Idempotent Structures in Optimisation

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Abstract

Consider the set $A = \mathbb{R} \cup \{+\infty\}$ with the binary operations $\oplus = \max$ and $\odot = +$ and denote by A^n the set of vectors $v = (v_1, ..., v_n)$ with entries in A. Let the generalised sum $u \oplus v$ of two vectors denote the vector with entries $u_j \oplus v_j$, and the product $a \odot v$ of an element $a \in A$ and a vector $v \in A^n$ denote the vector with the entries $a \odot v_j$. With these operations, the set A^n provides the simplest example of an idempotent semimodule. The study of idempotent semimodules and their morphisms is the subject of idempotent linear algebra, which has been developing for about 40 years already as a useful tool in a number of problems of discrete optimisation. Idempotent analysis studies infinite dimensional idempotent semimodules and is aimed at the applications to the optimisations problems with general (not necessarily finite) state spaces. We review here the main facts of idempotent analysis and its major areas of applications in optimisation theory, namely in multicriteria optimisation, in turnpike theory and mathematical economics, in the theory of generalised solutions of the Hamilton-Jacobi Bellman (HJB) equation, in the theory of games and controlled Marcov processes, in financial mathematics.

1 Introduction

Consider the set $A = \mathbb{R} \cup \{+\infty\}$ with the binary operations $\oplus = \max$ and $\odot = +$ and denote by A^n the set of vectors $v = (v_1, ..., v_n)$ with entries in A. Let the generalised sum $u \oplus v$ of two vectors denote the vector with entries $u_j \oplus v_j$, and the product $a \odot v$ of an element $a \in A$ and a vector $v \in A^n$ denote the vector with the entries $a \odot v_j$. With these operations, the set A^n provides the simplest example of an idempotent semimodule. The study of idempotent semimodules and their morphisms is the subject of idempotent linear algebra, which has been developing for about 40 years already as a useful tool in a number of problems of discrete optimisation, see e.g. [22], [44], [47], [48], citeVo1, [55], [35] for the first results in this direction and [4], [6], [16],[21], [43], [18] [59], [10] and references therein for recent developments. Idempotent analysis studies infinite dimensional idempotent semimodules and is aimed at the applications to the optimisations problems with general (not necessarily finite) state spaces. We review here the main facts of idempotent analysis and its major areas of applications in optimisation theory.

We do not give here all the proofs. For a more copmprehensive exposition and for historical guides, the reader is reffered to the books [41], [32], [33] and to the reviews [20], [37].

The plan of the paper is the following. Sections 2 and 3 are devoted to the main notions and facts of idempotent mathematics. Section 4 is devoted to the turnpikes in models of mathematical economics and to the recently developed theory of infinite extremals in dynamic optimisation with infinite planning horizon. Section 5 introduces the elements of nonlinear idempotent analysis presenting some facts from the (now actively developing) theory of nonexpansive homogeneous maps with their applications to the turnpike theorems for stochastic games and controlled Markov processes. Section 6 explains how the introduction of idempotent structures solves the problem of constructing and defining the generalised solutions to the Hamilton-Jacobi-Bellman (HJB) equation and also presents some results on the large time behavior of these solutions. The perturbation theory for the solutions of the deterministic Bellman equation perturbed by random noise is given in section 7. Application to the multicriteria optimisation is given in section 8, where, in particular, a new equation describing the dynamics of Pareto sets in dynamic optimisation is deduced by means of idempotent analysis. Some stochastic and infinite dimensional generalisations are discussed in sections 9 and 10.

In section 11 we discuss a deterministic approach to the option pricing theory in financial mathematics. This approach allows to obtain some generalisations of the standard Black-Sholes formula (characterised by more rough assumptions on the underlying common stocks evolution) and reduces the analysis of derivative securities pricing to the study of homogeneous nonexpansive maps, which however, unlike the situations discussed in section 5, act in infinite dimensional spaces.

To conclude the introduction, let us note that there is an interesting correspondence principle between probability theory and stochastic processes on the one hand, and optimization theory and decision processes on the other hand. In particular, the Markov causality principle corresponds to the Bellman optimality principle. For the development of the theory of optimization processes in the spirit of stochastic processes, where the notions of optimization martingales and optimization measurability play the main role, see [45], [3] and references therein. Formally, in the case of (space and time) homogeneous processes, the connection between Markov and Bellman processes is given by the Cramer transform [3], In general, Bellman processes present a kind of semiclassical approximation to the Markov processes [33], [29]. We are not going into details in this direction, but we shall explain here shortly the connection between probability and idempotent measures, which is given by the large deviation principle. To this end, let us recall first the following general definition from [53]. Let Ω be a topological space and $\mathcal{A}$ be the algebra of its Borel sets. One says that a family of probabilities (P_ε), $\varepsilon > 0$, on $(\Omega, \mathcal{A})$ obeys the *large deviation principle* if there exists a *rate function* $I : \Omega \mapsto [0, \infty]$ such that

1) I is lower semi-continuous and $\Omega_a = \{\omega \in \Omega : I(\omega) \le a\}$ is a compact set for any $a < \infty$,

2) -$\limsup_{\varepsilon \to 0} \varepsilon \log P_\varepsilon(C) \ge \inf_{\omega \in C} I(\omega)$ for each closed set $C \subset \Omega$,

3) -$\liminf_{\varepsilon \to 0} \varepsilon \log P_\varepsilon(U) \le \inf_{\omega \in U} I(\omega)$ for each open set $U \subset \Omega$.

Clearly $m(A) = \inf_{\omega \in A} I(\omega)$ is then a positive sigma-additive with respect to the operation $\oplus = \min$ function on $\mathcal{A}$. In idempotent analysis, such functions are called idempotent measures (see Section 3 below). Therefore, it is naturally to generalise the previous definition in the following way [1]. For any Borel set A let

$$P^{out} = \limsup_{\varepsilon \to 0} \varepsilon \log P_\varepsilon(A), \quad P^{in} = \liminf_{\varepsilon \to 0} \varepsilon \log P_\varepsilon(A).$$

One says that (P_ε) obeys *the weak large deviation principle*, if there exists a positive idempotent measure m on $(\Omega, \mathcal{A})$ such that

1) there exists a sequence (Ω_n) of compact subsets of Ω such that $m(\Omega_n^c) \to \mathbb{0} = +\infty$ as $n \to \infty$, where C^c stands for the complimentary set of C,

2) $m(C) \le -P^{out}(C)$ for each closed $C \subset \Omega$,

2) $m(U) \ge -P^{in}(U)$ for each open $U \subset \Omega$.

Using Theorem 1 (from Section 3 below) and its generalizations one can prove (see details in [1]) that the large deviation principle and its weak version are actually equivalent for some (rather general) "good" spaces Ω. One can obtain also an interesting correspondence between the tightness conditions for probability and idempotent measures.

2 Idempotent Semigroups and Semirings

Idempotent analysis is the analysis on the spaces of functions with values in idempotent semirings. In this section we give the definition of idempotent semirings and provide some examples.

An *idempotent semigroup* is a set M equipped with a commutative, associative operation $\oplus$ (generalized addition) that has a unit element $\mathbb{0}$ such that $\mathbb{0} \oplus a = a$ for each $a \in M$ and satisfies the idempotency condition $a \oplus a = a$ for any $a \in M$. There is a naturally defined partial order on any idempotent semigroup; namely, $a \le b$ if and only if $a \oplus b = a$. Obviously, the reflexivity of $\le$ is equivalent to the idempotency of $\oplus$, whereas the transitivity and the antisymmetricity ($a \le b$, $b \le a \implies a = b$) follow, respectively, from the associativity and the commutativity of the semigroup operation. The unit element

$\mathbb{0}$ is the greatest element; that is, $a \leq \mathbb{0}$ for all $a \in M$. The operation $\oplus$ is uniquely determined by the relation $\leq$, due to the formula

$$a \oplus b = \inf\{a, b\} \tag{1}$$

Furthermore, if every subset of cardinality 2 in a partially ordered set M has an infimum, then equation (1) specifies the structure of an idempotent semigroup on M.

An idempotent semigroup is called an *idempotent semiring* if it is equipped with yet another associative operation $\odot$ (generalized multiplication) that has a unit element $\mathbb{1}$, which distributes over $\oplus$ on the left and on the right, i.e.,

$$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c), \qquad (b \oplus c) \odot a = (b \odot a) \oplus (c \odot a),$$

and satisfies the property $\mathbb{0} \odot a = \mathbb{0}$ for all a. An idempotent semiring is said to be *commutative* (*abelian*) if the operation $\odot$ is commutative.

An idempotent semigroup (semiring) M is called an idempotent *metric* semigroup (semiring) if it is endowed with a metric $\rho\colon M \times M \to \mathbb{R}$ such that the operation $\oplus$ is (respectively, the operations $\oplus$ and $\odot$ are) uniformly continuous on any order-bounded set in the topology induced by ρ and any order-bounded set is bounded in the metric.

Let X be a set, and let $M = (M, \oplus, \rho)$ be an idempotent metric semigroup. The set $B(X, M)$ of bounded mappings $X \to M$ (i.e., mappings with order-bounded range) is an idempotent metric semigroup with respect to the pointwise addition $(\varphi \oplus \psi)(x) = \varphi(x) \oplus \psi(x)$, the corresponding partial order, and the uniform metric $\rho(\varphi, \psi) = \sup_x \rho(\varphi(x), \psi(x))$. If $A = (A, \oplus, \odot, \rho)$ is a semiring, then $B(X, A)$ bears the structure of an A-*semimodule*; namely, the multiplication by elements of A is defined on $B(X, A)$ by the formula $(a \odot \varphi)(x) = a \odot \varphi(x)$. This A-semimodule will also be referred to as the space of (bounded) A-valued functions on X. If X is a topological space, then by $C(X, A)$ we denote the subsemimodule of continuous functions in $B(X, A)$. If X is finite, $X = \{x_1, \dots, x_n\}$, $n \in \mathbb{N}$, then the semimodules $C(X, A)$ and $B(X, A)$ coincide and can be identified with the semimodule $A^n = \{(a_1, \dots, a_n) : a_j \in A\}$. Any vector $a \in A^n$ can be uniquely represented as a linear combination $a = \bigoplus_{j=1}^n a_j \odot e_j$, where $\{e_j,\ j = 1, \dots, n\}$ is the standard basis of A^n (the jth coordinate of e_j is equal to $\mathbb{1}$, and the other coordinates are equal to $\mathbb{0}$). As in the conventional linear algebra, we can readily prove that the semimodule of continuous homomorphisms $m\colon A^n \to A$ (in what follows such homomorphisms are called *linear functionals* on A^n) is isomorphic to A^n itself. Similarly, any endomorphism $H\colon A^n \to A^n$ (a linear operator on A^n) is determined by an A-valued $n \times n$ matrix.

1. $A = \mathbb{R} \cup \{+\infty\}$ with the operations $\oplus = \max$ and $\odot = +$, the unit elements $\mathbb{0} = +\infty$ and $\mathbb{1} = 0$, the natural order, and the metric

$$\rho(a, b) = |e^{-a} - e^{-b}|.$$

This is the simplest and the most important semiring, which we shall call the standard (min,+) semiring. This semiring is isomorphic to the semiring $\mathbb{R}_+$ with the operations $\oplus = \min$ and $\odot = \times$ (the usual multiplication). The isomorphism is given by the mapping $x \mapsto \exp(x)$. When considering the problems of maximization (instead of minimization), it is convenient to consider the standard semiring with inverted order, namely the standard (max,+) semiring, which is the set $\mathbb{R} \cup \{-\infty\}$ with the operations $\oplus = \max$, $\odot = +$.

2. The semiring of endomorphisms of A^n (or A-valued $n \times n$ matrices) with the pointwise addition $\oplus$ and the composition as the generalised multiplication. Here A can be any other idempotent semiring.

3. The compactified real line $\mathbb{R} \cup \{\pm\infty\}$ with the operations $\oplus = \min$ and $\odot = \max$, the unit elements $\mathbb{0} = +\infty$ and $\mathbb{1} = -\infty$, and the metric

$$\rho(a, b) = |\arctan a - \arctan b|.$$

4. The subsets of a given set form an idempotent semiring with respect to the operations $\oplus$ of set union and $\odot$ of set intersection. There are various ways to introduce metrics on semirings of sets. For example, if we deal with compact subsets of a metric space, then the Hausdorff metric is appropriate.

5. Pareto order ($a = (a_1, \dots, a_n) \leq b = (b_1, \dots, b_n)$ if and only if $a_i \leq b_i$ for all $i = 1, \dots, n$) defines in $\mathbb{R}^n_+$ the structure of an idempotent semigroup. For any subset $M \subset \mathbb{R}^k$, by $\mathrm{Min}(M)$ we denote the set of minimal elements of the closure of M in $\mathbb{R}^k$. Let $P(\mathbb{R}^k)$ denote the class of subsets $M \subset \mathbb{R}^k$ whose elements are pairwise incomparable,

$$P(\mathbb{R}^k) = \{M \subset \mathbb{R}^k : \mathrm{Min}(M) = M\}.$$

Obviously, $P(\mathbb{R}^k)$ is a semiring with respect to the operations $M_1 \oplus M_2 = \mathrm{Min}(M_1 \cup M_2)$ and $M_1 \odot M_2 = \mathrm{Min}(M_1 + M_2)$; the neutral element $\mathbb{0}$ with respect to addition in this semiring is the empty set, and the neutral element with respect to multiplication is the set whose sole element is the zero vector in $\mathbb{R}^k$. The semiring $P(\mathbb{R}^k)$ is isomorphic to the semiring of *normal sets*, that is, closed subsets $N \subset \mathbb{R}^k$ such that $b \in N$ implies $a \in N$ for any $a \geq b$; the sum and the product of normal sets are defined as their usual union and sum, respectively. Indeed, if N is normal, then $\mathrm{Min}(N) \in P(\mathbb{R}^k)$; conversely, with each $M \in P(\mathbb{R}^k)$ we can associate the normalization $\mathrm{Norm}(M) = \{a \in \mathbb{R}^k \mid \exists\, b \in M : a \geq b\}$.

6. Convolution semirings. If X is a topological group and A is the standard semiring (other semirings can be used here as well, but we shall need only this particular case), one can define an idempotent analog $\circledast$ of convolution on $B(X, A)$ by setting

$$(\varphi \circledast \psi)(x) = \inf_{y \in X} (\varphi(y) \odot \psi(x - y)). \qquad (2)$$

This operation turns $B(X, A)$ into an idempotent semiring, which will be denoted by $CS(X)$ and referred to as the *convolution semiring*. Some subsemirings of $CS(X)$ are of interest in studying multicriteria optimization. Namely, let L denote the hyperplane in $\mathbb{R}^k$ determined by the equation

$$L = \Big\{(a^j) \in \mathbb{R}^k : \sum a^j = 0\Big\},$$

and let us define a function $n \in CS(L)$ by setting $n(a) = \max_j(-a^j)$. Obviously, $n \circledast n = n$; that is, n is a multiplicatively idempotent element of $CS(L)$. Let $CS_n(L) \subset CS(L)$ be the subsemiring of functions h such that $n \circledast h = h \circledast n = h$. It is easy to see that $CS_n(L)$ contains the function identically equal to $\mathbb{0} = \infty$ and that the other elements of $CS_n(L)$ are just the functions that take the value $\mathbb{0}$ nowhere and satisfy the inequality $h(a) - h(b) \leq n(a-b)$ for all $a, b \in L$. In particular, for each $h \in CS_n(L)$ we have

$$|h(a) - h(b)| \leq \max_j |a^j - b^j| = \|a - b\|,$$

which implies that h is differentiable almost everywhere.

It turns out that the last two examples of semirings are closely connected, as shows the following proposition that is a specialization of a more general result stated in [49].

Proposition 1 *The semirings $CS_n(L)$ and $P(\mathbb{R}^k)$ are isomorphic.*

Proof. The main idea is that the boundary of each normal set in $\mathbb{R}^k$ is the graph of some real-valued function on L, and vice versa. More precisely, let us consider the vector $e = (1, \ldots, 1) \in \mathbb{R}^k$ normal to L and assign a function $h_M\colon L \to \mathbb{R}$ to each set $M \in P(\mathbb{R}^k)$ as follows. For $a \in L$, let $h_M(a)$ be the greatest lower bound of the set of $\lambda \in \mathbb{R}$ for which $a + \lambda e \in \operatorname{Norm}(M)$. Then the functions corresponding to singletons $\{\varphi\} \subset \mathbb{R}^k$ have the form

$$h_\varphi(a) = \max_j(\varphi^j - a^j) = \overline{\varphi} + n(a - \varphi_L), \tag{3}$$

where $\overline{\varphi} = k^{-1}\sum_j \varphi^j$ and $\varphi_L = \varphi - \overline{\varphi}e$ is the projection of φ on L. Since idempotent sums $\oplus$ of singletons in $P(\mathbb{R}^k)$ and of functions (3) in $CS_n(L)$ generate $P(\mathbb{R}^k)$ and $CS_n(L)$, respectively, we can prove the proposition by verifying that the $\odot$-multiplication of vectors in $\mathbb{R}^k$ passes into the convolution of the corresponding functions (3). Namely, one needs to show that

$$h_\varphi \circledast h_\psi = h_{\varphi\oplus\psi}.$$

By virtue of (3), it suffices to show that

$$n_\varphi \circledast n_\psi = n_{\varphi\oplus\psi},$$

where $n_\varphi(a) = n(a - \varphi_L)$, and the latter identity is valid since

$$n_\varphi \circledast n_\psi = n_0 \circledast n_{\varphi+\psi} = n \circledast n_{\varphi+\psi} = n_{\varphi+\psi}.$$

3 Main Facts of Idempotent Analysis

We present here the simplest versions of the main general facts of idempotent analysis, in particular, restricting our consideration to the case of standard (min,+) semiring A. Proofs, generalisations and references could be found in [33].

1) All idempotent measures are absolutely continuous; i.e., any such measure can be represented as the idempotent integral of a density function with respect to some standard measure. Let us formulate this fact more presicely. Let $C_{\mathbb{0}}(X, A)$ denote the space of continuous functions $f\colon X \to A$ on a locally compact normal space X vanishing at infinity (i.e. such that for any $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $\rho(\mathbb{0}, f(x)) < \varepsilon$ for all $x \in X \setminus K$). The topology on $C_{\mathbb{0}}(X, A)$ is defined by the uniform metric $\rho(f, g) = \sup_X \rho(f(x), g(x))$. The space $C_{\mathbb{0}}(X, A)$ is an idempotent semimodule. If X is a compact set, then the semimodule $C_{\mathbb{0}}(X, A)$ coincides with the semimodule $C(X, A)$ of all continuous functions from X to A. The homomorphisms $C_{\mathbb{0}}(X, A) \to A$ will be called *linear functionals* on $C_{\mathbb{0}}(X, A)$. The set of linear functionals will be denoted by $C^*_{\mathbb{0}}(X, A)$ and called the *dual semimodule* of $C_{\mathbb{0}}(X, A)$.

Theorem 1 *For any $m \in C^*_{\mathbb{0}}(X, A)$ there exists a unique lower semicontinuous and bounded below function $f\colon X \to A$ such that*

$$m(h) = \inf_x f(x) \odot h(x) \qquad \forall h \in C_{\mathbb{0}}(X, A). \tag{4}$$

Conversely, any function $f\colon X \to A$ bounded below defines an element $m \in C^*_{\mathbb{0}}(X, A)$ by formula (4). At last, the functionals m_{f_1} and m_{f_2} coincide if and only if the functions f_1 and f_2 have the same lower semicontinuous closures; that is, $\mathrm{Cl}\, f_1 = \mathrm{Cl}\, f_2$*, where*

$$(\mathrm{Cl}\, f)(x) = \sup\{\psi(x) : \psi \le f,\ \psi \in C(X, A)\}.$$

The Riesz–Markov theorem in functional analysis establishes a one-to-one correspondence between continuous linear functionals on the space of continuous real functions on a locally compact space X vanishing at infinity and regular finite Borel measures on X. Similar correspondence exists in idempotent analysis. We can define an idempotent measure μ_f on the subsets of X by the formuls $\mu_f(A) = \inf\{x : x \in A\}$. Clearly this measure is σ-additive (with respect to the operation $\oplus$). Equation (4) specifies a continuation of m_f to the set of A-valued functions bounded below. On analogy with conventional analysis, we say that such functions are integrable with respect to the measure μ_f and denote the values taken by m_f on these functions by the *idempotent integral*

$$m_f(h) = \int_X^{\oplus} h(x)\, d\mu_f(x) = \inf_x f(x) \odot h(x). \tag{5}$$

This is not difficult to show that this "integral" can be alternatively defined as the limits of the corresponding idempotent Riemannian (or Lebesgue) sums. Then Theorem 1 will be equivalent to the statement that all idempotent measures are absolutely continuous with respect to the standard idempotent measure $m_{\mathbb{1}}$.

It turns out that many of the properties of the conventional integral hold in this situation as well. One can develop the concept of weak convergence and the corresopnding theory of generalised functions. In particular, the δ-functions in idempotent analysis are given by the indicator functions of points:

$$\delta_y(x) = \begin{cases} \mathbb{1} & \text{for } x = y, \\ \mathbb{0} & \text{for } x \neq y. \end{cases}$$

Indeed,

$$m_{\delta_y}(x) = \inf(\delta_y(x) \odot h(x)) = h(y).$$

For $X = \mathbb{R}^n$, simple delta-shaped sequences can be constructed of smooth convex functions; for example, $\delta_y(x)$ is the weak limit of the sequence $f_n(x) = n(x-y)^2$. Thus, by virtue of the preceding, each linear functional (or operator) on $C_{\mathbb{0}}(\mathbb{R}^n)$ is uniquely determined by its values on smooth convex functions.

2) All linear operators on the semimodule of functions ranging in a semiring with idempotent addition are integral operators. To say more precise, let A and X be as above. An A-linear continuous operator on (or homomorphisms of) semimodules $C_{\mathbb{0}}(X)$ is, by definition, continuous mappings $B\colon C_{\mathbb{0}}(Y) \to C_{\mathbb{0}}(X)$ such that

$$B(a \odot h \oplus c \odot g) = a \odot B(h) \oplus c \odot B(g)$$

for each $a, c \in A$ and $h, g \in C_{\mathbb{0}}(Y)$.

Theorem 2 *Let $B\colon C_{\mathbb{0}}(Y) \to C_{\mathbb{0}}(X)$ be a continuous A-linear operator. Then there exists a unique function $b\colon X \times Y \to A$ (called the integral kernel of B) lower semicontinuous with respect to the second argument and such that*

$$(Bh)(x) = \inf_y b(x, y) \odot h(y). \tag{6}$$

This fact is actually a direct consequence of the previous theorem. One can describe exactly the properties of the kernel $b(x, y)$, which are necessary and sufficient for formula 6 to define a continuous linear operator. One can also specify the properties of b under which the correponding operator acts on the spaces of continious functions with a compact support, or on the space of functions having a limit at infinity. Moreover, one can describe the properties of the kernel b that ensures that the corresponsing operator is compact (or completely continuous), in the usual sense, i.e. it carries each set bounded in the metric to a precompact set. For example, one easily sees that if X is compact, then for any continuous b, formula 6 defines a completely continuous linear operator.

3) The group of invertible linear operators is very slender. For example, any invertible $n \times n$ matrix with entries in A is the composition of a diagonal matrix and a permutation of the standard basis of the free semimodule A^n. More generally, the following result holds.

Theorem 3 *Let*

$$B: C_{\mathbb{0}}(Y) \to C_{\mathbb{0}}(X) \text{ and } D: C_{\mathbb{0}}(X) \to C_{\mathbb{0}}(Y)$$

be mutually inverse A-linear operators. Then there exists a homeomorphism $\beta: X \to Y$ and continuous functions $\varphi: X \to A$ and $\psi: Y \to A$ nowhere assuming the value $\mathbb{0}$ such that $\varphi(x) \odot \psi(\beta(x)) \equiv \mathbb{1}$ and the operators B and D are given by the formulas

$$(Bh)(x) = \varphi(x) \odot h(\beta(x)), \quad (Dg)(y) = \psi(y) \odot g(\beta^{-1}(y)).$$

4) The spectrum of general compact linear opeators in idempotent analysis is also very slender. For example, the following statement holds.

Theorem 4 *Let X be a compact set, and let B be a A-linear operator on $C(X, A))$ with a continuous integral kernel b, which is nowhere equal to $\mathbb{0} = \infty$. Then B has a unique eigenvalue $\alpha \neq \mathbb{0}$ and a (not necessaryly unique) eigenfunction $h \in C(X, A))$, which is nowhere equal to $\mathbb{0}$, such that $Bh = \alpha \odot h = \alpha + h$.*

The application of this result to the optimation theory is based essentially on the following two corollaries.

Corollary 1 *Let an operator B satisfy the conditions of Theorem 4, so that its eigenvalue α is unique and the eigenvector h is nowhere equal to $\mathbb{0}$. Let $f \leq h+c$ with some constant c. Then*

$$\lim_{m\to\infty} \frac{B^m f(x)}{m} = \alpha. \tag{7}$$

Proof. The inequality

$$\inf_x (f - h) \leq B^m f(x) - B^m h(x) \leq \sup_x (f - h), \tag{8}$$

is clear for $m = 1$ and is obtained by trivial induction for all positive integral m. This implies (7), since $B^m(h) = m\alpha + h$.

Corollary 2 *Under the conditions of Corollary* 1*, if $\alpha > 0$, then the Neumann series*

$$f \oplus B(f) \oplus B^2(f) \oplus \cdots \tag{9}$$

is finite, that is, is equal to the finite sum

$$B \oplus B(f) \oplus \cdots \oplus B^k(f)$$

for some k.

Proof. It follows from (8) that

$$B^m f(x) - f(x) \geq \inf_x (f-h) + h(x) - f(x) + m\alpha,$$

whence

$$B^m f(x) \geq f(x) + \inf_x (f-h) + m\alpha - c.$$

Consequently, $B^m f(x) > f(x)$ for all x provided that m is sufficiently large. Hence, the series (9) is finite.

The calculations of the iterations B^m is needed in solving optimization problems of the form

$$\sum_{k=0}^{m-1} b(x_k, x_{k+1}) + g(x_m) \to \min, \quad x_0 \text{ is fixed},$$

by the dynamic programming technique. Namely, the desired minimum is $(B^m g)(x_0)$. Thus, Corollary 1 describes the asymptotic behavior of solutions of this problem for large m. Series (9) solves the equation $g = Bg \oplus f$ which stands for the stationary optimisation problem corresponding to the Bellman operator B.

5) The Fourier transformation for functions ranging in the standard semiring is the usual Legendre transformation. Indeed, the general Fourier transformation takes complex-valued functions on a commutative locally compact group G to functions on the dual group $\widehat{G}$ according to the formula

$$(Fh)(\chi) = \int_G \chi(x) h(x)\, dx,$$

where $\chi \in \widehat{G}$ is a character of G, that is, a continuous homomorphism of G into the unit circle S^1 considered as the multiplicative subgroup of unimodular numbers in $\mathbb{C}$.

In idempotent analysis, the characters of G can naturally be understood as the homomorphisms of G into the multiplicative group of the number semiring; then, for $G = \mathbb{R}^n$, the set of characters is the set of usual linear functionals on $\mathbb{R}^n$, naturally identified with $\mathbb{R}^n$. Next, we replace the integral by inf and the usual multiplication by the multiplication $\odot = +$ and obtain the following formula for the Fourier transform of an A-valued function h on $\mathbb{R}^n$:

$$(Fh)(p) = \inf_x (px + h(x)).$$

We see that $(Fh)(-p)$ is the usual Legendre transform with the opposite sign.

The usual Fourier transformation satisfies a commutation formula with convolution and is an eigenoperator of the translation operator. The same properties are valid for the Legendre–Fourier operator.

To conclude this section, let us prove a result concerning the uniqueness of the eigenfunctions of an idempotent linear operator.

Theorem 5 *Let X be a compact set, and let the integral kernel $b(x,y)$ of an operator B be a continuous function on $X \times X$ such that $b(x,y)$ is nowhere equal to $\mathbb{0} = +\infty$ and attains its minimum at a unique point (w,w), which lies on the diagonal in $X \times X$. Let $b(w,w) = 0$ (this assumption does not result in any loss in generality, since it can always be ensured by a shift by an appropriate constant). Then the eigenvalue of B is equal to $\mathbb{1} = 0$, and the iterations B^n with integral kernels $b^n(x,y)$ converge as $n \to \infty$ to the operator $\overline{B}$ with separated kernel*

$$\overline{b}(x,y) = \varphi(x) + \psi(y),$$

where $\varphi(x) = \lim_{n\to\infty} b^n(x,w)$ is the unique eigenfunction of B and $\psi(x) = \lim_{n\to\infty} b^n(w,x)$ is the unique eigenfunction of the adjoint operator, i.e. of the operator $\tilde{B}$ with the integral kernel $\tilde{b}(x,y) = b(y,x)$.

Proof. Let $\overline{y}_1, \dots, \overline{y}_{n-1}$ be the points at which the minimum is attained in the expression

$$b^n(x,z) = \min_{y_1,\dots,y_{n-1}} (b(x,y_1) + b(y_1,y_2) + \cdots + b(y_{n-1},z))$$

for the kernel of B^n.

Since $0 \le b^n(x,z) < b(x,w) + b(w,z)$, it follows that $b^n(x,z)$ is uniformly bounded with respect to x, z, and n and moreover, for any $\varepsilon > 0$ and any sufficiently large n, all but finitely many $\overline{y}_j$ lie in the ε-neighborhood U_ε of w. Since $b(x,y)$ is continuous, we see that

$$\forall\, \delta > 0\ \exists\, \varepsilon > 0: \quad b(t,z) < \delta \text{ for } t,z \in U_\varepsilon.$$

Let $\overline{y}_j \in U_\varepsilon$. Then for $m \ge 1$ we have

$$\begin{aligned} b^{n+m}(x,z) &\le b(x,\overline{y}_1) + \cdots + b(y_{j-1},\overline{y}_j) \\ &\qquad + b(\overline{y}_j, w) + b(w, \overline{y}_{j+1}) + \cdots + b(\overline{y}_{n-1}, z) \\ &\le b^n(x,z) + 2\delta. \end{aligned}$$

Consequently, the sequence $b^n(x,z)$ is "almost decreasing," that is,

$$\forall\, \delta > 0 \quad \exists\, N \quad \forall\, n > N: \quad b^n(x,z) \le b^N(x,z) + \delta.$$

In conjunction with boundedness, this property implies that the limit

$$\lim_{n\to\infty} b^n(x,z) = \beta(x,z)$$

exists. Since, obviously,

$$b^{2n}(x,z) = b^n(x,t(n)) + b^n(t(n),z)$$

for some $t(n) \to w$ as $n \to \infty$, we obtain, by passing to the limit,

$$\beta(x,z) = \beta(x,w) + \beta(w,z).$$

Thus, the kernel of the limit operator is separated, which, in particular, implies that the eigenfunction is unique. Let us prove that $\beta(x,w)$ is an eigenfunction of B with eigenvalue $\mathbb{1} = 0$. Indeed,

$$\begin{aligned} B(\beta(x,w)) &= \inf_y \Big(b(x,y) + \lim_{n\to\infty} b^n(y,w) \Big) \\ &= \lim_{n\to\infty} \inf_y (b(x,y) + b^n(y,w)) = \lim_{n\to\infty} b^{n+1}(x,w) = \beta(x,w). \end{aligned}$$

Let us also point out that the uniform continuity of $b(x,y)$ implies the continuity of $\beta(x,z)$ and that the convergence $b^n(x,z) \to \beta(x,z)$ is uniform with respect to (x,z).

Theorem 5 can readily be generalized to the case in which the performance function $b(x,y)$ has several points of minimum. It is only essential that these points lie on the diagonal in $X \times X$. In particular, the following result is valid.

Theorem 6 *Let X be a compact set, and let the integral kernel $b(x,y)$ of an operator B be a continuous function on $X \times X$ that is nowhere equal to $\mathbb{0} = +\infty$ and that attains its minimum λ at some points (w_j, w_j), $j = 1, \dots, k$, on the diagonal in $X \times X$. Then the eigenvalue of B is equal to λ, the functions $\varphi_j = \lim_{n\to\infty} b^n(x, w_j)$ (respectively, $\psi_j = \lim_{n\to\infty} b^n(w_j, x)$), $j = 1, \dots, k$, form a basis of the eigenspace of B (respectively, of the adjoint B'), and the iterations $(B - \lambda)^n$ converge to a finite-dimensional operator with separated kernel*

$$\bar{b}(x,y) = \bigoplus_{j=1}^{n} \varphi_j(x) \odot \psi_j(y) = \min_j (\varphi_j(x) + \psi_j(y)).$$

In more general cases, the connected components of the set of minima of $b(x,y)$ must be used instead of the points (w_j, w_j). Possible generalizations to problems with continuous time or infinite dimensional state space are given in are given in Sections 6 and 10.

4 Infinite Extremals and Turnpikes in Dynamic Optimisation and Mathematical Economics

In this section, we discuss possible applications of the idempotent spectral analysis to the study of dynamic optimisation problems with infinite planning horizon. In particular, we sketch the theory of infinite extremals, which is due essentially to S. Yakovenko [57], [58].

Let $\mathrm{extr}_n(b, f)$ be the set of solutions (extremals) to the finite-horizon optimization problem

$$\sum_{k=0}^{n-1} b(x_k, x_{k+1}) + f(x_n) \to \min . \tag{10}$$

Then it follows from Bellman's optimality principle [7] that

$$x_{k+1} \in \arg\min_{y \in X} (b(x_k, y) + (B^{n-k}) f(y))$$

for each $\{x_k\} \in \mathrm{extr}_n(b, f)$, where B is the Bellman operator with kernel $b(x, y)$, i.e.,

$$(Bf)(x) = \min_y (b(x, y) + f(y))$$

for any continuous real function f.

Let h be an eigenfunction of the operator B, that is, a solution of the equation $Bh = \lambda \odot h = \lambda + h$. An infinite trajectory $\kappa = \{x_k\}_{k=0}^{\infty}$ is called an *infinite extremal* (or an *h-extremal*) if

$$x_{k+1} \in \arg\min_{y \in X} (b(x_k, y) + h(y))$$

for each $k = 0, 1, \ldots$.

Let $\mathrm{extr}_\infty(B, h)$ denote the set of all (infinite) h-extremals, and let $\lambda = \mathrm{Spec}(B)$. It is easy to see that

$$\mathrm{extr}_\infty(B, \lambda \odot h) = \mathrm{extr}_\infty(B, h).$$

The following result, which shows that the notion introduced is meaningful, is a direct consequence of the definition, Bellman's optimality principle, and the spectral theorem 4.

Proposition 2 *Let B be a Bellman operator with continuous real kernel, and let $a \in X$ be an arbitrary initial state. Then the following assertions hold.*

(a) *There exists an infinite extremal $\kappa = \{x_k\}_{k=0}^{\infty}$ issuing from a.*

(b) *The relationship between B and the set of its extremals is conjugation-invariant: if $B = C^{-1} \circ B' \circ C$, where an invertible operator C is the composition of a diagonal operator with a "change of variables" $f(x) \mapsto f(\beta^{-1}(x))$ for some homomorphism β (see Theorem 3), then*

$$\kappa \in \mathrm{extr}(B, h) \iff \beta(\kappa) \in \mathrm{extr}_\infty(B', Ch),$$

where

$$\beta(\kappa) = \{\beta(x_k)\}_{k=0}^{\infty}.$$

(c) *If* $\kappa \in \mathrm{extr}_\infty(B,h)$, *then each segment* $\{x_k\}_{k=k'}^{k=k''}$ *is a finite extremal of the n-step optimization problem* (10) *with fixed initial point and with terminant* $f(x_n) = h(x_n)$ $(n = k'' - k')$. *In particular, this segment is a solution of the optimization problem*

$$\sum_{k=k'}^{k''-1} b(x_k, x_{k+1}) \to \min$$

with fixed endpoints.

One can introduce a weaker notion of an extremal, which is also invariantly related to B. Let $\lambda = \mathrm{Spec}(B)$. Then $\kappa = \{x_k\}_{k=0}^{\infty}$ is called a λ-*trajectory* if

$$\sum_{k=0}^{n-1} b(x_k, x_{k+1}) = n\lambda + O(1) \quad \text{as } n \to \infty.$$

It is easy to see that each infinite extremal is a λ-trajectory. However, unlike in the case of extremals, a trajectory differing from a λ-trajectory by a finite number of states is itself a λ-trajectory. Thus, the notion of λ-trajectories reflects limit properties of infinite extremals. In what follows we assume that $\mathrm{Spec}(B) = 0$. This can always be achieved by adding an appropriate constant.

Generally speaking, the eigenfunction of an operator is not unique, so there exist several various types of infinite extremals issuing from a given point. However, one can always single out an infinite extremal that is (in a sense) the limit as $n \to \infty$ of finite extremals of problem (10) with fixed terminant. More precisely, the following theorem is valid.

Proposition 3 *Let* B *be a Bellman operator with continuous kernel and with* $\mathrm{Spec}(B) = 0$. *Then there exists a unique "projection" operator* Ω *in* $C(X)$ *such that*

(a) Ω *is a linear operator in the semimodule* $C(X,A)$ (*here* $A = \mathbb{R} \cup \{+\infty\}$, $\oplus = \min$, *and* $\odot = +$);

(b) *the relation between* Ω *and* B *is conjugation-invariant, that is, if* $B = C^{-1} \circ B' \circ C$ *for some invertible operator* C *and* Ω' *is the projection operator corresponding to* B', *then* $\Omega' = C^{-1} \circ \Omega \circ C$;

(c) $\Omega f = f \iff f$ *is an eigenfunction of* B;

(d) Ωf *is an eigenfunction of* B *for any* f;

(e) $\Omega B = \Omega$.

Note that properties (c) and (d) are equivalent to the operator identities $B\Omega = \Omega$ and $\Omega^2 = \Omega$.

Proof. The existence and the properties of Ω readily follow from the explicit formula

$$\Omega f = \lim_{n\to\infty} \bigoplus_{n=N}^{\infty} B^n f = \lim_{n\to\infty} \inf_{n\geq N} B^n f.$$

Since $\mathrm{Spec}(B) = 0$, it follows that all $B^n f$ are bounded and the infimum exists. Let us prove the uniqueness. Suppose that $\widetilde{\Omega}$ satisfies the same conditions. Then

$$\Omega f = \widetilde{\Omega}\Omega f = \widetilde{\Omega}\left(\lim_{N\to\infty} \bigoplus_{n=N}^{\infty} B^n f \right)$$

$$= \lim_{N\to\infty} \bigoplus_{n=N}^{\infty} \widetilde{\Omega} B^n f = \lim_{N\to\infty} \widetilde{\Omega} f = \widetilde{\Omega} f.$$

Before going further, let us describe shortly the traditional approach to the definition of infinite extremals in the formal optimization problem

$$\sum_{k=0}^{\infty} b(x_k, x_{k+1}) \to \min,$$

where $b: X \times X \to A$ is a continuous function, X is a metric compactum, $x_0 = a$ is fixed, and $x_k \in X$, $k = 0, 1, \ldots$. One says that a trajectory $\kappa' = \{x'_k\}_{k=0}^{\infty}$ *overtakes* (respectively, *supertakes*) a trajectory $\kappa = \{x_k\}_{k=0}^{\infty}$ if $x_0 = x'_0$ and

$$\delta(\kappa', \kappa) = \varliminf_{n\to\infty} \sum_{k=0}^{n-1} (b(x'_k, x'_{k+1}) - b(x_k, x_{k+1})) \leq 0$$

(respectively, $\delta(\kappa', \kappa) < 0$). A trajectory is said to be *weakly optimal* if it is not supertaken by any other trajectory. A trajectory is said to be *overtaking* if it overtakes any other trajectory with the same starting point.

Although these notions are frequently used (e.g., see [50], where a variety of other possible definitions of the same type are discussed), the set of, say, weakly optimal trajectories is empty in quite a few reasonable optimization problems. However, if such trajectories do exist, they are infinite extremals in the sense defined above.

Consider now a special situation described in Theorem 5. This situation includes, for instance, an important case of a convex utility function. The λ-trajectories for such operators B possess the turnpike property.

Theorem 7 *Let the assumptions of Theorem* 5 *be satisfied. Then*

(a) *If* $\kappa = \{x_k\}_{k=0}^{\infty}$ *is a* λ*-trajectory for* B*, then*

$$\lim x_k = w. \tag{11}$$

(b) *Each infinite extremal is a weakly optimal trajectory.*

Proof. (a) Note that

$$\forall\,\varepsilon > 0 \quad \exists\,\delta > 0 : \quad \rho(x,w) \geq \delta \implies \forall\,y \quad b(x,w) > \varepsilon, \tag{12}$$

where ρ is the distance function on X. If (11) is violated, then, according to (12), the sum $\sum b(x_k, x_{k+1})$ along κ tends to $+\infty = \mathbb{0}$, which contradicts the fact that $\operatorname{Spec}(B) = 0 = \mathbb{1}$ and Eq. (7).

(b) Obviously, a λ-trajectory can only be overtaken by another λ-trajectory. Now assume that some λ-trajectory $\kappa' = \{x_k\}_{k=0}^{\infty}$ supertakes an infinite extremal $\kappa = \{x_k\}_{k=0}^{\infty}$, $x'_0 = x_0 = a$. Then, by definition, there exists a sequence N_j such that

$$\sum_{k=0}^{N_j} (b(x'_k, x'_{k+1}) - b(x_k, x_{k+1})) \leq -\varepsilon < 0. \tag{13}$$

But according to Proposition 2, the sum $\sum_{k=0}^{N-1} b(x_k, x_{k+1})$ is the minimum in the N-step optimization problem with fixed endpoints $x_0 = a$ and x_N. Since $x_N \to w$, it follows that we can choose a neighborhood U of the point w so that the minima in the N-step problems with fixed endpoints $x_0 = a$ and $x \in U$ are uniformly close to one another for all $x \in U$ and $N \geq N_0$. This contradicts (13), and so the proof is complete.

Needless to say, the point w is a turnpike in problems with fixed (but large) planning horizon. Let us state the related result in a more general form.

Theorem 8 *Let X be a locally compact metric space with metric ρ, and let continuous functions*

$$f\colon X \to \mathbb{R} \cup \{+\infty\}, \qquad b\colon X \times X \to \mathbb{R} \cup \{+\infty\}$$

bounded below be given (*thus, f and b are continuous bounded A-valued functions, where A is the standard idempotent semiring*)*. Let F and λ be the greatest lower bounds of $f(x)$ and $b(x,y)$, respectively. Suppose that the set*

$$W = \{w \in X : b(w,w) = \lambda\}$$

is not empty. Let $\kappa = \{x_k\}_{k=0}^{n}$ be an optimal trajectory for problem (10) *with fixed starting point $x_0 = a \in X$, and suppose that there exists a $w \in W$ such that $b(a,w) + f(w) \neq +\infty$. Then for any $\varepsilon > 0$ there exists a positive integer K such that for all positive integers n, however large, the inequality $\rho(x_k, W) < \varepsilon$ is violated at most at K points of the trajectory κ.*

Proof. Without loss of generality we can assume that $\lambda = 0$. Let $w \in W$ be an arbitrary point. Then the functional

$$\sum_{k=0}^{n-1} b(x_k, x_{k+1}) + f(x_n) \tag{14}$$

to be minimized attains the value $b(a, w) + f(w) \stackrel{\text{def}}{=} C$, independent of n, on the trajectory $\kappa_w = \{x_k\}_{k=0}^n$, where $x_0 = a$ and $x_j = w$, $j = 1, \dots, n$. Consequently, the value of problem (10) does not exceed c for all n. Furthermore, it follows from the continuity of $b(x, y)$ that for each $\delta > 0$ there exists an $\varepsilon > 0$ such that $b(x, y) \geq \delta > 0 = b(w, w)$ whenever $\rho(x, W) > \varepsilon$ or $\rho(y, w) > \varepsilon$. It follows that if the inequality $\rho(x_k, w) < \varepsilon$ is violated more than K times on some trajectory, then the value of the functional (14) on that trajectory exceeds $K\delta + F$, which is greater than C for $K > (C - F)/\delta$. Consequently, for these K the trajectory cannot be optimal.

To illustrate the utility of these results in mathematical economics, let us show how the well-known turnpike theorem for the classical von Neumann–Gale (NG) model can be derived from theorems 7 8. Generalisations of various kinds can be obtained from Theorem 8 and the results of section 10. First let us recall the definitions. An NG model is specified by a closed convex cone $\mathbb{Z} \subset \mathbb{R}_+^n \times \mathbb{R}_+^n$ such that $(0, y) \notin \mathbb{Z}$ for any $y \neq 0$ and the projection of $\mathbb{Z}$ on the second factor has a nonempty intersection with the interior of $\mathbb{R}_+^n$. The cone $\mathbb{Z}$ uniquely determines a set-valued mapping $a \colon \mathbb{R}_+^n \to 2^{\mathbb{R}_+^n}$ by the rule

$$y \in a(x) \iff (x, y) \in \mathbb{Z}.$$

In the economical interpretation, the mapping a describes possible transitions from one set of goods to another in one step of the production process under a prescribed technology. A triple (α, y, p), where $\alpha > 0$, $z = (y, \alpha y) \in \mathbb{Z}$, and $p \in \mathbb{R}_+^n \setminus \{0\}$ is called an *equilibrium state* of the NG model if

$$\alpha(p, x) \geq (p, v) \quad \forall\, (x, v) \in \mathbb{Z}.$$

If, moreover, $(p, y) > 0$, then the equilibrium is said to be nondegenerate, the coefficient $\alpha > 0$ is referred to as the *growth rate*, and p is known as the vector of equilibrium prices. A *trajectory* in the NG model is a sequence $\{x_k\}_{k=1}^T$, $T \in \mathbb{N}$, such that $(x_k, x_{k+1}) \in \mathbb{Z}$ for all k. For a given utility function $u \colon \mathbb{R}_+^n \to \mathbb{R}$, the planning problem on a time horizon $[0, T]$ is to find a trajectory $\{x_k\}_{k=1}^T$ on which the terminal performance functional $u(x_T)$ attains its maximal value. Such a trajectory is said to be optimal.

It will be convenient to use the angular metric $\rho(x, y) = |x/\|x\| - y/\|y\||$ on the set of rays in $\mathbb{R}_+^n$.

A ray $\{\alpha y : \alpha \in \mathbb{R}_+\}$, $y \in \mathbb{R}_+^n$, is called a *strong* (respectively, *weak*) *turnpike* if for each $\varepsilon > 0$ there exists a positive integer $K = K(\varepsilon)$ such that for each optimal trajectory $\{x_k\}_{k=1}^T$, regardless of the planning horizon $T \in \mathbb{N}$ and of the utility function from a given class U, the inequality $\rho(x_k, y) < \varepsilon$ can be violated only for the first K and the last K indices k (respectively, for at most K indices k). The optimization problem for the NG model is known to be reducible to a multistep optimization problem on a compactum. using this reduction, we obtain now, as a direct consequence of Theorem 8 (more precosely, of its analog in which min is replaced by max), the following well-known Radner's theorem about weak turnpikes.

Proposition 4 *Suppose that*

1) *An NG model is given, determined by a cone $\mathbb{Z}$ such that*

R1) *there exists an $\alpha > 0$ and a $y \in \mathbb{R}^n_+ \setminus \{0\}$ with $z = (y, \alpha y) \in \mathbb{Z}$;*

R2*) there exists a $p \in \mathbb{R}^n$ (a price vector) such that $\alpha(p, x) > (p, v)$ for any vector $(x, v) \in \mathbb{Z}$ that is not a multiple of $(y, \alpha y)$ (actually, this condition means that the cone $\mathbb{Z}$ is strictly convex in the vicinity of the point $(y, \alpha y)$);*

R3) *for each $x \in \mathbb{R}^n_+$ there exists an $L > 0$ such that $(x, Ly) \in \mathbb{Z}$ (this is a purely technical condition, which can be ensured by an arbitrarily small perturbation of the model and which means that the turnpike proportions can be achieved from an arbitrary initial state).*

2) *A class $U = \{u : \mathbb{R}^n_+ \to \mathbb{R}\}$ of utility functions is given such that each $u \in U$ satisfies the following conditions*:

R4) $u(x)$ *is continuous and nonnegative*;

R5) $u(\lambda x) = \lambda u(x)\ \forall\, x \in \mathbb{R}^n_+\ \forall\, \lambda > 0$;

R6) $u(y) > 0$ (*the consistency condition*);

R7) *there exists a $k > 0$ such that $u(y) \leq k(p, y)$.*

Then the ray $\{\alpha y : \alpha > 0\}$ is a weak turnpike.

Note that any optimal trajectory $\{x_k\}_{k=0}^T$ in an NG model satisfies the following *maximal expansion condition* at each step:

$$\max\{\mu : (x_{k-1}, \mu x_k) \in \mathbb{Z}\} = 1, \qquad k = 1, \ldots, T.$$

Thus, in seeking optimal trajectories, only trajectories satisfying this condition will be considered feasible.

Let us now consider a multistep optimization problem on the set

$$\Pi = \{x \in \mathbb{R}^n_+ : (p, x) = 1\}$$

equipped with the metric induced by the angular metric on the set of rays. We introduce the transition function

$$b(x, v) = \ln \max\{\lambda > 0 : (x, \lambda v) \in \mathbb{Z}\},$$

where $b = -\infty$ is assumed if the set in the braces is empty. It follows from conditions R1)–R2) that

$$\alpha = b(y, y) = \max\{b(x, v) : x, v \in \Pi\}.$$

To each trajectory $\{x_k\}_{k=0}^T$ of the NG model there corresponds a unique sequence $\{v_k\}_{k=0}^T$ of the points in Π such that v_k and x_k lie on the same ray, $k = 0, \dots, T$. Moreover, by condition R5) we have

$$\ln u(x_T) = \sum_{k=0}^{T-1} b(v_k, v_{k+1}) + \ln u(v_T)$$

on the trajectories satisfying the maximal expansion condition, and so the problem of constructing optimal trajectory in the NG model is equivalent to the multistep optimization problem with the performance functional (14). Properties R1)–R7) of the model and of the utility function ensure the validity of all assumptions in Theorem 8. In particular, the set W is a singleton (its unique element lies on the turnpike ray $\{\alpha y : \alpha > 0\}$).

Using the same reduction of the NG model to a general multistep optimisation problem and the concept of infinite extremals, described above, we obtain a natural definition of infinite extremals in the NG model, which coincides with the classical definition based on the Pareto order in $\mathbb{R}_+^n$.

5 Homogeneous Operators in Idempotent Analysis and Turnpikes for Stochastic Games

Additive and homogeneous operators are important generalizations of linear operators. This section deals with operators homogeneous in the sense of the semiring $A = \mathbb{R} \cup \{+\infty\}$, i.e., operators B on function spaces such that

$$B(\lambda + h) = \lambda + B(h)$$

for any number λ and any function h. We shall show that the theory of such operators is closely related to game theory and obtain an analog of the eigenvalue theorem for such operators. We apply this analog to construct turnpikes in stochastic games. For simplicity, we only consider the case of a finite state space $X = \{1, \dots, n\}$ in detail.

First, let us show whence homogeneous operators appear. Let us define an antagonistic game on X. Let $p_{ij}(\alpha, \beta)$ denote the probability of transition from state i to state j if the two players choose strategies α and β, respectively (α and β belong to some fixed metric spaces), and let $b_{ij}(\alpha, \beta)$ denote the income of the first player from this transition. The game is called a *game with value* if

$$\begin{aligned}\min_\alpha \max_\beta \sum_{j=1}^n p_{ij}(\alpha, \beta)(h^j + b_{ij}(\alpha, \beta)) \\ = \max_\beta \min_\beta \sum_{j=1}^n p_{ij}(\alpha, \beta)(h^j + b_{ij}(\alpha, \beta))\end{aligned} \tag{15}$$

for all $y \in \mathbb{R}^n$. In that case, the operator $B: \mathbb{R}^n \to \mathbb{R}^n$ such that $B_i(y)$ is equal to (15) is called the *Bellman operator* of the game. By the dynamic programming method [7], we can show that the value of the k-step game defined by the initial position i and the terminal income $h \in \mathbb{R}^n$ of the first player exists and is equal to $B_i^k(h)$.

It is clear that the operator B has the following two properties:

$$B(ae + h) = ae + B(h) \quad \forall a \in \mathbb{R},\ h \in \mathbb{R}^n,\ e = (1, \dots, 1) \in \mathbb{R}^n, \tag{16}$$

$$\|B(h) - B(g)\| \le \|h - g\| \quad \forall h, g \in \mathbb{R}^n, \tag{17}$$

where $\|h\| = \max |h^i|$. Interestingly, these two properties are characteristic of the game Bellman operator. Namely, as it was proven in [[23]], each map satisfying (16) and (17) can be represented in form (15). Another characterisation of homogeneous nonexpansive maps was obtained in [14]: if $B: \mathbb{R}^n \to \mathbb{R}^n$ satisfies (16), then B is nonexpansive (i.e. it satisfies (17)) if and only if it is order-preserving.

In studying homogeneous operators, it is useful to define the quotient space Φ of the space $\mathbb{R}^n$ by the one-dimensional subspace generated by the vector $e = (1, \dots, 1)$. Let $\Pi: \mathbb{R}^n \mapsto \Phi$ be the natural projection. The quotient norm on Φ is obviously defined by the formula

$$\|\Pi(h)\| = \inf_{a \in \mathbb{R}} \|h + ae\| = \frac{1}{2}\Big(\max_j h^j - \min_j h^j\Big).$$

It is clear that Π has a unique isometric (but not linear) section $S: \Phi \mapsto \mathbb{R}^n$. The image $S(\Phi)$ consists of all $h \in \mathbb{R}^n$ such that $\max_j h^j = -\min_j h^j$. By virtue of properties (16) and (17) of B, the continuous quotient map $\widetilde{B}: \Phi \mapsto \Phi$ is well defined.

To state the main result of this section, we need some additional technical assumptions on the transition probabilities:

$$\exists \delta > 0 : \forall i, j, \alpha \quad \exists \beta : p_{ij}(\alpha, \beta) \ge \delta, \tag{18}$$

$$\exists \delta > 0 : \forall i, j \quad \exists m : \forall \alpha, \beta : p_{im}(\alpha, \beta) > \delta,\ p_{jm}(\alpha, \beta) > \delta. \tag{19}$$

Let all $|b_{ij}(\alpha, \beta)|$ be bounded by some constant C. The proof of the following simple fact can be found in [23] or [33]

Proposition 5 A) *If* (18) *holds and $\delta < 1/n$, then $\widetilde{B}$ maps each ball of radius $R \ge C\delta^{-1}$ centered at the origin into itself.*

B) *If* (19) *holds, then*

$$\|\widetilde{B}(H) - \widetilde{B}(G)\| \le (1 - \delta)\|H - G\|, \qquad \forall H, G \in \Phi.$$

As a direct consequence of this proposition the fixed point theorems, one obtains the following result.

Theorem 9 A) *If* (18) *holds, then there exists a unique* $\lambda \in \mathbb{R}$ *and a vector* $h \in \mathbb{R}^n$ *such that*

$$B(h) = \lambda + h \tag{20}$$

and for all $g \in \mathbb{R}^n$ *we have*

$$\|B^m g - m\lambda\| \leq \|h\| + \|h - g\|, \tag{21}$$

$$\lim_{m\to\infty} \frac{B^m g}{m} = \lambda. \tag{22}$$

B) *If* (19) *holds, then* h *is unique* (*up to equivalence*), *and*

$$\lim_{m\to\infty} S \circ \Pi(B^m(g)) = S \circ \Pi(h) \quad \forall g \in \mathbb{R}^n. \tag{23}$$

Let $E(g)$ denote the set of equilibrium strategies for a vector g, i.e.

$$E(g) = \{(\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n) : B_i(g) = \sum_{j=1}^{n} p_{ij}(\alpha_i, \beta_i)(g^j + b_{ij}(\alpha_i, \beta_i))\}.$$

In particular, the set $E(h)$ with h being a solution of of (20), contains stationary strategies in the infinite-time game. Suppose that the set $E(g)$ depends continuously on g. Then Theorem 9 (more precisely, formula (23)) implies the following turnpike theorem for the game in question.

Theorem 10 *Let* (19) *hold. Then for an arbitrary* $\Omega > 0$ *and an arbitrary neighborhoud* $U(E(h))$ *of the set* $E(h)$, *there exists an* $M \in \mathbb{N}$ *such that if* $E(B^{T-t}g)$ *denote the set of equilibrium strategies in the* T*-step game,* $T > M$, *with terminal income of the first player defined by a vector* g *with* $\|\Pi(g)\| \leq \Omega$, *then*

$$E(B^{T-t}g) \subset U(E(h))$$

for all $t < T - M$.

Now suppose additionally that $E(h)$ contains only one pair of strategies $\{\widetilde{\alpha}_1, ..., \widetilde{\alpha}_n, \widetilde{\beta}_1, ..., \widetilde{\beta}_n\}$. Let $Q^* = (q_1^*, \ldots, q_n^*)$ denote the stationary distribution for the stationary Markov chain defined on the state space X by these strategies. Then one can obtain the the following turnpike theorem on the state space.

Theorem 11 ([23]) *For all* $\alpha > 0$ *and* $\Omega > 0$ *there exists an* $M \in \mathbb{N}$ *such that for each* T*-step game,* $T > 2M$, *with terminal income* $g \in \mathbb{R}^n$, $\|\Pi(g)\| < \Omega$, *of the first player we have*

$$\|Q(t) - Q^*\| < \varepsilon$$

for $t \in [M, T - M]$, *where* $Q(t) = (q_1(t), \ldots, q_n(t))$ *and* $q_i(t)$ *is the probability that the process is in the state* $i \in X$ *at time* t *if the game is carried out with the equilibrium strategies.*

In other words, q_j^* is the mean amount of time that each sufficiently long game with equilibrium strategies spends in position j.

Proof. It follows from Theorem 10 that for each $\varepsilon_1 > 0$ there exists an $M_1 \in \mathbb{N}$ such that for any t-step equilibrium game, $t > M_1$, with the first player's terminal income g, $\|\Pi(g)\| \leq \Omega$, the transition probabilities at the first $t - M_1$ steps are ε_1-close to the transition probabilities $p_{ij}(\tilde{\alpha}_i, \tilde{\beta}_i)$. Consequently,

$$Q(t) = Q^0(P + \delta_1) \cdots (P + \delta_t) = Q^0(P^t + \Delta(t)),$$

where the matrices δ_k (and, hence, $\Delta(t)$) are ε_1-close to zero. By a theorem on the convergence of probability distributions in homogeneous Markov chains to a stationary distribution, we have

$$\|Q^0 P^t - Q^*\| \leq (1 - \delta)^{t-1}.$$

Thus, we can successively choose M_2 and ε_1 so that

$$\|Q(M_2) - Q^*\| < \varepsilon \quad \text{for all } Q^0.$$

Then

$$\|Q(t) - Q^*\| < \varepsilon \quad \text{for all } t \in [M_2, T - M_1].$$

There are natural generalizations of conditions (18) and (19) under which the cited results can still be proved. Namely, one can require these conditions to be valid for some iteration of the operator B. This is the case of cyclic games. Some generalizations to n-person games were obtained in [30].

If each coordinate of a homogeneous nonexpansive operator B is convex, then B has the form of a Bellman operator of some controlled Markov process. Thus, Theorems 10 and 11, in particular, contain the turnpike theorems for Markov processes. Generalization of the obtained results to the case of continuous time and general state-space is given in [33].

6 Generalized Solutions of the HJB Equation

This section is devoted to a short presentation of the theory of generalised solutions to the Cauchy problem

$$\begin{cases} \dfrac{\partial S}{\partial t} + H\left(x, \dfrac{\partial S}{\partial x}\right) = 0, \\ S|_{t=0} = S_0(x) \end{cases} \tag{24}$$

of a Hamilton-Jacobi equation with Hamiltonian H being convex in p. More circumstancial exposition can be found in [32], [33]. Since any convex function can be whritten in the form

$$H(x,p) = \max_{u \in U}(pf(x,u) - g(x,u)) \tag{25}$$

with some functions f, g, equation in (24) can be written in the equivalent form

$$\frac{\partial S}{\partial t} + \max_{u\in U}\left(\left\langle \frac{\partial S}{\partial t}, f(x,u)\right\rangle - g(x,u)\right) = 0, \tag{26}$$

which is called the *nonstationary Bellman differential equation.*

Let us discuss first, what difficulties occur when one tries to give a reasonable definition of the solutions to problem (24)? First, as simple examples shows, the classical (i.e., smooth) solution of the Cauchy problem (24) does not exist for large time even for smooth H and S_0. All the more, one cannot hope to obtain smooth solutions for nonsmooth H and S_0. On the other hand, in contrast with the theory of linear equations, where generalized solutions can be defined in the standard way as functionals on the space of test functions, there is no such approach in the theory of nonlinear equations.

The most popular approach to the theory of generalized solutions of the HJB equation is the vanishing viscosity method. This means that one defines a generalized solution to problem (24) as the limit as $h \to 0$ of solutions of the Cauchy problem

$$\begin{cases} \dfrac{\partial w}{\partial t} + H\left(x, \dfrac{\partial w}{\partial x}\right) - \dfrac{h}{2}\dfrac{\partial^2 w}{\partial x^2} = 0, \\ w|_{t=0} = w_0(x) = S_0(x). \end{cases} \tag{27}$$

For continuous initial data and under some reasonable restrictions on the growth of H and S_0, one can prove that there exists a unique smooth solution $w(t,x,h)$ of problem (27) and that the limit $S(t,x) = \lim_{h\to 0} w(t,x,h)$ exists and is continuous. Furthermore, it turns out that the solutions thus obtained are selected from the set of continuous functions satisfying the Hamilton–Jacobi equation almost everywhere by some conditions on the discontinuities of the derivatives. In some cases, these conditions have a natural physical interpretation. They also can be used as a definition of a generalised solution. A detailed exposition of these approaches can be found in [12, 36, 51].

However, according to [36], this method cannot be used to construct a reasonable theory of generalized solutions to (24) for discontinuous initial functions. Furthermore, it is highly desirable to devise a theory of problems (24) on the basis of only intrinsic properties of HJB equations (i.e., regardless of the way in which the set of HJB equations is embedded in the set of higher-order equations). Such a theory, including solutions with discontinuous initial data, can be constructed for equations with convex Hamiltonians on the basis of idempotent analysis, using the new superposition principle for the solutions of (24) (which was first noted in [39],[40]) and the idempotent analogue of the inner product

$$\langle f, g\rangle_A = \inf_x f(x) \odot g(x), \tag{28}$$

replacing the usual L_2-product. We discuss it now in more detail, defining generalized solutions for the case of a smooth function H. For nonsmooth H, a limit procedure can be easily applied.

Let H satisfy the following conditions.

1) H is C^2 and the second derivatives of H are uniformly bounded:

$$\max\left(\sup_{x,p}\left\|\frac{\partial^2 H}{\partial x^2}\right\|, \sup_{x,p}\left\|\frac{\partial^2 H}{\partial x \partial p}\right\|, \sup_{x,p}\left\|\frac{\partial^2 H}{\partial p^2}\right\|\right) \leq c = \mathrm{const}\,.$$

2) H is strongly convex; that is, there exists a constant $\delta > 0$ such that the least eigenvalue of the matrix $\partial^2 H/\partial p^2$ is not less than δ for all (x,p).

By $(y(\tau,\xi,p_0), p(\tau,\xi,p_0))$ we denote the solution of Hamilton's system

$$\begin{cases} \dot{y} = \dfrac{\partial H}{\partial p}, \\ \dot{p} = -\dfrac{\partial H}{\partial x} \end{cases} \tag{29}$$

with the initial conditions $y(0) = \xi$, $p(0) = p_0$. Let $S(t,x,\xi)$ denote the greatest lower bound of the action functional

$$\int_0^t L(z(\tau), \dot{z}(\tau))\, d\tau \tag{30}$$

over all continuous piecewise smooth curves joining ξ with x in time t ($z(0) = \xi$ and $z(t) = x$). Here $L(x,v)$ is the Lagrangian of the variational problem associated with H, that is, the Legendre transform of H with respect to the variable p.

One can show (see e.g. [33]) that under the given assumptions on H the two-point function $S(t,x,\xi)$ is smooth for all x, ξ and $t \in (0, t_0)$ with some t_0, and strictly convex in both ξ and x. It follows that if the initial function $S_0(x)$ in the Cauchy problem (24) is convex, then the function

$$(R_t S_0)(x) = S(t,x) = \min_{\xi}(S_0(\xi) + S(t,x,\xi)) \tag{31}$$

is continuously differentiable for all $t \leq t_0$ and $x \in \mathbb{R}^n$. Indeed, the minimum in (31) is obviously attained at a unique point $\xi(t,x)$. It follows then from the calculus of variation that Eq. (31) specifies the unique classical solution of the Cauchy problem (24). To define generalised solution of this Cauchy problem with arbitrary initial data we proceed as follows.

As was noted in Section 2, smooth convex functions form a "complete" subset in $C_{\mathbb{0}}(\mathbb{R}^{2n})$, since they approximate the idempotent "δ-function"

$$\delta_\xi(x) = \lim_{n\to\infty} n(x-\xi)^2 = \begin{cases} \mathbb{1} = 0, & x = \xi, \\ \mathbb{0} = +\infty, & x \neq \xi. \end{cases}$$

Consequently, each functional $\varphi \in (C_{\mathbb{0}}(\mathbb{R}^n))^*$ is uniquely determined by its values on this set of functions.

The Cauchy problem

$$\begin{cases} \dfrac{\partial S}{\partial t} + H\left(x, -\dfrac{\partial S}{\partial x}\right) = 0, \\ S|_{t=0} = S_0(x) \end{cases} \tag{32}$$

with Hamiltonian $\widetilde{H}(x,p) = H(x,-p)$ will be called the *adjoint problem* to the Cauchy problem (24). This terminology is due to a simple observation that the classical resolving operator R_t^* of the Cauchy problem (32) is determined on smooth convex functions by the formula

$$(R_t^* S_0)(x) = \min_\xi (S_0(\xi) + S(t, \xi, x)), \tag{33}$$

is linear (with respect to the operations $\oplus = \min$ and $\odot = +$) on this set of functions, and is the adjoint of the resolving operator R_t (31) of the initial Cauchy problem with respect to the inner product (28). We are now in a position to define weak solutions of problem (24) by analogy with the theory of linear equations; we also take into account the fact that, by Theorem 1, the functionals $\varphi \in (C_0(\mathbb{R}^n))^*$ are given by usual functions bounded below.

Let $S_0 \colon \mathbb{R}^n \to A = \mathbb{R} \cup \{+\infty\}$ be a function bounded below, and let $m_{S_0} \in (C_0(\mathbb{R}^n))^*$ be the corresponding functional. Let us define the *generalized weak solution* of the Cauchy problem (24) as the function $(R_t S_0)(x)$ determined by the equation

$$m_{R_t S_0}(\psi) = m_{S_0}(R_t^* \psi),$$

or equivalently

$$\langle R_t S_0, \psi \rangle_A = \langle S_0, R_t^* \psi \rangle_A,$$

for all smooth strictly convex functions ψ.

The following theorem is a direct consequence of this definition, Theorem 1 and formulas (28), (31), (33).

Theorem 12 *Suppose that the Hamiltonian H satisfies the above-stated conditions 1),2). For an arbitrary function $S_0(x)$ bounded below, the generalized weak solution of the Cauchy problem* (24) *exists and can be found according to the formula*

$$(R_t S_0)(x) = \inf_\xi (S_0(\xi) + S(t, x, \xi)). \tag{34}$$

Various solutions have the same lower semicontinuous closure Cl*, so the solution in the class of semicontinuous functions is unique and is given by the formula*

$$\mathrm{Cl}(R_t S_0) = R_t \,\mathrm{Cl}\, S_0.$$

Let us discuss now shortly the corresponding notion of generalised solution to the stationary Hamilton-Jacobi -Bellman equation

$$H\left(x, \frac{\partial S}{\partial x}\right) + \lambda = 0. \tag{35}$$

Obviously, if a smooth function of the form $S(t,x) = S_0(x) + t\lambda$ is a solution of problem (24), then $S_0(x)$ is a classical solution of the stationary equation (35). Thus, it is natural to define a generalized solution of Eq. (35) as an eigenfunction (in the sense of idempotent analysis) of the resolving operator (34) of the nonstationary problem. Let the Lagrangian $L(x,v)$, defined as the Legendre transform of $H(x,p)$ with respect to p, satisfy $L(x,v) \to \infty$ as $\|x\|, \|v\| \to \infty$. Then the operator R_t (34) is a compact A-linear operator and has at most one eigenvalue by Theorem 4. . It turns out that in this case there is a natural method of constructing eigenfunctions of R_t (generalized solutions of Eq. (35)). Consider, for example, the important particular case in which the semimodule of generalized solutions is finite-dimensional. This situation is particularly important, because it occurs when constructing multiplicative asymptotics for the Schrödinger equation with several potential wells, see [33].

Theorem 13 ([23]) *Suppose that $L(x,v)$ has finitely many points of minimum $(\xi_1, 0), \ldots, (\xi_k, 0)$. Then $\lambda = \min_{x,v} L(x,v)$ is a value for which there exist generalized solutions of the stationary problem* (35). *The semimodule of these solutions* (*in the sense of idempotent structure*) *has a finite basis $\{S_1, \ldots, S_k\}$, where $S_j(x)$ is the infimum of the functional*

$$J_j(q(\,\cdot\,)) = \int_0^t (L(q(\tau), \dot{q}(\tau)) - \lambda)\, dt$$

over all $t > 0$ and all piecewise smooth curves joining ξ_j with x in time t. Moreover, the operator family $R_t - \lambda t$ converges as $t \to 0$ to the finite-dimensional operator B given by the formula

$$(Bh)(x) = \bigoplus_j \langle h, \widetilde{S}_j \rangle_A \odot S_j(x), \tag{36}$$

where $\{\widetilde{S}_1, \ldots, \widetilde{S}_k\}$ is a basis of the eigensemimodule for the adjoint operator.

Proof. Let us show that each S_j is an eigenfunction of the resolving operator (34). In fact,

$$\begin{aligned}(R_t S_j)(x) &= \inf_{\xi} \inf_{\tau \geq 0} (S(\tau, \xi, \xi_j) - \tau\lambda + S(t, x, \xi)) \\ &= \inf_{\tau \geq 0} (S(t+\tau, x, \xi_j) - \tau\lambda) = \inf_{\tau \geq t} (S(\tau, x, \xi_j) - \tau\lambda + t\lambda) \\ &= \inf_{\tau \geq 0} (S(\tau, x, \xi_j) + \tau\lambda) + t\lambda = S_j(x) + \lambda t.\end{aligned}$$

The limit equation (36) means that the family $R_t - \lambda t$ is convergent to the operator with the decomposable kernel

$$\bigoplus_{j=1}^{k} S_j(x) \odot \widetilde{S}_j(y),$$

which can be proved by analogy with Theorem 6.

To conclude the section, let us note that if a convex Hamiltonian H is representable as the limit (uniform on compact sets) of a sequence of Hamiltonians H_n satisfying the assumptions of Theorem 12, then it is clear that if we define generalized solutions of problem (24) as the limits of the corresponding solutions of the Cauchy problems with Hamiltonians H_n, then Theorem 12 will be valid for H.

7 Jump Stochastic Perturbations of Deterministic Optimization Problems.

Idempotent analysis studies resulted in including a series of important nonlinear differential equations (such as numerous optimal control equations and some quasilinear systems occurring in hydrodynamics) in the scope of linear methods, since these equations become linear in the new arithmetic. Idempotent analysis also implies a new approach to the study of a class of nonlinear (even in the new sense) equations, namely, equations "close" to equations linear in idempotent semimodules or semirings. It is natural to study such equations in the framework of the corresponding perturbation theory. Indeed, the theory of numerous important equations of mathematical physics was constructed on the basis of the fact that the nonlinearity is a small "correction" to a linear equation.

The main characteristics of a long-time optimal process are determined by the solutions (λ, h) (where λ is a number and h a function on the state space) of the equation

$$Bh = \lambda + h, \tag{37}$$

where B is the Bellman operator of the optimization problem. Namely, λ is the mean income per step of the process, whereas h specifies stationary optimal strategies or even turnpike control modes (see Section 4). For a usual deterministic control problem, in which B is linear in the sense of the operations $\oplus = \min$ or $\oplus = \max$ and $\odot = +$, Eq. (37) is the idempotent analog of an eigenvector equation in standard linear algebra. Thus, the solutions λ and h are actually called an *eigenvalue* and an *eigenvector* of B, respectively. In the case of stochastic control, the Bellman operator is no longer linear in the idempotent semimodule in general; it is only homogeneous in the sense of the operation $\odot = +$. However, if the influence exerted on the process by stochastic factors is small (the process is nearly deterministic), then the corresponding Bellman operator is close to an operator linear in the idempotent semimodule, and the solutions to Eq. (37) are naturally provided by perturbation theory.

In this section we consider a stochastic perturbations of Eq. (37), linear in the semiring with the operations $\oplus = \max$ and $\odot = +$, when the perturbation theory can be constructed. As a result, we obtain approximate formulas for the mean income and for the stationary optimal strategies in the corresponding controlled stochastic jump processes. These results can be used in approximately solving the Bellman equation corresponding to the controlled dynamics described by a stochastic differential equation, see [24], [33] or respectively [25] for examples of optimal controls of observed quantum mechanical systems described by stochastic equations of Poisson and diffusion types respectively.

The results of this section was obtained in [24] and the proofs can be found in [24] or [33].

Let the process state space X and the control set U be compact, and let v and V be two distinct points of X. Suppose that the process dynamics is determined by a continuous mapping $y\colon X \times U \times [0, \varepsilon_0] \to X$ and a continuous function $q\colon X \to \mathbb{R}_+$ as follows. If a control $u \in U$ is chosen when the process is in a state $x \in X$, then at the current step the transition into the state $y(x, u, \varepsilon)$ takes place with probability $1 - \varepsilon q(x)$, whereas with probability $\varepsilon q(x)$ the transition is into v. The income from residing in a state $x \in X$ is specified by a Lipschitz continuous function $b\colon X \to \mathbb{R}$. The Bellman operator B_ε obviously acts in the space of continuous functions on X according to the formula

$$(B_\varepsilon h)(x) = b(x) + (1 - \varepsilon q(x)) \max_{u \in U} h(y(x, u, \varepsilon)) + \varepsilon q(x) h(v).$$

Theorem 14 *Suppose that for each ε the deterministic dynamics is controllable in the sense that by moving successively from x to $y(x, u, \varepsilon)$ one can reach any point from any other point in a fixed number of steps. Suppose also that b attains its maximum at a unique point V, where $b(V) = 0$ and moreover,*

$$V \in \{y(V, u, \varepsilon) : u \in U\}.$$

Then Eq. (32) *is solvable, and the solution satisfies*

$$h^\varepsilon - h^0 = O(\varepsilon), \tag{38}$$

$$\lambda_\varepsilon = q(V) h^0(v) \varepsilon + o(\varepsilon), \tag{39}$$

where $\lambda_0 = 0$ and h^0 is the unique solution of Eq. (37). *at $\varepsilon = 0$.*

Let now X be a smooth compact manifold, and let $f(x, u, \varepsilon)$ be a vector field on X depending on the parameters $u \in U$ and $\varepsilon \in [0, \varepsilon_0]$ and Lipschitz continuous with respect to all arguments. Consider a special case of the process described above, in which $y(x, u, \varepsilon)$ is the point reached at time τ by the trajectory of the differential equation $\dot{z} = f(z, u, \varepsilon)$ issuing from x and the probability of the transition into v in one step of the process is equal to $\tau \varepsilon q(x)$. As $\tau \to 0$, this process becomes a jump process in continuous time; this process is described by a stochastic differential equation with stochastic differential of Poisson type.

Let $S_n^\varepsilon(t, x)$ be the mathematical expectation of the maximal income per n steps of the cited discrete process with time increment $\tau = (T - t)/n$ beginning at time t at a point x and with terminal income specified by a Lipschitz continuous function $S_T(x)$. Then $S_n^\varepsilon = (B_\varepsilon^\tau)^n$, where B_ε^τ is the Bellman operator corresponding to the discrete problem with step τ.

Theorem 15 *The sequence of continuous functions S_n^ε is uniformly convergent with respect to x and ε to a Lipschitz continuous (and hence, almost everywhere smooth) function $S^\varepsilon(t,x)$, which satisfies the functional-differential Bellman equation*

$$\frac{\partial S}{\partial t} + b(x) + \varepsilon q(x)(S(v) - S(x)) + \max_{u \in U}\left(\frac{\partial S}{\partial x}, f(x,u,\varepsilon)\right) = 0 \qquad (40)$$

at each point of differentiability.

The limit function S^ε may be called a generalized solution of the Cauchy problem for Eq. (40). This function specifies the mathematical expectation of the optimal income for the limit (as $t \to 0$) jump process in continuous time. For $\varepsilon = 0$, this solution coincides with that obtained in the framework of idempotent analysis in the previous section.

Theorems 14 and 15 imply the following result.

Theorem 16 *There exists a continuous function h^ε and a unique λ_ε such that the generalized solution of the Cauchy problem for Eq.* (40) *with terminal function $S_T^\varepsilon = h^\varepsilon$ has the form*

$$S^\varepsilon(t,x) = \lambda_\varepsilon(T-t) + h^\varepsilon(x),$$

λ_ε satisfies the asymptotic formula (39)*, and the generalized solution $S^\varepsilon(t,x)$ of Eq.* (40) *with an arbitrary Lipschitz continuous terminal function S_T^ε satisfies the limit equation*

$$\lim_{t\to-\infty} \frac{1}{T-t}\, S^\varepsilon(t,x) = \lambda_\varepsilon.$$

8 The Pontryagin Maximum Principle and The Bellman Differential Equation for Multicriteria Optimization Problems

Let us consider the controlled process in $\mathbb{R}^n$ specified by a controlled differential equation $\dot{x} = f(x,u)$ (where u belongs to a metric control space U) and by a continuous function $\varphi \in B(\mathbb{R}^n \times U, \mathbb{R}^k)$, which determines a vector-valued integral criterion

$$\Phi(x(\,\cdot\,)) = \int_0^t \varphi(x(\tau), u(\tau))\, d\tau$$

on the trajectories. Let us pose the problem of finding the Pareto set $\omega_t(x)$ for a process of duration t issuing from x with terminal set determined by some function $\omega_0 \in B(\mathbb{R}^n, \mathbb{R}^k)$, that is,

$$\omega_t(x) = \operatorname{Min} \bigcup_{x(\cdot)} (\Phi(x(\,\cdot\,)) \odot \omega_0(x(t))), \tag{41}$$

where $x(\,\cdot\,)$ ranges over all admissible trajectories issuing from x. By Proposition 1, we can encode the functions $\omega_t \in B(\mathbb{R}^n, P\mathbb{R}^k)$ by the functions

$$S(t, x, a) \colon \mathbb{R}_+ \times \mathbb{R}^n \times L \to \mathbb{R}.$$

The optimality principle permits us to write out the following equation, which is valid modulo $O(\tau^2)$ for small τ:

$$S(t, x, a) = \operatorname{Min}_u (h_{\tau\varphi(x,u)} \circledast S(t-\tau, x + \Delta x(u)))(a). \tag{42}$$

It follows from the representation (4) of $h_{\tau\varphi(x,u)}$ and from the fact that n is, by definition, the multiplicative unit in $CS_n(L)$ that

$$S(t, x, a) = \min_u (\tau\overline{\varphi}(x, u) + S(t-\tau, x + \Delta x(u), a - \tau\varphi_L(x, u))).$$

Let us substitute $\Delta x = \tau f(x, u)$ into this equation, expand S in a series modulo $O(\tau^2)$, and collect similar terms. Then we obtain the equation

$$\frac{\partial S}{\partial t} + \max_u \left(\varphi_L(x, u) \frac{\partial S}{\partial a} - f(x, u) \frac{\partial S}{\partial x} - \overline{\varphi}(x, u) \right) = 0. \tag{43}$$

Although the presence of a vector criterion has resulted in a larger dimension, this equation coincides in form with the usual Bellman differential equation. Consequently, the generalized solutions can be defined on the basis of the idempotent superposition principle, as in Section 6. We thus obtain the main result of this section.

Theorem 17 ([31], [34]) *The Pareto set $\omega_t(x)$ (41) is determined by a generalized solution $S_t \in B(\mathbb{R}^n, CS_n(L))$ of Eq. (43) with the initial condition*

$$S_0(x) = h_{\omega_0(x)} \in B(\mathbb{R}^n, CS_n(L)).$$

The mapping $R_{CS} \colon S_0 \mapsto S_t$ is a linear operator on $B(\mathbb{R}^n, CS_n(L))$.

Note that $B(\mathbb{R}^n, S_n(L))$ is equipped with the $CS_n(L)$-valued bilinear inner product

$$\langle h, g \rangle = \inf_x (h \circledast g)(x).$$

The application of Pontryagin's maximum principle to the problem in question is based on the following observation. Let R be the usual resolving operator for generalized solutions of the Cauchy problem for Eq. (43), so that R acts on the space $B(\mathbb{R}^n \times L, \mathbb{R} \cup \{+\infty\})$ of $\mathbb{R} \cup \{+\infty\}$-valued functions bounded below on $\mathbb{R}^n \times L$. Clealy, there is an embedding

$$\mathrm{in}\colon B(\mathbb{R}^n, CS_n(L)) \to B(\mathbb{R}^n \times L, \mathbb{R} \cup \{+\infty\}),$$

which is an idempotent group homomorphism, that is, preserves the operation $\oplus = \min$. The diagram

$$\begin{array}{ccc} B(\mathbb{R}^n, CS_n(L)) & \xrightarrow{R_{CS}} & B(\mathbb{R}^n, CS_n(L)) \\ \downarrow \mathrm{in} & & \downarrow \mathrm{in} \\ B(\mathbb{R}^n \times L, \mathbb{R} \cup \{+\infty\}) & \xrightarrow{R} & B(\mathbb{R}^n \times L, \mathbb{R} \cup \{+\infty\}) \end{array}$$

commutes. Indeed, for smooth initial data this follows from the fact that a smooth solution of Eq. (43) always defines optimal synthesis. However, this implies commutativity for general initial conditions, since the operators R_{CS} and R are uniquely defined by their action on smooth functions and by the property that they are homomorphisms of the corresponding idempotent semigroups, that is, preserve the operation $\oplus = \min$. This implies the following assertion.

Proposition 6 *$S(t, x, a)$ is the minimum of the functional*

$$\int_0^t \overline{\varphi}(x(\tau), u(\tau))\, d\tau + h_{\omega_0(x(t))}(a(t)) \tag{44}$$

defined on the trajectories of the system

$$\dot{x} = f(x, u), \qquad \dot{a} = -\varphi_L(x, u) \tag{45}$$

in $\mathbb{R}^n \times L$ issuing from (x, a) with free right endpoint and fixed time t.

Let us state a similar result for the case in which the time is not fixed. Namely, the problem is to find the Pareto set

$$\omega(x) = \mathrm{Min} \bigcup_{x(\cdot)} \Phi(x(\,\cdot\,)), \tag{46}$$

where Min is taken over the set of all trajectories of the equation $\dot{x} = f(x, u)$ joining a point $x \in \mathbb{R}^n$ with a given point ξ. For the corresponding function $S(x, a)$, we now obtain the stationary Bellman equation

$$\max_u \left(\varphi_L \frac{\partial S}{\partial a} - f(x, u) \frac{\partial S}{\partial x} - \overline{\varphi}(x, u) \right) = 0.$$

By analogy with the preceding case, we obtain the following assertion.

Proposition 7 *The Pareto set* (46) *is determined* (*by virtue of the isomorphism in Proposition* 1) *by the function* $S: \mathbb{R}^n \times L \to \mathbb{R}$ *such that* $S(x, a)$ *is the infimum of the functional*

$$\int_0^t \overline{\varphi}(x, u)\, d\tau + n(a(t))$$

defined on the trajectories of system (45) *issuing from* (x, a) *and satisfying the boundary condition* $x(t) = \xi$.

In [33], [34], one can find a simple example of a variational problem with fixed endpoints and with two quadratic Lagrangians, where the Pareto set can be calculated explicitly using the theory developed above.

9 Stochastic Optimisation and HJB Equation

The method of constructing generalised solutions to the HJB equation considered in Section 3, can be used in more general situations, for example for stochastic or infinite dimensional generalisations, as we explain in this and the following sections. Here we formulate some results obtained in [27], where one can find the proofs as well as the applications of these results to the construction of WKB-type asymptotics of stochastic pseudodifferential equations. Namely, we consider the equation

$$dS + H\left(t, x, \frac{\partial S}{\partial x}\right) dt + \left(c(t, x) + g(t, x)\frac{\partial S}{\partial x}\right) \circ dW = 0, \tag{47}$$

where $x \in \mathcal{R}^n$, $t \geq 0$, $W = (W^1, \ldots, W^n)$ is the standard n-dimensional Brownian motion ($\circ$, as usual, denotes the Stratonovich stochastic differential), $S(t, x, [W])$ is an unknown function, and the Hamiltonian $H(t, x, p)$ is convex with respect to p. This equation can naturally be called the stochastic Hamilton–Jacobi–Bellman equation. First, we explain how this equation appears in the theory of stochastic optimization. Then we develop the stochastic version of the method of characteristics to construct classical solutions of this equation, and finally, on the basis of the methods of idempotent analysis (and on analogy with the deterministic case), we construct a theory of generalized solutions of the Cauchy problem for this equation. Let the controlled stochastic dynamics be defined by the equation

$$dx = f(t, x, u)\, dt + g(t, x) \circ dW,$$

where the control parameter u belongs to some metric space U and the functions f and g are continuous in t and u and Lipschitz continuous in x. Let the income along the trajectory $x(\tau)$, $\tau \in [t, T]$, defined by the starting point $x = x(0)$ and the control $[u] = u(\tau)$, $\tau \in [t, T]$, be given by the integral

$$I_t^T(x, [u], [W]) = \int_t^T b(\tau, x(\tau), u(\tau))\, d\tau + \int_t^T c(\tau, x(\tau)) \circ dW.$$

We are looking for an equation for the cost (or Bellman) function

$$S(t, T, x, [W]) = \sup_{[u]} (I_t^T(x, [u], [W]) + S_0(x(T))),$$

where the supremum is taken over all piecewise smooth (or equivalently, piecewise constant) controls $[u]$ and S_0 is some given function (terminal income). Our argument is based on the following well-known fact: if we approximate the noise W in some stochastic Stratonovich equation by smooth functions, then the solutions of the corresponding classical (deterministic) equations will tend to the solution of the given stochastic equation. For smooth functions W, we have the dynamics

$$\dot{x} = f(\tau, x, u) + g(\tau, x)\dot{W}(\tau)$$

and the integral income

$$\int_t^T [b(\tau, x(\tau), u(\tau)) + c(\tau, x(\tau))\dot{W}(\tau)]\, d\tau.$$

On writing out the Bellman equation for the corresponding deterministic (nonhomogeneous) optimization problem, we obtain

$$\frac{\partial S}{\partial t} + \sup_u \left(b(t, x, u) + f(t, x, u)\frac{\partial S}{\partial x} \right) + \left(c(t, x) + g(t, x)\frac{\partial S}{\partial x} \right) \dot{W}(t) = 0.$$

By rewriting this equation in the stochastic Stratonovich form, we obtain (47) with

$$H(t, x, p) = \sup_u (b(t, x, u) + pf(t, x, u)).$$

Let us indicate the following two particular cases.

(i) $c = 0$ and $g = g(t)$ is independent of x. Then, by differentiating (47), we obtain

$$d\,\frac{\partial S}{\partial x} + \left(\frac{\partial H}{\partial x} + \frac{\partial H}{\partial p}\,\frac{\partial^2 S}{\partial x^2} \right) dt + g\,\frac{\partial^2 S}{\partial x^2} \circ dW = 0,$$

and using the relationship

$$v \circ dW = v\, dW + \frac{1}{2}\, dv\, dW$$

between the Itô and the Stratonovich differentials, we obtain the equation for S in the Itô form

$$dS + H\left(t, x, \frac{\partial S}{\partial x} \right) dx + \frac{1}{2}\, g^2 \frac{\partial^2 S}{\partial x^2}\, dt + g\,\frac{\partial S}{\partial x}\, dW = 0.$$

For the mean optimal cost function $\widetilde{S}$, this implies the standard second-order Bellman equation of the stochastic control theory:

$$\frac{\partial \widetilde{S}}{\partial t} + H\left(t, x, \frac{\partial \widetilde{S}}{\partial x}\right) + \frac{1}{2}\, g^2 \frac{\partial^2 \widetilde{S}}{\partial x^2} = 0.$$

(ii) $g = 0$. Then Eq. (47) acquires the form

$$dS + H\left(t, x, \frac{\partial S}{\partial x}\right) dt + c(t, x)\, dW = 0, \tag{48}$$

since in this case the Itô and the Stratonovich differential forms coincide.

For simplicity, we reduce our study to the case of Eq. (48) with H and c that do not explicitly depend on t. Our main tool is the stochastic Hamiltonian system

$$\begin{cases} dx = \dfrac{\partial H}{\partial p}\, dt, \\ dp = -\dfrac{\partial H}{\partial x}\, dt - c'(x)\, dW. \end{cases} \tag{49}$$

Let us define the two-point stochastic action

$$S_W(t, x, \xi) = \inf \int_0^t (L(q, \dot{q})\, d\tau - c(q)\, dW), \tag{50}$$

where the infimum is taken over all piecewise smooth curves $q(\tau)$ such that $q(0) = \xi$ and $q(t) = x$, and the Lagrangian L is, as usual, the Legendre transform of the Hamiltonian H with respect to the last argument.

Proposition 8 *Suppose that all second derivatives of the functions H and c are uniformly bounded, the matrix $\operatorname{Hess}_p H$ of the second derivatives of H with respect to p is uniformly positive (i.e., $\operatorname{Hess}_p H \geq \lambda E$ for some constant λ), and for any fixed x_0 all matrices $\operatorname{Hess}_p H(x_0, p)$ commute. Then there exists t_0 such that for all $t \leq t_0$ and all ξ, x there exists almost surely a unique solution $(q(\tau), p(\tau))$ of system (49) such that $q(0) = \xi$, $q(t) = x$ and*

$$S_W(t, x, \xi) = \int_0^t (p(\tau)\, dq(\tau) - H(q(\tau), p(\tau))\, dt - c(q(\tau))\, dW(\tau)).$$

Moreover,

(i) $p(t) = \dfrac{\partial S}{\partial x}, \quad p_0 = -\dfrac{\partial S}{\partial \xi}$;

(ii) *S satisfies Eq. (48) as a function of x;*

(iii) *$S(t, x, \xi)$ is convex in x and ξ.*

Finally, let the function $S_0(x)$ be smooth and convex. Then for $t \leq t_0$ there exists almost surely a unique classical (i.e., everywhere smooth) solution to the

Cauchy problem for Eq. (48) *with the initial function* $S_0(x)$*. This solution is given by the formula*

$$R_t S_0(x) = S(t,x) = \min_{\xi}(S_0(\xi) + S_W(t,x,\xi)). \tag{51}$$

Now one can directly apply the method for constructing the generalized solution of the deterministic Bellman equation to the stochastic case, thus obtaining the following theorem.

Theorem 18 *For an arbitrary initial function* $S_0(x)$ *bounded below, there exists a unique generalized solution of the Cauchy problem for Eq.* (48)*, which is given by* (51) *for all* $t \geq 0$*.*

Approximating nonsmooth Hamiltonians by smooth functions and defining the generalized solutions as the limits of the solutions corresponding to the smooth Hamiltonians, we find (on analogy with the deterministic case) that formula (51) for generalized solutions remains valid for nonsmooth Hamiltonians.

10 Turnpikes for the Infinite-Dimensional HJB Equation

In this section, we give an infinite-dimensional generalization of the results of Section 4 concerning turnpike properties in dynamic optimization problems, as well as of the results of Section 6 concerning the large-time behavior of solutions of the Hamilton–Jacobi–Bellman equation.

Let X be a metric space with metric ρ, and let B_t be the semigroup of operators acting on functions $f\colon X \to A = \mathbb{R} \cup \{+\infty\}$ bounded below according to the formula

$$(B_t f)(x) = \inf_y (b(t,x,y) + f(y)), \tag{52}$$

or, in terms of the idempotent operations on A,

$$(B_t f)(x) = \int_X^{\oplus} b(t,x,y) \odot f(y)\, d\mu_{\mathbb{1}}(y),$$

where $t \in \mathbb{R}_+$ (continuous time) or $t \in \mathbb{Z}_+$ (discrete time).

Theorem 19 *Assume that the function family* $b(t,x,y)$ *in Eq.* (52) *has the following properties*:

(i) $b(t,x,y) \geq 0 \ \forall\, t,x,y$;

(ii) *there exist* $\xi_1,\dots,\xi_k \in X$ *such that* $b(t,x,y) = 0$ *if and only if* $x = y = \xi_j$ *for some* $j \in \{1,\dots,k\}$;

(iii) *for any* $x \in X$ *and* $j \in \{1, \ldots, k\}$, *there exists a* t *such that* $b(t, x, \xi_j) \neq +\infty$ *and* $b(t, \xi_j, x) \neq +\infty$;

(iv) *there exists a* t_0 *such that the functions* $b(t_0, \xi_j, x)$ *and* $b(t_0, x, \xi_j)$ *are continuous in* x *at* $x = \xi_j$ *for each* j;

(v) *for any neighborhoods* $U_j \subset X$ *of the points* ξ_j *in* X, *we have*

$$\inf \left\{ b(t_0, x, y) : (x, y) \notin \bigcup_{j=1}^{k} U_j \times U_j \right\} > 0.$$

Then

(i) *the functions* $b(t, x, \xi_j)$ *and* $b(t, \xi_j, x)$ *have the limits*

$$b_j(x) = \lim_{t \to \infty} b(t, x, \xi_j), \qquad \tilde{b}_j(x) = \lim_{t \to \infty} b(t, \xi_j, x); \tag{53}$$

(ii) *the operator family* B_t *is convergent to an operator with a factorizable kernel*; *namely,*

$$\lim_{t \to \infty} b(t, x, y) = \min_j (b_j(x) + \tilde{b}_j(y)). \tag{54}$$

Remark 1. The statement of the theorem does not include the metric. Actually, the theorem in this form is valid for an arbitrary topological space X. However, to verify the main technical condition (v) (which holds automatically for continuous functions $b(t, x, y)$ on a compact space X), we need some analytical estimates. For example, condition (v) is satisfied if

$$b(t_0, x, y) \geq C \max \left(\min_j \rho^{\alpha}(y, \xi_j), \min_j \rho^{\alpha}(x, \xi_j) \right)$$

with some positive constants c and α.

Remark 2. It follows from the theorem that the idempotent operators B_t have the unique eigenvalue $\lambda = 0 = \mathbb{1}$ for all t and that the corresponding eigenspace is finite-dimensional and has the basis $\{b_j(x)\}_{j=1,\ldots,k}$.

Proof. (i) It follows from the semigroup property of the operators B_t that

$$b(t + \tau, x, y) = \inf_{\eta} (b(t, x, \eta) + b(\tau, \eta, y)) \tag{55}$$

for any t and τ. Hence,

$$b(t + \tau, x, \xi_j) \leq b(t, x, \xi_j) + b(\tau, \xi_j, \xi_j) = b(t, x, \xi_j)$$

according to (ii). Thus, properties (i) and (ii) imply that the functions $b(t, x, \xi_j)$ and $b(t, \xi_j, x)$ are bounded below and nonincreasing with respect to t, whence assertion (i) of the theorem follows.

(ii) The semigroup property (55) implies

$$b(2t,x,y) \le b(t,x,\xi_j) + b(t,\xi_j,y)$$

for any t, j. Consequently,

$$\overline{\lim_{t\to\infty}}\, b(t,x,y) \le \min_j (b_j(x) + \tilde{b}_j(y)). \tag{56}$$

Furthermore, let $N(t)$ denote the maximum integer in t/t_0. By the semigroup property, we have

$$b(t,x,y) = \inf\{B(t,\eta_1,\dots,\eta_{N(t)}) : \eta_1,\dots,\eta_{N(t)} \in X\},$$

where

$$B(t,\eta_1,\dots,\eta_{N(t)}) = b(t_0,x,\eta_1) + b(t_0,\eta_1,\eta_2) + \cdots \\ + b(t_0,\eta_{N(t)-1},\eta_{N(t)}) + b(t - t_0 N(t), \eta_{N(t)}, y).$$

Let us say that a tuple $\eta_1,\dots,\eta_{N(t)}$ is ε-*optimal* if

$$|b(t,x,y) - B(t,\eta_1,\dots,\eta_{N(t)})| \le \varepsilon.$$

Consider arbitrary neighborhoods U_j of the points ξ_j. It follows from (56) and from condition (55) that for each ε-optimal tuple all points $\eta_1,\dots,\eta_{N(t)}$ except for a finite number $K(\varepsilon,\{U_j\})$ (which depends on ε and $\{U_j\}$ but is independent of t) of such points lie in the union $\bigcup_{j=1}^k U_j$. In particular, one can construct functions $T(t) \in [t/3, 2t/3]$ and $\eta(t) \in \bigcup_{j=1}^k U_j$ such that

$$|b(t,x,y) - ((b(T(t),x,\eta(t)) + (b(t - T(t),\eta(t),y))| < \varepsilon. \tag{57}$$

Using property (iv), let us choose U_j so that

$$b(t_0,\eta,\xi_j) \le \varepsilon, \qquad b(t_0,\xi_j,\eta) \le \varepsilon$$

for each $\eta \in U_j$.

Using the semigroup property once more, let us write

$$b(T(t),x,\eta(t)) > b(T(t)+t_0,x,\xi_j) - b(t_0,\eta(t),\xi_j), \\ b(t - T(t),\eta(t),y) > b(t - T(t) + t_0,\xi_j,y) - b(t_0,\xi_j,\eta(t)).$$

Consequently,

$$\underline{\lim_{t\to\infty}}\, b(T(t),x,\eta(t)) + b(t - T(t),\eta(t),y) \ge \min_j (b_j(x) + \tilde{b}_j(y)) - \varepsilon.$$

It follows from this and from Eq. (55) that

$$\underline{\lim_{t\to\infty}}\, b(t,x,y) \ge \min_j (b_j(x) + \tilde{b}_j(y)) - 2\varepsilon, \tag{58}$$

where $\varepsilon > 0$ is arbitrary. Equations (56) and (58) imply Eq. (53). The theorem is proved.

Let us now proceed to the infinite-dimensional differential HJB equation. Namely, let Φ be an arbitrary locally convex space with a countable base of neighborhoods of zero, so that the topology of Φ can be specified by a translation-invariant metric ρ, and let Φ' be the dual space, that is, the space of continuous linear functionals on Φ. Let $H\colon \Phi \times \Phi' \to \mathbb{R} \cup \{+\infty\}$ be a function convex with respect to the second argument. The equation

$$\frac{\partial S}{\partial t} + H\left(x, \frac{\partial S}{\partial t}(t, x)\right) = 0 \tag{59}$$

will be called an infinite-dimensional Hamilton–Jacobi–Bellman equation, and the function $L\colon \Phi \times \Phi \to \mathbb{R} \cup \{+\infty\}$ given by the formula

$$L(x, v) = \sup_p ((p, v) - H(x, p))$$

will be referred to as the *Lagrangian* corresponding to the Hamiltonian H. Let

$$b(t, x, y) = \inf \int_0^t L(q, \dot{q})\, d\tau,$$

where the infimum is taken over all continuous piecewise smooth curves $q(\tau)$ such that $q(0) = y$ and $q(t) = x$. The function

$$(R_t S_0)(x) = \inf_y (S_0(y) + b(t, x, y)) \tag{60}$$

will be called the generalized solution of the Cauchy problem for Eq. (59) with the initial function $S_0(x)$.

Under various assumptions about the Hamiltonian H, this definition of a generalized solution can be justified in several ways: one can either construct a sufficient supply of classical solutions following Section 6, or use the results of Section 6 to define generalized solutions of the infinite-dimensional equation (59) as limits of solutions of its finite-dimensional approximations, or construct semimodules of generalized functions with a natural action of the differentiation operator following [42], or use the infinite-dimensional version of the vanishing viscosity method [13]. Here we do not dwell on this justification, since in numerous problems (for example, in dynamic optimization problems) formula (60) is the primary one (namely, Eq. (59) occurs as a corollary of (60)). Instead, we study the behavior of the function (60) for large t. The proof of the following theorem is quite similar to the proof of Theorem 19

Theorem 20 *Suppose that the Lagrangian L has the following properties*:

(i) $L(x, v) \geq 0$ *for any* x, $v \in \Phi$;

(ii) *there exist points* $\xi_1, \dots, \xi_k$ *such that* L *vanishes only at* $(\xi_j, 0)$, $j = 1, \dots, k$;

(iii) $L(x, v)$ *is bounded in some neighborhoods of* $(\xi_j, 0)$ *and continuous at these points*;

(iv) *there exist neighborhoods* U_j *of the points* ξ_j *in* Φ *such that* $L_k(x, v) \geq c > 0$, *with some constant* c, *for all* $x \in \bigcup_{j=1}^k U_j$ *and all* v *and that*

$$L(x, v) \geq c\rho^\alpha(x, \xi_j)$$

for all $x \in U_j$ *with some constants* c *and* α.

Then the operator family R_t *given by Eqs.* (60) *is a semigroup* (*with continuous time* $t \in \mathbb{R}_+$) *and converges as* $t \to \infty$ *to an operator with factorizable kernel. Thus, the kernel family* $b(t, x, y)$ *satisfies conclusions* (i) *and* (ii) *for Theorem* 19.

11 Homogeneous maps and option pricing

The famous Black-Sholes (BS) and Cox-Ross-Rubinstein (CRR) formulas are basic results in the modern theory of option pricing in financial mathematics. They are usually deduced by means of stochastic analysis; various generalisations of these formulas were proposed using more sophisticated stochastic models for common stocks pricing evolution. The systematic deterministic approach to the option pricing leads to a different type of generalisations of BS and CRR formulas characterised by more rough assumptions on common stocks evolution (which are therefore easier to verify). This approach reduces the analysis of the option pricing to the study of certain homogeneous nonexpansive maps, which however, unlike the situations described in Sections 4 and 5, are "strongly" infinite dimensional: they act on the spaces of functions defined on sets, which are not (even locally) compact.

Following our paper [28], we shall show here what type of generalisations of the standard CRR and BS formulas can be obtained using the deterministic (actually game-theoretic) approach to option pricing and what class of homogeneous nonexpansive maps appear in these formulas, considering first a simplest model of financial market with only two securities in discrete time, then its generalisation to the case of several common stocks, and then the continuous limit. One of the objective of this exposition is to show that the infinite dimensional generalisation of the theory of homogeneous nonexpansixe maps (which does not exists at the moment) would have direct applications to the analysis of derivative securities pricing. On the other hand, this exposition, which uses neither martingales nor stochastic equations, makes the whole apparatus of the standard game theory appropriate for the study of option pricing.

A simplest model of financial market deals with only two securities: the risk-free bonds (or bank account) and common stocks. The prices of the units of these securities, $B = (B_k)$ and $S = (S_k)$ respectively, change in discrete moments of time $k = 0, 1, ...$ according to the recurrent equations $B_{k+1} = \rho B_k$, where $\rho \geq 1$ is a fixed number, and $S_{k+1} = \xi_{k+1} S_k$, where ξ_k is an (a priori unknown) sequence taking value in a fixed compact set $M \in \mathcal{R}$. We denote by u and d respectively the exact upper and lower bounds of M (u and d stand for up and down) and suppose that $0 < d < \rho < u$. We shall be interested especially in two cases:

(i) M consists of only two elements, its upper and lower bounds u and d,

(ii) M consists of the whole closed interval $[d, u]$.

No probability assumptions on the sequence ξ_k are specified. Case (i) corresponds to the CRR model and case (ii) stands for the situation when only minimal information on the future evolution of common stocks pricing is available, namely, the rough bounds on its growth per unit of time.

An investor is supposed to control the growth of his capital in the following way. Let X_{k-1} be his capital at the moment $k-1$. Then the investor chooses his portfolio defining the number γ_k of common stock units held in the moment $k-1$. Then one can write

$$X_{k-1} = \gamma_k S_{k-1} + (X_{k-1} - \gamma_k S_{k-1}),$$

where the sum in brackets corresponds to the part of the capital laid on the bank account (and which will thus increases deterministically). All operations are friction-free. The control parameter γ_k can take all real values, i.e. short selling and borrowing are allowed. In the moment k the value ξ_k becomes known and thus the capital becomes equal to

$$X_k = \gamma_k \xi_k S_{k-1} + (X_{k-1} - \gamma_k S_{k-1})\rho.$$

The strategy of the investor is by definition any sequence of numbers $\Gamma = (\gamma_1, ..., \gamma_n)$ such that each γ_j can be chosen using the whole previous information: the sequences $X_0, ..., X_{j-1}$ and $S_0, ...S_{j-1}$. It is supposed that the investor, selling an option by the price $C = X_0$ should organise the evolution of this capital (using the described procedure) in a way that would allow him to pay to the buyer in the prescribed moment n some premium $f(S_n)$ depending on the price S_n. The function f defines the type of the option under consideration. In the case of the standard European call option, which gives to the buyer the right to buy a unit of the common stocks in the prescribed moment of time n by the fixed price K, the function f has the form

$$f(S_n) = \max(S_n - K, 0). \tag{61}$$

Thus the income of the investor will be $X_n - f(S_n)$. The strategy $\gamma_1, ..., \gamma_n$ is called a hedge, if for any sequence $\xi_1, ..., \xi_n$ the investor is able to meet his obligations, i.e. $X_n - f(S_n) \geq 0$. The minimal value of the initial capital X_0 for which the hedge exists is called the hedging price C_h of an option. The hedging price C_h will be called correct (or fair), if moreover, $X_n - f(S_n) = 0$ for any hedge and any sequence ξ_j. The correctness of the price is equivalent to the impossibility of arbitrage, i.e. of a risk-free premium for the investor. It was in fact proven in [15] (using some additional probabilistic assumptions on the sequence ξ_j) that for case (i) the hedging price C_h exists and is correct. On the other hand, it is known that when the set M consists of more than two points, the hedging price will not be correct anymore. We shall show now using exclusively deterministic arguments that both for cases (i) and (ii) the hedge exists and is the same for both cases whenever the function f is nondecreasing and convex (possibly not strictly).

When calculating prices, one usually introduces the relative capital Y_k defined by the equation $Y_k = X_k/B_k$. Since the sequence B_k is positive and deterministic, the problem of the maximisation of the value $X_n - f(S_n)$ is equivalent to the maximisation of $Y_n - f(S_n)/B_n$. Consider first the last step of the game. If the relative capital of the investor at moment $n-1$ is equal to $Y_{n-1} = X_{n-1}/B_{n-1}$, then his relative capital at the next moment will be

$$Y_n(\gamma_n, \xi_n) - \frac{f(\xi_n S_{n-1})}{B_n} = Y_{n-1} + \gamma_n \frac{S_{n-1}}{B_n}(\xi_n - \rho) - \frac{1}{B_n} f(\xi_n S_{n-1}).$$

Therefore, it is clear that the guaranteed income (in terms of relative capital) in the last step can be written as

$$Y_{n-1} - \frac{1}{B_{n-1}}(\mathcal{B}f)(S_{n-1}),$$

where the Bellman operator $\mathcal{B}$ is defined by the formula

$$(\mathcal{B}f)(z) = \frac{1}{\rho} \min_{\gamma} \max_{\xi \in M} [f(\xi z) - \gamma z(\xi - \rho)]. \tag{62}$$

We suppose further the function f to be nondecreasing and convex (perhaps, not strictly), having in mind the main example, which corresponds to the standard European call option and where this assumption is satisfied. Then the maximum in (62) is evidently attained on the end points of M and thus

$$(Bf)(z) = \frac{1}{\rho} \min_{\gamma} \max\,[f(dz) - \gamma z(d - \rho),\ f(uz) - \gamma z(u - \rho)]. \tag{63}$$

One sees directly that for $\gamma \geq \gamma^h$ (resp. $\gamma \leq \gamma^h$), the first term (resp. the second) under max in (63) is maximal, where

$$\gamma^h = \gamma^h(z, [f]) = \frac{f(uz) - f(dz)}{z(u - d)}. \tag{64}$$

It implies that the minimum in (63) is given by $\gamma = \gamma^h$, which yields

$$(\mathcal{B}f)(z) = \frac{1}{\rho}\left[\frac{\rho - d}{u-d}f(uz) + \frac{u-\rho}{u-d}f(dz)\right]. \tag{65}$$

The mapping $\mathcal{B}$ is a linear operator on the space of continuous functions on the positive line that preserves the set of nondecreasing convex functions. Using this property and induction in k one gets that the guaranteed relative income of the investor to the moment of time n is given by the formula $Y_0 - B_0^{-1}(\mathcal{B}^n f)(S_0)$ and thus his guaranteed income is equal to

$$\rho^n(X_0 - (\mathcal{B}^n f)(S_0)). \tag{66}$$

The hedge strategy (the use of which guarantees him this guaranteed income) is $\Gamma^h = (\gamma_1^h, ..., \gamma_n^h)$, where each γ_j^h is calculated step by step using formula (64). The minimal value of X_0 for which this income is not negative (and which by definition is the hedge price C_h of the corresponding option contract) is therefore given by the formula

$$C_h = (\mathcal{B}^n f)(S_0). \tag{67}$$

Using (64) one easily finds for C_h the following CRR formula [15]:

$$C_h = \rho^{-n}\sum_{k=0}^{n} C_n^k \left(\frac{\rho-d}{u-d}\right)^k \left(\frac{u-\rho}{u-d}\right)^{n-k} f(u^k d^{n-k} S_0), \tag{68}$$

where C_n^k are standard binomial coefficients. When f is defined by (61), this yields

$$C_h = S_0 \mathcal{P}_\mu\left(\frac{u}{\rho}\frac{\rho-d}{u-d}\right) - K\rho^{-n}\mathcal{P}_\mu\left(\frac{\rho-d}{u-d}\right),$$

where the function $\mathcal{P}_k$ is defined by the formula

$$\mathcal{P}_k(q) = \sum_{j=k}^{n} C_n^j q^j (1-q)^{n-j},$$

the integer μ is the minimal integer k such that $u^k d^{n-k} S_0 > K$, and it is supposed that $\mu \leq n$.

If the investor uses his hedge strategy $\Gamma^h = (\gamma_1^h, ..., \gamma_n^h)$, then the two terms under max in expression (63) are equal (for each step $j = 1, ..., n$). Therefore, in the case (i) (when the set M consists of only two elements), if $X_0 = C_h$, the resulting income (66) does not depend on the sequence $\xi_1, ..., \xi_n$ and vanishes always, whenever the investor uses his hedge strategy, i.e. the prize C_h is correct in that case (Cox-Ross-Rubinstein theorem).

In general case it is not so anymore. Let us give first the exact formula for the maximum of the possible income of the investor in the general case supposing that he uses his hedge strategy. Copying the previous arguments one sees that this maximal income is given by the formula

$$\rho^n(X_0 - (\mathcal{B}^n_{min} f)(S_0)), \tag{69}$$

where

$$(\mathcal{B}_{min} f)(z) = \frac{1}{\rho} \min_{\xi \in M} [f(\xi z) - \gamma z(\xi - \rho)]|_{\gamma=\gamma^h}. \tag{70}$$

Thus, in the case of general M, the income of the investor playing with his hedge strategy will consists of the sum of the guaranteed income (66) and some unpredictable surplus (risk-free premium), which does not exceed the difference between expressions (65) and (70). Hence, a reasonable price for the option should belong to the interval $[C_{min}, C_h]$ with C_h given by (67) and

$$C_{min} = (\mathcal{B}^n_{min} f)(S_0). \tag{71}$$

Since the value $\mathcal{B}^n_{min}$ is essentially more difficult to calculate than $\mathcal{B}^n$, it may be useful to have some simple reasonable estimate for it. Taking $\xi = \rho$ in (70) yields $(\mathcal{B}^n_{min} f)(z) \leq \rho^{-1} f(\rho z)$ and therefore by induction

$$(\mathcal{B}^n_{min} f)(z) \leq \rho^{-n} f(\rho^n z). \tag{72}$$

Looking at the evolution of the capital X_k as at the game of the investor with the nature (γ_k and ξ_k are their respective controls) one can say that (for the hedge strategy of the investor) the nature plays against the investor, when its controls ξ_k lie near the boundary $[d, u]$ of the set M (then the investor gets his minimal guaranteed income (66)) and conversely, it plays for the investor, when its controls ξ_k are in the middle of M, say, near ρ. If it is possible to estimate roughly the probability p that ξ_k would be near the boundaries of M, one can estimate the mean income of the investor (who uses his hedge strategy) by

$$\rho^n(X_0 - ((\mathcal{B}_{mean})^n f)(S_0)),$$

where

$$(\mathcal{B}_{mean} f)(z) = p(\mathcal{B} f)(z) + (1-p)\frac{1}{\rho} f(\rho z)$$

or equivalently

$$(\mathcal{B}_{mean} f)(z) = \frac{1}{\rho}\left[p\frac{u-\rho}{u-d} f(dz) + (1-p) f(\rho z) + p\frac{\rho - d}{u-d} f(uz)\right], \tag{73}$$

which gives for the mean price the approximation

$$C_{mean} = ((\mathcal{B}_{mean})^n f)(S_0).$$

One can easily obtain for C_{mean} more explicit expression, similar to (68), see [28].

Consider now a more general situation, when there are several types of common stocks on a market. Say, for simplicity, the number of common stocks is two, whose prices S_k^1 and S_k^2, $k = 0, 1, ...$, satisfy the recurrent equations $S_k^i = \xi_k^i S_{k-1}^i$, where ξ_j^i take values in compact sets M_i, $i = 1, 2$, with bounds d_i and u_i respectively. The investor controls his capital by choosing in each moment of time $k-1$ his portfolio consisting of γ_k^i units of common stocks of the type i, the rest of the capital being laid on the risk-free bank account. His capital at the next time k becomes therefore

$$X_k = \gamma_k^1 \xi_k^1 S_{k-1}^1 + \gamma_k^2 \xi_k^2 S_{k-1}^2 + \rho(X_{k-1} - \gamma_k^1 S_{k-1}^1 - \gamma_k^2 S_{k-1}^2).$$

The premium to the buyer of the option at a fixed time n will be now $f(S_n^1, S_n^2)$, where f is a given nondecreasing convex continuous function on the positive octant $\mathcal{R}_+^2$. For instance, the analog of the standard European option is given by the function

$$f(z_1, z_2) = \max(\max(0, z_1 - K_1), \max(0, z_2 - K_2)), \tag{74}$$

which describes the option contract that permits to the buyer to purchase one unit of the common stocks belonging to any type $1, 2$ by his choice. Similarly to the case of only one type of common stocks, one obtains now the formula

$$Y_{n-1} - \frac{1}{B_{n-1}} (\mathcal{B}f)(S_{n-1}^1, S_{n-1}^2),$$

for the guaranteed relative income of the investor in the last step of the game starting from the relative capital Y_{n-1} at the time $n-1$. Here the Bellman operator $\mathcal{B}$ has the form

$$(\mathcal{B}f)(z_1, z_2) = \frac{1}{\rho} \min_{\gamma^1, \gamma^2} \max_{\xi^1 \in M_1, \xi^2 \in M_2} [f(\xi^1 z_1, \xi^2 z_2) - \gamma^1 z_1(\xi^1 - \rho) - \gamma^2 z_2(\xi^2 - \rho)]. \tag{75}$$

In order to give an explicit formula for this operator (similar to (65)), one should make additional assumptions on the function f. We say that a nondecreasing function f on $\mathcal{R}_+^2$ is nice, if the expression

$$f(d_1 z_1, u_2 z_2) + f(u_1 z_1, d_2 z_2) - f(d_1 z_1, d_2 z_2) - f(u_1 z_1, u_2 z_2)$$

is nonnegative everywhere. One easily sees for instance, that any function of the form $f(z_1, z_2) = \max(f_1(z_1), f_2(z_2))$ is nice for any nondecreasing functions f_1, f_2 and any numbers $d_i < u_i$, $i = 1, 2$. In particular, function (75) is nice. Clear the nice functions constitute a linear space and the set of continuous nondecreasing convex nice functions is a convex subset in this space, which we denote NS (nice set). The proof of the following statement uses only elementary manipulations and will not be given here.

Proposition 9 *Let*

$$\kappa = \frac{(u_1u_2 - d_1d_2) - \rho(u_1 - d_1 + u_2 - d_2)}{(u_1 - d_1)(u_2 - d_2)}. \tag{76}$$

If $f \in NS$ *and* $\kappa \geq 0$, *then* $(\mathcal{B}f)(z_1, z_2)$ *equals*

$$\frac{1}{\rho}\left[\frac{\rho - d_1}{u_1 - d_1} f(u_1z_1, d_2z_2) + \frac{\rho - d_2}{u_2 - d_2} f(d_1z_1, u_2z_2) + \kappa f(d_1z_1, d_2z_2)\right] \tag{77}$$

and the γ^{h1}, γ^{h2} *giving minimum in* (75) *are equal to*

$$\gamma^{h1} = \frac{f(u_1z_1, d_2z_2) - f(d_1z_1, d_2z_2)}{z_1(u_1 - d_1)}, \quad \gamma^{h2} = \frac{f(d_1z_1, u_2z_2) - f(d_1z_1, d_2z_2)}{z_2(u_2 - d_2)}.$$

If $\kappa \leq 0$ *(and again* $f \in NS$*), then* $(\mathcal{B}f)(z_1, z_2)$ *equals*

$$\frac{1}{\rho}\left[\frac{u_1 - \rho}{u_1 - d_1} f(d_1z_1, u_2z_2) + \frac{u_2 - \rho}{u_2 - d_2} f(u_1z_1, d_2z_2) + |\kappa| f(u_1z_1, u_2z_2)\right],$$

and

$$\gamma^{h1} = \frac{f(u_1z_1, u_2z_2) - f(d_1z_1, u_2z_2)}{z_1(u_1 - d_1)}, \quad \gamma^{h2} = \frac{f(u_1z_1, u_2z_2) - f(u_1z_1, d_2z_2)}{z_2(u_2 - d_2)}.$$

It follows that the operator $\mathcal{B}$ preserves NS and by the same induction as in the previous section one proves that if the premium is defined by a function $f \in NS$, then the hedge price for the option contract exists and is equal to

$$C_h = (\mathcal{B}^n f)(S_0^1, S_0^2). \tag{78}$$

One can write down a more explicit expression. For example, if $\kappa = 0$,

$$C_h = \frac{1}{\rho^n} \sum_{k=0}^{n} C_n^k \left(\frac{\rho - d_1}{u_1 - d_1}\right)^k \left(\frac{\rho - d_2}{u_2 - d_2}\right)^{n-k} f(d_1^{n-k}u_1^k z_1, d_2^k u_2^{n-k} z_2). \tag{79}$$

For the most important particular case, when the function f is of form (74) formula (79) can be written even more explicitly, see [28].

Though the obtained formula for C_h is similar to the one obtained above for a market with only one type of common stocks, there is a principle difference, namely: even if each M_i consists of only two points, this hedge price is not correct.

If $M_i = [d_i, u_i]$, the maximal income of the investor who uses his hedge strategy is given by the formula

$$\rho^n(X_0 - (\mathcal{B}^n_{min} f)(S_0^1, S_0^2)),$$

where $(\mathcal{B}_{min}f)(z_1,z_2)$ equals

$$\frac{1}{\rho}\min_{\xi^1\in M_1}\min_{\xi^2\in M_2}[f(\xi^1 z_1,\xi^2 z_2)-\gamma^1 z_1(\xi^1-\rho)-\gamma^2 z_2(\xi^2-\rho)]|_{\gamma^1=\gamma^{h1},\gamma^2=\gamma^{h2}}, \quad (80)$$

and the corresponding minimal price of the option is

$$C_{min}=((\mathcal{B}_{min})^n f)(S_0^1,S_0^2). \quad (81)$$

Supposing, as in the case of only one type of common stocks, that one can estimate the probability p of the numbers ξ_k^i to be near the boundaries of the corresponding sets M_i, one gets for the mean price of the option the formula

$$C_{mean}=((\mathcal{B}_{mean})^n f)(S_0^1,S_0^2), \quad (82)$$

where (when supposing $\kappa=0$ as above) $(\mathcal{B}_{mean}f)(z_1,z_2)$ is equal to

$$\frac{1}{\rho}\left[p\frac{\rho-d_1}{u_1-d_1}f(u_1z_1,d_2z_2)+(1-p)f(\rho z_1,\rho z_2)+p\frac{\rho-d_2}{u_2-d_2}f(d_1z_1,u_2z_2)\right].$$

As was shown in [15], the binomial CRR formula for option prices (68) tends to the famous Black-Sholes formula under certain probabilistic assumptions on the random variables ξ_j. We find similar limits for "two-dimensional" formulas (79)-(81), without any use of probability theory. The only "trace" of the geometric Brownian motion model of Black-Sholes will be the assumption (which is clearly more rough than the usual assumptions of the standard Black-Sholes model) that the logarithm of the relative growth of the stock prices is proportional to $\sqrt{\tau}$ for small intervals of time τ. More exactly, if τ is the time between the successive evaluations of common stock prices, then the bounds d_i, u_i of M_i are given by the formulas $\log u_i=\sigma_i\sqrt{\tau}+\mu_i\tau$ and $\log d_i=-\sigma_i\sqrt{\tau}+\mu_i\tau$, where the coefficients $\mu_i>0$ stand for the systematic growth and the coefficients σ_i (so called volatilities) stand for "random oscillations". Moreover, as usual, $\log\rho$ is proportional to τ, i.e. $\log\rho=r\tau$ for some constant $r\geq 1$. Let $\mathcal{B}(\tau)$ denote the corresponding operator (75). Under these assumptions, the calculation of the coefficient κ and the strategies γ^h from the Proposition 9 for small τ yields

$$\kappa=\frac{1}{2}\left(\frac{\sigma_1+\sigma_2}{2}+\frac{\mu_1-r}{\sigma_1}+\frac{\mu_2-r}{\sigma_2}\right)\sqrt{\tau}+O(\tau^{3/2}),$$

$$\gamma^{hj}=\frac{\partial f}{\partial z_j}(z_1,z_2)(1+O(\tau), \quad j=1,2.$$

Expanding now the corresponding expression for $\mathcal{B}(\tau)f$ from (75) in a series in small times, using Taylor formula (see the corresponding simple calculations in [28]), one sees that all terms proportional to $\sqrt{\tau}$ vanish and one obtains the differential equation

$$\frac{\partial F}{\partial t}=\frac{1}{2}\sigma_1^2z_1^2\frac{\partial^2 F}{\partial z_1^2}+\frac{1}{2}\sigma_2^2z_2^2\frac{\partial^2 F}{\partial z_2^2}+rz_1\frac{\partial F}{\partial z_1}+rz_2\frac{\partial F}{\partial z_2}-rF \quad (83)$$

for the function

$$F_h(t, z_1, z_2) = \lim_{n\to\infty} (\mathcal{B}^n(t/n)f)(z_1, z_2),$$

with initial condition $F(0, z_1, z_2) = f(z_1, z_2)$. Rewriting this equation in terms of the function R defined by the formula

$$F(t, z_1, z_2) = e^{-rt} R(t, rt + \log z_1, rt + \log z_2) \tag{84}$$

yields a linear diffusion equation with constant coefficients

$$\frac{\partial R}{\partial t} = \frac{1}{2}\sigma_1^2 \left(\frac{\partial^2 R}{\partial p_1^2} - \frac{\partial R}{\partial p_1}\right) + \frac{1}{2}\sigma_2^2 \left(\frac{\partial^2 R}{\partial p_2^2} - \frac{\partial R}{\partial p_2}\right).$$

It allows to write the solution of the Cauchy problem for Eq. (83) explicitly, which yields the two-dimensional version of the Black-Sholes formula for hedging option price in continuous time

$$F_h = e^{-rt}(2\pi)^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} du_1 du_2 \exp\{-(u_1^2 + u_2^2)/2\}$$

$$\times f(S_0^1 \exp\{u_1\sigma_1\sqrt{t} + (r - \sigma_1^2/2)t\}, S_0^2 \exp\{u_2\sigma_2\sqrt{t} + (r - \sigma_2^2/2)t\}).$$

For the function f of form (74), this takes the form

$$F_h = \frac{1}{2\pi}\int\int_{A_1(t)} \left(S_0^1 e^{-(u_1-\sigma_1\sqrt{t})^2/2} + K_1 e^{-rt} e^{-u_1^2/2}\right) e^{-u_2^2/2}\, du_1 du_2$$

$$+\frac{1}{2\pi}\int\int_{A_2(t)} \left(S_0^2 e^{-(u_2-\sigma_2\sqrt{t})^2/2} + K_2 e^{-rt} e^{-u_2^2/2}\right) e^{-u_1^2/2}\, du_1 du_2,$$

where the sets $A_1(t), A_2(t)$ are defined by the formulae

$$A_i(t) = \{(u_1, u_2) : S_0^i e^{\sigma_i y_1\sqrt{t}+(r-\sigma_i^2/2)t} - K_i \geq \max(0, S_0^j e^{\sigma_j y_1\sqrt{t}+(r-\sigma_j^2/2)t} - K_j)\}$$

with j being equal to 2 for $i = 1$ and conversely.

The continuous limit for prices (81), (82) can be found in the same way. For the function

$$F_{mean}(t, z_1, z_2) = (\mathcal{B}_{mean}^t f)(z_1, z_2) = \lim_{n\to\infty} (\mathcal{B}_{mean}^n(t/n)f)(z)$$

one obtains the same equation (83) but with volatilities $\sqrt{p}\sigma_1$, $\sqrt{p}\sigma_2$ instead of σ_1 and σ_2 respectively. For the continuous limit of the minimal price

$$F_{min}(t, z_1, z_2) = (\mathcal{B}_{min}^t f)(z_1, z_2) = \lim_{n\to\infty} (\mathcal{B}_{min}^n(t/n)f)(z)$$

(which is therefore equal to the difference between the hedge price F_h and the maximal unpredictable surplus of an investor) one obtains a more difficult, essentially nonlinear, equation

$$\frac{\partial F}{\partial t} = \frac{1}{2}\max_{s_1\in[0,\sigma_1]} s_1^2 z_1^2 \frac{\partial^2 F}{\partial z_1^2} + \frac{1}{2}\max_{s_2\in[0,\sigma_2]} s_2^2 z_2^2 \frac{\partial^2 F}{\partial z_2^2} + rz_1\frac{\partial F}{\partial z_1} + rz_2\frac{\partial F}{\partial z_2} - rF,$$

which under transformation (84) reduces to

$$\frac{\partial R}{\partial t} = \frac{1}{2} \max_{s_1\in[0,\sigma_1]} \left(\frac{\partial^2 R}{\partial p_1^2} - \frac{\partial R}{\partial p_1} \right) + \frac{1}{2} \max_{s_2\in[0,\sigma_2]} \left(\frac{\partial^2 R}{\partial p_2^2} - \frac{\partial R}{\partial p_2} \right).$$

This nonlinear diffusion equation is a two-dimensional version of the equation obtained in [38] by means of stochastic analysis and under certain probabilistic assumptions on the evolution of the underlying common stocks.

One sees that all three types of prices, C_h, C_{min}, C_{mean}, are expressed in terms of the iterations of some homogeneous nonexpansive maps (in the sense of Section 5), which act however not in a finite dimensional space but in the space of continuous functions on the real line or on the plane (actually it is defined on a subspace of this space). All reasonable generalisations of the model lead to the same result. For example, it was supposed above (which is a commonly used assumption) that the number of stock units γ, which an investor chooses in every moment of time, is arbitrary (no restrictions are posed, this number can even be negative). However, in reality, the boundaries on possible values of γ seem to exist either due to the general boundary on the existing common stock units (one should suppose then that $\gamma \leq \gamma_0$ for some fixed γ_0), or due to the bounds on the possibilities of an investor to make (friction-free) borrowing (one should suppose then the restrictions of the type $\gamma_k \leq X_k/S_k$, say, when no borrowing is allowed). On the other hand, one can omit the assumption of the friction-free exchange of the market securities. In all cases, one proves the existence of hedge strategies and the formula of type (78), (82) for the hedging or minimal price by the same arguments, and in all cases, the Bellman operator $\mathcal{B}$ is a nonexpansive homogeneous mapping on the space of continuous functions on some metric space. However, the formula for this $\mathcal{B}$ would be more complicated. Therefore, in order to be able to find the asymptotic formulas for hedging or minimal prices in various situations one needs to expand the theory of nonexpansive maps iterations to the infinite dimensional case.

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