General flows of deterministic and stochastic replicator dynamics

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Replicator Dynamics (RD) displays beautiful mathematical features and is successful in modeling real life behavior in biology and economics (like some lizards following the RD prediction of the Rock-Paper-Scissors game). In some cases, however, the behavior of species in real life situations are quite different from the predictions of RD and its various modifications. This suggest the existence of an underlying more fundamental structure. We argue that this structure is provided by Markov processes of interacting particles. From this point of view the RD becomes just a dynamic law of large numbers for these random systems. And the fluctuation from this limit can be studied by means of an appropriate dynamic Central Limit Theorem.

We aim to overview a method of general derivation of the dynamic law of large numbers from Markov models of interaction. This allows one to present in a unified way the deduction of both a variety of the basic equations from statistical mechanics (say Smoluchovski, Boltzman, Vlasov, etc), as well as RD equations in very general situations. In the talk we mostly stress the formal scheme. Mathematical details can be found in references given at the end.

In the talk we assume some familiarity with Markov chains in continuous time and with related notions of Markov semigroups and their generators.

1. State space of systems of interacting particles (set-up)

X a locally compact metric space

X^0 a one-point space, $X^j = X \times ... \times X$ (j-times) (with product topology),

$\mathcal{X} = \mathcal{X} = \cup_{j=0}^{\infty} X^j$ (disjoint union)

X specifies the state space of one particle and $\mathcal{X} = \cup_{j=0}^{\infty} X^j$ stands for the state space of a random number of similar particles.

$C_{sym}(\mathcal{X})$ the Banach spaces of symmetric bounded continuous functions on $\mathcal{X}$ and by $C_{sym}(X^k)$ the corresponding spaces of functions on the finite power X^k

$\mathcal{M}_{sym}(\mathcal{X})$ symmetric measures

The elements of $\mathcal{M}^+_{sym}(\mathcal{X})$ and $C_{sym}(\mathcal{X})$ are respectively the *(mixed) states and observables* for a Markov process on $\mathcal{X}$.

Bold letters, e.g. $\mathbf{x}$, $\mathbf{y}$ denote the elements of $\mathcal{X}$. For a finite set $I = \{i_1, ..., i_k\}$, $\mathbf{x}_I$ is the collection of the variables $x_{i_1}, ..., x_{i_k}$ and $d\mathbf{x}_I = dx_{i_1}...dx_{i_k}$.

Factor spaces SX^k and $S\mathcal{X}$ are obtained by the factorization of X^k and $\mathcal{X}$ with respect to all permutations, which allows for the identifications:

$C_{sym}(\mathcal{X}) = C(S\mathcal{X})$, $S\mathcal{X}$ the set of all finite subsets of X.

Basic inclusion $S\mathcal{X} \to \mathcal{M}^+(X)$:

$$\mathbf{x} = (x_1, ..., x_l) \mapsto \delta_{x_1} + ... + \delta_{x_l}, \tag{1}$$

is a bijection between $S\mathcal{X}$ and the space $\mathcal{M}_\delta^+(X)$ of finite linear combinations of δ-measures.

Clearly each $f \in C_{sym}(\mathcal{X})$ is defined by its components (restrictions) f^k on X^k so that for $\mathbf{x} = (x_1, ..., x_k) \in X^k \subset \mathcal{X}$, say, one can write $f(\mathbf{x}) = f(x_1, ..., x_k) = f^k(x_1, ..., x_k)$. Similar notations are for measures. In particular, the pairing between $C_{sym}(\mathcal{X})$ and $\mathcal{M}(\mathcal{X})$ can be written as

$$(f, \rho) = \int f(\mathbf{x})\rho(d\mathbf{x})$$

$$= f^0\rho_0 + \sum_{n=1}^{\infty} \int f(x_1, ..., x_n)\rho(dx_1...dx_n),$$

$$f \in C_{sym}(\mathcal{X}),\ \rho \in \mathcal{M}(\mathcal{X})$$

so that $\|\rho\| = (\mathbf{1}, \rho)$ for $\rho \in \mathcal{M}^+(\mathcal{X})$.

A useful class of measures (and mixed states) on $\mathcal{X}$ is given by the decomposable measures of the form $Y^{\otimes}$, which are defined for an arbitrary finite measure $Y(dx)$ on X by their components

$$(Y^{\otimes})_n(dx_1...dx_n) = Y(dx_1)...Y(dx_n)$$

Similarly the decomposable observables (multiplicative or additive)) are defined for an arbitrary $Q \in C(X)$ as

$$(Q^{\otimes})^n(x_1, ..., x_n) = Q(x_1)...Q(x_n)$$

and

$$(Q^{\oplus})(x_1, ..., x_n) = Q(x_1) + ... + Q(x_n)$$

($Q^{\oplus}$ vanishes on X^0). In particular, if $Q = \mathbf{1}$, then $Q^{\oplus} = \mathbf{1}^{\oplus}$ is the number of particles: $\mathbf{1}^{\oplus}(x_1, ..., x_n) = n$.

Here we talk about pure jump processes on $\mathcal{X}$ (generalized Markov chains), whose semigroup and the generator preserves the space C_{sym} of continuous symmetric functions and hence are given by symmetric transition kernels $q(\mathbf{x}; d\mathbf{y})$ that could be thus considered as kernels on the factor space $S\mathcal{X}$.

2. Pure jump Markov models of interacting particles

Assume

$$P^2(x_1, x_2; d\mathbf{y}) = \{P^2_m(x_1, x_2; dy_1...dy_m)\}$$

a continuous transition kernel $SX^2 \to S\mathcal{X}$ such that $P^2(\mathbf{x}; \{\mathbf{x}\}) = 0$ for all $\mathbf{x} \in X^2$, with the intensity

$$P^2(x_1, x_2) = \int_{\mathcal{X}} P^2(x_1, x_2; d\mathbf{y})$$

$$= \sum_{m=0}^{\infty} \int_{X^m} P^2_m(x_1, x_2; dy_1...dy_m).$$

The intensity defines the rate of decay of any pair of particles x_1, x_2 and the measure $P^k(x_1, x_2; d\mathbf{y})$ defines the distribution of possible outcomes.

Supposing that any randomly chosen pair of particles from a given set of n particles can interact, leads to the following *generator of binary interacting particles defined by the kernel* P^2:

$$(G_2 f)(x_1, ..., x_n) = \sum_{I \subset \{1,...,.n\}, |I|=2}$$

$$\int (f(\mathbf{x}_{\bar{I}}, \mathbf{y}) - f(x_1, ..., x_n)) P^2(x_I, d\mathbf{y}).$$

Probabilistic interpretation in terms of exponential waiting times.

Similarly, a k-ary interaction of a pure jump type is specified by a transition kernel

$$P^k(x_1, ..., x_k; d\mathbf{y}) = \{P^k_m(x_1, ..., x_k; dy_1...dy_m)\} \tag{2}$$

from SX^k to $S\mathcal{X}$ such that $P^k(\mathbf{x}; \{\mathbf{x}\}) = 0$ for all $\mathbf{x} \in \mathcal{X}$, having the intensity

$$P^k(x_1, ..., x_k) = \int P^k(x_1, ..., x_k; d\mathbf{y})$$

$$= \sum_{m=0}^{\infty} \int P^k_m(x_1, ..., x_k; dy_1...dy_m). \tag{3}$$

This kernel defines the following generator of k-ary interacting particles:

$$(G_k f)(x_1, ..., x_n) = \sum_{I \subset \{1,...,.n\}, |I|=k}$$

$$\int (f(\mathbf{x}_{\bar{I}}, \mathbf{y}) - f(x_1, ..., x_n)) P^k(x_I, d\mathbf{y}). \tag{4}$$

Changing the state space by (1) yields the corresponding Markov process on $\mathcal{M}^+_\delta(X)$.

Choosing a positive parameter h, we shall perform now the following scaling: we scale the empirical measures $\delta_{x_1}+...+\delta_{x_n}$ by a factor h and the operator of k-ary interactions by a factor h^{k-1} (similar to the scaling used in the theory of superprocesses, which in our notations corresponds to the 'interaction free' case of $k=1$). This leads to the operator

$$\Lambda_k^h f(h\delta_{\mathbf{x}}) = h^{k-1} \sum_{I\subset\{1,...,n\},|I|=k}$$

$$\int [f(h\delta_{\mathbf{x}}-\sum_{i\in I} h\delta_{x_i}+h\delta_{\mathbf{y}})-f(h\nu)]P(\mathbf{x}_I;d\mathbf{y}) \quad (5)$$

(where we denoted $\delta_{\mathbf{y}} = \delta_{y_1} + ... + \delta_{y_m}$ for $\mathbf{y} = (y_1, ..., y_m)$), acting on the space of continuous functions on the set $\mathcal{M}_{h\delta}^{+}(X)$ of measures of the form $h\nu = h\delta_{\mathbf{x}} = h\delta_{x_1}+...+h\delta_{x_n}$.

The above scaling (usually applied in statistical mechanics) is not the only reasonable one. For the theory of evolutionary games (or other biological model) a more natural scaling is by normalizing on the number of particles, i.e. by division of k-ary interaction by $n^{k-1} = (\|h\nu\|/h)^{k-1}$. This leads (instead of (5)) to the operator

$$\tilde{\Lambda}_k^h f(h\delta_{\mathbf{x}}) = h^{k-1} \sum_{I\subset\{1,\ldots,n\},|I|=k}$$

$$\int [f(h\nu - \sum_{i\in I} h\delta_{x_i} + h\delta_{\mathbf{y}}) - f(h\nu)] \frac{P(\mathbf{x}_I; d\mathbf{y})}{\|h\delta_{\mathbf{x}}\|^{k-1}}. \quad (6)$$

3. Heuristic derivation of kinetic equations

Using the obvious formula

$$\sum_{I\subset\{1,\dots,n\},|I|=2} f(\mathbf{x}_I) =$$

$$\frac{1}{2}\int\int f(z_1,z_2)\delta_{\mathbf{x}}(dz_1)\delta_{\mathbf{x}}(dz_2)-\frac{1}{2}\int f(z,z)\delta_{\mathbf{x}}(dz) \tag{7}$$

(valid for $f \in C^{sym}(X^2)$ and $\mathbf{x} = (x_1,\dots,x_n) \in X^n$),

the operator Λ_2^h can be written in the form

$$\Lambda_2^h f(h\delta_{\mathbf{x}}) = -\frac{1}{2}\int_{\mathcal{X}}\int_X [f(h\delta_{\mathbf{x}}-2h\delta_z+h\delta_{\mathbf{y}})-f(h\delta_{\mathbf{x}})]$$

$$P(z,z;d\mathbf{y})(h\delta_{\mathbf{x}})(dz)$$

$$+\frac{1}{2h}\int_{\mathcal{X}}\int_{X^2}[f(h\delta_{\mathbf{x}}-h\delta_{z_1}-h\delta_{z_2}+h\delta_{\mathbf{y}})-f(h\delta_{\mathbf{x}})]$$

$$P(z_1,z_2;d\mathbf{y})(h\delta_{\mathbf{x}})(dz_1)(h\delta_{\mathbf{x}})(dz_2).$$

For a linear function $f_g(\mu)=\int g(y)\mu(dy)$,

$$\Lambda_2^h f_g(h\delta_{\mathbf{x}})=\frac{1}{2}\int_{\mathcal{X}}\int_{X^2}[g^{\oplus}(z_1,z_2)-g^{\oplus}(\mathbf{y})]$$

$$P(z_1,z_2;d\mathbf{y})(h\delta_{\mathbf{x}})(dz_1)(h\delta_{\mathbf{x}})(dz_2)$$

$$-\frac{1}{2}h\int_{\mathcal{X}}\int_{X}[g^{\oplus}(z,z)-g^{\oplus}(\mathbf{y})]P(z,z;d\mathbf{y})(h\delta_{\mathbf{x}})(dz)$$

It follows that if $h\to 0$ and $h\delta_{\mathbf{x}}$ tends to some finite measure μ (large number of particles limit with a finite "whole mass"), the evolution equation $\dot{f}=\Lambda_2^h f$ for linear $f=f_g$ tends to the equation

$$\frac{d}{dt}\int_X g(z)\mu_t(dz)$$

$$=\frac{1}{2}\int_{\mathcal{X}}\int_{X^2}(g^{\oplus}(\mathbf{y})-g^{\oplus}(\mathbf{z}))P^2(\mathbf{z};d\mathbf{y})\mu_t^{\otimes 2}(d\mathbf{z}) \tag{8}$$

with $\mathbf{z}=(z_1,z_2)$, which represents the *general kinetic equation for binary interactions of pure jump type* in the *weak form*.

The famous Boltzman equation and the Smoluchovski coagulation equation are particular cases of (8).

Similar procedure with k-ary interactions leads to the *general kinetic equation for k-ary interactions of pure jump type in the weak form*:

$$\frac{d}{dt}\int_X g(z)\mu_t(dz)$$

$$= \frac{1}{k}\int_{\mathcal{X}}\int_{X^k}(g^{\oplus}(\mathbf{y}) - g^{\oplus}(\mathbf{z}))P^k(\mathbf{z};d\mathbf{y})\mu_t^{\otimes k}(d\mathbf{z}) \tag{9}$$

with $\mathbf{z} = (z_1, ..., z_k)$. On the other hand, using the alternative scaling, leads to the equation

$$\frac{d}{dt}\int_X g(z)\mu_t(dz) = \frac{1}{k}\int_{\mathcal{X}}\int_{X^k}(g^{\oplus}(\mathbf{y}) - g^{\oplus}(\mathbf{z}))$$

$$P^k(\mathbf{z};d\mathbf{y})\left(\frac{\mu_t}{\|\mu_t\|}\right)^{\otimes k}(d\mathbf{z})\|\mu_t\|. \tag{10}$$

In biological context one traditionally writes the dynamics in terms of normalized (probability) measures.

For positive μ, (10) implies

$$\frac{d}{dt}\|\mu_t\| = -\frac{1}{k}\int_{X^k} Q(\mathbf{z})\left(\frac{\mu_t}{\|\mu_t\|}\right)^{\otimes k}(d\mathbf{z})\|\mu_t\|, \tag{11}$$

where

$$Q(\mathbf{z}) = -\int_{\mathcal{X}}\int_{X^k}(\mathbf{1}^{\oplus}(\mathbf{y}) - \mathbf{1}^{\oplus}(\mathbf{z}))P^k(\mathbf{z}; d\mathbf{y}). \tag{12}$$

Consequently, rewriting equation (10) in terms of normalized measure $\nu_t = \mu_t/\|\mu_t\|$ yields

$$\frac{d}{dt}\int_X g(z)\nu_t(dz)$$

$$= \frac{1}{k}\int_{\mathcal{X}}\int_{X^k}(g^{\oplus}(\mathbf{y}) - g^{\oplus}(\mathbf{z}))P^k(\mathbf{z}; d\mathbf{y})(\nu_t)^{\otimes k}(d\mathbf{z})$$

$$+\frac{1}{k}\int_X g(z)\nu_t(dz)\int_{\mathcal{X}}\int_{X^k} Q(\mathbf{z})(\nu_t)^{\otimes k}(d\mathbf{z}). \tag{13}$$

4. Simple well-posedness results for kinetic equations and convergence of Markov approximation

The central notion for the theory of kinetic equations is the criticality. The transition kernel $P = P^k(\mathbf{x}; d\mathbf{y})$ from (2) is called *subcritical* (respectively *critical*), if

$$\int_{\mathcal{X}} (\mathbf{1}^{\oplus}(\mathbf{y}) - \mathbf{1}^{\oplus}(\mathbf{x})) P^k(\mathbf{x}; d\mathbf{y}) \leq 0$$

for all $\mathbf{x} \in X^k$ (respectively if the equality holds).

Theorem 1. *Suppose the transition kernel is subcritical and its intensity is uniformly bounded. Then for any non-negative measure $\mu \in \mathcal{M}^+(X)$ there exists a unique global solution to the basic kinetic equations above (with each of two scalings considered). Moreover this solution is positive, $\|\mu_t\| \leq \|\mu\|$ for all t, and the mapping $t \mapsto \mu_t$ is strongly differentiable.*

Proof. Key point is positivity preservation.

A worth noting feature of equation (9) is that its evolution preserves L_1 spaces. Namely, the following holds.

Theorem 2. *Under assumptions of the previous Theorem suppose we are given a reference (not necessarily finite, but σ-finite and positive) measure M on X. Let μ has a non-negative density f with respect to M, i.e. $f \in L_1(X, M)$. Then the solution μ_t will also have a non-negative density $f_t \in L_1(X, M)$ satisfying the weak equation*

$$\frac{d}{dt}\int_X g(z) f_t(z) M(dz) = \frac{1}{k}\int_{\mathcal{X}}\int_{X^k} (g^{\oplus}(\mathbf{y}) - g^{\oplus}(\mathbf{z}))$$

$$P^k(\mathbf{z}; d\mathbf{y}) \prod_{i=1}^{k} f_t(z_i) M^{\otimes k}(d\mathbf{z}). \qquad (14)$$

Proof. It repeats literarily the proof of the previous theorem. Namely, one first get the local existence and uniqueness and then extends to all times by positivity.

Theorem 3. *Under the assumptions of the previous Theorem assume additionally that X is compact. Then the operator* (5) *(respectively* (6)*) generates a uniquely defined nonexplosive Markov process Z_t^h (respectively $\tilde{Z}_t^h$) on $\mathcal{M}_{h\delta}^+$ for any $h > 0$. Moreover, if the initial conditions Z_0^h converge weakly to a finite measure μ on X as $h \to 0$, then the processes Z_t^h (respectively $\tilde{Z}_t^h$) converge in the sense of distribution to the solution μ_t (a deterministic process) of the equation* (9) *(respectively* (10)*) constructed in Theorem 1.*

5. Evolutionary games and Replicator dynamics

Recall: a *k-person game (in normal form)* is specified by a collection of k compact spaces $X_1, ..., X_k$ of possible pure strategies for the players and a collection of continuous payoff functions H_1,...,H_k on $X_1 \times \cdots \times X_k$.

One step of such a game is played according to the following rule. Each player i, $i = 1, ..., k$, chooses independently a strategy $x_i \in X_i$ and then receives the payoff $H_i(x_1, ..., x_k)$. The collection $\{x_1, ..., x_k\}$ is called a *profile (or situation) of the game*. In elementary models X_i are finite sets.

A game is called *symmetric* if $X_i = X$ do not depend on i and the payoffs are symmetric in the sense that they are specified by a single function $H(x; y_1, ..., y_{k-1})$ on X^k symmetric with respect to the last $k-1$ variables $y_1, ..., y_{k-1}$ via the formula

$$H_i(x_1, ..., x_k) = H(x_i, x_1, ..., x_{i-1}, x_{i+1}, ..., x_k).$$

In symmetric games the label of the player is irrelevant, only the strategy is important.

By the *mixed strategy extension* of a game with strategy spaces X_i and payoffs H_i, $i = 1, ..., k$, we mean a k-person game with the spaces of strategies $\mathcal{P}(X_i)$ (considered as a compact space in its weak topology), $i = 1, ..., k$, and the payoffs

$$H_i^\star(P) = \int_{X^k} H_i(x_1, ..., x_k) P(dx_1 \cdots dx_k),$$

$$P = (p_1, ..., p_k) \in \mathcal{P}(X_1) \times \cdots \times \mathcal{P}(X_k).$$

Playing a *mixed strategy* p_i is interpreted as choosing pure strategies randomly with probability law p_i.

The key notion in the theory of games is that of Nash equilibrium. Let

$$H_i^\star(P \| x_i) = \int_{X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n}$$

$$H_i(x_1, ..., x_n)\, dp_1 \cdots dp_{i-1} dp_{i+1} \cdots dp_n. \quad (15)$$

A situation $P = (p_1, ..., p_k)$ is called a *Nash equilibrium*, if

$$H_i^\star(P) \geq H_i^\star(P \| x_i) \quad (16)$$

for all i and $x_i \in X_i$.

For symmetric games and *symmetric profiles* $P = (p, ..., p)$, which are of particular interest for evolutionary games,

$$H_i^\star(P) = H^\star(P) = \int_{X^k} H(x_1, ..., x_k) p^{\otimes k}(dx_1 \cdots dx_k)$$

and

$$H_i^\star(P\|y) = H^\star(P\|y)$$

$$= \int_{X^{k-1}} H(y, x_1, ..., x_{k-1})\, p^{\otimes(k-1)}(dx_1 \cdots dx_{k-1})$$

do not depend on i and the condition of equilibrium is

$$H^\star(P) \geq H^\star(P\|x), \quad x \in X. \qquad (17)$$

The *replicator dynamics (RD)* of evolutionary game theory is supposed to model the process of approaching the equilibrium from a given initial state by decreasing the losses produced by deviating from the equilibrium (adjusting the strategy to the current situation).

More precisely, assuming a mixed profile is given by a density f_t with respect to a certain reference measure M on X (f_t can be interpreted as the fraction of a large population using strategy x), the replicator dynamics is defined as

$$\dot{f}_t(x) = f_t(x)(H^\star(f_t M \| x) - H^\star(f_t M)). \quad (18)$$

We want to show how this evolution appears as a simple particular case of the general dynamic law of large number limit (kinetic equation) described above.

A key feature distinguishing the evolutionary game setting in the general framework: the species produce new species of their own kind (with inherited behavioral patterns). In the usual model of evolutionary game theory it is assumed that any k randomly chosen species can occasionally meet and play a k-person symmetric game specified by a payoff function $H(x; y_1, ..., y_{k-1})$ on X^k, where the payoff measures fitness expressed in terms of expected number of offspring.

To specify a Markov model we need to specify the game a bit further. We shall assume that X is a compact set and that the result of the game for player x playing against $y_1, ..., y_{k-1}$ is given by the probability rates $H^m(x; y_1, ..., y_{k-1})$, $m = 0, 1, ...$, of the number m of particles of type x that would appear in place of x after this game (one interaction). To fit into the original model, the H^m can be chosen arbitrary, as long as the average change equals the original function H:

$$H(x; y_1, ..., y_{k-1}) = \sum_{m=0}^{\infty} (m-1) H^m(x; y_1, ..., y_{k-1}). \tag{19}$$

The simplest model is one, in which a species can either die or produce another species of the same kind with given rates H^0, H^2; the probabilities are therefore $H^0/(H^0 + H^2)$ and $H^2/(H^0+H^2)$. Under these assumptions equation (19) reduces to

$$H(x; y_1, ..., y_{k-1}) = (H^2 - H^0)(x; y_1, ..., y_{k-1}). \tag{20}$$

In any case, we have transition kernels of the form

$$P_m^k(z_1,...,z_k;d\mathbf{y}) = H^m(z_1;z_2,...,z_k)\prod_{j=1}^m \delta_{z_1}(dy_j)$$

$$+H^m(z_2;z_1,...,z_k)\prod_{j=1}^m \delta_{z_2}(dy_j)+\cdots$$

$$+H^m(z_k;z_1,...,z_{k-1})\prod_{j=1}^m \delta_{z_k}(dy_j) \qquad (21)$$

so that

$$\int_{\mathcal{X}}(g^{\oplus}(\mathbf{y})-g^{\oplus}(\mathbf{z}))P^k(\mathbf{z};d\mathbf{y})$$

$$=g(z_1)H(z_1;z_2,...,z_k)+\cdots+g(z_k)H(z_k;z_1,...,z_{k-1}).$$

Due to the symmetry of H, equation (10) takes the form

$$\frac{d}{dt}\int_X g(x)\mu_t(dx) = \frac{\|\mu_t\|}{(k-1)!}\int_{X^k}$$

$$g(z_1)H(z_1;z_2,...,z_k)\left(\frac{\mu_t}{\|\mu_t\|}\right)^{\otimes k}(dz_1\cdots dz_k). \qquad (22)$$

Hence for the normalized measure $\nu_t = \mu_t / \|\mu_t\|$ one gets the evolution

$$\frac{d}{dt} \int_X g(x) \nu_t(dx)$$

$$= \frac{1}{(k-1)!} \int_X (H^\star(\nu_t \| x) - H^\star(\nu_t)) g(x) \nu_t(dx), \tag{23}$$

which represents the *replicator dynamics* in weak form for a symmetric k-person game with an arbitrary compact space of strategies. It is obtained here as a simple particular case of (13).

If a reference probability measure M on X is chosen, equation (23) can be rewritten in terms of the densities f_t of ν_t with respect to M as (18).

6. Equilibria

Proposition 1. *(i) If ν defines a symmetric Nash equilibrium for symmetric k-person game (its mixed strategy extension) specified by payoff $H(x; y_1, ..., y_{k-1})$ on X^k (X again a compact space), then ν is a fixed point for RD* (23). *If ν is such that any open set in X has a positive ν measure (pure mixed profile), then the inverse statement holds.*

Proof. (i)

$$H^\star(\nu\|x) \leq H^\star(\nu) \tag{24}$$

for an equilibrium ν and all $x \in X$. The set $M = \{x : H^\star(\nu\|x) < H^\star(\nu)\}$ should have ν-measure zero (otherwise integrating (24) would lead to a contradiction). This implies that

$$\int_X (H^\star(\nu\|x) - H^\star(\nu))g(x)\nu_t(dx) = 0. \tag{25}$$

for all g. (ii) Conversely assuming (25) holds for all g implies (taking into account here that ν is purely mixed profile)

$$H^\star(\nu\|x) = H^\star(\nu)$$

on a open dense subset of X and hence everywhere, due to the continuity of H.

Proposition 2. *Consider a mixed strategy extension of a two-person symmetric game with a compact space of pure strategies X of each player and a payoff matrix being an antisymmetric function H on X^2, i.e. $H(x,y) = -H(y,x)$. Assume there exists a positive finite measure M on X such that $\int H(x,y)M(dy) = 0$ for all x. Then M specifies a symmetric Nash equilibrium. Moreover, the function*

$$L(f) = \int \ln f_t(x)M(dx)$$

is the first integral (i.e. it is constant on the trajectories) of the RD on densities with respect to M.

Proof. Is a straightforward generalization of the discrete case.

Note however that the existence of the first integral is not enough to make a conclusion on the stability in this infinite-dimensional setting.

7. Stochastic replicator dynamics

Kinetic equations and RD described above specify the law of large numbers limits for Markov processes of interacting particles when this limit is a deterministic process. However there are natural situations when the law of large numbers is itself random. Studying such random limits leads to general measure-valued processes with (possibly infinite dimensional) pseudo-differential generators. The simplest case of these limits obtained from branching (noninteracting particle systems) yields a popular class of processes called superprocesses.

Let us briefly introduce those limiting processes turning to two-person games with only a finite set $X = \{1, ..., d\}$ of pure strategies.

N_j the number of individuals playing the strategy j

$N = \sum_{j=1}^{d} N_j$ the whole size of the population.

The outcome of a game between players with strategies i and j is a probability distribution $A_{ij} = \{A_{ij}^m\}$ of the number of offsprings $m \geq -1$ ($m = -1$ means the death of the individual) of the players ($\sum_{m=-1}^{\infty} A_{ij}^m = 1$).

If the intensity of the reproduction per time unit a_{ij} are given, this yields the Markov chain on $\mathbf{Z}_+^d$ with the generator

$$Gf(N) = \sum_{j=1}^{d} N_j \sum_{m=-1}^{\infty}$$

$$\left(B_j^m + \sum_{k=1}^{d} a_{jk} A_{jk}^m \frac{N_k}{|N|} \right) (f(N + me_j) - f(N)) \tag{26}$$

(where B_j^m describe the background reproduction process), which is a version of the generators of binary interactions introduced above.

Assuming a natural dependence of probabilities A^m_{jk} on a small parameter h (inverse of the average number of particles) leads in $h \to 0$ limit to the process on $\mathbf{R}^d_+$ with the generator of the type

$$\Lambda = \sum_{j=1}^d x_j \left(\phi_j + \sum_{k=1}^d \frac{x_k}{\mu(x)} \phi_{jk} \right), \qquad (27)$$

where ϕ_j and ϕ_{jk} generate one-dimensional Lévy processes:

$$\phi_{jk} f(x) = g_{jk} \frac{\partial^2 f}{\partial x_j^2}(x) + \beta_{jk} \frac{\partial f}{\partial x_j}(x)$$

$$+ \int \left(f(x + y e_j) - f(x) - \mathbf{1}_{y \le 1}(y) \frac{\partial f}{\partial x_j}(x) y_j \right) \nu_{jk}(dy), \qquad (28)$$

$$\phi_j f(x) = g_j \frac{\partial^2 f}{\partial x_j^2}(x) + \beta_j \frac{\partial f}{\partial x_j}(x)$$

$$+ \int \left(f(x + y e_j) - f(x) - \mathbf{1}_{y \le 1}(y) \frac{\partial f}{\partial x_j}(x) y \right) \nu_j(dy),$$

where all ν_{jk}, ν_j are Borel measures on $(0, \infty)$ such that the function $\min(y, y^2)$ is integrable with respect to these measures, g_j and g_{jk} are non-negative.

The interpretation of the approximations of the three terms in (28) (diffusion, drift and integral part):

The first term stands for a quick game ("death or birth" game), which describes some sort of fighting for reproduction, whose outcome is that an individual either dies or produces an offspring;

The second term (approximating drift) describes games for death or for life depending on the sign of β_{jk};

Third term corresponds to games with a slow reproduction of the large number of offsprings.

Developments: the fluctuations around the LLN (kinetic equations or replicator dynamics) are described by the dynamic CLT.

Remark. Underlying probabilistic models should be taken into consideration when discussing alternative evolutionary dynamics.

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