THE RATE OF CONVERGENCE

OF THE BLOCK COUNTING PROCESS

OF EXCHANGEABLE COALESCENTS WITH DUST

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Exchangeable coalescents

- Exchangeable coalescents are Markov processes $\Pi = (\Pi_t)_{t\geq 0}$ with state space $\mathcal{P}$, the set of partitions (equivalence relations) on $\mathbb{N} := \{1, 2, \ldots\}$.
- During each transition, blocks (equivalence classes) merge together. Simultaneous multiple collisions of blocks are allowed.
- Schweinsberg (2000) characterizes exchangeable coalescents via a finite measure Ξ on the infinite simplex

$$\Delta \;:=\; \{u = (u_1, u_2, \ldots) \;:\; u_1 \geq u_2 \geq \cdots \geq 0, |u| := \sum_{r\in\mathbb{N}} u_r \leq 1\}.$$

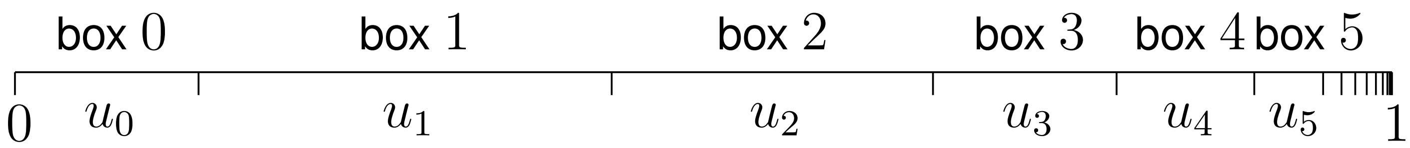
- These processes are therefore also called Ξ-coalescents.
- The subclass of Λ-coalescents is obtained if Ξ is concentrated on $\{u \in \Delta : u_2 = 0\}$. In this case $\Lambda(B) = \Xi(B \times \{0\} \times \{0\} \times \cdots)$ for all Borel sets $B \subseteq [0, 1]$.

An urn model, Kingman's paintbox

Karlin (1967), Dutko (1989), Gnedin, Hansen and Pitman (2007)

Fix $u = (u_1, u_2, \ldots) \in \Delta$. Note that $|u| := \sum_{r \in \mathbb{N}} u_r \leq 1$. Define $u_0 := 1 - |u|$.

Imagine a countable infinite number of boxes having labels $r \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$.



Balls are allocated successively to these boxes. It is assumed that every ball goes to box $r \in \mathbb{N}_0$ with probability u_r independently of the other balls.

Let $X_r(i, u)$ be the number of balls in box r after $i \in \mathbb{N}$ balls have been allocated.

Then $(X_0(i, u), X_1(i, u), \ldots)$ has an infinite multinomial distribution with parameters i and $(u_0, u_1, u_2, \ldots)$.

The block counting process

Let $N_t^{(n)}$ denote the number of blocks of $\Pi_t^{(n)}$, the restriction of Π_t to a sample of size n.

The block counting process $N^{(n)} := (N_t^{(n)})_{t\geq 0}$ moves from state i to state $j < i$ at the rate

$$\boxed{q_{ij} \;=\; \Xi(\{0\})\binom{i}{2}\delta_{j,i-1} + \int_\Delta \mathbb{P}(Y(i,u)=j)\,\nu(\mathrm{d}u)}$$

where $\nu(\mathrm{d}u) := \Xi(\mathrm{d}u)/(u,u)$ with $(u,u) := \sum_{r\in\mathbb{N}} u_r^2$ and

$$\begin{aligned} Y(i,u) \;&:=\; \text{number of balls in box } 0 \text{ plus number of other non-empty boxes}\\ &=\; X_0(i,u) + \sum_{r\in\mathbb{N}} 1_{\{X_r(i,u)>0\}}. \end{aligned}$$

Remark. Note that $\mathbb{P}(Y(i,u)=j) = \sum_{k=1}^j f_{ijk}(u)$, where

$$f_{ijk}(u) \;:=\; \frac{u_0^{j-k}}{(j-k)!} \sum_{\substack{i_1,\ldots,i_k\geq 1\\ i_1+\cdots+i_k=i-j+k}} \frac{i!}{i_1!\cdots i_k!} \sum_{1\leq r_1<\cdots<r_k} u_{r_1}^{i_1}\cdots u_{r_k}^{i_k}.$$

The fixation line (Hénard (2015), Gaiser and M. (2016))

For $n \in \mathbb{N}$ and $t \geq 0$ define $L_t^{(n)} := \sup\{k \in \mathbb{N} : N_t^{(k)} \leq n\}$.

The fixation line $L^{(n)} := (L_t^{(n)})_{t\geq 0}$ moves from state i to state $j > i$ at the rate

$$\gamma_{ij} \;=\; \Xi(\{0\})\binom{j}{2}\delta_{j,i+1} + \int_\Delta \mathbb{P}(Y(j,u)=i, Y(j+1,u)=i+1)\,\nu(\mathrm{d}u)$$

with $Y(.,u)$ and ν as before.

Remark. The probability below the integral can be provided explicitly as

$$\mathbb{P}(Y(j,u)=i, Y(j+1,u)=i+1) \;=\; \sum_{k=1}^{i} g_{ijk}(u), \text{ where}$$

$$g_{ijk}(u) := \frac{u_0^{i-k}}{(i-k)!} \sum_{\substack{i_1,\ldots,i_k\geq 1\\ i_1+\cdots+i_k=j-i+k}} \frac{j!}{i_1!\cdots i_k!} \sum_{1\leq r_1<\cdots<r_k} u_{r_1}^{i_1}\cdots u_{r_k}^{i_k}(1-(u_{r_1}+\cdots+u_{r_k})).$$

Siegmund duality

Let Π be a Ξ-coalescent and let $N^{(n)} = (N_t^{(n)})_{t\geq 0}$ and $L^{(n)} = (L_t^{(n)})_{t\geq 0}$ denote the block counting process and the fixation line of Π respectively.

Theorem 1. (Gaiser and M., 2016)

The block counting process is Siegmund dual to the fixation line, that is

$$\mathbb{P}(N_t^{(i)} \leq j) \;=\; \mathbb{P}(L_t^{(j)} \geq i)$$

for all $i, j \geq 1$ and $t \geq 0$.

Exchangeable coalescents with dust

Definition. A Ξ-coalescent $\Pi = (\Pi_t)_{t\geq 0}$ has proper frequencies (no dust) if, for all times $t \geq 0$, the frequency S_t of singletons of Π_t satisfies $S_t = 0$ almost surely.

Proposition 1. (Schweinsberg, 2000)
A coalescent Π has dust if and only if $\Xi(\{0\}) = 0$ and $\boxed{\int_\Delta |u|\,\nu(\mathrm{d}u) < \infty}$, where $|u| := \sum_{r\in\mathbb{N}} u_r$ and $\nu(\mathrm{d}u) := \Xi(\mathrm{d}u)/(u,u)$ with $(u,u) := \sum_{r\in\mathbb{N}} u_r^2$.

Remark. If Π has dust then $Z = (Z_t)_{t\geq 0} := (-\log S_t)_{t\geq 0}$ is a drift-free subordinator (Lévy process with non-decreasing paths) with Laplace exponent

$$\boxed{\Phi(q) \;:=\; \int_\Delta (1-(1-|u|)^q)\,\nu(\mathrm{d}u)}, \qquad q \geq 0.$$

Note that $\mathbb{E}(S_t^q) = \mathbb{E}(e^{-qZ_t}) = e^{-t\Phi(q)}$ for $t, q \geq 0$.

Asymptotics for large sample size

Theorem 2. (Gaiser and M., 2016)
Let Π be a Ξ-coalescent with dust, i.e. $\Xi(\{0\}) = 0$ and $\int_\Delta |u|\,\nu(\mathrm{d}u) < \infty$. Then the following two assertions hold.

a) As $n \to \infty$ the scaled block counting process $(N_t^{(n)}/n)_{t\ge 0}$ converges in $D_{[0,1]}[0,\infty)$ to the frequency of singleton process $S = (S_t)_{t\ge 0} = (e^{-Z_t})_{t\ge 0}$.

b) As $n \to \infty$ the scaled fixation line $(L_t^{(n)}/n)_{t\ge 0}$ converges in $D_{[1,\infty]}[0,\infty)$ to the reciprocal frequency of singleton process $(1/S_t)_{t\ge 0} = (e^{Z_t})_{t\ge 0}$.

Remarks

- Proof of part a) based on the method of moments and some general weak convergence machinery. Proof of part b) uses duality.
- For Ξ-coalescents with dust, both processes $(\log n - \log N_t^{(n)})_{t\geq 0}$ and $(\log L_t^{(n)} - \log n)$ converge in $D_{[0,\infty]}[0,\infty)$ to the drift-free subordinator Z.
- Let Ξ be concentrated on $\Delta^* := \{u \in \Delta\,:\,|u| = 1\}$.
  Then Π has dust if and only if ν is finite.
  In this case $S_t \stackrel{d}{=} 1_{\{T_f>t\}}$, where T_f is exponentially distributed with parameter $\nu(\Delta^*)$.
  Examples are Dirichlet coalescents and Poisson–Dirichlet coalescents.

A Bernstein function

Let Π be a Ξ-coalescent with dust. For $q \geq 0$ define

$$\widetilde{\Phi}(q) \ := \int_\Delta \sum_{r \in \mathbb{N}} (1 - (1 - u_r)^q) \, \nu(\mathrm{d}u).$$

Properties:

- In general $\widetilde{\Phi}$ differs from Φ. For Λ-coalescents $\widetilde{\Phi}$ coincides with Φ.
- $\widetilde{\Phi}(0) = 0$, $\widetilde{\Phi}(1) = \int_\Delta |u| \, \nu(\mathrm{d}u)$, $\Phi(n) \leq \widetilde{\Phi}(n)$ for all $n \in \mathbb{N}$
- $\widetilde{\Phi}$ is a Bernstein function (infinitely often differentiable on $(0, \infty)$ with $(-1)^{k-1}\widetilde{\Phi}^{(k)} \geq 0$ for all $k \in \mathbb{N}$ and $q > 0$).
- Lévy–Khintchine representation: $$\widetilde{\Phi}(q) \ = \int_{(0,1]} (1 - (1 - x)^q) \, \widetilde{\nu}(\mathrm{d}x),$$

  where $\widetilde{\nu}(B) := \int_\Delta \sum_{r \in \mathbb{N}} 1_B(u_r) \, \nu(\mathrm{d}u)$ for all Borel sets $B \subseteq (0, 1]$.

Rate of convergence

Notation: $E := [0,1]$, $E_n := \{k/n : k \in \{1,\ldots,n\}\}$.

$\pi_n : B(E) \to B(E_n)$, $\pi_n f(x) := f(x)$ for $f \in B(E)$ and $x \in E_n$.

Theorem 3. (Rate of convergence of the block counting process, M., 2019)

Let $\Pi = (\Pi_t)_{t\geq 0}$ be a Ξ-coalescent with dust and let A_n and A denote the generators of the scaled block counting process $(N_t^{(n)}/n)_{t\geq 0}$ and the frequency of singleton process $(S_t)_{t\geq 0}$ respectively. Then, for all $n \in \mathbb{N}$ and $f \in C^2([0,1])$,

$$\boxed{\|A_n\pi_n f - \pi_n A f\| \;:=\; \sup_{x\in E_n} |A_n\pi_n f(x) - \pi_n A f(x)| \;\leq\; C_f r(n)},$$

where $C_f := \|f'\| + 2\|f''\|$ and $\boxed{r(n) := \frac{\widetilde{\Phi}(n)}{n}}$.

We call $r(n)$ the rate function.

Remark: The rate $r(n)$ in Theorem 3 is optimal.

Sketch of proof

Define $u_0 := 1 - |u|$. Proof of Theorem 3 uses the paintbox representation

$$A_n \pi_n f(x) - \pi_n A f(x) \;=\; \int_\Delta \left(\mathbb{E}\left(f\left(\frac{Y(nx,u)}{n} \right) \right) - f(xu_0) \right) \nu(\mathrm{d}u),$$

the Taylor expansion

$$f\left(\frac{Y(nx,u)}{n} \right) - f(xu_0) \;=\; f'(xu_0)x\widetilde{Y} + f''(\xi)x^2\widetilde{Y}^2,$$

for some random point ξ taking values between $Y(nx,u)/n$ and xu_0, and the concentration inequality $\boxed{\mathbb{E}(\widetilde{Y}^2) \le 2\mathbb{E}(\widetilde{Y}),}$ where

$$\widetilde{Y} \;:=\; \widetilde{Y}(nx,u) \;:=\; \frac{Y(nx,u)}{nx} - u_0.$$

Remark. For Λ-coalescents the sharper concentration inequality $\boxed{\mathbb{E}(\widetilde{Y}^2) \le \mathbb{E}(\widetilde{Y})}$ holds. We conjecture that this sharper inequality holds for all Ξ-coalescents.

Rate of convergence (continued)

Corollary 1. (Rate of convergence, semigroup version, M., 2019)

In the situation of Theorem 3, let $(T_t^{(n)})_{t\geq 0}$ and $(T_t)_{t\geq 0}$ denote the semigroups of the scaled block counting process and the frequency of singleton process $(S_t)_{t\geq 0}$ respectively. Then, for all $t \geq 0$, $n \in \mathbb{N}$, and $f \in C^2(E)$,

$$\boxed{\|T_t^{(n)}\pi_n f - \pi_n T_t f\| \ \leq\ tC_f r(n)},$$

where C_f is the constant from Theorem 3 and the rate $r(n)$ is defined as before.

Rate of convergence of the fixation line

Notation: $F := [1, \infty]$, $F_n := \{k/n : k \in \{n, n+1, \ldots\}\} \cup \{\infty\}$.

$\tau_n : B(F) \to B(F_n)$, $\tau_n g(y) := g(y)$ for $g \in B(F)$ and $y \in F_n$.

Conjecture. (Rate of convergence of the fixation line; work in progress)

Let Π be a Ξ-coalescent with dust and let B_n and B denote the generators of the scaled fixation line $(L_t^{(n)}/n)_{t \geq 0}$ and the reciprocal frequency of singleton process $(1/S_t)_{t \geq 0}$ respectively. Then, for all $n \in \mathbb{N}$ and $g \in C_c^2([1, \infty])$,

$$\|B_n \tau_n g - \tau_n B g\| := \sup_{y \in F_n} |B_n \tau_n g(y) - \tau_n B g(y)| \leq D_g r(n)$$

with rate $r(n)$ as before and constant $D_g := \|f'\| + 2\|f''\|$, where $f(x) := g(1/x)$.

Remark. Conjecture holds for Λ-coalescents, even with improved constant $D_g := \|f'\| + \|f''\|$. Some technical gaps in the proof for the Ξ-coalescent.

Example 1: Dirac coalescent

Let $\nu = \delta_a$ be the Dirac measure at $a = (a_1, a_2, \ldots) \in \Delta \setminus \{0\}$. Then

$$\boxed{\widetilde{\Phi}(q) \;=\; \sum_{i\in\mathbb{N}}(1-(1-a_i)^q) \;=\; q\int_0^\infty e^{-qx}u(x)\,\mathrm{d}x}, \qquad q>0,$$

where $u(x) := \mu([1-e^{-x},1])$ for $x \geq 0$ with $\mu := \sum_{i\geq 1}\delta_{a_i}$.

Thus, $\dfrac{\widetilde{\Phi}(q)}{q}$ coincides with the Laplace transform of u.

Asymptotics of $\widetilde{\Phi}(q)$ difficult to describe in general.

Set $U(x) := \int_0^x u(t)\,\mathrm{d}t$. If $U(x) \sim x^{1-\alpha}\ell(1/x)$, $x \to 0$, for some $\alpha \in [0,1]$ and some function ℓ slowly varying at infinity, then, by a Tauberian argument,

$$\boxed{\widetilde{\Phi}(q) \;\sim\; \Gamma(2-\alpha)q^\alpha\ell(q)}, \qquad q \to \infty.$$

Dirac coalescent (continued)

Examples with $\alpha = 0$ (slow variation), $0 < \alpha < 1$ and $\alpha = 1$ (rapid variation):

a_i	Parameter	Constant α	Asymptotics of $\widetilde{\Phi}(q)$
cp^i	$0 < p < 1$	0	$\mu \log q$ with $\mu := -1/\log p$
$ci^{-\beta}$	$\beta > 1$	$\beta^{-1} \in (0,1)$	$\Gamma(1-\alpha)c^\alpha q^\alpha$
$\dfrac{c}{i(\log i)^\beta}$	$\beta > 1$	1	$\dfrac{c}{\beta - 1} \dfrac{q}{(\log q)^{\beta-1}}$

Constant $c > 0$ chosen sufficiently small such that $|a| \leq 1$

Example 2: Dirichlet coalescent

Let $(X_1, \ldots, X_N) \stackrel{d}{=} D_N(\alpha)$ be symmetric Dirichlet distributed with parameters $N \in \mathbb{N}$ and $\alpha > 0$.

Let $X_{(1)} \geq \cdots \geq X_{(N)}$ denote the order statistics.

$\nu :=$ distribution of $(X_{(1)}, \ldots, X_{(N)}, 0, 0, \ldots)$.

The associated exchangeable coalescent is called the Dirichlet coalescent.

This coalescent neither comes down from infinity nor stays infinite, since $\mathbb{P}(N_t = \infty) = \mathbb{P}(T_f > t) = e^{-t}$ for all $t \geq 0$.

Dirichlet coalescent (continued)

Notation. $[x|y]_n := \prod_{k=0}^{n-1}(x+ky), \quad (x|y)_n := \prod_{k=0}^{n-1}(x-ky),$

$[x]_n := [x|1]_n, \quad (x)_n := (x|1)_n$

Rates of the block counting process: $q_{ij} \;=\; \dfrac{(N\alpha|\alpha)_j}{[N\alpha]_i} S_\alpha(i,j), j < i$

Rates of the fixation line: $\gamma_{ij} \;=\; \dfrac{(N\alpha|\alpha)_{i+1}}{[N\alpha]_{j+1}} S_\alpha(j,i), i < j$

$S_\alpha(i,j) := S(i,j;-1,\alpha,0)$ is the generalized Stirling number as defined in Hsu and Shiue (1998).

Dirichlet coalescent (continued)

Define $\Delta_N := \{u \in \Delta : u_1 + \cdots + u_N = 1\}$. Then

$$
\begin{aligned}
\widetilde{\Phi}(q) &= \int_{\Delta_N} \sum_{r=1}^{N} (1-(1-u_r)^q)\,\nu(\mathrm{d}u) \\
&= \int_{\mathbb{R}^N} \sum_{r=1}^{N} (1-(1-u_r)^q)\,D_N(\alpha)(\mathrm{d}u_1,\ldots,\mathrm{d}u_N) \\
&= N\mathbb{E}(1-(1-X_1)^q),
\end{aligned}
$$

If $N=1$ then $X_1 \equiv 1$ and $\widetilde{\Phi} = \Phi$. If $N>1$ then X_1 is beta distributed with parameters α and $N\alpha - \alpha$ and

$$
\boxed{\widetilde{\Phi}(q) = N\left(1 - \frac{\Gamma(N\alpha)\Gamma(N\alpha-\alpha+q)}{\Gamma(N\alpha-\alpha)\Gamma(N\alpha+q)}\right) \sim N}, \qquad q \to \infty,
$$

differs from $\Phi(q) = 1$, $q > 0$.

Example 3: Poisson–Dirichlet coalescent

(Sagitov (2003), M. (2010), Gaiser and M. (2016))

This is the coalescent where ν is the Poisson–Dirichlet distribution with parameters $0 \le \alpha < 1$ and $\theta > -\alpha$.

Rates of the block counting process: $$q_{ij} \;=\; c_{j,\alpha,\theta}\frac{\Gamma(\theta+\alpha j)}{\Gamma(\theta+i)}s_\alpha(i,j),\, j<i$$

Normalizing constant: $$c_{j,\alpha,\theta} := \prod_{k=1}^{j}\frac{\Gamma(\theta+1+(k-1)\alpha)}{\Gamma(\theta+k\alpha)}$$

Rates of the fixation line: $$\gamma_{ij} \;=\; c_{i,\alpha,\theta}\frac{\Gamma(\theta+\alpha i+1)}{\Gamma(\theta+j+1)}s_\alpha(j,i),\, i<j$$

$s_\alpha(i,j) := S(i,j;-1,-\alpha,0)$ is the generalized absolute Stirling number of the first kind as defined in Hsu and Shiue (1998).

Poisson–Dirichlet coalescent (continued)

By a result of Handa (2009), applied with $f(x) := 1-(1-x)^q$,

$$\widetilde{\Phi}(q) \;=\; \int_\Delta \sum_{r\in\mathbb{N}} f(u_r)\,\nu(\mathrm{d}u) \;=\; \int_{\mathbb{R}} (1-(1-x)^q)\,\mu_1(\mathrm{d}x), \qquad q \ge 0,$$

where μ_1 denotes the correlation measure associated with the Poisson–Dirichlet coalescent. The density of μ_1 is explicitly known (see Handa, 2009), and it follows that

$$\widetilde{\Phi}(q) \;=\; c_{1,\alpha,\theta} \int_0^1 (1-(1-x)^q)x^{-\alpha-1}(1-x)^{\theta+\alpha-1}\,\mathrm{d}x, \qquad q \ge 0,$$

with normalizing constant $c_{1,\alpha,\theta} := \mathrm{B}(1-\alpha,\theta+\alpha)$. For $\alpha > 0$ this leads to

$$\boxed{\widetilde{\Phi}(q) \;=\; \frac{\theta+q}{\alpha}\frac{\Gamma(\theta+\alpha+q)\Gamma(\theta+1)}{\Gamma(\theta+1+q)\Gamma(\theta+\alpha)} - \frac{\theta}{\alpha} \;\sim\; \frac{\Gamma(\theta+1)}{\Gamma(\theta+\alpha)}\frac{q^\alpha}{\alpha}}, \qquad q \to \infty.$$

For $\alpha = 0$ it follows that $\boxed{\widetilde{\Phi}(q) = \theta(\Psi(q+\theta)-\Psi(\theta)) \sim \theta\log q}$ as $q \to \infty$, where $\Psi := \Gamma'/\Gamma$. The associated subordinator is the Ψ-subordinator. In all cases $\widetilde{\Phi}$ differs from Φ.

Example 4: A symmetric coalescent

Let $(m_k)_{k\in\mathbb{N}}$ be a sequence of non-negative real numbers satisfying $\sum_{k\in\mathbb{N}} m_k/k < \infty$.
Suppose ν assigns for each $k \in \mathbb{N}$ mass m_k to $u^{(k)} := (1/k, \ldots, 1/k, 0, 0, \ldots) \in \Delta^*$.
This coalescent occurs in González Casanova, Miró Pina and Siri-Jégousse (2019).

Rates of the block counting process: $$q_{ij} \;=\; S(i,j)\sum_{k\in\mathbb{N}}\frac{(k)_j}{k^i}m_k,\; j<i$$

Rates of the fixation line: $$\gamma_{ij} \;=\; S(j,i)\sum_{k\in\mathbb{N}}\frac{(k)_{i+1}}{k^{j+1}}m_k,\; i<j$$

[$(x)_i := x(x-1)\cdots(x-i+1)$, $S(.,.)$ are the Stirling number of the second kind]

A symmetric coalescent (continued)

The symmetric coalescent has dust if and only if $\nu(\Delta) = \sum_{k\in\mathbb{N}} m_k < \infty$. In this case

$$\boxed{\widetilde{\Phi}(q) \;=\; \sum_{k\in\mathbb{N}} k m_k \left(1 - \left(1 - \frac{1}{k}\right)^q\right)}, \qquad q \geq 0.$$

For example, if $m_k = k^{-\alpha}$ with $\alpha > 1$, then, as $q \to \infty$,

$$\widetilde{\Phi}(q) \;\sim\; \begin{cases} \zeta(\alpha - 1) & \text{if } \alpha > 2, \\ \log q & \text{if } \alpha = 2, \\ -\Gamma(\alpha - 2) q^{2-\alpha} & \text{if } \alpha \in (1,2). \end{cases}$$

$r(n) = O(\frac{1}{n})$ for $\alpha > 2$, $r(n) = O(\frac{\log n}{n})$ for $\alpha = 2$, $r(n) = O(\frac{1}{n^{\alpha-1}})$ for $\alpha \in (1,2)$.

Thank you very much for your attention!

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