Λ.

Proof. Let us consider only the case $l_2 > k_2 - 1$ and $l_1 > k_1 - 1$ (other cases are obtained quite similarly by using the corresponding formulas from Propositions 3.2 and 3.3).

For an arbitrary δ and a continuous function ϕ, one has

$$\int_0^\infty (\phi(x+\xi) - \phi(x))\nu(d\xi) \le \max_{y\in[x,x+\delta]} g'(y) \int_0^\delta \xi\nu(d\xi) + 2\max_{y\ge x} g(y) \int_\delta^\infty \nu(d\xi).$$

Using (3.7) and (3.9) respectively for l_1 and l_2 one gets for any positive x_0 and all j that

$$\max_{x\le x_0} |x^j (N_j \mathcal{L}_\lambda f)(x)|$$

$$\le \|f\| \left[K x_0^{j-(k_1-1)} (1 + O(x) + O(\lambda^{-1/2})) \int_0^\delta \xi\nu_j(d\xi) + \frac{2x_0^j}{\lambda} \int_\delta^\infty \nu_j(d\xi) \right] \tag{4.4}$$

with some constant K and

$$\max_{x\ge x_0} |x^j (N_j \mathcal{L}_\lambda f)(x)| \le \frac{2}{b_{l_2}} \|f\| x_2^{j-l_2} (1 + O(x^{-1}) + O(\lambda^{-1/2})) \int_0^\infty \xi\nu_j(d\xi). \tag{4.5}$$

As $j \le l_2$ and due to (4.1), one can choose x_0 large enough, so that the sum over all j of the r.h.s. of (4.5) does not exceeds $1-\epsilon$ for some $\epsilon > 0$. By choosing δ small enough, one can ensure that the sum of the first terms of the r.h.s. of (4.4) is arbitrary small. At last, by choosing appropriate Λ, one makes the sum of the second terms on the r.h.s. of (4.4) arbitrary small, and hence (4.3) follows.

Proof of Theorem 4.1. Since A generates a semigroup, the statement of the theorem is an immediate consequence of Proposition 4.1, the standard perturbation theory arguments and the Hille-Yosida theorem (see e.g. [EK]), because (4.3) ensures the invertibility of the operator $1 + L\mathcal{L}_\lambda$ and thus the surjectivity of $\lambda - A - L$.

Let us give a similar result for Lévy measures ν_j without a finite first moment.

Theorem 4.2. *Let a differential operator A be given by (1.7), (1.8), where $b_{l_2} > 0$ whenever $l_2 = k_2 - 1$. Let N be given by (1.9), (1.11), where all ν_j satisfy (1.5). Suppose that if $l_2 \geq k_2$ (respectively $l_2 < k_2$), then ν_j do not vanish only for $k_1 \leq j \leq k_2$ (respectively $k_1 \leq j \leq k_2 - 1$); moreover, suppose that if $l_2 = k_2$, then there exists $\delta > 0$ such that*

$$\frac{1}{a_{l_2}} \int_0^\delta \xi^2 \nu_{l_2}(d\xi) + \frac{1}{|b_{l_2}|} \int_\delta^\infty \xi \nu_{l_2}(d\xi) < \frac{1}{4}. \tag{4.6}$$

At last, suppose $\nu_{k_1} = 0$, if $l_1 = k_1 - 1$ and $b_{l_1} \neq -l_1 a_{l_1}$. Then the operator $A + N$ is the generator of a contraction semigroup on the same space as A, and hence of a Markov process on $(0, \infty)$.

Proof. As above, this is a consequence of the following statement.

Proposition 4.2 *The assumptions of Theorem 4.2 imply (4.3) for some Λ.*

Proof. It is quite similar to the above. Consider only the case $l_2 \geq k_2$. By Proposition 3.2 (iv), for an arbitrary x_0 and $\delta > 0$

$$\max_{x \geq x_0} |x^j (N_j \mathcal{L}_\lambda f)(x)| \leq \|f\| [\frac{4}{a_{k_2}} x_0^{j-k_2} \int_0^\delta \xi^2 \nu_j(d\xi) + \frac{4}{|b_{l_2}|} x_2^{j-l_2} \int_\delta^\infty \xi \nu_j(d\xi)], \tag{4.7}$$

(up to the terms of the lower order). For $j < k_2$ this can be made arbitrary small by choosing x_0 large enough. If $j = k_2$ and $l_2 > k_2$, one can choose δ small in such a way that the first term become arbitrary small and then again by choosing x_0 one can make the second term arbitrary small. At last, in case $j = k_2 = l_2$, one take δ from (4.6) to make the r.h.s. of (4.7) to be less than $1 - \epsilon$ with some $\epsilon > 0$. Next, we use the elementary inequality $|F'(x)| \leq B\|F\| + B^{-1} \max_{\xi \geq x} |F''(\xi)|$ (that is valid for any real twice differentiable function F and any number $B > 0$) to estimate the first derivative of the resolvent and hence to obtain from (3.8) (Proposition 3.3.) that

$$\max_{x \leq x_0} |x^j (N_j \mathcal{L}_\lambda f)(x)| \leq K \|f\| [x_0^{j-k_1} (\int_0^\delta \xi^2 \nu_j(d\xi) + \frac{1}{B} \int_\delta^\infty \xi \nu_j(d\xi)) + x_0^j \frac{B}{\lambda} \int_\delta^\infty \nu_j(d\xi)], \tag{4.8}$$

with some constant K. Hence, choosing first small δ, then large B and then large λ we can make sequentially the first, the second and the third term here arbitrary small.

5. K-nary interaction of indistinguishable particles.

Consider a system of identical indistinguishable particles with interaction of k-nary type, $k = 1, ..., K$ (we exclude the simple spontaneous inputs with $k = 0$ for simplicity of some notations), which is a Markov process on $0, 1, ...$ with the generator of the form $G = \sum_{k=0}^{K} G_k$ with G_k given by (1.1). Occasionally we shall use two additional assumptions on the coefficients g_n^k:

(i) subcriticality condition:

$$\sum_{m=-k}^{\infty} g_m^k m \leq 0 \quad \text{for all} \quad k. \tag{5.1}$$

(ii) non-existence of long left jumps:

$$g_m^k = 0 \quad \text{for all} \quad m < -1. \tag{5.2}$$

The restriction (5.2) provides the generators with some additional nice properties (see below), and at the same time in all practically used models of coagulation ($m < 0$ in (1.1)) one usually excludes (see e.g. [Al]) the possibility of the coagulation of more than two particles in one go.

The q-matrix of the process under consideration is clearly defined by

$$Q = \sum_{k=0}^{K} Q^k, \quad Q_{nj}^k = C_n^k g_{j-n}^k, \quad j \neq n \tag{5.3}$$

(in particular, $Q_{nj}^k = 0$ for $n < k$). As usual one defines $Q_n = -Q_{nn} = \sum_{j \neq n} Q_{nj}$. Clearly Q is conservative and a Feller q-matrix (the latter means that $Q_{nj} \to 0$ as $n \to \infty$ for any j). This implies that Q is also a Reuter matrix, i.e. $\sum_{l=j}^{\infty} Q_{nl} \to 0$ as $n \to \infty$ for any j. For unification of some formula we shall consider (5.3) to hold for all j, n by putting $g_0^k = -\sum_{m \neq 0} g_m^k$ (this convention clearly does not change the generator (1.1)).

It is well known that to any conservative q-matrix there corresponds a unique Markov chain, called the minimal Markov chain of q (see e.g. [An]), defined by transition probabilities being the minimal solution of the corresponding Kolmogorov backward equation (or, equivalently, as a Markov chain corresponding to q with a minimum life-time). Let $Z_m(t)$ be the minimal Markov chain starting at m and corresponding to the q-matrix Q from (5.3).

The significance of condition (5.1) is revealed by the following result.

Proposition 5.1 *The subcriticality condition (5.1) implies that the process $Z_m(t)$ is a positive supermartingale for any m.*

Proof. By Dynkin's formula for Markov chains (see e.g. [Br]), the process $f(Z_m(t)) - \int_0^t (Gf)(Z_m(s))\, ds$ is a martingale for any m and all non-negative f, for which the integral is defined and has a finite expectation. Take $f(x) = x$. By (5.1), $Gf(x) \leq 0$ for all x. Hence the expectation of $Z_m(t)$ does not increase in time and $Z_m(t)$ is a supermartingale.

Corollary 5.2. *Under (5.1) the matrix Q defines a unique Markov process, which is a regular jump Markov process without explosions at finite times.*

Proof. Uniqueness follows from non-explosion of the minimal chain, and non-explosion follows from Proposition 5.1 and Doob's martingale inequality.

Proposition 5.3. *If (5.2) holds, then the matrix (5.3) is stochastically monotone, i.e.*

$$\sum_{j\geq l} Q_{nj} \leq \sum_{j\geq l} Q_{n+1,j} \tag{5.4}$$

holds for all n,l such that $l \neq n+1$.

Proof. By linearity, it is enough to prove the statement for each k separately. If $n < k-1$, both sides of (5.4) vanish. If $n = k-1$, then the l.h.s. of (5.4) vanishes, and (5.4) is obvious (for $l \neq n+1 = k$). Thus, we need to prove (6.4) for $n \geq k$. In this case, (5.4) takes the form

$$\sum_{j\geq l} C_n^k g_{j-n}^k \leq \sum_{j\geq l} C_{n+1}^k g_{j-n-1}^k. \tag{5.5}$$

Due to the identity $C_{n+1}^k = C_n^k + C_n^{k-1}$, the r.h.s. of (5.5) equals

$$\sum_{j\geq l-1} C_{n+1}^k g_{j-n}^k = \sum_{j\geq l} C_n^k g_{j-n}^k + \sum_{j\geq l} C_n^{k-1} g_{j-n}^k + C_{n+1}^k g_{l-1-n}^k.$$

Thus (5.5) takes the form

$$k\sum_{j\geq l} g_{j-n}^k + (n+1) g_{l-1-n}^k \geq 0.$$

For $l > n+1$ all terms on the l.h.s. are non-negative. For $l < n$, the sum on the l.h.s. equals k. It remains to consider the case $l = n$, which is equivalent to $-kg_{-1}^k + (n+1)g_{-1}^k \geq 0$, and this clearly holds for $n \geq k$.

As a corollary from Proposition 5.3, the Reuter property of Q (mentioned above), and the general theory (see e.g. Theorem 3.4 and Corollary 4.3 from [An]), we get another uniqueness result (independent on assumption (6.1)):

Corollary 5.4. *Under (5.2), the minimal process corresponding to Q is stochastically monotone, i.e. its transition probabilities $P_{ij}(t)$ are such that $\sum_{j\geq l} P_{nj}(t)$ is a non-decreasing function of n for any l and t. Moreover, the minimal process is the unique stochastically monotone process corresponding to Q.*

Let us recall (see Proposition 4.2 from [An]) that if Q is a stochastically monotone Reuter q-matrix, then $\tilde{Q}$ defined by

$$\tilde{Q}_{nj} = \sum_{l=n}^{\infty} (Q_{jl} - Q_{j-1,l}), \tag{5.6}$$

where $Q_{-1,l} = 0$, is a conservative Feller matrix such that the transition probability $\tilde{P}$ of the minimal Markov process defined by $\tilde{Q}$ is connected with the (unique) stochastic monotone process P defined by Q by the duality equation

$$\sum_{l\leq j} \tilde{P}_{nl}(t) = \sum_{l\geq n} P_{jl}(t), \tag{5.7}$$

which in fact justifies the uniqueness result used in the above corollary.

Proposition 5.5. *If Q is given by (5.3), and if (5.2) holds, then*

$$\tilde{Q}_{i,i+q} = C^k_{i+q} g^k_{-q} + C^{k-1}_{i+q-1} \sum_{l=1}^{\infty} g^k_{l-q} \tag{5.8}$$

for $k - i \le q \le 1$ and vanishes otherwise ($\tilde{Q}_{i,i+q} = g^0_{-q}$ for $k = 0$). In particular,

(i) if $g^k_{-1} = 0$, then $\tilde{Q}_{i,i+q}$ vanishes for q outside $[k-i, 0]$;

(ii) if g^k_l vanishes for $|l| > 1$, then $\tilde{Q}_{i,i+q}$ also vanishes for $|q| > 1$ and

$$\tilde{Q}_{i,i+1} = C^k_i g^k_{-1}, \quad \tilde{Q}_{i,i-1} = C^k_{i-1} g^k_1. \tag{5.9}$$

Proof. From (5.6) and (5.3)

$$\tilde{Q}_{im} = \sum_{l=i}^{\infty} C^k_m g^k_{l-m} - \sum_{l=i+1}^{\infty} C^k_{m-1} g^k_{l-m} = C^k_m g^k_{i-m} + C^{k-1}_{m-1} \sum_{l=i+1}^{\infty} g^k_{l-m}$$

(where we again used $C^k_m = C^k_{m-1} + C^{k-1}_{m-1}$), and (5.8) follows. Bounds for q that allow non-vanishing $\tilde{Q}_{im}$ follow from the bounds for coefficients C^k_n and g^k_n.

6. Limits of k-nary interacting particle systems and duality.

We shall show now that the one-dimensional Feller processes on $(0, \infty)$ with polynomial pseudo-differential generators, i.e. generators of form $A + N$ given by (1.7), (1.8) and (1.9) are obtained as limits of scaled systems of k-nary interacting particle systems described in the previous section.

Let ν be a measure on $(0, \infty)$ satisfying (1.5) and let $\tilde{\nu}$ be the corresponding finite measure: $\tilde{\nu}(dy) = \min(y, y^2)\nu(dy)$. Let α, β_1 and β_2 be positive constants, and $\omega \in (0, 1)$. Consider the operator

$$\begin{aligned}(G^h_k f)(hn) = h^k C^k_n \big[&\frac{\alpha}{h^2}(f(hn+h) + f(hn-h) - 2f(hn)) \\ &+ \frac{\beta_1}{h}(f(hn+h) - f(hn)) + \frac{\beta_2}{h}(f(hn-h) - f(hn)) \\ + \sum_{l=[h^{-\omega}]}^{\infty} &\big(f(nh+lh) - f(nh) + lh \frac{f(nh-h) - f(nh)}{h} \big) v(l,h) \big],\end{aligned} \tag{6.1}$$

where $[h^{-\omega}]$ denotes the integer part of $h^{-\omega}$ and where

$$v(l,h) = \max\left(\frac{1}{hl}, \frac{1}{h^2 l^2}\right) \tilde{\nu}[lh, lh+h).$$

Remark. One could possibly use $\nu[hl, hl+h)$ instead of $v(l,h)$ in (6.1); however, more cumbersome $v(l,h)$ give a better approximation and turn out to be simpler to deal with.

Clearly, operator (6.1) is a scaled version of type (1.2) of the corresponding operator of type (1.1) describing the k-nary interaction of identical indistinguishable particles. Clearly the corresponding process on $\mathbf{Z}_+$ always satisfies (5.2), and it satisfies (5.1) if and only if $\beta_1 - \beta_2 \leq 0$. The extension of (6.1) to the functions on $\mathbf{R}_+$ (like in (1.3)) yields the operator

$$(G_k^h f)(x) = \frac{x(x-h)...(x-(k-1)h)}{k!}\Big[\frac{\alpha}{h^2}(f(x+h)+f(x-h)-2f(x))$$
$$+\frac{\beta_1}{h}(f(x+h)-f(x)) + \frac{\beta_2}{h}(f(x-h)-f(x))$$
$$+\sum_{l=[h^{-\omega}]}^{\infty}\left(f(x+lh)-f(x)+lh\frac{f(x-h)-f(x)}{h}\right)v(l,h)\Big]. \tag{6.2}$$

Proposition 6.1. *Operator (6.2) tends to the operator*

$$(L_k f)(x) = \frac{x^k}{k!}(\alpha f''(x) + (\beta_1-\beta_2)f'(x) + \int_0^\infty (f(x+y)-f(x)-f'(x)y)\nu(dy)) \tag{6.3}$$

as $h \to 0$. More precisely, if $f \in \tilde{C}[0,\infty]$ (the linear span of constants and the smooth functions with a compact support on $[0,\infty)$) and as $h \to 0$, one has

$$|(L_k - G_k^h)f(x)| \leq o(1)(x^k + hx^{k-1})\max_{y\geq x}(|f'''(y)|+|f''(y)|+|f'(y)|) + O(h)x^{-1}L_k f(x), \tag{6.4}$$

where the last term can be dropped if $k \leq 1$.

Remark. If a measure ν satisfies stronger assumption $\int \xi\nu_j(d\xi) < \infty$, one can take instead of the sum in (6.1) a simpler expression

$$\sum_{l=[h^{-\omega}]}^{\infty}(f(nh+lh)-f(nh))\nu[lh, lh+h). \tag{6.5}$$

Then the corresponding integral term in the limiting operator (6.3) will have the form $\int_0^\infty (f(x+lh)-f(x))\nu(dy)$, and in estimate (6.4) one can write $O(h^\omega)$ instead of just $o(1)$. In such a form of the approximating system, condition (5.1) is equivalent to the condition

$$\beta_1 - \beta_2 + \int_0^\infty \xi\nu(d\xi) \leq 0.$$

Proof of Proposition 6.1. The following estimate is obvious:

$$|\sum_{n=1}^{\infty} g(nh)\tilde{\nu}[nh, nh+h) - \int_0^\infty g(x)\tilde{\nu}(dx)| \leq h\max_{x\geq 0}|g'(x)|\int_0^\infty \tilde{\nu}(dy). \tag{6.6}$$

Since

$$(f(x-h)-f(x)) = -hf'(x) + \frac{1}{2}h^2 f''(x-\theta), \quad \theta \in [0,h],$$

and

$$\sum_{l=[h^{-\omega}]}^{\infty} lh^2\nu[lh, lh+h) \le h\int_{h[h^{-\omega}]}^{\infty} y\nu(dy) \le h\int_1^{\infty} y\nu(dy) + h^{\omega}\int_0^1 y^2\nu(dy),$$

it follows that the difference between the sum in (6.2) and the integral in (6.3) can be written as

$$\sum_{l=[h^{-\omega}]}^{\infty}(f(x+lh)-f(x)-f'(x)hl)v(l,h) - \int_{[h^{-\omega}]}^{\infty}(f(x+y)-f(x)-f'(x)y)\nu(dy))$$

$$+\max_{y\in[-h,1]}|f''(x+y)|\left(O(1)\int_0^{h^{1-\omega}} y^2\nu(dy) + \sum_{l=[h^{-\omega}]}\frac{1}{2}lh^2 v(l,h)\right). \tag{6.7}$$

and the last sum in (6.7) can be estimated as

$$O(h^{\omega})\int_0^{\infty}\min(y^2,y)\nu(dy).$$

Dividing now the difference of the sum and the integral in the first line of (6.7) into two parts by restricting the measure ν to $[0,1]$ and $[1,\infty)$ respectively and then estimating the first difference by means of (6.6) with $g(y) = y^{-2}(f(x+y)-f(x)-f'(x)y)$,

$$g'(y) = y^{-4}\left((f'(x+y)-f'(x))y^2 - 2y(f(x+y)-f(x)-f'(x)y)\right)$$

$$= \frac{1}{2}f'''(x+\theta_1) - \frac{1}{3}f'''(x+\theta_2), \quad \theta_1,\theta_2 \in [0,y],$$

and the second difference by means of (6.6) with $g(y) = y^{-1}(f(x+y)-f(x))$,

$$g'(y) = \frac{f'(x+y)y - f(x+y) + f(x)}{y^2} = f''(x+\theta_1) - \frac{1}{2}f''(x+\theta_2), \quad \theta_1,\theta_2 \in [0,y],$$

yields for (6.7) the estimate

$$O(h)\max_{y\ge x}|f'''(y)|)\int_0^1 y^2\nu(dy) + O(h)\max_{y\ge x}|f''(y)|\int_1^{\infty} y\nu(dy)$$

$$+\max_{y\in[-h,1]}|f''(x+y)|\left(O(1)\int_0^{h^{1-\omega}} y^2\nu(dy) + O(h^{\omega})\int_0^{\infty}\min(y^2,y)\nu(dy)\right). \tag{6.8}$$

Clearly (6.4) follows from (6.8) and the observation that

$$\frac{x(x-h)...(x-(k-1)h)}{k!} = \frac{x^k}{k!} + O(h)x^{k-1},$$

where the last term can be dropped for $k \le 1$.

Proposition 6.2. *The dual generator $\tilde{G}_k^h$ constructed from the q-matrix of the process corresponding to (6.1) tends to the operator given by*

$$(\tilde{L}_k f)(x) = \frac{x^{k-1}}{(k-1)!}\left(\alpha f''(x) + \int_0^x (f(x-y)-f(x))\nu(y,\infty)\,dy\right) + \frac{x^k}{k!}\alpha f''(x)$$

$$+\frac{x^k}{k!}\left((\beta_2-\beta_1)f'(x) + \int_0^x (f(x-y)-f(x)+yf'(x))\nu(dy) + f'(x)\int_x^\infty y\nu(dy)\right) \quad (6.9)$$

in the sense that for $f \in \tilde{C}[0,\infty]$ and as $h \to 0$ one has

$$|(\tilde{L}_k f)(x) - (\tilde{G}_k^h f)(x)| \le o(1)(x^k + hx^{k-1})\max_{y\le x}(|f'''(y)|+|f''(y)|+|f'(y)|) + O(h)x^{-1}L_k f(x), \quad (6.10)$$

where the last term can be dropped for $k \le 1$.

Proof. By (5.9), the dual to the local part of operator (6.1), i.e. the part that corresponds to the first three terms in (6.1) is

$$\frac{(x-h)...(x-kh)}{k!}\left(\frac{\alpha}{h^2}+\frac{\beta_1}{h}\right)(f(x-h)-f(x))$$

$$+\frac{x(x-h)...(x-kh+h)}{k!}\left(\frac{\alpha}{h^2}+\frac{\beta_2}{h}\right)(f(x+h)-f(x))$$

$$= \frac{(x-h)...(x-kh+h)}{k!}$$

$$\times\left((x-kh)\left(\frac{\alpha}{h^2}+\frac{\beta_1}{h}\right)(f(x-h)-f(x)) + x\left(\frac{\alpha}{h^2}+\frac{\beta_2}{h}\right)(f(x+h)-f(x))\right)$$

which tends to the sum

$$\alpha\frac{x^k}{k!}f''(x) + (\alpha\frac{x^{k-1}}{(k-1)!} - (\beta_1-\beta_2)\frac{x^k}{k!})f'(x)$$

of the local terms of (6.9). Also by Proposition 5.5, the part of the dual operator corresponding to the sum in (6.1) is

$$\sum_{q=1}^{[x/h]-k}[\frac{(x-qh)...(x-qh-(k-1)h)}{k!}v(q,h)$$

$$+h\frac{(x-qh-h)...(x-qh-(k-1)h)}{(k-1)!}\sum_{l=\max([h^{-\omega}],q)}^{\infty}v(l,h)\big](f(x-qh)-f(x))$$

$$+\frac{x(x-h)...(x-kh+h)}{k!}(f(x+h)-f(x))\sum_{l=[h^{-\omega}]}^{\infty}lv(l,h),$$

which tends to the sum of the three integral terms in (6.9). The estimate (6.10) is obtained analogously to (6.4).

At last, we are able to prove our main results.

Proof of Theorems 1.1 and 1.2. From Proposition 6.1 it follows (by standard results on convergence of contracting semigroups, see e.g. [EK], Theorem 6.1 from Ch.1) that if operator (6.3) generates a strongly continuous semigroup (and hence a Markov process) and the space $\tilde{C}[0,\infty]$ forms a core for this semigroup, then the family of the semigroups defined by (6.1) or (6.2) converges as $h \to 0$ to the semigroup defined by (6.3), and hence the corresponding Markov processes converge. The existence of the process with the generator (1.4) under conditions of Theorem 1.1 was obtained in Section 4. Moreover, since the process is constructed using a perturbation argument from a diffusion process, a core for its diffusion part is also a core for (1.4). It is known (see [Man]) that $\tilde{C}[0,\infty]$ is a core for the diffusion operators (1.7), (1.8). On functions from this core, the r.h.s. of (6.4) is finite. Hence the part (i) of Theorem 1.1 and Theorem 1.2 follow. Moreover, from the stochastical monotonicity of the approximating systems and the existence of the dual process one gets directly the monotonicity of the limiting process (see [KFS] for general definitions of stochastic monotonicity) and the existence of its dual, i.e. the parts (ii) and (iii) of Theorem 1.1.

Appendix. Heat kernel estimates.

For the construction of the semiclassical approximation for the heat kernel of a diffusion equation, it is convenient to have a bounded diffusion coefficient. Hence we change the variable to $y = a^{-1/2}\ln x$ in (1.12) which leads to the equation

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial y^2} - f(y)\frac{\partial v}{\partial y}, \qquad (Ap1)$$

for $v(y) = u(e^{\sqrt{a}y})$, where

$$f(y) = \frac{1}{2}(\sqrt{a} + \frac{b}{\sqrt{a}}) + \frac{c}{2\sqrt{a}}e^{\sqrt{a}y} \qquad (Ap2)$$

The characteristic properties of function (Ap2) which will be used below are the following: it is (strictly) positive, monotone, convex, and exponentially increasing.

In the semiclassical method one attaches to equation (Ap1) the classical mechanical motion described by the Hamiltonian function $H(y,p) = \frac{1}{2}p^2 + f(y)p$, the corresponding Hamiltonian system being $\dot{y} = p + f(y)$, $\dot{p} = -f'(y)p$. We shall denote by $Y(t,y_0,p_0)$, $P(t,y_0,p_0)$ the solution to the Cauchy problem for this Hamiltonian system with initial condition $y(0)=y_0, p(0)=p_0$. The elementary property of this Hamiltonian flow (proved by inspection) are collected in the following

Proposition Ap1. *(i) if $p_0 \geq -f_0 = -f(y_0)$, then $\dot{y}(t) > 0$ for all $t > 0$ and $Y(t, y_0, p_0)$ explodes (i.e. reaches $+\infty$) at time*

$$t = \int_{y_0}^{\infty} \frac{d\xi}{\sqrt{f^2(\xi) + 2H(x_0, p_0)}};$$

(ii) if $-2f_0 < p_0 < -f_0$ then $Y(t, y_0, p_0)$ decreases till $\tilde{t}$ with $\tilde{y} = Y(\tilde{t}, y_0, p_0)$ given by

$$\tilde{t} = \int_{\tilde{y}}^{y_0} \frac{d\xi}{\sqrt{f^2(\xi) + 2H(x_0, p_0)}}, \quad f(\tilde{y}) = \sqrt{-2H} = \sqrt{|2H|},$$

and after $\tilde{t}$ the trajectory behaves like in (i); (iii) if $p_0 \leq -2f_0$, then $Y(t, y_0, p_0)$ decreases for all $t > 0$, is defined for all finite $t > 0$, and tends to $-\infty$ as $t \to \infty$.

The nice property of this Hamiltonian flow is that though it is exploding in finite times, the boundary value problem is always uniquely solvable.

Proposition Ap2. *(i) The estimate $\frac{\partial Y}{\partial p_0}(t, y_0, p_0) \geq t$ holds for all x, x_0 and all $t > 0$ before the explosion. (ii) For arbitrary $y_0, y \in \mathbf{R}$, and $t > 0$, there exists a unique $p_0 = p_0(t, y, y_0)$ such that $Y(t, y_0, p_0) = y$.*

Proof. (i) Differentiating the Hamiltonian equations with respect to the initial condition p_0 one gets that the derivative $z = \frac{\partial Y}{\partial p_0}(t, y_0, p_0)$ satisfies the equation

$$\ddot{z} = (f'' f + (f')^2)(Y(t, y_0, p_0))z, \quad z(0) = 0, \quad \dot{z}(0) = 1,$$

which easily implies the estimate (i), because f, f', f'' are supposed to be non-negative. (ii) It follows from (i), because (i) implies that $Y(t, y_0, p_0)$ is (strictly) increasing in p_0 for all given t, y_0 and from Proposition Ap1 it follows that $Y(t, y_0, p_0) \to \pm\infty$ respectively as $p_0 \to \pm\infty$.

Due to the statement (i), one can define globally the smooth Jacobian J, the amplitude ϕ and the two-point function S of our Hamiltonian flow by the formulas

$$J(t, y, y_0) = \frac{\partial Y}{\partial p_0}(t, y_0, p_0), \quad \phi(t, y, y_0) = (J(t, y, y_0) P(t, y_0, p_0)/p_0)^{-1/2}, \qquad (Ap3)$$

$$S(t, y, y_0) = \int_0^t (P(s, y_0, p_0)\dot{Y}(s, y_0, p_0) - H(Y(s, y_0, p_0), P(s, y_0, p_0)))\, ds \qquad (Ap4)$$

where $p_0 = p_0(t, y, y_0)$. Then the next statement follows directly from the general theory of the semiclassical approximation for diffusions (see, e.g., Chapter 3 from [Kol2]).

Proposition Ap3. *The function*

$$v_{as}(t, y, y_0) = (2\pi)^{-1/2} \phi(t, y, y_0) \exp\{-S(t, y, y_0)\} \qquad (Ap5)$$

is an asymptotic Green function for equation (Ap1) in the sense that it satisfies the initial condition $v_{as}(0, y, y_0) = \delta(y - y_0)$ *and it satisfies the equation (Ap1) approximately up to an additive remainder term*

$$F(t, y, y_0) = \frac{1}{2}(2\pi)^{-1/2}\frac{\partial^2\phi(t, y, y_0)}{\partial y^2}\exp\{-S(t, y, y_0)\}. \qquad (Ap6)$$

When justifying this asymptotics, i.e. when proving that v_{as} is close to the exact Green function of equation (Ap1), the most difficult part (unlike the case of non-degenerate diffusion with bounded coefficients, where this part is very simple) consists in obtaining certain estimates for the amplitude ϕ and the two-point function S. These estimates are given by the following

Proposition Ap4. *(i) Uniformly for all* y, y_0 *and* $t \in (0, T]$ *for any given* T,

$$C^{-1}t^{-1/2} \le \phi(t, y, y_0) \le Ct^{-1/2} \qquad (Ap7)$$

with some constant $C > 1$ *and*

$$\frac{\partial \phi}{\partial y} = O(\phi), \quad \frac{\partial^2\phi}{\partial y^2} = O(\phi); \qquad (Ap8)$$

(ii) for y *from any compact set*

$$S(t, y, y_0) \sim \frac{1}{t}(y - y_0)^2, \quad y_0 \to \pm\infty,$$

(iii) for any y_0, $S(t, y, y_0)$ *behaves like* c/t *with some positive* $c = c(y_0)$ *for* $y \to +\infty$, *and like* $(y - y_0)^2/t$ *for* $y \to -\infty$.

The proof is based on a tedious but direct analysis of all parts of the phase portrait of our Hamiltonian flow as described in Proposition Ap1. Let us prove (Ap7) for one half of the phase portrait, namely the part with $\dot{y}(0) \ge 0$ (other estimates are obtained analogously). We shall write $\phi \sim \psi$ for functions ϕ and ψ such that $C^{-1}\psi \le \phi \le C\psi$ with some constant $C > 1$. Hence we need to prove that $\phi^{-2} \sim t$.

Step 1. We shall need the following formulas connecting momentum p, position y, the Jacobian J and the amplitude ϕ on a trajectory $y = Y(t, y_0, p_0)$ such that $\dot{y}_0 = p_0 + f_0 \ge 0$ or, equivalently, $p_0 \ge -f_0$:

$$t = \int_{y_0}^{y} \frac{d\xi}{\sqrt{f^2(\xi) + 2H}}, \quad p = \sqrt{f^2(y) + 2H} - f(y) \qquad (Ap9)$$

$$J = (p_0 + f_0)\sqrt{f^2(y) + 2H}\int_{y_0}^{y} \frac{d\xi}{(f^2(\xi) + 2H)^{3/2}}, \quad p > -f_0, \qquad (Ap10)$$

$$\phi^{-2} = \frac{p_0 + f_0}{p_0}p\sqrt{f^2(y) + 2H}\int_{y_0}^{y} \frac{d\xi}{(f^2(\xi) + 2H)^{3/2}}, \quad p > -f_0, \qquad (Ap11)$$

The formula for the momentum in (Ap9) follows from the conservation of energy law (i.e. because $2H = p^2 + 2pf(y)$ is constant on any trajectory). In turn, this formula implies that $\dot{y} = \sqrt{f^2(y) + 2H}$, which gives the first formula in (Ap9) by integrating. Next, (Ap10) is obtained from the first formula in (Ap9) by differentiating with respect to y_0, and (Ap11) follows from (Ap3) and (Ap10).

Step 2. (i) If $2H \geq a^2 f^2(y)$ (for any given $a > 0$), then $p \sim \sqrt{2H}$; more precisely,

$$\sqrt{2H}(\sqrt{1 + a^{-1}} - a^{-1}) \leq p \leq \sqrt{2H}.$$

(ii) If $|2H| \leq a^2 f^2(y)$, then $|p| \sim |H|/f(y)$; more precisely, if $0 \leq 2H < a^2 f^2(y)$, then

$$2\frac{\sqrt{1 + a^2} - 1}{a^2}\frac{H}{f(y)} \leq p \leq \frac{H}{f(y)};$$

if $H < 0$ (and hence $|2H| \leq f^2(y)$), then $p < 0$ and $|H|/f(y) \leq |p| \leq 2|H|/f(y)$. These estimates follow directly from the formula for p from (Ap9) and elementary estimates of square roots. Namely, to get (i), one writes

$$p = \sqrt{2H}(\sqrt{1 + \frac{f^2(y)}{2H}} - \frac{f(y)}{\sqrt{2H}})$$

and uses the fact that the function $\sqrt{1 + \omega^2} - \omega$ is a decreasing function of ω. Similarly, to get (ii) one writes

$$p = f(y)\left(\sqrt{1 + \frac{2H}{f^2(y)}} - 1\right).$$

In case $H \leq 0$, one then uses the estimate $1 - \omega \leq \sqrt{1 - \omega} \leq 1 - \omega/2$ for $\omega \in [0, 1]$; in case $H \geq 0$, one uses the estimate $\sqrt{1 + \omega} \geq 1 + \frac{\sqrt{1+a^2}-1}{a^2}$ for $\omega \in [0, a^2]$.

Step 3. Let us prove (Ap7) assuming $p_0 \geq af_0$ with some $a > 0$. Let us stress that we are looking for estimates that are uniform with respect to f_0 and depend only on a given constant $a > 0$. Let $x = x(y_0, p_0)$ be defined by the equation $f^2(x) = 2H$.

If $y \in [y_0, x]$, then $2H \geq f^2(y) \geq f^2(y_0)$ and

$$\frac{p_0 + f_0}{p_0} \in [1, 1 + a^{-1}].$$

Hence from (Ap11) and Step 2 (i) one gets

$$\phi^{-2} \sim 2H \int_{y_0}^{y} \frac{d\xi}{(f^2(\xi) + 2H)^{3/2}} \sim \frac{y - y_0}{\sqrt{2H}} \sim t.$$

If $y > x$, then by Step 2 (ii)

$$\phi^{-2} \sim \frac{\max(0, x - y_0)}{\sqrt{2H}} + H \int_{x}^{y} \frac{d\xi}{(f^2(\xi) + 2H)^{3/2}}$$

$$\sim \frac{\max(0, x-y_0)}{\sqrt{2H}} + \frac{\max(y-x, 1)}{f(x)} \sim \frac{\max(0, x-y_0) + \max(y-x, 1)}{f(x)},$$

where we have used that f increases exponentially at infinity. One gets then the same expression for t proving (Ap7) for this case.

Step 4. Let us prove (Ap7) assuming $-f_0 b \leq p_0 \leq a f_0$ for arbitrary given $a > 0$, $b \in (0,1)$. In this case $(p_0 + f_0)H/p_0$ is of the order f_0^2 and one gets

$$\phi^{-2} \sim f_0^2 \int_{y_0}^{y} \frac{d\xi}{(f^2(\xi) + 2H)^{3/2}} \sim f_0^{-1} \max(1, y - y_0)$$

and the same expression for t.

Step 5. Let $p_0 = -f_0(1-\epsilon)$ with $\epsilon \in (0, 1/2)$. Then $2H = -(1-\epsilon^2) f_0^2$. We shall prove (Ap7) uniformly for all these ϵ, which implies (Ap7) also for $\epsilon = 0$.

Suppose first that $y \leq y_0 + 1$. Then

$$f^2(\xi) + 2H \sim f_0^2(\epsilon^2 + \xi - y_0)$$

uniformly for all $\xi \in [y_0, y]$ and $f(y) \sim f_0$. Hence

$$t \sim \frac{1}{f_0} \int_0^{y-y_0} \frac{d\xi}{\sqrt{\xi + \epsilon^2}} \sim \frac{y - y_0}{f_0}$$

and by Step 2 (ii)

$$\phi^{-2} \sim \frac{\epsilon}{f_0} \sqrt{\epsilon^2 + y - y_0} \int_0^{y-y_0} (\epsilon^2 + \xi)^{-3/2}$$

$$= \frac{1}{2f_0} (\sqrt{\epsilon^2 + y - y_0} - \epsilon) \sim \frac{y - y_0}{f_0} \sim t.$$

Suppose now that $y > y_0 + 1$. Clearly $f^2(\xi) + 2H \sim f^2(\xi)$ for $\xi > y_0 + 1$. Hence

$$t \sim \frac{1}{f_0} + \int_{y_0+1}^{y} f^{-1}(\xi)\, d\xi \sim \frac{1}{f_0}(1 + \max(1, y - y_0 - 1)) \sim \frac{1}{f_0},$$

and similar expression holds for ϕ^{-2}.

When the estimates from Proposition Ap4 are obtained, the justification of the asymptotics (Ap5) is done precisely by the same well known arguments as for the case of diffusions with bounded coefficients, see e.g. Chapter 3 of book [Kol2]. This leads to the following result.

Theorem Ap1. *The Green functions v_G and u_G of the Cauchy problems for equations (Ap1) and (1.12) respectively exist as smooth functions and for small t enjoy the estimates*

$$v_G(t, y, y_0) = O(t^{-1/2}) \exp\{-S(t, y, y_0)\}, \qquad (Ap12)$$

$$u_G(t, x, x_0) = O(t^{-1/2}) x_0^{-1} \exp\{-S(t, b^{-1/2} \ln x, b^{-1/2} \ln x_0)\}. \qquad (Ap13)$$

Proof. Formula (Ap13) follows from (Ap12) by the change of the variables. As we mentioned, (Ap12) follows by rather standard arguments. Namely, a standard formula of the perturbation theory gives for the exact Green function v_G the following (formal) series representation:

$$v_G(t,y,\xi) = v_{as}(t,y,\xi) + \sum_{k=0}^{\infty} \int_0^t d\tau \int_{-\infty}^{\infty} d\xi \, v_{as}(t-\tau,y,\eta)\mathcal{F}^k F(\tau,\eta,\xi), \qquad (Ap14)$$

where F is given by (Ap6) and $\mathcal{F}^k$ is the k-th power of the integral operator $\mathcal{F}$ defined by the formula

$$\mathcal{F}\psi(t,y,\xi) = \int_0^t d\tau \int_{-\infty}^{\infty} d\eta \, F(t-\tau,x,\eta)\psi(\tau,\eta,\xi).$$

In order to prove (Ap12) one needs to prove the convergence of series (Ap14) and to show that the sum is dominated by the first term v_{as}. This is done by the Laplace method. In fact, let us show how the first integral in (Ap14) is estimated, namely the integral

$$\int_0^t d\tau \int_{-\infty}^{\infty} d\xi \, v_{as}(t-\tau,y,\eta)F(\tau,\eta,\xi).$$

By (Ap6) - (Ap8), this integral can be estimated by

$$O(1)\int_0^t \frac{1}{\sqrt{t-\tau}\sqrt{\tau}}\, d\tau \int_{-\infty}^{\infty} \exp\{-S(t-\tau,y,\eta) - S(\tau,\eta,\xi)\}\, d\xi. \qquad (Ap15)$$

To apply the Laplace method one looks for the stationary points of the phase, i.e. for the solutions η_0 of the equation

$$\frac{\partial S(t-\tau,y,\eta)}{\partial \eta} + \frac{\partial S(\tau,\eta,\xi)}{\partial \eta} = 0,$$

and it is well known (and is easily seen) that the solutions are given by $\eta_0 = Y(\tau,\xi,p_0)$, where $Y(t,\xi,p_0) = y$, i.e. η_0 lies on a trajetory of the Hamiltonian flow that joins ξ and y in time t. Due to Proposition Ap2 this trajectory is uniquely defined, and hence η_0 is uniquely defined, and hence the phase in the Laplace integral (Ap15) has a unique minimal point η_0 such that

$$S(t-\tau,y,\eta_0) + S(\tau,\eta_0,\xi) = S(t,y,\xi).$$

Consequently (and due to the estimates of Proposition Ap4), the application of the Laplace method to integral (Ap15) yields for this integral the estimate $O(\sqrt{t})\exp\{-S(t,y,\xi)\}$, which is of order $O(t)v_{as}(t,y,\xi)$, i.e. smaller than v_{as} for small t as we claimed. Similar estimates of other terms (obtained by induction) prove the convergence and the required estimate of series (Ap14), which completes the proof of Theorem Ap1.

Remark. For small $x-x_0$ one can obtain from this theorem a more precise multiplicative asymptotics for the heat kernel u_G. Moreover, it is not difficult to extend the estimates

(Ap12), (Ap13) to all finite (not necessarily small) times using the Chapman-Kolmogorov equation and again the Laplace method.

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