Inspection games and crime prevention

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February 8, 2010

Reference

V.N. Kolokoltsov, O.A. Malafeyev.

Understanding Game Theory

To appear in World Scientific 2009.

Tax payer against the tax man

This is a game between a tax payer (player I) and the tax police (player II). The player I has 2 pure strategies: to hide part of the taxes (H) or to pay them in full (P). Player II has also 2 strategies: to check the player I (C) and to rest (R). The player I gets the income r if he pays the tax in full. If he chooses the action (H), he gets the additional surplus l. But if he is caught by the player II, he has to pay the fine f.

In the profile (C,H) the player II can discover the unlawful action of player I with the probability $p(\overline{p} = 1 - p)$, so that p can be called the efficiency of the police. Choosing (C), the player II spends c on the checking procedure. Of course $l, r, f, c > 0$.

Hence we defined a bi-matrix game given by the table

P II (Police)

P I		Check (C)	Rest (R)
	Hide (H)	$r+\overline{p}l-pf, -c+pf-\overline{p}l$	$r+l, -l$
	Pay (P)	$r, -c$	$r, 0$

or shortly by the payoff matrix

$$\begin{pmatrix} r+\overline{p}l-pf, -c+pf-\overline{p}l & r+l, -l \\ r, -c & r, 0 \end{pmatrix}$$

The candidates to the mixed equilibrium are the strategies $(\beta, \overline{\beta})$, $(\alpha, \overline{\alpha})$, where

$$\alpha = \frac{a_{22}-a_{12}}{a_{11}-a_{12}-a_{21}+a_{22}} = \frac{l}{p(l+f)} > 0$$

$$\beta = \frac{b_{22}-b_{21}}{b_{11}-b_{12}-b_{21}+b_{22}} = \frac{c}{p(l+f)} > 0.$$

In order to have these strategies well defined, it is necessary to have $\alpha < 1$ and $\beta < 1$ respectively.

Proposition *1) If $c \geq p(f+l)$, the pair (H,R) is an equilibrium, and moreover the strategy (R) is dominant for the police (even strictly, if the previous inequality is strict). 2) If $c < p(f+l)$ and $fp \leq \overline{p}l$, the pair (H,C) is an equilibrium and the strategy (H) is dominant (strictly if the previous inequality is strict). 3) If $c < p(f+l)$, $fp > \overline{p}l$, then the unique Nash equilibrium is the profile of mixed strategies $(\beta, \overline{\beta}), (\alpha, \overline{\alpha})$.*

Consequently, in cases 1) and 2) the actions of the police are not effective.

The equilibrium in case 3) is stable.

It is more interesting to analyze the game obtained by extending the strategy space of the player I by allowing him to choose the amount l of tax evasion: $l \in [0, l_M]$, where l_M is the full tax due to the player I. For example, we shall assume that the fine is proportional to l, i.e. $f(l) = nl$.

For example, in the Russian tax legislation $n = 0.4$.

Under these assumptions the table takes the form

P II (Police)

P I		Check (C)	Rest (R)
	Hide (H)	$r + \overline{p}l - pln, -c + pln - \overline{p}l$	$r + l, -l$
	Pay (P)	$r, -c$	$r, 0$

and the key coefficients α, β become

$$\alpha = \frac{1}{p(n+1)}, \quad \beta = \frac{c}{l} \frac{1}{p(n+1)}.$$

Let $H_I(l)$ denote the payoff to player I in the equilibrium when l is chosen.

Result of analysis:

Case 1: $p > \frac{1}{n+1} \Longleftrightarrow \alpha < 1$. Let

$$l_1 = \frac{c}{p(n+1)}.$$

Then $H_I(l > l_1) < H_I(l < l_1)$ and player I will avoid tax on the amount $l = l_1$.

Case 2: $p < \frac{1}{n+1} \Longleftrightarrow \alpha > 1$. If

$$\frac{l_1}{1 - p(n+1)} \leq l_M, \qquad (1)$$

the equilibrium strategy for player I is $l = l_M$ and otherwise $l = l_1$.

Conclusion: in both cases it is profitable to avoid tax on the amount l_1, but as the efficiency of tax man increases, it becomes unreasonable to avoid tax on a higher amount.

Let us see which condition in the second case would ensure the inequality (1) when the amount of tax avoidance is l_M in the equilibrium. Plugging l_1 in (1) yields

$$\frac{c}{p(n+1)(1-p(n+1)} \leq l_M.$$

Denoting $x = p(n+1) < 1$ one can rewrite it as

$$x^2 - x + \frac{c}{l_M} \leq 0. \qquad (2)$$

The roots of the corresponding equation are

$$x_{1,2} = \frac{1 \pm \sqrt{1 - \frac{4c}{l_M}}}{2}.$$

Hence for $c > l_M/4$ inequality (1) does not hold for any p, and for $c \leq l_M/4$ the solution to (2) is

$$x \in \left[\frac{1 - \sqrt{1 - \frac{4c}{l_M}}}{2}; \frac{1 - \sqrt{1 + \frac{4c}{l_M}}}{2} \right].$$

Thus for

$$c \leq \frac{l_M}{4}, \quad p \in \left[\frac{1-\sqrt{1-\frac{4c}{l_M}}}{2(n+1)}; \frac{1-\sqrt{1+\frac{4c}{l_M}}}{2(n+1)} \right] \tag{3}$$

it is profitable to avoid tax payment on the amount l_M.

Let us consider a numeric example with $n = 0.4$, $c = 1000$, $l_M = 100000$. Then $c \leq l_M/4$.

1) Suppose $p < 0.714$. By (3), for $p \in [0.007; 0.707]$ it is profitable to avoid tax on the whole amount, i.e. 100000.

2) If $p > 0.714$ it is profitable to avoid tax on the amount $l_1 = 714.29$.

Hence if the efficiency of tax payment checks is $p < 0.707$, it is profitable to avoid tax on the whole amount of 100000, and if $p > 0.707$, then not more than on 1010.

Multi-step games

Let us consider the n-step game $\Gamma_{k,m}(n)$, where during this time the player I can break the law maximum k times and the player II can organize the check maximum m times. Assume that after the end of each period (step), the result becomes known to both players. Total payoff in n steps equals the sum of payoffs in each step. It is also assumed that all this information (rules of the game) is available to both players.

Let $(u_{k,m}(n), v_{k,m}(n))$ be the value of this game. We get the following system of recurrent equations:

$$(u_{k,m}(n), v_{k,m}(n)) = Val\left(A(u_{k-\delta,m-\delta}, v_{k-\delta,m-\delta})\right)$$

($\delta = 0, 1$), if all $\Gamma_{k,m}(n)$ have values, i.e. their equilibrium payoffs are uniquely defined.

The boundary conditions $(m, n, k \geq 0)$ are

$$(u_{0,m}(n), v_{0,m}(n)) = (nr, 0);$$

$$(u_{k,0}(n), v_{k,0}(n)) = (nr + ks, -kl); \quad k \leq n,$$

reflecting the following considerations: if the trespasser is unable to break the law, the pair of solutions (R,R) will be repeated over all periods; and if the inspector is unable to check, the trespasser will commit the maximum number of violation available.

Some explicit formulas are available.

Other models of inspection games can be found in

R. Avenhaus. Applications of inspection games. Math. Model. Anal. **9:3** (2004).

R. Avenhaus, M. J. Canty. Playing for time: a sequential inspection game. European J. Oper. Res. **167:2** (2005), 475-492.

T. Ferguson, C. Melolidakis. On the inspection game. Naval Res. Logist. **45:3** (1998), 327-334.

Appl.: arms control inspection effort

To conclude, on the next page 'the heavy hand of the law' is presented (graphics of A.T. Fomenko)