SDEs driven by nonlinear Lévy noise (based partially on 'The Lévy-Khintchine type operators with variable Lipschitz continuous coefficients generate linear or nonlinear Markov processes and semigroups' submitted to PTRF)

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PLAN:

1) What is 'SDEs driven by nonlinear Lévy noise'? How usual SDEs (with and without jumps) fit to this general scheme?

2) Main motivation: 'The Lévy-Khintchine type operators with variable Lipschitz continuous coefficients generate linear or nonlinear Markov processes and semigroups': Reconciling the theory of SDEs with the theory of Markov semigroups.

3) Example of further applications: curvilinear OU processes

SDEs DRIVEN BY NONLINEAR LÉVY NOISE

$$dX_t = dY_t(X_t) \Longleftrightarrow X_t = x + \int_0^t dY_s(X_s)\, ds. \qquad (1)$$

or more generally

$$dX_t = dY_t(X_t, \mathcal{L}(X_t))$$

$$\Longleftrightarrow X_t = x + \int_0^t dY_s(X_s, \mathcal{L}(X_s))\, ds,$$

where $\mathcal{L}(\xi)$ denotes the law of a r.v. ξ and where $Y_t(z, \mu)$ is a family of Lévy processes specified by their generators

$$L[z,\mu]f(x) = \frac{1}{2}(G(z,\mu)\nabla, \nabla)f(x) + (b(z,\mu), \nabla f(x))$$

$$+ \int (f(x+y) - f(x) - (\nabla f(x), y))\nu(z, \mu; dy).$$

One can construct solutions from the Euler type approximation scheme. For instance, take equation (1). Let $Y_\tau^l(x)$ be a collection (depending on $l = 0, 1, 2, ...$) of independent families of the Lévy processes $Y_\tau(x)$ depending measurably on x. Define the approximations $X^{\mu,\tau}$ by:

$$X_t^{\mu,\tau} = X_{l\tau}^{\mu,\tau} + Y_{t-l\tau}^l(X_{l\tau}^{\mu,\tau}), \quad \mathcal{L}(X_\mu^\tau(0)) = \mu,$$

for $l\tau < t \le (l+1)\tau$, where $\mathcal{L}(X)$ means the probability law of X.

Stochastic equation of the standard form

$$X_t = x + \int_0^t \sigma(X_{s-})dB_s + \int_0^t b(X_{s-})ds$$

$$+ \int_0^t \int F(X_{s-}, y)\tilde{N}(dsdy),$$

where F is a measurable mapping $\mathbf{R}^n \times \mathbf{R}^d \mapsto \mathbf{R}^n$ and σ maps $\mathbf{R}^n$ to $n \times d$-matrices and $\tilde{N}(dsdx)$ is the corresponding compensated Poisson measure of jumps of the Lévy process Y with the generator

$$Lf(x) = + \int [f(x+y) - f(x) - (y, \nabla)f(x)]\nu(dy),$$

having $\int |y|^2 \nu(dy) < \infty$.

It complies with the above general scheme for

$$Y_t(z) = \sigma(z)B_t + b(z)t + \int_0^t \int F(z, y)\tilde{N}(dsdy). \tag{2}$$

Unlike the general case the processes Y_t are given as functionals of a single process.

MARKOV PROCESSES WITH A GIVEN GENERATOR

It is well known (the Courrège theorem) that the generator L of a conservative (i.e. preserving constants) Feller semigroup in $\mathbf{R}^d$ is conditionally positive ($f \geq 0, f(x) = 0 \implies Lf(x) \geq 0$) and if its domain contains the space $C_c^2(\mathbf{R}^d)$, then it has the following Lévy-Khintchine form with variable coefficients:

$$Lf(x) = \frac{1}{2}(G(x)\nabla, \nabla)f(x) + (b(x), \nabla f(x))$$

$$+ \int (f(x+y) - f(x) - (\nabla f(x), y)\mathbf{1}_{B_1}(y))\nu(x, dy), \tag{3}$$

where $G(x)$ is a symmetric non-negative matrix and $\nu(x,.)$ a Borel measure on $\mathbf{R}^d$ (called Lévy measure) such that

$$\int_{\mathbf{R}^n} \min(1, |y|^2)\nu(x; dy) < \infty, \quad \nu(\{0\}) = 0. \tag{4}$$

The inverse question on whether a given operator of this form (or better to say its closure) actually generates a Feller semigroup is nontrivial and attracted lots of attention.

Theorem 1 *Under the assumption*

$$\left[tr(\sqrt{G(x_1)} - \sqrt{G(x_2)})^2\right]^{1/2} + |b(x_1) - b(x_2)|$$

$$+W_2(\nu(x_1, .), \nu(x_2, .)) \leq \kappa_2 \|x_1 - x_2\|. \quad (5)$$

(i) for any $\mu \in \mathcal{P}(\mathbf{R}^d) \cap \mathcal{M}_2(\mathbf{R}^d)$ there exists a limit process X_t^μ for the approximations $X_t^{\mu,\tau}$ such that

$$\sup_\mu \sup_{s \in [0,t]} W_2^2 \left(X^{\mu,\tau_k}_{[s/\tau_k]\tau_k}, X_t^\mu \right) \leq c(t)\tau_k,$$

and even stronger

$$\sup_\mu W^2_{2,t,un} (X^{\mu,\tau_k}, X^\mu) \leq c(t)\tau_k,$$

(ii) the distributions $\mu_t = \mathcal{L}(X_t^\mu)$ depend 1/2-Hölder continuous on t in the metric W_2 and Lipschitz continuously on the initial condition:

$$W_2^2(X_t^\mu, X_t^\eta) \leq c(T) W_2^2(\mu, \eta);$$

(iii) the processes

$$M(t) = f(X_t^\mu) - f(x) - \int_0^t (Lf(X_s^\mu)\, ds$$

are martingales for any $f \in C^2(\mathbf{R}^d)$*, where*

$$Lf(x) = \frac{1}{2}(G(x)\nabla, \nabla)f(x) + (b(x), \nabla f(x))$$

$$+ \int [f(x+y) - f(x) - (y, \nabla)f(x)]\nu(x, dy),$$

in other words, the process X_t^μ *solves the corresponding martingale problem;*

(iv) the operators $T_t f(x) = \mathbf{E} f(X_t^x)$ *form a conservative Feller semigroup preserving the space of Lipschitz continuous functions and with the domain of generator containing* $C_\infty^2(\mathbf{R}^d)$*, where it is given by* (3).

CURVILINEAR OU PROCESSES

The analog of the free motion $\dot{x} = p, \dot{p} = 0$ on a Riemannian manifold (M, g) is called the *geodesic flow* on M and is defined as the Hamiltonian system on the cotangent bundle $T^{\star}M$ specified by the Hamiltonian

$$H(x,p) = \frac{1}{2}(G(x)p, p), \quad G(x) = g^{-1}(x),$$

describing the kinetic energy in a curvilinear space, the geodesic flow equations being

$$\begin{cases} \dot{x} = G(x)p \\ \dot{p} = -\frac{1}{2}(\frac{\partial G}{\partial x}p, p) \end{cases} \tag{6}$$

In physics, stochastics often appears as a model of heat bath obtained by adding a homogeneous noise to the second equation of a Hamiltonian system, i.e. by changing the above to

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} \\ dp = -\dfrac{\partial H}{\partial x} dt + dY_t, \end{cases} \tag{7}$$

where Y_t is a Lévy process. In the most studied models Y_t stands for the BM with the variance proportional to the square root of the temperature. To balance the energy pumped into the system by the noise, one often adds friction to the system, i.e. a non conservative force proportional to the velocity. In case of initial free motion this yields the system

$$\begin{cases} \dot{x} = p \\ dp = -\alpha p \, dt + dY_t \end{cases} \tag{8}$$

with a nonnegative matrix α, called the Ornstein-Uhlenbeck (OU) system driven by the Lévy noise Y_t. Especially well studied are the cases when Y_t is BM or a stable process.

If a random force is not homogeneous, as should be the case on a manifold or in a non-homogeneous media, one is led to consider Y_t to be a family of processes depending on the position $x \in M$ leading naturally to SDEs studied above. In particular, the curvilinear analog of the OU system is the process in $T^\star M$ specified by the equation

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} \\ dp = -\dfrac{\partial H}{\partial x} dt - \alpha(x) p\, dt + dY_t(x) \end{cases} \tag{9}$$

where $H(x,p) = (G(x)p,p)/2$. Assuming for simplicity that all Y_t are zero mean Lévy processes with the Lévy measure being absolutely continuous with respect to the invariant Lebesgue measure on $T_x^\star M$ and having finite outer first moment ($\int_{|y|>1} |y| \nu(dy) < \infty$), the generator of Y_t can be written in the form

$$L_Y^x f(p) = \frac{1}{2}(A(x)\nabla, \nabla) f(p)$$

$$+ \int [f(p+q) - f(p) - \nabla f(p) q] \frac{(\det G(x))^{1/2} dq}{\omega(x,q)}$$

with a certain positive $\omega(x,q)$. Hence the corresponding full generator of the process given by (9) has the form

$$Lf(x,p)=\frac{\partial H}{\partial p}\frac{\partial f}{\partial x}-\frac{\partial H}{\partial x}\frac{\partial f}{\partial p}-(\alpha(x)p,\frac{\partial f}{\partial p})$$

$$+\frac{1}{2}\mathrm{tr}\left(\mathrm{A(x)}\frac{\partial^2\mathrm{f(x,p)}}{\partial\mathrm{p}^2}\right)$$

$$+\int[f(x,p+q)-f(x,p)-\frac{\partial f(x,p)}{\partial p}q]\frac{(\det G(x))^{1/2}dp}{\omega(x,p)}.$$

Of course in order to have a true system on a manifold, this expression should be invariant under the change of coordinate, which requires certain transformation rules for the coefficient α, A, ω. This issue is settled in the next statement.

Proposition 1 *Operator L is invariant under the change of coordinates*

$$x \mapsto \tilde{x}, \quad p \mapsto \tilde{p} = \left(\frac{\partial x}{\partial \tilde{x}}\right)^T p,$$

if and only if ω is a function on $T^\star M$, α is $(1,1)$-tensor and A is a $(0,2)$-tensor, i.e.

$$\begin{aligned} \tilde{\omega}(\tilde{x},\tilde{p}) &= \omega(x(\tilde{x}), p(\tilde{x},\tilde{p})), \\ \tilde{\alpha}(\tilde{x}) &= \left(\frac{\partial x}{\partial \tilde{x}}\right)^T \alpha(x(\tilde{x})) \left(\frac{\partial \tilde{x}}{\partial x}\right)^T, \\ \tilde{A}(\tilde{x}) &= \left(\frac{\partial x}{\partial \tilde{x}}\right)^T A(x(\tilde{x}))\frac{\partial x}{\partial \tilde{x}}. \end{aligned} \tag{10}$$

Of particular interest are processes depending only on the Riemannian structure. For instance (9) defines the *curvilinear OU process of diffusion type if Y_t has the generator*

$$L_Y^x f(p) = \frac{1}{2}(g(x)\nabla, \nabla) f(p)$$

and of the β- stable type, $\beta \in (0,2)$, if Y_t has the generator

$$L_Y^x f(p) = \int [f(p+q) - f(p) - \nabla f(p) q] \frac{(\det G(x))^{1/2} dq}{(q, G(x)q)^{(\beta+1)/2}}.$$

An alternative way to extend OU processes to manifolds is via embedding to Euclidean spaces. Namely, observe that one can write $dY_t = \frac{\partial}{\partial x} x dY_t$ in $\mathbf{R}^n$ meaning that adding a Lévy noise force is equivalent to adding the singular nonhomogeneous potential $-x\dot{Y}_t$ (position multiplied by the noise) to the Hamiltonian function. Assume now that a Riemannian manifold (M, g) is embedded to the Euclidean space R^n via a smooth mapping $r : M \mapsto \mathbf{R}^n$ and that the random environment in $\mathbf{R}^n$ is modeled by the Lévy process Y_t. The position of a point x in $\mathbf{R}^n$ is now $r(x)$ so that the analog of xY_t is the product $r(x)Y_t$, and the term Y_t from (7) should have as the curvilinear modification the term

$$\left(\frac{\partial r}{\partial x}\right)^T dY_t = \{\sum_{j=1}^n \frac{\partial r^j}{\partial x^i} dY_t^j\}_{i=1}^d,$$

that yields the projection of the 'free noise' Y_t on the cotangent bundle to M at x.

In particular, the stochastic (or stochastically perturbed) geodesic flow induced by the embedding r can be defined by the stochastic system

$$\begin{cases} \dot{x} = G(x)p \\ dp = -\frac{1}{2}(\frac{\partial G}{\partial x}p, p)dt + \left(\frac{\partial r}{\partial x}\right)^T dY_t \end{cases} \tag{11}$$

which represents simultaneously the natural stochastic perturbation of the geodesic flow (6) *and the curvilinear analog of the stochastically perturbed free motion $\dot{x} = p, dp = dY_t$.*