NONLINEAR MARKOV GAMES*

Vassili N. Kolokoltsov†

I will discuss a new class of stochastic games that I call nonlinear Markov games, as they arise as a (competitive) controlled version of nonlinear Markov processes (an emerging field of intensive research). This class of games can model a variety of situation for economics and epidemics, statistical physics, and pursuit - evasion processes. Further discussion of this topic is given in my monograph 'Nonlinear Markov processes and kinetic equations', CUP 2010.

Nonlinear Markov process: future depends on the past not only via its present position, but also its present distribution.

A nonlinear Markov semigroup can be considered as a nonlinear deterministic dynamic system, though on a weird state space of measures.

Thus, as the stochastic control theory is a natural extension of the deterministic control, we extend it further by turning back to deterministic control, but of measures.

Nonlinear Markov chains (discrete time)

A *nonlinear Markov semigroup* Φ^k, $k \in \mathbf{N}$, is specified by an arbitrary continuous mapping $\Phi : \Sigma_n \to \Sigma_n$, where the simplex

$$\Sigma_n = \{\mu = (\mu_1, ..., \mu_n) \in \mathbf{R}_+^n : \sum_{i=1}^n \mu_i = 1\}$$

represents the set of probability laws on the finite state space $\{1, ..., .n\}$.

In order to get a process one has to choose a *stochastic representation* for Φ, i.e. to write it down in the form

$$\Phi(\mu) = \{\Phi_j(\mu)\}_{j=1}^n = \{\sum_{i=1}^n P_{ij}(\mu)\mu_i\}_{i=1}^n, \tag{1}$$

where $P_{ij}(\mu)$ is a family of stochastic matrices ($\sum_{j=1}^d P_{ij}(\mu) = 1$ for all i), depending on μ (nonlinearity!), whose elements specify the *nonlinear transition probabilities*.

One can get nonlinear analogs of many results from the usual Markov chains. For example, let us present the following simple fact on the long time behavior.

Proposition 1. *(i) For any continuous* $\Phi : \Sigma_n \to \Sigma_n$ *there exists a stationary distribution, i.e. a measure* $\mu \in \Sigma_n$ *such that* $\Phi(\mu) = \mu$. *(ii) If a representation* (1) *for* Φ *is chosen in such a way that there exists a* $j_0 \in [1, n]$, *a time* $k_0 \in \mathbf{N}$ *and a positive* δ *such that*

$$P_{ij_0}^{k_0}(\mu) \geq \delta \tag{2}$$

for all i, μ, *then* $\Phi^m(\mu)$ *converges to a stationary measure for any initial* μ.

Proof. Statement (i) is a consequence of the Brower fixed point principle. Statement (ii) follows from the representation (given above) of the corresponding nonlinear Markov chain as a time non-homogeneous Markov process.

Nonlinear Markov chains (continuous time)

A *nonlinear Markov semigroup* with the finite state space $\{1, ..., n\}$ is a semigroup Φ^t, $t \geq 0$, of continuous transformations of Σ_n. As in the case of discrete time the semigroup itself does not specify a process. A *continuous family of nonlinear transition probabilities* on $\{1, ..., n\}$ is a family $P(t, \mu) = \{P_{ij}(t, \mu)\}_{i,j=1}^n$ of stochastic matrices depending continuously on $t \geq 0$ and $\mu \in \Sigma_n$ such that the following *nonlinear Chapman-Kolmogorov equation* holds:

$$\sum_{i=1}^n \mu_i P_{ij}(t+s, \mu) = \sum_{k,i} \mu_k P_{ki}(t, \mu) P_{ij}(s, \sum_{l=1}^n P_{l.}(t, \mu)\mu_l). \tag{3}$$

This family is said to yield a *stochastic representation* for the Markov semigroup Φ^t whenever

$$\Phi_j^t(\mu) = \sum_i \mu_i P_{ij}(t, \mu), \quad t \geq 0, \mu \in \Sigma_n. \tag{4}$$

If (4) holds, the equation (3) represents just the semigroup identity $\Phi^{t+s} = \Phi^t \Phi^s$.

Once a stochastic representation (4) for the semigroup Φ^k is chosen one can define the corresponding stochastic process started at $\mu \in \Sigma_n$ as a time nonhomogeneous Markov chain with the transition probabilities from time s to time t being

$$p_{ij}(s, t, \mu) = P_{ij}(t - s, \Phi^s(\mu)).$$

To get the existence of a stochastic representation (4) one can use the same idea as for the discrete time case and define

$$\tilde{P}_{ij}(t, \mu) = \Phi^t_j(\mu).$$

However, this is not a natural choice from the point of view of stochastic analysis. The natural choice should correspond to a reasonable generator.

Namely, assuming the semigroup Φ^t is differentiable in t one can define the *(nonlinear) infinitesimal generator* of the semigroup Φ^t as the nonlinear operator on measures given by

$$A(\mu) = \frac{d}{dt}\Phi^t|_{t=0}(\mu).$$

The semigroup identity for Φ^t implies that $\Phi^t(\mu)$ solves the Cauchy problem

$$\frac{d}{dt}\Phi^t(\mu) = A(\Phi^t(\mu)), \quad \Phi^0(\mu) = \mu. \qquad (5)$$

As follows from the invariance of Σ_n under this dynamics, the mapping A is *conditionally positive* in the sense that $\mu_i = 0$ for a $\mu \in \Sigma_n$ implies $A_i(\mu) \geq 0$ and is also *conservative* in the sense that A maps the measures from Σ_n to the space of the signed measures

$$\Sigma_n^0 = \{\nu \in \mathbf{R}^n : \sum_{i=1}^n \nu_i = 0\}.$$

We shall say that such an A has a *stochastic representation* if it is written in the form

$$A_j(\mu) = \sum_{i=1}^{n} \mu_i Q_{ij}(\mu) = (\mu Q(\mu))_j, \qquad (6)$$

where $Q(\mu) = \{Q_{ij}(\mu)\}$ is a family of infinitesimally stochastic matrices (also referred to as Q-matrices or Kolmogorov's matrices) depending on $\mu \in \Sigma_n$. Thus in stochastic representation the generator has the form of a usual Markov chain generator, though additionally depending on the present distribution. The existence of a stochastic representation for the generator is not obvious, but is not difficult to get.

In practice, the converse problem is of more importance: not to construct the generator from a given semigroup, but to construct a semigroup (i.e. a solution to (5)) from a given operator A, which in applications is usually given directly in its stochastic representation.

Examples: Lotka-Volterra, replicator dynamics, epidemics

The nonlinear Markov semigroups are present in abundance among the popular models of natural and social sciences.

The replicator dynamics of the evolutionary game arising from the classical paper-rock-scissors game has the form

$$\begin{cases} \frac{dx}{dt} = (y-z)x \\ \frac{dy}{dt} = (z-x)y \\ \frac{dz}{dt} = (x-y)z \end{cases} \tag{7}$$

Its generator has a clear stochastic representation with

$$Q(\mu) = \begin{pmatrix} -z & 0 & z \\ x & -x & 0 \\ 0 & y & -y \end{pmatrix} \tag{8}$$

where $\mu = (x, y, z)$.

The famous LotkaVolterra equations describing a biological systems with two species, a predator and its prey, have the form

$$\begin{cases} \dfrac{dx}{dt} = x(\alpha - \beta y) \\ \dfrac{dy}{dt} = -y(\gamma - \delta x) \end{cases} \tag{9}$$

where $\alpha, \beta, \gamma, \delta$ are some positive parameters. The generator of this model is conditionally positive, but not conservative, as the total mass $x + y$ is not preserved. However, due to the existence of the integral of motion $\alpha \log y - \beta y + \gamma \log x - \delta x$, the dynamics (9) is path-wise equivalent to the dynamics (7), i.e. there is a continuous mapping taking the phase portrait of system (9) to the one of system (7).

One of the simplest deterministic models of epidemics can be written in the form of the system of 4 differential equations:

$$\begin{cases} \dot{X}(t) = -\lambda X(t)Y(t) \\ \dot{L}(t) = \lambda X(t)Y(t) - \alpha L(t) \\ \dot{Y}(t) = \alpha L(t) - \mu Y(t) \\ \dot{Z}(t) = \mu Y(t) \end{cases} \tag{10}$$

where $X(t), L(t), Y(t)$ and $Z(t)$ denote respectively the numbers of susceptible, latent, infectious and removed individual at time t and the positive coefficients λ, α, μ (which may actually depend on X, L, Y, Z) reflect the rates at which susceptible individuals become infected, latent individuals become infectious and infectious individuals become removed. Written in terms of the proportions $x = X/\sigma$, $y = Y/\sigma$, $l = L/\sigma$, $z = Z/\sigma$, i.e. normalized on the total mass $\sigma = X + L + Y + Z$, this system becomes

$$\begin{cases} \dot{x}(t) = -\sigma\lambda x(t)y(t) \\ \dot{l}(t) = \sigma\lambda x(t)y(t) - \alpha l(t) \\ \dot{y}(t) = \alpha l(t) - \mu y(t) \\ \dot{z}(t) = \mu y(t) \end{cases} \tag{11}$$

with $x(t)+y(t)+l(t)+z(t)=1$. Subject to the often made assumption that $\sigma\lambda$, α and μ are constants, the r.h.s. is an infinitesimal generator of a nonlinear Markov chain in Σ_4. This generator depends again quadratically on its variable and has an obvious stochastic representation (6) with the infinitesimal stochastic matrix

$$Q(\mu) = \begin{pmatrix} -\lambda y & \lambda y & 0 & 0 \\ 0 & -\alpha & \alpha & 0 \\ 0 & 0 & -\mu & \mu \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (12)$$

where $\mu = (x, l, y, z)$, yielding a natural probabilistic interpretation to the dynamics (11).

Discrete nonlinear Markov games and controlled processes

A nonlinear Markov semigroup is after all just a deterministic dynamic system (with some special features). Thus, as the stochastic control theory is a natural extension of the deterministic control, we are going to further extend it by turning back to deterministic control, but of measures, thus exemplifying the usual spiral development of science. The next 'turn of the screw' would lead to stochastic measure-valued games.

We shall work in the competitive control setting (game theory), in discrete time and finite original state space $\{1, ..., n\}$. The full state space is then chosen as a set of probability measures Σ_n on $\{1, ..., n\}$.

Suppose we are given two metric spaces U, V of the control parameters of two players, a continuous transition cost function $g(u, v, \mu)$, $u \in U$, $v \in V$, $\mu \in \Sigma_n$ and a transition law $\nu(u, v, \mu)$ prescribing the new state $\nu \in \Sigma_n$ obtained from μ once the players had chosen their strategies $u \in U, v \in V$. The problem of the corresponding one-step game (with sequential moves) consists in calculating the Bellman operator

$$(BS)(\mu) = \min_u \max_v [g(u, v, \mu) + S(\nu(u, v, \mu))] \tag{13}$$

for a given final cost function S on Σ_n.

In case of no competition (only one control parameter), this turns to

$$(BS)(\mu) = \min_u [g(u, v, \mu) + S(\nu(u, v, \mu))] \tag{14}$$

MAX-PLUS linear!

According to the dynamic programming principle, the dynamic multi-step game solution is given by the iterations $B^k S$.

Long horizon problem: behavior of the optimal cost $B^k S(\mu)$ as $k \to \infty$.

The function $\nu(u, v, \mu)$ can be interpreted as the controlled version of the mapping Φ specifying a nonlinear discrete time Markov semigroup.

Assume a stochastic representation is chosen:

$$\nu_j(u, v, \mu) = \sum_{i=1}^{n} \mu_i P_{ij}(u, v, \mu)$$

with stochastic matrices P_{ij}. If g describes the averages over the random transitions, then

$$g(u, v, \mu) = \sum_{i,j=1}^{n} \mu_i P_{ij}(u, v, \mu) g_{ij}$$

with certain real coefficients g_{ij} and

$$(BS)(\mu) = \min_u \max_v \Big[\sum_{i,j=1}^{n} \mu_i P_{ij}(u, v, \mu) g_{ij} + S\left(\sum_{i=1}^{n} \mu_i P_{i\cdot}(u, v, \mu) \right) \Big]. \qquad (15)$$

We can now identify the (not so obvious) place of the usual stochastic control theory in this nonlinear setting. Namely, assume P_{ij} above do not depend on μ. But even then the set of the linear functions $S(\mu) = \sum_{i=1}^n s_i\mu^i$ on measures (identified with the set of vectors $S = (s_1, ..., s_n)$) is not invariant under B. Hence we are not automatically reduced to the usual stochastic control setting, but to a game with incomplete information, where the states are probability laws on $\{1, ..., n\}$, i.e. when choosing a move the players do not know the position precisely, but only its distribution. Only if we allow only Dirac measures μ as a state space (i.e. no uncertainty on the state), the Bellman operator would be reduced to the usual one of the stochastic game theory:

$$(\bar{B}S)_i = \min_u \max_v \sum_{j=1}^n P_{ij}(u,v)(g_{ij} + S_j). \quad (16)$$

Example of a nonlinear result: analog of the result on the existence of the average income for long lasting games.

Proposition 2. *If the mapping ν is a contraction uniformly in u, v, i.e. if*

$$\|\nu(u,v,\mu^1) - \nu(u,v,\mu^2)\| \leq \delta\|\mu^1 - \mu^2\| \quad (17)$$

with a $\delta \in (0,1)$, where $\|\nu\| = \sum_{i=1}^n |\nu_i|$, and if g is Lipschitz continuous, i.e.

$$\|g(u,v,\mu^1) - g(u,v,\mu^2)\| \leq C\|\mu^1 - \mu^2\| \quad (18)$$

with a constant $C > 0$, then there exists a unique $\lambda \in \mathbf{R}$ and a Lipschitz continuous function S on Σ_n such that

$$B(S) = \lambda + S, \quad (19)$$

and for all $g \in C(\Sigma_n)$ we have

$$\lim_{m\to\infty} \frac{B^m g}{m} = \lambda. \quad (20)$$

One can extend the other results for stochastic multi-step games to this nonlinear setting, say, the turnpike theorems (from Kolokoltsov 1992).

Continuous state spaces. SDEs driven by nonlinear Lévy noise

Weak equations of the form

$$\frac{d}{dt}(f,\mu_t) = (L_{\mu_t} f, \mu_t), \quad \mu_t \in \mathcal{P}(\mathbf{R}^d), \quad \mu_0 = \mu, \tag{21}$$

(that should hold, say, for all $f \in C_c^2(\mathbf{R}^d)$), where

$$L_\mu f(x) = \frac{1}{2}(G(x,\mu)\nabla,\nabla)f(x) + (b(x,\mu),\nabla f(x))$$

$$+\int (f(x+y) - f(x) - (\nabla f(x), y)\mathbf{1}_{B_1}(y))\nu(x,\mu,dy), \tag{22}$$

play indispensable role in the theory of interacting particles (mean field approximation) and exhaust all positivity preserving evolutions on measures subject to certain mild regularity assumptions. I call them *general kinetic equations* as they include Vlasov, Boltzmann, Smoluchovski, Landau-Fokker-Planck equations, McKean diffusions and many other models. The strong form is of course

$$\dot{\mu} = L_\mu^\star \mu.$$

A resolving semigroup $U_t : \mu \mapsto \mu_t$ of the Cauchy problem for this equation specifies a so called *generalized or nonlinear Markov process* $X(t)$, whose distribution μ_t at time t can be determined by the formula $U_{t-s}\mu_s$ from its distribution μ_s at any previous moment s.

In the case of diffusions (when ν vanishes) the theory of the corresponding semigroups is well developed, as well as pure jump case.

We exploit the idea of nonlinear integrators combined with a certain coupling of Lévy processes in order to push forward the probabilistic construction in a way that allows the natural Lipschitz continuous dependence of the coefficients G, b, ν on x, μ. Thus obtained extension of the standard SDEs with Lévy noise represents a probabilistic counterpart of the celebrated extension of the Monge mass transformation problem to the generalized Kantorovich one.

Wasserstein-Kantorovich metrics W_p, $p \geq 1$, on the set of probability measures $\mathcal{P}(\mathbf{R}^d)$ on $\mathbf{R}^d$ are defined as

$$W_p(\nu_1, \nu_2) = \left(\inf_\nu \int |y_1 - y_2|^p \nu(dy_1 dy_2)\right)^{1/p}, \tag{23}$$

where inf is taken over the class of probability measures ν on $\mathbf{R}^{2d}$ that couple ν_1 and ν_2, i.e. that satisfy

$$\int\int (\phi_1(y_1) + \phi_2(y_2))\nu(dy_1 dy_2) = (\phi_1, \nu_1) + (\phi_2, \nu_2) \tag{24}$$

for all bounded measurable ϕ_1, ϕ_2.

The Wasserstein distances between the distributions in the Skorohod space $D([0,T], \mathbf{R}^d)$:

$$W_{p,T}(X_1, X_2) = \inf \left(\mathbf{E} \sup_{t \leq T} |X_1(t) - X_2(t)|^p\right)^{1/p}, \tag{25}$$

where inf is taken over all couplings of the distributions of the random paths X_1, X_2.

To compare the Lévy measures, we extend these distances to unbounded measures with a finite moment.

For simplicity, we present the arguments for L_μ having the form

$$L_\mu f(x) = \frac{1}{2}(G(x,\mu)\nabla,\nabla)f(x)+(b(x,\mu),\nabla f(x))$$
$$+\int(f(x+z)-f(x)-(\nabla f(x),z))\nu(x,\mu;dz) \tag{26}$$

with $\nu(x,\mu;.) \in \mathcal{M}_2(\mathbf{R}^d)$. Let $Y_\tau(z,\mu)$ be a family of Lévy processes depending measurably on the points z and probability measures μ in $\mathbf{R}^d$ and specified by their generators

$$L[z,\mu]f(x) = \frac{1}{2}(G(z,\mu)\nabla,\nabla)f(x)+(b(z,\mu),\nabla f(x))$$
$$+\int(f(x+y)-f(x)-(\nabla f(x),y))\nu(z,\mu;dy) \tag{27}$$

where $\nu(z,\mu) \in \mathcal{M}_2(\mathbf{R}^d)$.

Our approach to solving (21) is via the solution to the following nonlinear distribution dependent stochastic equation with nonlinear Lévy type integrators:

$$X(t) = X + \int_0^t dY_s(X(s), \mathcal{L}(X(s))), \ \mathcal{L}(X) = \mu, \tag{28}$$

with a given initial distribution μ and a random variable X independent of $Y_\tau(z,\mu)$.

We shall define the solution through the Euler type approximation scheme, i.e. by means of the approximations X_μ^τ:

$$X_\mu^\tau(t) = X_\mu^\tau(l\tau) + Y_{t-l\tau}^l(X_\mu^\tau(l\tau), \mathcal{L}(X_\mu^\tau(l\tau))), \tag{29}$$

$\mathcal{L}(X_\mu^\tau(0)) = \mu$, where $l\tau < t \leq (l+1)\tau$, $l = 0, 1, 2, ...$, and $Y_\tau^l(x,\mu)$ is a collection (depending on l) of independent families of the Lévy processes $Y_\tau(x,\mu)$ introduced above. Clearly these approximation processes are cadlag.

Theorem 1. *Let an operator L_μ have form* (26)*. Moreover assume that*

$$\|\sqrt{G(x,\mu)} - \sqrt{G(z,\eta)}\| + |b(x,\mu) - b(z,\eta)|$$

$$+W_2(\nu(x,\mu;.),\nu(z,\eta;.)) \leq \kappa(|x-z|+W_2(\mu,\eta)), \tag{30}$$

holds true with a constant κ and

$$\sup_{x,\mu}\left(\sqrt{G(x,\mu)} + |b(x,\mu)| + \int |y|^2\nu(x,\mu,dy)\right) < \infty. \tag{31}$$

Then for any $\mu \in \mathcal{P}(\mathbf{R}^d)\cap\mathcal{M}_2(\mathbf{R}^d)$ the approximations $X_\mu^{\tau_k}$, $\tau_k = 2^{-k}$, converge to a process $X_\mu(t)$ in W^2_{2,t_0} and the resolving operators $U_t : \mu \mapsto \mu_t$ of the Cauchy problem (21) *form a nonlinear Markov semigroup, i.e. they are continuous mappings from $\mathcal{P}(\mathbf{R}^d) \cap \mathcal{M}_2(\mathbf{R}^d)$ (equipped with the metric W_2) to itself such that U_0 is the identity mapping and $U_{t+s} = U_tU_s$ for all $s,t \geq 0$. If $L[z,\mu]$ do not depend explicitly on μ the operators $T_tf(x) = \mathbf{E}f(X_x(t))$ form a conservative Feller semigroup preserving the space of Lipschitz continuous functions.*

For example, assumption on ν is satisfied if one can decompose the Lévy measures $\nu(x;.)$ in the countable sums $\nu(x;.) = \sum_{n=1}^{\infty} \nu_n(x;.)$ of probability measures so that

$$W_2(\nu_i(x;.), \nu_i(z;.)) \leq a_i|x - z|$$

and the series $\sum a_i^2$ converges. It is well known that the optimal coupling of probability measures (Kantorovich problem) can not always be realized via a mass transportation (a solution to the Monge problem), thus leading to the examples when the construction of the process via standard stochastic calculus would not work.

Another important particular situation is that of a common star shape of the measures $\nu(x;.)$, i.e. if they can be represented as

$$\nu(x; dy) = \nu(x, s, dr)\, \omega(ds), \quad r = |y|, s = y/r, \tag{32}$$

with a certain measure ω on S^{d-1} and a family of measures $\nu(x, s, dr)$ on $\mathbf{R}_+$. This allows to reduce the general coupling problem to a much more easily handled one-dimensional one.

Controlled Markov processes and games

Suppose first that L does not depend on μ explicitly, but there is additional controllable drift $f(x, \alpha, \beta)$ and an integral payoff given by $g(x, \alpha, \beta)$. This leads to the HJB equation

$$\frac{\partial S}{\partial t} + H(x, \frac{\partial S}{\partial x}) + LS = 0 \qquad (33)$$

with

$$H(x,p) = \max_{\alpha} \min_{\beta} \left(f(x, \alpha, \beta) \frac{\partial S}{\partial x} + g(x, \alpha, \beta) \right).$$

Theorem 2. *Suppose $H(x,p)$ is Lipshitz in p uniformly in x with a Lipshitz constant κ, and the process generated by L has a heat kernel (Green's function) $G(t, x, \xi)$, which is of class C^1 with respect to all variables for $t > 0$. Moreover*

$$\sup_x \int_0^t \int \left| \frac{\partial}{\partial x} G(s, x, \xi) \right| ds d\xi < \infty$$

for $t > 0$. Then for any $S_0 \in C^1(\mathbf{R}^d)$ there exists a unique classical solution for the Cauchy problem for equation (33) yielding also the solution to the corresponding optimal control problem.

Proof (sketch). It is based of course on the fixed point argument for the mapping

$$\Phi_t(S) = \int G(t,x,\xi)S_0(\xi)d\xi$$

$$+\int_0^t \int G(t-s,x,\xi)H(\xi,\frac{\partial S_s}{\partial x})dsd\xi,$$

which is applicable, because for S^1, S^2 with $S_0^1 = S_0^2$

$$\left\| \frac{\partial \Phi_t(S^1)}{\partial x} - \frac{\partial \Phi_t(S^2)}{\partial x} \right\|$$

$$\leq \kappa \int_0^t \frac{\partial G}{\partial x}(t-s,x,\xi)d\xi ds \sup_{s\leq t} \left\| \frac{\partial S^1}{\partial x} - \frac{\partial S^2}{\partial x} \right\|$$

implying the contraction property of Φ for small enough t.

Example: controlled stable-like processes with the generator $\Delta^{\alpha(x)}$ or more generally

$$\int_{S^{d-1}} |(\nabla, s)|^{\alpha(x)} \mu(ds).$$

Example of an application: extension of Nash Certainty Equivalence (NCE) principle of P. Caines et al (obtained for interacting diffusions) to stable-like processes.

Controlled nonlinear Markov processes and games

Returning to L depending on μ consider a single control variable u. Assume that μ only is observable, so that the control is based on μ. This leads to the following infinite-dimensional HJB equation

$$\frac{\partial S}{\partial t} + \max_u \left(L_{\mu,u} \frac{\delta S}{\delta \mu} + g_u, \mu \right) = 0. \qquad (34)$$

If the Cauchy problem for the corresponding kinetic equation $\dot{\mu} = L^{\star}_{\mu,u}\mu$ is well posed (see book [3]) uniformly for controls u from a compact set, with a solution denoted by $\mu^t(\mu, u)$ this can be resolved via discrete approximations

$$S_k(t-s) = B^k S(t), \quad k = (t-s)/\tau,$$

$$BS(\mu) = \max_u [S(\mu^\tau(\mu, u) + (g_u, \mu)].$$

Convergence proof (yielding a Lipshitz continuous function for a Lipshitz continuous initial one) is the same as in book [1], Section 3.2, yielding a resolving operator $R_s(S)$ for the inverse Cauchy problem (34) as a *linear operator in the* max-*plus algebra*, i.e. satisfying the condition

$$R_s(a_1 \otimes S_1 \oplus a_2 \otimes S_2) = a_1 \otimes R_s(S_1) \oplus a_2 \otimes R_s(S_2)$$

with $\oplus = \max$, $\otimes = +$. This linearity allows for effective numeric schemes.

Extensions to a competitive control case (games) is settled via the approach with generalized dynamic systems as presented in Section 11.4 of book [2].

Selected bibliography (papers):

[1] V.N. Kolokoltsov. Kinetic equations for the pure jump models of k-nary interacting particle systems. Markov Processes and Related Fields **12** (2006), 95-138.

[2] V. N. Kolokoltsov. *Nonlinear Markov Semigroups and Interacting Lévy Type Processes*. Journ. Stat. Physics **126:3** (2007), 585-642.

[3] V. N. Kolokoltsov. Generalized Continuous-Time Random Walks (CTRW), Subordination by Hitting Times and Fractional Dynamics. arXiv:0706.1928v1[math.PR] 13 June 2007. Theory of Probability and its Applications **53:4** (2009), 594-609.

[4] V.N. Kolokoltsov. The Lévy-Khintchine type operators with variable Lipschitz continuous coefficients generate linear or nonlinear Markov processes and semigroups. arXiv:0911.5688v1 [Math.PR] (2009). To appear in PTRF.

Other authors:

[1] T.D. Frank. Nonlinear Markov processes. Phys. Lett. A **372:25** (2008), 4553-4555.

[2] M. Zak. Quantum Evolution as a Nonlinear Markov Process. Foundations of Physics Letters **15:3** (2002), 229-243.

Bibliography (monographs):

[1] V.N. Kolokoltsov, V.P. Maslov. *Idempotent analysis and its applications*. Kluwer Publishing House, 1997.

[2] V.N. Kolokoltsov, O.A. Malafeyev. Understanding Game Theory. World Scientific 2010.

[3] V. N. Kolokoltsov. Nonlinear Markov processes and kinetic equations. To appear in Cambridge University Press, August 2010.