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Convexity and Smoothness of Scale Functions and de Finetti's Control Problem

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Abstract We continue the recent work of Avram et al. (Ann. Appl. Probab. 17:156–180, 2007) and Loeffen (Ann. Appl. Probab., 2007) by showing that whenever the Lévy measure of a spectrally negative Lévy process has a density which is log-convex then the solution of the associated actuarial control problem of de Finetti is solved by a barrier strategy. Moreover, the level of the barrier can be identified in terms of the scale function of the underlying Lévy process. Our method appeals directly to very recent developments in the theory of potential analysis of subordinators and their application to convexity and smoothness properties of the relevant scale functions.

Keywords Potential analysis · Special Bernstein function · Scale functions for spectrally negative Lévy processes · Control theory

Mathematics Subject Classification (2000) Primary 60J99 · Secondary 93E20 · 60G51

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1 Introduction

Recently there has been a growing body of literature concerning actuarial mathematics which explores the interaction of classical models of risk and fine properties of Lévy processes with a view to gaining new results on both sides (see, for example, [2, 7, 12–15, 17, 19, 21, 24]). At the same time, there has been considerable progress in the potential analysis of subordinators, in particular with the identification of a natural class of subordinators known as special subordinators (see, for example, [22, 23]). In this paper, we shall marry some of these developments together. We will use new potential analytic considerations found in, for example, [22, 23] to understand better smoothness properties of scale functions for spectrally negative Lévy processes. This builds on other recent developments which closely link the theory of scale functions to potential analysis of subordinators, see [11] and [18]. In turn, this will allow us to solve de Finetti's classical actuarial control problem for a much larger class of driving spectrally negative Lévy processes than previously known. For the remainder of this introduction, we shall elaborate on the latter in more detail before moving on to our results and their proofs.

Henceforth we assume that $X=(X_t:t\geq 0)$ is a spectrally negative Lévy process with Lévy triplet given by (γ,σ,Π), where $\gamma\in\mathbb{R}$, $\sigma\geq 0$ and Π is a measure on $(0,\infty)$ satisfying

$$\int_0^\infty (1\wedge x^2)\Pi(dx)<\infty.$$

The Laplace exponent of X is given by

$$\psi(\theta)=\log\bigl(\mathbb{E}\bigl(e^{\theta X_1}\bigr)\bigr)=\gamma\theta+\frac{1}{2}\sigma^2\theta^2-\int_0^\infty\bigl(1-e^{-\theta x}-\theta x\mathbf{1}_{\{0<x<1\}}\bigr)\Pi(dx).$$

The reader will note that, for convenience, we have arranged the representation of the Laplace exponent in such a way that the support of the Lévy measure is positive even though the process experiences only negative jumps. In this article, we shall only consider the case that Π is absolutely continuous with respect to Lebesgue measure, in which case we shall refer to its density as π.

Let $\Phi(0)$ be the largest real root of ψ and recall that $\Phi(0)>0$ if and only if X drifts to $-\infty$, or equivalently $\psi'(0+)<0$. The restriction $\psi:[\Phi(0),\infty)\to[0,\infty)$ is a bijection whose inverse will be denoted by Φ. Now let ϕ be the Laplace exponent of the descending ladder height subordinator $\widehat{H}=(\widehat{H}_s,s\geq 0)$ associated to X. Standard theory dictates that ϕ and ψ are related by the Wiener–Hopf factorization

$$\psi(\theta)=\bigl(\theta-\Phi(0)\bigr)\phi(\theta),\quad \theta\geq 0,$$

where ϕ satisfies

$$\phi(\theta)=\kappa+\mathrm{d}\theta+\int_0^\infty\bigl(1-e^{-\theta x}\bigr)\Upsilon(x)\,dx,\quad \theta\geq 0, \tag{1.1}$$

with $d = \sigma^2/2$, $\kappa \geq 0$, $\kappa\Phi(0) = 0$ and $\Upsilon : (0, \infty) \to (0, \infty)$ a function such that $\int_0^\infty (1 \wedge x)\Upsilon(x)\,dx < \infty$. Moreover,

$$\overline{\Pi}(x) := \int_x^\infty \Pi(dx) \quad \text{and}$$
$$\overline{\Upsilon}(x) := \int_x^\infty \Upsilon(z)\,dz = e^{\Phi(0)x} \int_x^\infty e^{-\Phi(0)z}\overline{\Pi}(z)\,dz, \qquad x > 0,$$

where the last equality is also a well established fact. The Wiener–Hopf factorization for ψ, in its Laplace transform form, also states that ψ, Φ and the Laplace exponent of the bivariate descending ladder processes, say $\widehat{\kappa} : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}$, are related by the equation

$$\widehat{\kappa}(\alpha, \beta) = c\frac{\alpha - \psi(\beta)}{\Phi(\alpha) - \beta}, \quad \alpha, \beta \geq 0, \tag{1.2}$$

where $c > 0$ is an arbitrary constant depending on the normalization of local time at the infimum. Without loss of generality, we can and will suppose that it is equal to 1.

A key object in the fluctuation theory of spectrally negative Lévy processes and its applications is the *scale functions*. For each $q \geq 0$ the so called q-scale function of X, $W^{(q)} : \mathbb{R} \to [0, \infty)$, is the unique function such that $W^{(q)}(x) = 0$ for $x < 0$ and on $[0, \infty)$ is a strictly increasing and continuous function whose Laplace transform is given by

$$\int_0^\infty e^{-\theta x} W^{(q)}(x)\,dx = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q).$$

In the last 10 years or so, the use of scale functions has proved to be of great importance in a wide variety of applied probability models driven by spectrally negative Lévy processes. We refer to [11, 16] and [18] for a recent overview of their presence in the literature. As alluded to above, we are concerned here in particular with their importance in one of the most classical problems of modern actuarial mathematics – de Finetti's control problem.

Recall that the classical Cramér–Lundberg risk process corresponds to a spectrally negative Lévy process X taking the form of a compound Poisson process with arrival rate $\lambda > 0$ and negative jumps, corresponding to claims, having common distribution function F with finite mean $1/\mu$ as well as a drift $c > 0$, corresponding to a steady income due to premiums. It is usual to assume the *net profit condition* $c - \lambda/\mu > 0$ which says nothing other than $\psi'(0+) > 0$.

An offshoot of the classical ruin problem for the Cramér–Lundberg process was introduced by de Finetti [6]. His intention was to make the study of ruin under the Cramér–Lundberg dynamics more realistic by introducing the possibility that dividends are paid out to shareholders up to the moment of ruin. Further, the payment of dividends should be made in such a way as to optimize the expected net present value of the total dividends paid to the shareholders from time zero until ruin. Mathematically speaking, de Finetti's dividend problem amounts to solving a control problem which we state in the next paragraph but within the framework of the general Lévy insurance risk process. The latter process is nothing more than a general spectrally negative Lévy process which respects the analogue of the net profit condition, namely $\psi'(0+) > 0$ (although the latter is not necessary in what follows).

Suppose that X is a general spectrally negative Lévy process (no assumption is made on its long term behavior) with probabilities $\{\mathbb{P}_x : x \in \mathbb{R}\}$ such that under $\mathbb{P}_x$ we have $X_0 = x$ with probability one. (For convenience, we shall write $\mathbb{P}_0 = \mathbb{P}$.) Let $\xi = \{L_t^\xi : t \geq 0\}$ be a dividend strategy consisting of a left-continuous non-negative non-decreasing process adapted to the (completed and right continuous) filtration $\{\mathcal{F}_t : t \geq 0\}$ of X. The quantity L_t^ξ thus represents the cumulative dividends paid out up to time t by the insurance company whose risk process is modeled by X. The controlled risk process when taking into account of the dividend strategy ξ is thus $U^\xi = \{U_t^\xi : t \geq 0\}$ where $U_t^\xi = X_t - L_t^\xi$. Write $\sigma^\xi = \inf\{t > 0 : U_t^\xi < 0\}$ for the time at which ruin occurs when the dividend payments are taken into account. A dividend strategy is called admissible if at any time before ruin a lump sum dividend payment is smaller than the size of the available reserves; in other words, $L_{t+}^\xi - L_t^\xi \leq \max\{U_t^\xi, 0\}$ for $t < \sigma^\xi$. Denoting the set of all admissible strategies by Ξ, the expected value discounted at rate $q > 0$ of the dividend policy $\xi \in \Xi$ with initial capital $x \geq 0$ is given by

$$v_\xi(x) = \mathbb{E}_x\left(\int_{[0,\sigma^\xi]} e^{-qt}\, dL_t^\xi\right),$$

where $\mathbb{E}_x$ denotes expectation with respect to $\mathbb{P}_x$ and $q > 0$ is a fixed rate. *De Finetti's dividend problem* consists of solving the following stochastic control problem: characterize

$$v^*(x) := \sup_{\xi \in \Xi} v_\xi(x) \tag{1.3}$$

and, further, if it exists, establish a strategy ξ^* such that $v^*(x) = v_{\xi^*}(x)$.

This problem was considered by Gerber [8] who proved that, for the Cramér–Lundberg model with exponentially distributed jumps, the optimal value function is the result of a *barrier strategy*. That is to say, a strategy of the form $L_t^a = a \vee \overline{X}_t - a$ for some $a \geq 0$ where $\overline{X}_t := \sup_{s \leq t} X_s$. In that case, the controlled process $U_t^a = X_t - L_t^a$ is a spectrally negative Lévy process reflected at the barrier a.

This result has been re-considered very recently in [3] for Cramér–Lundberg processes with a general jump distribution. In the latter paper, it was shown that for an appropriate choice of jump distribution, the above described barrier strategy is not optimal. In much greater generality, the paper [2] focuses on the spectrally negative case and finds sufficient conditions for the optimal strategy to consist of a simple barrier strategy. It is in the latter paper that we first begin to see the connection with scale functions, as the sufficient conditions given in [2] are phrased in terms of a variational inequality involving the value of a barrier strategy which itself can be expressed in terms of the associated scale function $W^{(q)}$. Indeed, when the optimal strategy belongs to the class of barrier strategies and the optimal barrier level is $a^* \geq 0$, then [2] showed that

$$v^*(x) = \begin{cases} W^{(q)}(x)/W^{(q)\prime}(a^*), & -\infty < x \leq a, \\ x - a^* + W^{(q)}(a^*)/W^{(q)\prime}(a^*), & \infty > x > a^*. \end{cases}$$

In a remarkable development shortly thereafter, Loeffen [19] made a decisive statement connecting the shape of the scale function $W^{(q)}$ to the existence of an optimal barrier strategy. Loeffen's result begins by requiring that the scale function $W^{(q)}$ is *sufficiently smooth* meaning that it belongs to $C^1(0,\infty)$ if X is of bounded variation and otherwise belongs to $C^2(0,\infty)$. Loeffen's theorem reads as follows.

Theorem 1.1 *Suppose that X is such that its scale functions are sufficiently smooth. Let*

$$a^* = \sup\bigl\{a \geq 0 : W^{(q)\prime}(a) \leq W^{(q)\prime}(x) \text{ for all } x \geq 0\bigr\},$$

(*which is necessarily finite*) *where we understand $W^{(q)\prime}(0) = W^{(q)\prime}(0+)$. Then the barrier strategy at a^* is an optimal strategy if*

$$W^{(q)\prime}(a) \leq W^{(q)\prime}(b) \quad \text{for all } a^* \leq a \leq b < \infty. \tag{1.4}$$

The condition (1.4) is tantamount to saying that the scale function $W^{(q)}$ is convex beyond the global minimum of its first derivative. An intriguing result in itself, it is, however, arguably not a particularly practical condition to verify. Nonetheless, [19] makes one further striking step by providing a very natural class of Lévy risk processes for which (1.4) holds. More precisely, it is shown that (1.4) holds when the Lévy measure Π is absolutely continuous with a completely monotone density.

Thanks then to Theorem 1.1 a clear mandate is set with regard to finding as broad a class of Lévy processes as possible for which the barrier strategy is optimal through smoothness and convexity properties of the scale functions $W^{(q)}$. Motivated by this problem, the current paper aims at establishing a larger class of Lévy processes for which the barrier strategy is optimal in de Finetti's control problem. Specifically, our main result reads as follows.

Theorem 1.2 *Suppose that X has a Lévy density π that is log-convex then the barrier strategy at a^** (*specified in Theorem 1.1*) *is optimal for* (1.3).

A key element which enables us to prove the above result is the following technical conclusion regarding smoothness and convexity of the scale function.

Theorem 1.3 *Suppose that X has a Lévy density π which is log-convex* (*and hence non-increasing*). *Then for $q > 0$ the functions $W^{(q)}$ and $W^{(q)\prime}$ are continuous and strictly convex in the interval (a^*, ∞), where*

$$a^* = \sup\bigl\{a \geq 0 : W^{(q)\prime}(a) \leq W^{(q)\prime}(y) \text{ for all } y \geq 0\bigr\} < \infty.$$

Moreover, if in addition X has a Gaussian component then $W^{(q)} \in C^2(0,\infty)$.

We close this section with a brief summary of the remainder of the paper. The next section deals with the proof of Theorem 1.3. In the section following that, we make some general remarks on Theorem 1.2. In particular, we discuss in more detail the point that Theorem 1.2 does not necessarily follow directly from the amalgamation of the above theorem together with Theorem 1.1 on account of the fact that the issue of sufficient smoothness of $W^{(q)}$ is not entirely addressed. In the final section, we give the proof of Theorem 1.2 which, for the most part, requires a more sensitive analysis of the Hamilton–Jacobi–Bellman inequality and the stochastic calculus that it entails on account of the fact that $W^{(q)}$ is not sufficiently smooth. Here the analytical information provided in Theorem 1.3 turns out to be a subtle but key element of the reasoning.

2 Convexity and Smoothness of Scale Functions

The proof of Theorem 1.3 follows as a trivial corollary of the following theorem which, although seemingly more elaborate, can, in fact, be established with no extra cost in terms of the length or complexity of its proof. Indeed, the below theorem merely lists a number of other interesting facts which one has to pass through, or gets for free, in proving Theorem 1.3.

Theorem 2.1 *If the function*

$$\overline{\Pi}(x) := \int_x^\infty \pi(s)\, ds, \quad x > 0$$

is log-convex, then for any $q \geq 0$ *the function* $g_q(x) := e^{-\Phi(q)x} W^{(q)}(x)$, $x > 0$, *is concave.*

(i) *If* $\Phi(0) = 0$ (*equiv.*, $\psi'(0+) \geq 0$) *and* $q = 0$ *then* W' *is convex. Furthermore, if* X *has a Gaussian component then* $W \in C^2(0, \infty)$.
(ii) *If* $\Phi(0) > 0$ (*equiv.*, $\psi'(0+) < 0$) *or* $q > 0$, *and if furthermore the function* π *is log-convex* (*and hence non-increasing*) *then* $\overline{\Pi}(x)$ *is log-convex, the first derivative of* g_q *is non-increasing and convex, and the functions* $W^{(q)}$ *and* $W^{(q)\prime}$ *are continuous and strictly convex in the interval* (a^*, ∞), *where*

$$a^* = \sup\left\{a \geq 0 : W^{(q)\prime}(a) \leq W^{(q)\prime}(y) \text{ for all } y \geq 0\right\} < \infty.$$

Moreover, if in addition X *has a Gaussian component then* $W^{(q)} \in C^2(0, \infty)$.

As will become clear from its proof, the key innovation is the connection between scale functions, potential densities of subordinators and older work on Volterra equations found in [9, 10]. In this light, recall that a subordinator H is said to be special if there exists another subordinator H^*, the so-called conjugate, such that if h and h^* are their respective Laplace exponents then

$$\theta = h(\theta) h^*(\theta), \quad \theta \geq 0.$$

We refer to [23] for a recent account of properties of this subclass of subordinators.

The proof of Theorem 2.1 relies on the following two technical lemmas. Their proofs will be postponed to the Appendix.

Lemma 2.2 *Let H be a subordinator whose Lévy density, say $\Upsilon(x)$, $x>0$, is log-convex (and hence non-increasing) then the restriction of its potential measure to $(0,\infty)$ has a non-increasing and convex density. If furthermore, the drift of H is strictly positive then the density is in $C^1(0,\infty)$.*

In the next lemma, the term $q/\Phi(q)$ is to be understood in the limiting sense, namely $\psi'(0+)$, when $q=0$ and $\Phi(0)=0$.

Lemma 2.3 *For each $q\geq 0$, the function $\widehat{\kappa}(q,\cdot)$ is a Bernstein function and its killing term is given by*

$$\widehat{\kappa}(q,0)=\frac{q}{\Phi(q)},$$

its drift term is given by

$$\lim_{\theta\to\infty}\frac{\widehat{\kappa}(q,\theta)}{\theta}=\mathrm{d}$$

and the tail of its Lévy measure is given by

$$\overline{\Upsilon}_q(x):=e^{\Phi(q)x}\int_x^\infty e^{-\Phi(q)y}\overline{\Pi}(y)\,dy,\quad x>0.$$

Furthermore, if π is non-increasing then for $q\geq 0$, the Lévy density associated to $\widehat{\kappa}(q,\cdot)$ is non-increasing.

We now feed these two lemmas into the proof of Theorem 2.1

Proof of Theorem 2.1 Let $q\geq 0$. We have by assumption that the function $\overline{\Pi}$ is log-convex, which implies that $e^{-\Phi(q)x}\overline{\Pi}(x)$, $x>0$ is also log-convex. Hence, it follows from the first paragraph in the proof of Theorem 2 in [9] that $\overline{\Upsilon}_q$ as defined in Lemma 2.3 is log-convex, and thus by Theorem 2.4 in [23] we have that the potential density associated to the Bernstein function $\widehat{\kappa}(q,\cdot)$ has a non-increasing density in $(0,\infty)$. We denote the latter by u_q. It follows from Lemmas 1 and 2 in [18] that the function $\widehat{\kappa}(q,\Phi(q)+\cdot)$ is still a Bernstein function such that its potential measure admits the function $e^{-\Phi(q)x}u_q(x)$ as its density in $(0,\infty)$. It now follows that the later function is non-increasing and $\lim_{x\to\infty}e^{-\Phi(q)x}u_q(x)=0$.

It is well known that the q-scale function $W^{(q)}$ satisfies $W^{(q)}(x)=e^{\Phi(q)x}W_{\Phi(q)}(x)$, $x>0$, where $W_{\Phi(q)}$ is the 0-scale function of the spectrally negative Lévy process with Laplace exponent given by $\psi_q(\theta)=\psi(\theta+\Phi(q))-q$, $\theta\geq 0$; see, e.g., Lemma 8.4 in [16] for a proof of this fact. By the Wiener–Hopf factorization, we have that ψ_q is given by

$$\psi_q(\theta)=\theta\widehat{\kappa}\big(q,\Phi(q)+\theta\big),\quad \theta\geq 0.$$

This implies in turn that

$$\frac{\theta}{\psi_q(\theta)}=\frac{1}{\widehat{\kappa}(q,\Phi(q)+\theta)}=\mathrm{d}_q^*+\int_0^\infty e^{-\theta x}e^{-\Phi(q)x}u_q(x)\,dx,\quad \theta\geq 0,$$

where $\mathrm{d}_q^* = \lim_{\theta\to\infty} 1/\widehat{\kappa}(q, \Phi(q)+\theta) \geq 0$. By the definition of 0-scale functions and integration by parts in the latter equation, it follows that

$$\frac{1}{\psi_q(\theta)} = \int_0^\infty e^{-\theta x} W_{\Phi(q)}(x)\,dx = \int_0^\infty e^{-\theta x}\left(\mathrm{d}_q^* + \int_0^x e^{-\Phi(q)z} u_q(z)\,dz\right) dx,$$
$$\theta \geq 0.$$

Thus the uniqueness of the Laplace transform implies that

$$W_{\Phi(q)}(x) = \mathrm{d}_q^* + \int_0^x e^{-\Phi(q)z} u_q(z)\,dz, \quad x \geq 0.$$

Now the first claim immediately follows from the earlier mentioned facts that $W_{\Phi(q)}(x) = e^{-\Phi(q)x} W^{(q)}(x)$ and $e^{-\Phi(q)x} u_q(x)$ is decreasing. Note also that in the case that $\Phi(0) = q = 0$ we see that $W' = u_0$ on $(0,\infty)$ and we may also invoke Lemma 2.2 and deduce the statement in (i).

For the proof of (ii), we henceforth assume that $q > 0$ or $q = 0$ and $\Phi(0) > 0$. The statement about $\overline{\Pi}$ is again a consequence of the first paragraph in the proof of Theorem 2 in [9]. For the remaining statements we note that under the assumption that π is log-convex Lemma 2.3 implies that the Lévy density of the Bernstein function $\widehat{\kappa}(q,\cdot)$ is

$$\Upsilon_q(x) = \overline{\Pi}(x) - \Phi(q)e^{\Phi(q)x}\int_x^\infty e^{-\Phi(q)y}\overline{\Pi}(y)\,dy, \quad x > 0.$$

We claim that the latter is log-convex. Indeed, since by assumption π is a log-convex Lévy density (and hence non-increasing), we know that for $\beta \geq 0$ the functions $x \mapsto e^{-\beta x}\pi(x)$, and (yet again, by the first paragraph of the proof of Theorem 2 in [9]) the function $\int_x^\infty e^{-\beta z}\pi(z)\,dz$, $x > 0$, are log-convex. It then follows that the function $x \mapsto e^{\beta x}\int_x^\infty e^{-\beta z}\pi(z)\,dz$, $x > 0$, is log-convex. Choosing $\beta = \Phi(q)$ and comparing the latter function against the expression for $\Upsilon_q(x)$ after integrating it by parts, the claim follows.

We may now apply Lemma 2.2 to deduce that u_q is a non-increasing convex function and, whenever the Gaussian coefficient, equivalently the linear term in $\widehat{\kappa}$, is strictly positive, we have $u_q \in C^1(0,\infty)$. By elementary arguments, it follows that $e^{-\Phi(q)x}u_q(x)$, $x > 0$, satisfies the same properties. Thus g_q is a concave function whose first derivative is convex and continuous in $(0,\infty)$.

To prove the claim about the convexity of $W^{(q)}$ and $W^{(q)\prime}$, we observe that as $W^{(q)\prime}$ is given by

$$W^{(q)\prime}(x) = \Phi(q)W^{(q)}(x) + u_q(x), \quad x > 0; \tag{2.1}$$

and, since u_q is convex, we will automatically get that $W^{(q)\prime}$ is continuous and ultimately convex once we have proved that $W^{(q)}$ is ultimately convex. Indeed, we have that

$$W^{(q)\prime\prime}(x) = \big(\Phi(q)\big)^2 W^{(q)}(x) + \Phi(q)u_q(x) + u_q'(x), \quad \text{a.e. } x > 0. \tag{2.2}$$

Then as u_q' increases and $W^{(q)}$ grows exponentially fast it follows that ultimately $W^{(q)\prime\prime} > 0$. Hence, $W^{(q)}$ and $W^{(q)\prime}$ are ultimately strictly convex. Furthermore, because $W^{(q)\prime}$ tends to infinity as x tends to ∞, it follows that $a^* < \infty$. Now, let $\alpha_1 < \alpha_2$ be points at which $W^{(q)\prime}$ reaches a local minimum. Because of the convexity of u_q we know that the right and left derivatives of u_q exist everywhere and they satisfy that $u_q'^{-}(\alpha_1) \le u_q'^{+}(\alpha_1) \le u_q'^{-}(\alpha_2) \le u_q'^{+}(\alpha_2)$. As a consequence the right and left derivatives of $W^{(q)\prime}$ exist everywhere and satisfy

$$W^{(q)\prime\prime -}(\alpha_i) = \Phi(q) W^{(q)\prime}(\alpha_i) + u_q'^{-}(\alpha_i) \le 0,$$

$$W^{(q)\prime\prime +}(\alpha_i) = \Phi(q) W^{(q)\prime}(\alpha_i) + u_q'^{+}(\alpha_i) \ge 0,$$

for $i = 1, 2$. These facts together imply that

$$0 \le \Phi(q)\big(W^{(q)\prime}(\alpha_1) - W^{(q)\prime}(\alpha_2)\big) + u_q'^{+}(\alpha_1) - u_q'^{-}(\alpha_2),$$

and hence $W^{(q)\prime}(\alpha_1) - W^{(q)\prime}(\alpha_2) \ge 0$. This implies that the last place where $W^{(q)\prime}$ reaches a local minimum is also the last place where it hits its global minimum. Moreover, for $x > 0$ we have that $W^{(q)\prime}(a^*) \le W^{(q)\prime}(x)$. It thus follows that the following inequalities

$$0 \le \Phi(q) W^{(q)\prime}\big(a^*\big) + u_q'^{+}\big(a^*\big) \le \Phi(q) W^{(q)\prime}(x) + u_q'^{-}(x) \le \Phi(q) W^{(q)\prime}(x) + u_q'^{+}(x),$$

hold for $x > a^*$. Actually, the second inequality is a strict one. Indeed, if there existed $x^* > a^*$ such that $\Phi(q) W^{(q)\prime}(a^*) + u_q'^{+}(a^*) = \Phi(q) W^{(q)\prime}(x^*) + u_q'^{-}(x^*)$, then since $u_q'^{-}(x^*) - u_q'^{+}(a^*) \ge 0$, we would have that $W^{(q)\prime}(a^*) \ge W^{(q)\prime}(x^*)$, which would be a contradiction to the fact that a^* is the largest value where $W^{(q)\prime}$ attains its global minimum. It follows that $W^{(q)\prime}$ is strictly increasing for $x > a^*$. That is, $W^{(q)}$ is strictly convex in (a^*, ∞) and, from (2.1), we deduce that $W^{(q)\prime}$ is also strictly convex for $x > a^*$. Finally, (2.2) proves also that, when furthermore the Gaussian coefficient is strictly positive, then $W^{(q)} \in C^2(0, \infty)$ as in this case we already proved that $u_q \in C^1(0, \infty)$. □

3 Remarks on Theorem 1.2

Before proceeding to the proof of Theorem 1.2, let us first make some remarks.

1. In principle, the proof of Theorem 1.2 follows directly from Theorem 1.3 and Theorem 1.1 if one can verify that $W^{(q)}$ is sufficiently smooth. This is possible in most cases, but not all. The outstanding case is the focus of the proof of Theorem 1.2 and we identify it below by excluding the cases for which sufficient smoothness can be established.

   If $a^* = 0$ then necessarily, either $\sigma > 0$, or $\sigma = 0$ and $\Pi(0, \infty) < \infty$ simultaneously. Other types of spectrally negative Lévy processes are not possible when $a^* = 0$ since then necessarily $W^{(q)}(0+) = \infty$. In the case $\sigma > 0$, we see that

$W^{(q)} \in C^2(0,\infty)$ by Theorem 1.3, and when $\sigma = 0$ and $\Pi(0,\infty) < \infty$ simultaneously we also see that $W^{(q)} \in C^1(0,\infty)$ from the same theorem. Thus, when $a^* = 0$, we have that $W^{(q)}$ is sufficiently smooth.

Suppose now that $a^* > 0$. If X is of bounded variation or $\sigma > 0$ then, similar to the previous paragraph, we may again deduce from Theorem 1.3 that $W^{(q)}$ is sufficiently smooth.

The outstanding case is thus given by $a^* > 0$, X is of unbounded variation and $\sigma = 0$.

2. Recall that Theorem 3 of [19] states that if X has a Lévy density π which is completely monotone then $W^{(q)\prime}$ is convex on $(0,\infty)$ and hence the barrier strategy at a^* is an optimal strategy. Theorem 1.2 is an improvement on this result on account of the fact that any completely monotone function density is log-convex. Below are some examples of Lévy densities which meet the criteria of Theorem 1.2 but not of Theorem 3 of [19].

   Suppose that f and g both map $(0,\infty)$ to $[0,\infty)$ and that they are both non-increasing and log-convex. Suppose moreover that for some (and hence every) $\varepsilon > 0$, $\int_0^\varepsilon x^2 f(x)\,dx < \infty$ and $\int_\varepsilon^\infty g(x)\,dx < \infty$. Further, for some fixed $\alpha > 0$, we have $f(\alpha) = g(\alpha)$ and

$$\frac{f'^{-}(\alpha)}{f(\alpha)} \le \frac{g'^{+}(\alpha)}{g(\alpha)}.$$

   Then

$$\pi(x) := \begin{cases} f(x), & x \in (0,\alpha), \\ g(x), & x \in [\alpha,\infty) \end{cases}$$

   is an example of a decreasing, log-convex function which is not completely monotone, in general. Specific cases in which π is not completely monotone may be taken to be

   (i) $f(x) = x^{-(1+\lambda_1)}$, $g(x) = x^{-(1+\lambda_2)}$, $\alpha = 1$ where $0 < \lambda_2 < \lambda_1 < 2$.
   (ii) $f(x) = e^{2-x}$, $g(x) = e^{1-\lambda x}$, $\alpha = 1/(1-\lambda)$ where $0 < \lambda < 1$.

3. The proof of Theorem 1.2, given in the next section, is lengthy, requiring some auxiliary results first. Scanning the proof, it is not immediately clear where the need for convexity on (a^*,∞) is needed. The precise point at which this property is required is embedded in the proof of Lemma 4.3 below and we have indicated that in the proof. Note also that we have used convexity properties of the scale function in, for example, Lemma 4.1 below.

4 Proof of Theorem 1.2

Following the first remark in the previous section, we shall assume throughout this section that $a^* > 0$, X is of unbounded variation and $\sigma = 0$. Moreover, we shall assume that the conditions of Theorem 1.2 are in force.

We define an operator $(\Gamma, \mathcal{D}(\Gamma))$ as follows. $\mathcal{D}(\Gamma)$ stands for the family of functions $f \in C^1(0,\infty)$ such that the integral

$$\int_{(0,\infty)} \big[f(x-y) - f(x) + yf'(x)\mathbf{1}_{\{y\le 1\}}\big]\Pi(dy)$$

is absolutely convergent for all $x > 0$. For any $f \in \mathcal{D}(\Gamma)$, we define

$$\Gamma f(x) = \gamma f'(x) + \int_{(0,\infty)} \big[f(x-y) - f(x) + yf'(x)\mathbf{1}_{\{y\le 1\}}\big]\Pi(dy).$$

Recall that for any $a > 0$, the expected value discounted at rate $q > 0$ of the barrier strategy at level a is given by

$$v_a(x) := \mathbb{E}_x\left(\int_{[0,\sigma^a]} e^{-qt}\, dL_t^a\right) = \begin{cases} W^{(q)}(x)/W^{(q)\prime}(a), & -\infty < x \le a, \\ x - a + W^{(q)}(a)/W^{(q)\prime}(a), & \infty > x > a \end{cases}$$

where $\sigma^a = \inf\{t > 0 : U_t^a < 0\}$. The second equality is taken from [2].

Lemma 4.1 *For any* $a > 0$, $v_a \in \mathcal{D}(\Gamma)$. *Furthermore, the function* $x \mapsto \Gamma v_a(x)$ *is continuous in* $(0,a)$.

Proof We have proved in Sect. 2 that $W^{(q)}$ is in $C^1(0,\infty)$, hence we know that v_a is in $C^1(0,\infty)$. To show that $v_a \in \mathcal{D}(\Gamma)$, we only need to show that the integral in the definition of Γv_a is absolutely convergent for all $x > 0$. It is easy to check that this is true for $x > a$, so we are going to concentrate on $x \in (0,a)$. Note that it suffices to consider $W^{(q)}$ instead of v_a. For each $x \in (0,a)$ we may write the integral in the definition of $\Gamma W^{(q)}$ as

$$\int_{(\varepsilon,\infty)} \big(W^{(q)}(x-y) - W^{(q)}(x) + yW^{(q)\prime}(x)\mathbf{1}_{\{y\le 1\}}\big)\Pi(dy) + \int_{(0,\varepsilon)} \big(W^{(q)}(x-y) - W^{(q)}(x) + yW^{(q)\prime}(x)\big)\Pi(dy) \tag{4.1}$$

where the value of $\varepsilon = \varepsilon(x) \in (0,1)$ is chosen for each x such that $x - 2\varepsilon > 0$. The absolute convergence of the first integral as well as its continuity in x follows in a straightforward way as a consequence of the continuity and boundedness of $W^{(q)\prime}$ on bounded intervals and dominated convergence in the case of continuity. With regard to the second integral, recall that $W^{(q)\prime}(x) = \Phi(q)W^{(q)}(x) + u_q(x)$. Using the mean value theorem and the fact that u_q is convex and non-increasing, we get that for all $y \in (0,\varepsilon)$

$$\begin{aligned} &\big|W^{(q)}(x-y) - W^{(q)}(x) + yW^{(q)\prime}(x)\big| \\ &\quad = y\big|W^{(q)\prime}(x) - W^{(q)\prime}\big(x-\xi(y)\big)\big| \quad \text{where } \xi(y) \in (0,y) \\ &\quad \le \Phi(q)y\big|W^{(q)}(x) - W^{(q)}\big(x-\xi(y)\big)\big| + y\big|u_q(x) - u_q\big(x-\xi(y)\big)\big| \end{aligned}$$

$$\le \Phi(q) y^2 \sup_{z\in[-\varepsilon,\varepsilon]} W^{(q)\prime}(x+z) + y\int_{x-\xi(y)}^{x} \left|u_q'(y)\right| dy$$

$$\le \Phi(q) y^2 \sup_{z\in[-\varepsilon,\varepsilon]} W^{(q)\prime}(x+z) + y^2\left|u_q'(x-\varepsilon)\right|$$

$$\le y^2 \sup_{z\in[-\varepsilon,\varepsilon]} \left(\Phi(q) W^{(q)\prime}(x+z) + \left|u_q'(x+z)\right|\right).$$

This estimate shows both that the second integral is uniformly integrable in (4.1) and continuous in x by dominated convergence. □

Lemma 4.2 *For any $a > 0$ we have*

$$(\Gamma - q) v_a(x) = 0, \quad x \in (0, a).$$

Proof It is well known that $e^{-q(t\wedge\tau_a^+\wedge\tau_0^-)} W^{(q)}(X_{t\wedge\tau_a^+\wedge\tau_0^-})$ is a $\mathbb{P}_x$-martingale for each $x \in (0, a)$ (cf. [1]), thus $e^{-q(t\wedge\tau_a^+\wedge\tau_0^-)} v_a(X_{t\wedge\tau_a^+\wedge\tau_0^-})$ is a $\mathbb{P}_x$-martingale for each $x \in (0, a)$. Appealing to the Meyer–Itô formula (cf. Theorem 70 of [20]), we have on $\{t < \tau_a^+ \wedge \tau_{1/n}^-\}$

$$v_a(X_t) - v_a(x) = m_t + \int_0^t \Gamma v_a(X_s)\, ds + \int_{\mathbb{R}} v_a''(y)\ell(y,t)\, dy \quad \mathbb{P}_x\text{-a.s.,} \qquad (4.2)$$

where $\ell(y,\cdot)$ is the *semimartingale local time* at y of X and, with $X^{(1)}$ as the martingale part of X consisting of compensated jumps of size less than or equal to unity,

$$m_t = \sum_{s\le t}\left[\Delta v_a(X_s) - \Delta X_s v_a'(X_{s-})\mathbf{1}_{\{|\Delta X_s|\le 1\}}\right]$$

$$- \int_0^t \int_{(0,\infty)} \left[v_a(X_{s-} - y) - v_a(X_{s-}) + y v_a'(X_{s-})\mathbf{1}_{\{y\le 1\}}\right]\Pi(dy)\, ds$$

$$+ \int_0^t v_a'(X_{s-})\, dX_s^{(1)}$$

is a local martingale which is also a true martingale on account of the fact that $W^{(q)\prime}$ is bounded on $[1/n, a]$. Note that we have used that the integral part of $\Gamma v_a(y)$ is absolutely convergent for each $y \in (0, a)$ in order to meaningfully write down the compensation in the expression for the martingale m_t. The occupation formula for the semimartingale local time of X says that

$$\int_{\mathbb{R}} \ell(y,t) g(y)\, dy = \sigma^2 \int_0^t g(X_s)\, ds \quad \mathbb{P}_x\text{-a.s.,}$$

where g is a bounded Borel measurable function. This implies that for Lebesgue almost every y, $\ell(y,\cdot)$ is identically zero almost surely. Taking this into account, the last integral in (4.2) is almost surely zero. Using the semimartingale decomposition in (4.2), one may now use stochastic integration by parts for semimartingales to deduce that on $\{t < \tau_a^+ \wedge \tau_{1/n}^-\}$

$$e^{-qt} v_a(X_t) - v_a(x) = \lambda_t + \int_0^t e^{-qs}(\Gamma - q)v_a(X_s)\,ds \quad \mathbb{P}_x\text{-a.s.},$$

where $\lambda_t = \int_0^t e^{-qs}\,dm_s$ is a martingale.

Next, use uniform boundedness of $(\Gamma - q)v_a(x)$ on $(\alpha, \beta) \subset [0, a]$ and the martingale property of $e^{-q(t\wedge\tau_\beta^+\wedge\tau_\alpha^-)} v_a(X_{t\wedge\tau_\beta^+\wedge\tau_\alpha^-})$ to get

$$0 = \mathbb{E}_x\left[\int_0^{\tau_\beta^+\wedge\tau_\alpha^-} e^{-qs}(\Gamma - q)v_a(X_s)\,ds\right] = \int_{(\alpha,\beta)} (\Gamma - q)v_a(y)u^{(q)}(x, y)\,dy, \quad (4.3)$$

where $u^{(q)}(x, y)\,dy = \int_0^\infty e^{-qs}\mathbb{P}_x(X_s \in dy; t < \tau_\beta^+ \wedge \tau_\alpha^-)$ is the strictly positive resolvent density of the process X killed on exiting (α, β) (see [5] for more details). As (α, β) is arbitrary and $(\Gamma - q)v_a(x)$ is continuous, it follows that the latter is identically zero on $(0, a)$. This is due to a classical argument by contradiction. If the claim is false then by continuity of $(\Gamma - q)v_a$ and strict positivity of $u^{(q)}$, there exists an interval $(\alpha', \beta') \subset [0, a]$ such that (without loss of generality) $(\Gamma - q)v_a(x) > 0$ on (α', β'). Then the equality (4.3) would be violated. □

For convenience, we use v to denote the function v_{a^*}, U to denote U^{a^*} and L to denote L^{a^*}. Then we have the following result.

Lemma 4.3 *For any $x > 0$ we have $(\Gamma - q)v(x) \le 0$.*

Proof There is nothing to prove when $x \in (0, a^*)$ because of Lemma 4.2 applied to the case $a = a^*$. Thanks to the continuity given by Lemma 4.1, this maybe extended to $(0, a^*]$. Finally, the inequality can be proved to hold on (a^*, ∞) by following verbatim the arguments in the proof of Theorem 2 in [19], although it is not necessary to replicate the behavior of second derivatives in that proof, since we have $\sigma = 0$.

It is important to note that the use of the convexity of $W^{(q)}$ on (a^*, ∞) appears in Theorem 2 of [19] and therefore in this paper the use of convexity is hidden in the latter part of the proof. □

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2 Recall that we are assuming that $a^* > 0$, X is of unbounded variation and $\sigma = 0$. Also recall we use v to denote the function v_{a^*}, U to denote U^{a^*} and L to denote L^{a^*}. The idea of the proof are similar to that of [2] and [19]; however, it is necessary to revisit the main line of reasoning and provide more sensitive arguments that accommodate for the fact that, in the present case $W^{(q)}$ is not *sufficiently smooth*, it is twice continuously differentiable almost everywhere but is not in $C^2(0, \infty)$.

Now let ξ be an admissible strategy and let $\overline{L}^\xi$ be the cadlag modification of the process L^ξ. Note that it is still adapted as the usual conditions have been assumed on the natural filtration, and hence $\overline{U}^\xi = X - \overline{L}^\xi$ is a semimartingale. Since the integral in the definition of Γv is absolutely convergent for every $x > 0$ and the second derivative of v is well defined Lebesgue almost everywhere, we can apply the Meyer–Itô formula (cf. Theorem 70 of [20]) to the process $v(\overline{U}^\xi)$ to get, after some straightforward manipulation, that on $\{t < \sigma^\xi\}$,

$$
\begin{aligned}
v\big(\overline{U}_t^\xi\big) - v\big(\overline{U}_0^\xi\big) = M_t^\xi &+ \int_0^t \Gamma v\big(\overline{U}_{s-}^\xi\big)\,ds \\
&+ \sum_{s\le t} \mathbf{1}_{\{\Delta\overline{L}_s^\xi>0\}}\big\{v\big(\overline{U}_{s-}^\xi + \Delta X_s - \Delta\overline{L}_s^\xi\big) - v\big(\overline{U}_{s-}^\xi + \Delta X_s\big)\big\} \\
&- \int_{(0,t]} v'\big(\overline{U}_{s-}^\xi\big)\,d\overline{L}_s^{\xi,c} + \frac{1}{2}\int_{\mathbb{R}} v''(x)\ell^\xi(x,t)\,dx
\end{aligned}
\tag{4.4}
$$

such that

$$
\begin{aligned}
M_t^\xi = &\sum_{s\le t} \mathbf{1}_{\{|\Delta X_s|>0\}}\big[v\big(\overline{U}_{s-}^\xi + \Delta X_s\big) - v\big(\overline{U}_{s-}^\xi\big) - \Delta X_s v'\big(\overline{U}_{s-}^\xi\big)\mathbf{1}_{\{|\Delta X_s|\le 1\}}\big] \\
&- \int_0^t \int_{(0,\infty)} \big[v\big(\overline{U}_{s-}^\xi - y\big) - v\big(\overline{U}_{s-}^\xi\big) + yv'\big(\overline{U}_{s-}^\xi\big)\mathbf{1}_{\{y\le 1\}}\big]\Pi(dy)\,ds \\
&+ \int_0^t v'\big(\overline{U}_{s-}^\xi\big)\,dX_s^{(1)}
\end{aligned}
$$

is a local martingale with $M_0^\xi = 0$, where $X^{(1)}$, the martingale part of X, consists of compensated jumps of size less than or equal to unity. Moreover, $\overline{L}^{\xi,c}$ is the continuous part of $\overline{L}^\xi$ and $\ell^\xi(x,\cdot)$ is the *semimartingale local time* at x of $\overline{U}^\xi$. We have used, in particular, the absolute convergence of the integral part of Γv in order to make sense of M_t^ξ as a compensated stochastic integral. Similarly as before, the occupation formula for the semimartingale local time of $\overline{U}^\xi$ reads

$$
\int_{\mathbb{R}} \ell^\xi(x,t)g(x)\,dx = \sigma^2 \int_0^t g\big(\overline{U}_s^\xi\big)\,ds,
$$

where g is a bounded Borel measurable function. Also similarly as before, since $\sigma = 0$ this implies that, for Lebesgue almost every x, $\ell^\xi(x,\cdot)$ is identically zero almost surely. Taking this into account, the last integral in (4.4) is almost surely zero. Stochastic integration by parts now gives us on $\{t < \sigma^\xi\}$

$$
\begin{aligned}
&e^{-qt}v\big(\overline{U}_t^\xi\big) - v\big(\overline{U}_0^\xi\big) \\
&\quad = \Lambda_t^\xi + \int_0^t e^{-qs}(\Gamma - q)v\big(\overline{U}_{s-}^\xi\big)\,ds \\
&\qquad + \sum_{s\le t} \mathbf{1}_{\{\Delta\overline{L}_s^\xi>0\}} e^{-qs}\big\{v\big(\overline{U}_{s-}^\xi + \Delta X_s - \Delta\overline{L}_s^\xi\big) - v\big(\overline{U}_{s-}^\xi + \Delta X_s\big)\big\} \\
&\qquad - \int_{(0,t]} e^{-qs}v'\big(\overline{U}_{s-}^\xi\big)\,d\overline{L}_s^{\xi,c},
\end{aligned}
\tag{4.5}
$$

where $\Lambda_t^\xi = \int_0^t e^{-qs}\,dM_s^\xi$ is a local martingale.

Now note that by inspection, using the properties of a^*, we see $v'(x) \geq 1$ and moreover, on $\{\Delta\overline{L}_s^\xi > 0\}$,

$$v\big(\overline{U}_{s-}^\xi + \Delta X_s - \Delta\overline{L}_s^\xi\big) - v\big(\overline{U}_{s-}^\xi + \Delta X_s\big) = -\int_{\overline{U}_{s-}^\xi + \Delta X_s - \Delta\overline{L}_s^\xi}^{\overline{U}_{s-}^\xi + \Delta X_s} v'(x)\,dx \leq -\Delta\overline{L}_s^\xi.$$

Note also that

$$\int_{(0,t]} e^{-qs}\,d\overline{L}_s^\xi = \int_{[0,t]} e^{-qs}\,dL_s^\xi - L_{0+}^\xi,$$

and, by the mean value theorem and the fact that $v'(x) \geq 1$, we also have under $\mathbb{P}_x$ that $v(\overline{U}_0^\xi) \leq v(x) - L_{0+}^\xi$. Recalling the property that $(\Gamma - q)v \leq 0$ for all $x > 0$ it follows that for any appropriate localization sequence of stopping times $\{T_n : n \geq 1\}$ we have under $\mathbb{P}_x$

$$\begin{aligned} v(x) - L_{0+}^\xi &\geq -\Lambda_{\sigma^\xi \wedge T_n}^\xi + \int_{(0,\sigma^\xi \wedge T_n]} e^{-qs}\,d\overline{L}_s^\xi + e^{-q(\sigma^\xi \wedge T_n)} v\big(\overline{U}_{\sigma^\xi \wedge T_n}^\xi\big) \\ &\geq -\Lambda_{\sigma^\xi \wedge T_n}^\xi + \int_{[0,\sigma^\xi \wedge T_n]} e^{-qs}\,dL_s^\xi - L_{0+}^\xi. \end{aligned}$$

Taking expectation and then limits as $n \uparrow \infty$ and recalling that ξ is an arbitrary strategy in Ξ, we thus deduce that

$$v(x) \geq \sup_{\xi \in \Xi} \mathbb{E}_x\left(\int_{[0,\sigma^\xi]} e^{-qt}\,dL_t^\xi\right) = v^*(x).$$

On the other hand, thanks to the expression

$$v(x) := \mathbb{E}_x\left(\int_{[0,\sigma^{a^*}]} e^{-qt}\,dL_t^{a^*}\right),$$

the upper bound is attained by the barrier strategy at a^* and the proof is complete. □

Appendix: Proof of Lemmas 2.2 and 2.3

Proof of Lemma 2.2 As has been repeatedly used in Sect. 2, we may appeal to the arguments in the first paragraph in the proof of Theorem 2 in [9] and get that $\overline{\Upsilon}(x) = \int_x^\infty \Upsilon(y)\,dy,\ x > 0$, is log-convex as a consequence of the same being true of Υ. Then, by Theorem 2.4 in [23], we know that H is a special subordinator, and therefore the restriction of its potential measure to $(0,\infty)$ has a non-increasing density. It is also know, for example from [22], that the function u satisfies the following equation

$$\mathrm{d}u(t) + \int_0^t \big\{\overline{\Upsilon}(t-s) + \kappa\big\} u(s)\,ds = 1, \quad t > 0,$$

where $\mathrm{d} \geq 0$ is the drift of $\widehat{H}$ and κ is the killing rate. Now, when $\mathrm{d} = 0$, we can apply Theorem 3 of [10] to conclude that the function u is convex. When $\mathrm{d} > 0$, we can apply Theorem 2 of [9] combined with the first two sentences of Sect. 4 in [9] to conclude that u is convex and in $C^1(0, \infty)$. □

Proof of Lemma 2.3 We recall that the right continuous inverse of the local time at 0 for X reflected at its supremum, L^{-1}, and that of X reflected at its infimum, say $\widehat{L}^{-1}$, are possibly killed subordinators whose Laplace exponents are given by $\Phi(\cdot)$, and $\widehat{\kappa}(\cdot, 0)$, respectively. It follows by the time–space Wiener–Hopf factorization that

$$q = \Phi(q)\widehat{\kappa}(q, 0), \quad q \geq 0,$$

see, e.g., [4] Chap. VII. Thus $\widehat{\kappa}(q, 0) = q/\Phi(q)$, $q \geq 0$. We recall that $\widehat{\kappa}(\cdot, \cdot)$ is the Laplace exponent of the bivariate descending ladder subordinator, and hence it can be represented as

$$\widehat{\kappa}(\lambda, \beta) = \kappa(\lambda, 0) + \mathrm{d}_1\beta + \int_{(0,\infty)^2} \mu_-(dt, dh)\big(e^{-\lambda t} - e^{-\lambda t - \beta h}\big), \quad \beta, \lambda \geq 0,$$

where $\mathrm{d}_1 \geq 0$ and μ_- is the Lévy measure of the bivariate descending ladder subordinator. It follows that, for $q \geq 0$, fixed $\widehat{\kappa}(q, \cdot)$ is a Bernstein function. Since $\widehat{\kappa}(0, \beta) = \phi(\beta)$ for $\beta \geq 0$ and the formula in the last display holds for every $\lambda \geq 0$, we get that $\mathrm{d}_1 = \mathrm{d}$. That is, the drift term of the Bernstein function $\widehat{\kappa}(q, \cdot)$ is equal to d. Moreover, it has been proved in Corollary 6 in [7] that the measure μ_- can be written as

$$\mu_-(dt, dh) = \int_{[0,\infty)} \mathcal{U}_+(dt, ds)\Pi(dh + s), \quad t, h > 0,$$

where $\mathcal{U}_+$ denotes the potential measure of the ascending ladder subordinator. In our case, X is spectrally negative, and hence, due to the absence of positive jumps, this formula becomes

$$\mu_-(dt, dh) = \int_{[0,\infty)} \mathcal{U}_+(dt, ds)\Pi(dh + s) = \int_{[0,\infty)} ds\mathbb{P}\big(L_s^{-1} \in dt\big)\Pi(dh + s),$$

see, e.g., Exercise 7.5 in [16]. This allows us to write

$$\begin{aligned}
&\int_{(0,\infty)^2} \mu_-(dt, dh)\big(e^{-qt} - e^{-qt-\beta h}\big) \\
&\quad = \iiint_{(0,\infty)^3} ds\mathbb{P}\big(L_s^{-1} \in dt\big)\Pi(dh + s)\big(e^{-qt} - e^{-qt-\beta h}\big) \\
&\quad = \iint_{(0,\infty)\times(0,\infty)} ds\Pi(dh + s)\big(e^{-s\Phi(q)} - e^{-s\Phi(q)-\beta h}\big) \\
&\quad = \int_0^\infty \big(1 - e^{-\beta h}\big) \int_0^\infty e^{-s\Phi(q)}\Pi(dh + s)\, ds, \quad \beta \geq 0.
\end{aligned}$$

As a consequence, for $q \geq 0$ fixed, the tail of the Lévy measure of $\widehat{\kappa}(q, \cdot)$ is given by

$$\Upsilon_q(z, \infty) := \int_z^\infty \int_0^\infty e^{-s\Phi(q)} \Pi(dh + s)\, ds = e^{\Phi(q)z} \int_z^\infty due^{-\Phi(q)u} \Pi(u, \infty),$$

which $z > 0$.

This proves the claim about the tail of the Lévy measure. Using it we get that the Lévy measure of $\widehat{\kappa}(q, \cdot)$ has a density given by

$$\upsilon_q(x) := \overline{\Pi}(x) - \Phi(q) e^{\Phi(q)x} \int_x^\infty e^{-\Phi(q)y} \overline{\Pi}(y) \mathrm{d}y, \quad x > 0.$$

To prove that υ_q is non-increasing we observe first that an integration by parts leads to the equality

$$\begin{aligned} \upsilon_q(x) &= \overline{\Pi}(x) - \left(\overline{\Pi}(x) - e^{\Phi(q)x} \int_x^\infty e^{-\Phi(q)z} \pi(z)\, dz \right) \\ &= e^{\Phi(q)x} \int_x^\infty e^{-\Phi(q)z} \pi(z)\, dz, \end{aligned} \tag{5.1}$$

for $x > 0$. Owing to the fact that π is non-increasing, thanks to the assumption that it is a log-convex Lévy density, we have that for $0 < x < y$

$$\begin{aligned} &\upsilon_q(x) - \upsilon_q(y) \\ &\quad = \Phi(q) e^{\Phi(q)x} \int_x^y e^{-\Phi(q)z} \pi(z)\, dz + \left(e^{\Phi(q)x} - e^{\Phi(q)y}\right) \Phi(q) \int_y^\infty e^{-\Phi(q)z} \pi(z)\, dz \\ &\quad \geq \pi(y) e^{\Phi(q)x} \left(e^{-\Phi(q)x} - e^{-\Phi(q)y}\right) + \pi(y) \left(e^{\Phi(q)x} - e^{\Phi(q)y}\right) e^{-\Phi(q)y} = 0, \end{aligned}$$

that is, υ_q is non-increasing. □

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