J. Appl. Prob. **58,** 254–273 (2021)
*doi:*10.1017/jpr.2020.86

DOUBLE HYPERGEOMETRIC LÉVY PROCESSES AND SELF-SIMILARITY

ANDREAS E. KYPRIANOU,* *University of Bath*
JUAN CARLOS PARDO,** *Centro de Investigación en Matemáticas*
MATIJA VIDMAR,*** *University of Ljubljana*

Abstract

Motivated by a recent paper (Budd (2018)), where a new family of positive self-similar Markov processes associated to stable processes appears, we introduce a new family of Lévy processes, called the double hypergeometric class, whose Wiener–Hopf factorisation is explicit, and as a result many functionals can be determined in closed form.

Keywords: Lévy processes; Wiener–Hopf factorisation; self-similar Markov processes; complete Bernstein functions; Pick functions

2010 Mathematics Subject Classification: Primary 60G51
Secondary 60G18; 60G52

1. Introduction

Real-valued Lévy processes are the class of stochastic processes which have stationary and independent increments and paths that are right-continuous with left-limits. We may include in this definition the possibility that the Lévy process is killed and sent to a cemetery state, in which case the killing time must necessarily take the form of an independent and exponentially distributed random time. As a consequence of this definition alone, we can proceed to show that they are strong Markov processes whose transitions are entirely characterised by a single quantity, the characteristic exponent. That is to say, if $(\xi_t, t \geq 0)$ is a one-dimensional Lévy process with probabilities $(\mathbf{P}_x, x \in \mathbb{R})$, then, for all $t \geq 0$ and $\theta \in \mathbb{R}$, there exists a function $\Psi : \mathbb{R} \to \mathbb{C}$ such that

$$\mathbf{E}_0[\mathrm{e}^{\mathrm{i}\theta\xi_t}] = \mathrm{e}^{-\Psi(\theta)t}, \qquad t \geq 0,$$

where Ψ can be described by the so-called Lévy–Khintchine formula, i.e.

$$\Psi(z) = q + \mathrm{i}az + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (1 - \mathrm{e}^{\mathrm{i}zx} + \mathrm{i}zx\mathbf{1}_{(|x|<1)})\, \Pi(\mathrm{d}x), \qquad z \in \mathbb{R},$$

Received 12 April 2019; revision received 30 July 2020.
* Postal address: Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, United Kingdom. Email: a.kyprianou@bath.ac.uk
** Postal address: Centro de Investigación en Matemáticas, Apartado Postal 402, CP 36000, Calle Jalisco s/n, Mineral de Valencianam Guanajuato, Gto. Mexico. Email: jcpardo@cimat.mx
*** Postal address: Department of Mathematics, Faculty of Mathematics and Physics, University of Ljubljana, Jadranska ulica 21, 1000 Ljubljana, Slovenia. Email: matija.vidmar@fmf.uni-lj.si

with $q \geq 0$, $a \in \mathbb{R}$, $\sigma^2 \geq 0$, and Π is a measure (called the Lévy measure) concentrated on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$. Here, q is the rate of the aforementioned exponential killing time, $-a$ is linear drift, $\sigma := \sqrt{\sigma^2}$ is the Gaussian coefficient, and $\Pi(\mathrm{d}x)$ determines the rate at which jumps of size $x \in \mathbb{R}\backslash\{0\}$ occur.

For each non-zero characteristic exponent (as described above) there exists a unique factorisation (the so-called spatial Wiener–Hopf factorisation) such that

$$\Psi(\theta) = \Phi^+(-\mathrm{i}\theta)\Phi^-(\mathrm{i}\theta), \qquad \theta \in \mathbb{R}, \tag{1.1}$$

where Φ^+, Φ^- are Bernstein functions or, in other words, the Laplace exponents of subordinators; uniqueness being up to multiplying and dividing Φ^+ and Φ^-, respectively, with a strictly positive constant. The factorisation has a physical interpretation in the sense that we can identify $\Phi^+(\lambda) = -t^{-1}\log \mathbf{E}_0[\mathrm{e}^{-\lambda S_t^+}]$, $\lambda \geq 0$, $t > 0$, where $(S_t^+, t \geq 0)$, the increasing ladder heights process, is a subordinator whose range agrees with that of $(\sup_{s \leq t} \xi_s, t \geq 0)$. In parallel with the uniqueness of Φ^+ only up to premultiplication by a strictly positive constant, S^+ is only meaningful up to a scaling of the time axis. The role of Φ^- is similar to that of Φ^+, albeit in relation to the dual process $-\xi$.

Vigon's theory of philanthropy [29] is arguably a milestone in the theory of Lévy processes in that it provides precise conditions through which we may fashion a Lévy process by defining it through a selected pair Φ^+, Φ^-, as opposed to, e.g., its transition law, Lévy measure, or characteristic exponent. Following the appearance of a number of new Wiener–Hopf factorisations in [3, 5, 7], this observation was made in [17, 19] and explored further in [14, 16, 22].

In particular, it is argued in [19] that a natural choice of Bernstein functions (equivalently subordinators) to fulfill the roles of Φ^+, Φ^- are those of the so-called *beta class*, taking the form

$$\Phi(\lambda) = \frac{\Gamma(\lambda + \beta + \gamma)}{\Gamma(\lambda + \beta)}, \qquad \lambda \geq 0, \tag{1.2}$$

where $0 \leq \alpha \leq \beta + \gamma$ and $\gamma \in [0, 1]$. When selecting Φ^+ and Φ^- from the class of beta subordinators within the restrictions permitted by Vigon's theory of philanthropy, the resulting family of Lévy processes was called the *hypergeometric class*. The given name of this class reflects the fact that the associated Lévy measure of the Lévy process fulfilling (1.1) can be expressed in a quite simple way using the Gauss hypergeometric function (often written ${}_2F_1$).

Selecting Φ^+ and Φ^- from the class of beta subordinators not only brings about a degree of tractability to a number of associated problems, but it also brings the hypergeometric class in line with a number of other attractive theories such as that of Lévy processes with completely monotone Wiener–Hopf factors; cf. [26]. As alluded to above, a broader definition for the hypergeometric class was also explored in later papers.

Our objective here is to consider yet another deepening of the definition of the hypergeometric class of Lévy processes. As such, we are interested in understanding the extent to which we may select the factors Φ^+, Φ^- from the class of Bernstein functions taking the form

$$\Phi(\lambda) = \mathrm{B}(\alpha, \beta, \gamma, \delta; \lambda) := \frac{\Gamma(\lambda + \alpha)\Gamma(\lambda + \beta)}{\Gamma(\lambda + \gamma)\Gamma(\lambda + \delta)}, \qquad \lambda \geq 0, \tag{1.3}$$

for an appropriate configuration of constants α, β, γ, δ. This will lead to a class of Lévy processes that we will call the *double hypergeometric class*. More generally, our work will show that it is in principle possible to select the factors Φ^+, Φ^- from a class of Bernstein functions taking the form

$$\Phi(\lambda) = \prod_{i=1}^{n} \frac{\Gamma(\lambda + \alpha_i)}{\Gamma(\lambda + \gamma_i)}, \qquad \lambda \geq 0,$$

again, for appropriate configurations of constants $\alpha_1, \ldots, \alpha_n$ and $\gamma_1, \ldots, \gamma_n$ and $n \in \mathbb{N}$. That said, we will stick to the setting of (1.3).

The generalisation we pursue in this paper is motivated through a specific example of a new Lévy process that emerged recently in the literature concerning planar maps, whose characteristic exponent is represented as a product of two functions of the form (1.3) for a suitable choice of parameters. In that setting, [4] introduces a path transformation of a two-sided jumping stable process in which, with a point of issue in $(0, \infty)$, each crossing of the origin is 'ricocheted' back into the positive half-line with a positive probability and otherwise killed. The resulting process is a positive self-similar Markov process (pssMp) and accordingly enjoys a representation as a space–time-changed Lévy process via the so-called Lamperti transform. When examining the Lamperti transform of the ricocheted stable process, for appropriate parameter choices, a double hypergeometric Lévy process emerges. Then a natural question arises: *is the form of the characteristic exponent of such a Lévy process a proper Wiener–Hopf factorisation*? We make these comments more precise later in this paper, at the point where we deal with our second main objective. The latter is to examine the interaction of double hypergeometric Lévy processes and path transformations of the stable process, both in the context of pssMp but also real-valued self-similar Markov processes.

The rest of the paper is organised as follows. In the next section we introduce the class of double hypergeometric Lévy processes and explore their advantages in terms of the Wiener–Hopf factorisation and their integrated exponential functionals. In Section 3 we briefly remind the reader of how self-similar Markov processes are related to Lévy processes or, more generally, Markov additive processes. This then allows us in Section 4 to discuss how the class of double hypergeometric Lévy processes appears naturally in the context of certain types of self-similar Markov processes that are built from a path perturbation of a stable process. The final Section 5 is concerned with the proofs of the main results.

2. Double beta subordinators and double hypergeometric Lévy processes

Let us start by introducing some notation. As already alluded to above, ${}_2F_1$ stands for the Gauss hypergeometric function. However, we will, more generally, use ${}_pF_q$ for the generalised hypergeometric function:

$${}_pF_q\left[\begin{matrix} a_1 \cdots a_p \\ b_1 \cdots b_q \end{matrix}; z\right] s = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k!},$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ denotes the rising factorial power and is frequently called the Pochhammer symbol in the theory of special functions.

Next, suppose that A and B are two countably infinite subsets of $\mathbb{R}$, bounded from above and discrete. Then B is said to interlace with A if $\max A \geq \max B$ and if for the order-preserving enumerations $(a_n)_{n\in\mathbb{Z}_{\leq 0}}$ and $(b_n)_{n\in\mathbb{Z}_{\leq 0}}$ of A and B, respectively, we have $a_{n-1} < b_n < a_n$ for all $n \in \mathbb{Z}_{\leq 0}$, the non-positive integers. If A and B interlace, then necessarily they are disjoint.

Below we give our first key result of this paper. It identifies a class of subordinators having a Laplace transform of the form (1.3). Before stating it, recall that (i) a real meromorphic function is a meromorphic function that maps the real line excluding the poles into the real line, (ii) a Pick (or Nevanlinna–Pick) function is an analytic map defined on a subset of $\mathbb{C}$ containing the open upper complex half-plane and that maps the latter into itself [27, Definition 6.7], and (iii) the non-constant complete Bernstein functions are precisely the Pick functions that are non-negative on $(0, \infty)$ [27, Theorem 6.9] (the latter omits the qualification 'non-constant', but without it, it is clearly false).

Theorem 2.1. *Let $\{\alpha, \beta, \gamma, \delta\} \subset (0, \infty)$; assume $\{\alpha-\beta, \gamma-\delta, \gamma-\alpha, \delta-\alpha, \gamma-\beta, \delta-\beta\} \cap \mathbb{Z} = \emptyset$, and suppose that $\{-\alpha, -\beta\} + \mathbb{Z}_{\leq 0}$ interlaces with $\{-\gamma, -\delta\} + \mathbb{Z}_{\leq 0}$, which, when without loss of generality $\alpha < \beta$ and $\gamma < \delta$, is equivalent to assuming that either*

(I) there is a $k \in \mathbb{N}_0$ with $\gamma + k < \alpha + k < \delta < \beta < \gamma + k + 1$, or

(II) there is a $k \in \mathbb{N}$ with $\alpha + k - 1 < \gamma + k < \beta < \delta < \alpha + k$.

Then:

(a) $\gamma + \delta < \alpha + \beta < \gamma + \delta + 1$;

(b) $\mathtt{B}(\alpha, \beta, \gamma, \delta; \cdot)$ defined in (1.3) is the Laplace exponent of a killed, infinite activity, pure jump subordinator with cemetery state ∞, whose Lévy measure possesses a completely monotone density $f : (0, \infty) \to [0, \infty)$ given by (for $s \in (0, \infty)$)

$$f(s) = -\frac{e^{-\alpha s}\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\delta-\alpha)} {}_2F_1\left[\begin{matrix} 1+\alpha-\gamma, 1+\alpha-\delta \\ 1+\alpha-\beta \end{matrix}; e^{-s}\right] - \frac{e^{-\beta s}\Gamma(\alpha-\beta)}{\Gamma(\gamma-\beta)\Gamma(\delta-\beta)} {}_2F_1\left[\begin{matrix} 1+\beta-\gamma, 1+\beta-\delta \\ 1+\beta-\alpha \end{matrix}; e^{-s}\right]; \tag{2.1}$$

(c) $\mathtt{B}(\alpha, \beta, \gamma, \delta; \cdot)$ is a non-constant complete Bernstein function, indeed it is a real meromorphic Pick function;

(d) the associated potential measure $u : (0, \infty) \to [0, \infty)$ (whose Laplace transform is given by $1/\mathtt{B}(\alpha, \beta, \gamma, \delta; \cdot)$) also admits a density which is completely monotone, and it is given, for $x \in (0, \infty)$, by

$$u(x) = \frac{e^{-\gamma x}\Gamma(\delta-\gamma)}{\Gamma(\alpha-\gamma)\Gamma(\beta-\gamma)} {}_2F_1\left[\begin{matrix} 1-\alpha+\gamma, 1-\beta+\gamma \\ 1-\delta+\gamma \end{matrix}; e^{-x}\right] + \frac{e^{-\delta x}\Gamma(\gamma-\delta)}{\Gamma(\alpha-\delta)\Gamma(\beta-\delta)} {}_2F_1\left[\begin{matrix} 1-\alpha+\delta, 1-\beta+\delta \\ 1-\gamma+\delta \end{matrix}; e^{-x}\right]. \tag{2.2}$$

Conversely, when, ceteris paribus, *the interlacing property of $\{-\alpha, -\beta\} + \mathbb{Z}_{\leq 0}$ with $\{-\gamma, -\delta\} + \mathbb{Z}_{\leq 0}$ fails, then the function $\mathtt{B}(\alpha, \beta, \gamma, \delta; \cdot)$ in (1.3) is not Pick, and so is not a non-constant complete Bernstein function on $(0, \infty)$.*

Finally, suppose merely $\{\alpha, \beta, \gamma, \delta\} \subset [0, \infty)$, and either

(i) *there is a $k \in \mathbb{N}_0$ with $\gamma + k \leq \alpha + k \leq \delta \leq \beta \leq \gamma + k + 1$, or*

(ii) *there is a $k \in \mathbb{N}$ with $\alpha + k - 1 \leq \gamma + k \leq \beta \leq \delta \leq \alpha + k$, or*

(iii) *one of the preceding holds once the substitutions $\alpha \leftrightarrow \beta$ and/or $\gamma \leftrightarrow \delta$ have been effected.*

Then $\mathtt{B}(\alpha, \beta, \gamma, \delta; \cdot)$ of (1.3) is still a complete Bernstein function.

We say that a subordinator S having the Laplace exponent $\mathtt{B}(\alpha, \beta, \gamma, \delta; \cdot)$ as in (1.3) is a *double beta subordinator*. The class of quadruples $(\alpha, \beta, \gamma, \delta) \in [0, \infty)^4$ satisfying conditions (i)–(iii) of Theorem 2.1 is denoted $\mathfrak{G}$. Note, we obtain the beta subordinators of (1.2) as a special case of double beta subordinators in the class $\mathfrak{G}$ by taking $\alpha = \gamma = 0$ and substituting $\beta \to \beta + \gamma$, $\delta \to \beta$ in (1.3) (it satisfies (i) with $k = 0$). The class of those quadruples $(\alpha, \beta, \gamma, \delta) \in (0, \infty)^4$ that satisfy the same conditions (i)–(iii) except that $\leq$ is replaced by $<$ is denoted $\mathfrak{G}^\circ$; it identifies *precisely* those quadruples $(\alpha, \beta, \gamma, \delta) \in (0, \infty)^4$ for which $\{\alpha - \beta, \gamma - \delta, \gamma - \alpha, \delta - \alpha, \gamma - \beta, \delta - \beta\} \cap \mathbb{Z} = \emptyset$ and that make $\mathtt{B}(\alpha, \beta, \gamma, \delta; \cdot)$ of (1.3) into a non-constant complete Bernstein function.

We should point out that our focus in Theorem 2.1 on double beta subordinators corresponding to (1.3) with $(\alpha, \beta, \gamma, \delta) \in \mathfrak{G}^\circ$, rather than more generally $(\alpha, \beta, \gamma, \delta) \in \mathfrak{G}$, is a matter of convenience: to describe the resulting subordinators for all the possible constellations that $\mathfrak{G}$ admits would be prohibitive in scope. For instance, as already remarked, $\mathfrak{G}$ includes the beta subordinators, but also the trivial (zero) subordinators killed at an independent exponential random time, etc. Besides, the conditions '$\alpha, \beta, \gamma, \delta > 0$ and $\{\alpha - \beta, \gamma - \delta, \gamma - \alpha, \delta - \alpha, \gamma - \beta, \delta - \beta\} \cap \mathbb{Z} = \emptyset$' serve to ensure that there are no 'cancellations or reductions' of the gamma factors in (1.3), i.e. $\mathtt{B}(\alpha, \beta, \gamma, \delta; \cdot)$ is 'truly' a quotient of four gammas. On the other hand, our insistence on the interlacing property, beyond merely demanding that $\alpha, \beta, \gamma, \delta > 0$ and $\{\alpha - \beta, \gamma - \delta, \gamma - \alpha, \delta - \alpha, \gamma - \beta, \delta - \beta\} \cap \mathbb{Z} = \emptyset$, i.e. on the 'non-constant complete Bernstein' property, is more than just convenience: it is not clear to us how to establish the Bernstein property of $\mathtt{B}(\alpha, \beta, \gamma, \delta; \cdot)$ from (1.3) when the complete Bernstein property fails (if this can in fact happen; see the comments below Proposition 4.1).

From Vigon's theory of philanthropy [29], and in particular the fact that double beta subordinators corresponding to $\mathfrak{G}$ are completely monotone, thereby having associated Lévy measures that are absolutely continuous with non-increasing densities, we can now define a new class of Lévy processes from the double beta subordinator class as follows; see [20, Section 6.6].

Corollary 2.1. *Let* $\{(\alpha, \beta, \gamma, \delta), (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta})\} \subset \mathfrak{G}$. *Then there exists a Lévy process with characteristic exponent* Ψ *that respects the spatial Wiener–Hopf factorisation in the form* $\Psi(\theta) = \mathtt{B}(\alpha, \beta, \gamma, \delta; -\mathrm{i}\theta)\mathtt{B}(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}; \mathrm{i}\theta)$, $\theta \in \mathbb{R}$.

In light of the above, we call a Lévy process, ξ, of the type introduced in the preceding corollary a *double hypergeometric Lévy process*. The following proposition gathers some basic properties of our new class of double hypergeometric Lévy processes (and accordingly justifies the name 'double hypergeometric').

Proposition 2.1. *Let* $\{(\alpha, \beta, \gamma, \delta), (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta})\} \subset \mathfrak{G}$ *and* ξ *be a double hypergeometric Lévy process with these parameters as in Corollary 2.1. Then:*

(i) Assume even $\{(\alpha, \beta, \gamma, \delta), (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta})\} \subset \mathfrak{G}^\circ$. *Then* ξ*, once stripped away of its killing, is a meromorphic Lévy process in the terminology of [16]. The Lévy measure* Π *of* ξ *has a density* π *that is given by*

$$\pi(x) = -\frac{\Gamma(\alpha+\hat{\alpha})\Gamma(\alpha+\hat{\beta})\Gamma(\beta-\alpha)\mathrm{e}^{-\alpha x}}{\Gamma(\alpha+\hat{\gamma})\Gamma(\alpha+\hat{\delta})\Gamma(\gamma-\alpha)\Gamma(\delta-\alpha)}\,{}_4F_3\left[\begin{matrix}\alpha+\hat{\alpha}, \alpha+\hat{\beta}, 1+\alpha-\gamma, 1+\alpha-\delta \\ 1+\alpha-\beta, \alpha+\hat{\gamma}, \alpha+\hat{\delta}\end{matrix}; \mathrm{e}^{-x}\right]$$
$$-\frac{\Gamma(\beta+\hat{\beta})\Gamma(\beta+\hat{\alpha})\Gamma(\alpha-\beta)\mathrm{e}^{-\beta x}}{\Gamma(\beta+\hat{\gamma})\Gamma(\beta+\hat{\delta})\Gamma(\gamma-\beta)\Gamma(\delta-\beta)}\,{}_4F_3\left[\begin{matrix}\beta+\hat{\beta}, \beta+\hat{\alpha}, 1+\beta-\gamma, 1+\beta-\delta \\ 1+\beta-\alpha, \beta+\hat{\gamma}, \beta+\hat{\delta}\end{matrix}; \mathrm{e}^{-x}\right]$$

for $x \in (0, \infty)$. *For* $x \in (-\infty, 0)$ *it is the same, except that the quantities with and without a hat get interchanged for their respective counterparts and* $-x$ *replaces* x.

(ii) *There is a Gaussian component if and only if* $\alpha+\beta=\gamma+\delta+1$ *and* $\hat{\alpha}+\hat{\beta}=\hat{\gamma}+\hat{\delta}+1$, *in which case the diffusion coefficient* $\sigma^2=2$.

(iii) *Recall that we may assume without loss of generality that* $\gamma\le\delta$, $\alpha\le\beta$, $\hat{\gamma}\le\hat{\delta}$, $\hat{\alpha}\le\hat{\beta}$, *and assume (for simplicity: if* $\delta=0$ ($\hat{\delta}=0$)*, then automatically* $\alpha=\gamma=0$ ($\hat{\alpha}=\hat{\gamma}=0$)*) that* $\delta\hat{\delta}>0$. *Then* ξ *has infinite lifetime if and only if either* $\gamma=\hat{\gamma}=0<\alpha\hat{\alpha}$, *in which case* ξ *oscillates; or* $\gamma=0<\hat{\gamma}\alpha$, *in which case* ξ *drifts to* ∞*; or* $\hat{\gamma}=0<\gamma\hat{\alpha}$, *in which case* ξ *drifts to* $-\infty$.

The exclusion of the parameters corresponding to $\mathfrak{G}\backslash\mathfrak{G}^\circ$ in the statement of Proposition 2.1(i) is not without cause: for instance, the functions 1 and $\mathrm{id}_{[0,\infty)}$ are of the double beta class, but they do not yield a meromorphic Lévy process, at least not in the sense of [16]; similarly, the expression of Proposition 2.1(i) for the Lévy density does not hold true across the whole class of double hypergeometric Lévy processes corresponding to $\mathfrak{G}$.

In the next section we shall briefly discuss examples of how the double hypergeometric Lévy process emerges in the setting of certain self-similar Markov processes. As we will see in the exposition there, for the theory of the latter processes, an important quantity of interest is the exponential functional of Lévy processes. Below we show that it is possible to characterise the exponential functional of double hypergeometric Lévy processes via an explicit Mellin transform. Our method is based on the 'verification' result of [19, Section 8.1]. In the following we use $\mathbf{P}$ to denote the law of the double hypergeometric Lévy process.

Proposition 2.2. *Let* $\{(\alpha,\beta,\gamma,\delta),(\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\delta})\}\subset\mathfrak{G}$, *and* ξ *be a double hypergeometric Lévy process with these parameters as in Corollary 2.1. Assume* $\hat{\gamma}\le\hat{\delta}$, $\hat{\alpha}\le\hat{\beta}$, *and* $0<\hat{\gamma}<\hat{\alpha}$ *(in fact the very last condition is automatic if even* $(\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\delta})\in\mathfrak{G}^\circ$*, while the first two are without loss of generality). Then, for a given* $c\in(0,\infty)$*, with* ζ *being the lifetime of* ξ*, for* $s\in\mathbb{C}$ *with* $\mathrm{Re}(s)\in(0,1+\hat{\gamma}c)$,

$$\mathbf{E}\left[\left(\int_0^\zeta \mathrm{e}^{-\xi_t/c}\,\mathrm{d}t\right)^{s-1}\right]=C\Gamma(s)\frac{G(c\gamma+s;c)G(c\delta+s;c)}{G(c\alpha+s;c)G(c\beta+s;c)}\frac{G(1+c\hat{\alpha}-s;c)G(1+c\hat{\beta}-s;c)}{G(1+c\hat{\gamma}-s;c)G(1+c\hat{\delta}-s;c)}, \tag{2.3}$$

where C *is a normalisation constant such that the left- and right-hand sides agree at* $s=1$ *(i.e., both are equal to unity), and* G *is Barnes' double gamma function (see [19, p. 121], and further references therein, for its definition and basic properties).*

Remark 2.1. In fact, in the proof only the functional form of the characteristic exponent of ξ, that ξ is killed or else drifting to ∞, and the inequality $\gamma+\delta-\alpha-\beta+\hat{\alpha}+\hat{\beta}-\hat{\gamma}-\hat{\delta}<6$ will be used. Hence, if it is known a priori that a possibly killed Lévy process has the Laplace exponent as given in Corollary 2.1 with $\gamma+\delta-\alpha-\beta+\hat{\alpha}+\hat{\beta}-\hat{\gamma}-\hat{\delta}<6$, and that it is killed or drifts to ∞, then the condition '$\{(\alpha,\beta,\gamma,\delta),(\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\delta})\}\subset\mathfrak{G}$' can be dropped in favour of all of the coefficients $\alpha,\ldots,\hat{\delta}$ being just non-negative.

As indicated in the introduction, before moving to the proofs of the main results in this section, we will first look at how double hypergeometric Lévy processes appear naturally in the setting of a special family of self-similar Markov processes. Our first step in this direction is to briefly introduce the latter processes.

3. Self-similar Markov processes

Let $Y=(Y_s, s\geq 0)$ be a pssMp with self-similarity index $\alpha\in(0,\infty)$ and probabilities $(\mathsf{P}_y, y>0)$. Then Y enjoys a bijection with the class of Lévy processes (as presented in the introduction of this article) via the Lamperti transform. Lamperti's result gives a bijection between the class of pssMps and the class of Lévy processes, possibly killed at an independent exponential time with cemetery state $-\infty$, such that, under P_y, $y>0$, with the convention $\xi_\infty=-\infty$,

$$Y_t=\exp(\xi_{\varphi_t}), \qquad t\in[0,\infty), \tag{3.1}$$

where $\varphi_t=\inf\{s>0: \int_0^s \exp(\alpha\xi_u)\,\mathrm{d}u>t\}$ and the Lévy process ξ is started in $\log y$.

In recent work, e.g. [8-10, 12, 15, 18, 30], effort has been invested in extending the theory of pssMps to the setting of $\mathbb{R}$. Analogously to Lamperti's representation, for a real self-similar Markov process (rssMp), say Y, there is a Markov additive process $((\xi_t, J_t), t\geq 0)$ on $\mathbb{R}\times\{-1,1\}$ such that, again under the convention $\xi_\infty=-\infty$,

$$Y_t=J_{\varphi_t}\exp\big(\xi_{\varphi_t}\big), \qquad t\in[0,\infty), \tag{3.2}$$

where $\varphi_t=\inf\{s>0: \int_0^s \exp(\alpha\xi_u)\,\mathrm{d}u>t\}$ and $(\xi_0, J_0)=(\log|Y_0|, [Y_0])$ with

$$[z]=\begin{cases}1 & \text{if } z>0,\\ -1 & \text{if } z<0.\end{cases}$$

The representation (3.2) is known as the Lamperti–Kiu transform. Here, by a Markov additive process (MAP) we mean the regular strong Markov process with probabilities $\mathbf{P}_{x,i}$, $x\in\mathbb{R}$, $i\in\{-1,1\}$, such that $(J_t, t\geq 0)$ is a continuous-time Markov chain on $\{-1,1\}$ (called the modulating chain) and, for any $i\in\{-1,1\}$ and $t\geq 0$, given $\{J_t=i\}$, the pair $(\xi_{t+s}-\xi_t, J_{t+s})_{s\geq 0}$ is independent of the past up to t and has the same distribution as $(\xi_s, J_s)_{s\geq 0}$ under $\mathbf{P}_{0,i}$. If the MAP is killed, then ξ is sent to the cemetery state $-\infty$. All background results for MAPs that relate to the present article can be found in the appendix of [10].

The analogue of the characteristic exponent of a Lévy process for the above MAP is provided by the matrix-valued function $\boldsymbol{\Psi}$ such that, for all $t\in[0,\infty)$, $\{i,j\}\subset\{1,-1\}$, $\theta\in\mathbb{R}$, we have $\mathbf{E}_{0,i}[\mathrm{e}^{\mathrm{i}\theta\xi_t}; J_t=j]=(\mathrm{e}^{\boldsymbol{\Psi}(\theta)t})_{ij}$ and $\boldsymbol{\Psi}(\theta)=-\mathrm{diag}(\Psi_1(\theta), \Psi_{-1}(\theta))+Q\circ G(\theta)$, where Q is the generator matrix of J; G is the matrix of the characteristic functions of the extra jumps that are inserted into the path of the MAP ξ when J transitions, the diagonal elements of G being set to unity; and Ψ_1 and Ψ_{-1} are the characteristic exponents of the Lévy processes that govern the evolution of the MAP component ξ when it is in states 1 and -1, respectively. The symbol $\circ$ denotes element-wise multiplication (Hadamard product).

The mechanism behind the Lamperti–Kiu representation is thus simple. The modulation J governs the sign of Y and, on intervals of time for which there is no change in sign, the Lamperti–Kiu representation effectively plays the role of the Lamperti representation of a pssMp, multiplied with -1 if Y is negative. In a sense, the MAP formalism gives a concatenation of signed Lamperti representations between times of sign change. Typically we can assume the Markov chain J to be irreducible, as otherwise the corresponding self-similar Markov process only switches signs at most once and can therefore be treated using the theory of pssMp.

4. Richocheted stable processes

Let us now turn to the setting of stable processes and path perturbations thereof that fall under the class of self-similar Markov processes and give rise to double hypergeometric Lévy processes in their Lamperti representation.

Let $(X_t, t \geq 0)$ with probabilities $(\mathbb{P}_x, x \in \mathbb{R})$ (as usual $\mathbb{P} := \mathbb{P}_0$) be a *strictly* α*-stable process* (henceforth just written 'stable process'), that is to say an unkilled Lévy process which also satisfies the *scaling property* that under $\mathbb{P}$, for every $c > 0$, the process $(cX_{tc^{-\alpha}})_{t\geq 0}$ has the same law as X. It is known that α necessarily belongs to $(0, 2]$, and the case $\alpha = 2$ corresponds to Brownian motion, which we exclude. The Lévy–Khintchine representation of such a process is as follows: $\sigma = 0$, Π is absolutely continuous with density given by $\pi(x) := c_+ x^{-(\alpha+1)}\mathbf{1}_{(x>0)} + c_-|x|^{-(\alpha+1)}\mathbf{1}_{(x<0)}$, $x \in \mathbb{R}$, where $c_+,\ c_- \geq 0$.

For consistency with the literature that we shall appeal to in this article, we shall always parametrise our α-stable process such that

$$c_+ = \Gamma(\alpha+1)\frac{\sin(\pi\alpha\rho)}{\pi} \quad \text{and} \quad c_- = \Gamma(\alpha+1)\frac{\sin(\pi\alpha\hat{\rho})}{\pi},$$

where $\rho = \mathbb{P}(X_1 \geq 0)$ is the positivity parameter, and $\hat{\rho} = 1 - \rho$. Note when $\alpha = 1$, then $c_+ = c_-$. Moreover, we may also identify the exponent as taking the form

$$\Psi(\theta) = |\theta|^\alpha(\mathrm{e}^{\pi \mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta>0)} + \mathrm{e}^{-\pi \mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta<0)}), \qquad \theta \in \mathbb{R}. \tag{4.1}$$

With this normalisation, we take the point of view that the class of stable processes is parametrised by α and ρ; the reader will note that all the quantities above can be written in terms of these parameters. We shall restrict ourselves a little further within this class by excluding the possibility of having only one-sided jumps. In particular, this rules out the possibility that X is a subordinator or the negative of a subordinator, which occurs when $\alpha \in (0, 1)$ and either $\rho = 1$ or 0.

For this class of two-sided jumping stable processes, it is well known that, when $\alpha \in (0, 1]$, the process never hits the origin (irrespective of the point of issue) and $\lim_{t\to\infty} |X_t| = \infty$. When $\alpha \in (1, 2)$, the process hits the origin with probability 1 (for all points of issue). In both cases, the process will cross the origin infinitely often.

In [4, Section 6] the following example of a positive self-similar Markov process, say $Y^* = (Y^*_t, t \geq 0)$, called a ricocheted stable process, was introduced in the context of the theory of planar maps. Fix a $\mathfrak{p} \in [0, 1]$ and let $\tau_0^- = \inf\{t > 0 : X_t < 0\}$. The stochastic dynamics of Y^* are as follows. From its point of issue in $(0, \infty)$, Y^* evolves as an X until its first passage into $(-\infty, 0)$, i.e. τ_0^-. At that time an independent coin is flipped with probability $\mathfrak{p}$ of heads. If heads is thrown, then the process Y^* is immediately transported to $-X_{\tau_0^-}$ (i.e. it is 'ricocheted' across 0). If tails is thrown, then it is sent to 0 and the process is killed. In the event Y^* is ricocheted, it continues to evolve as an independent copy of X, flipping a new coin on the first pass into $(-\infty, 0)$, and so on.

The process Y^* can be viewed as a way of resurrecting (with probability $\mathfrak{p}$) the process $(X_t\mathbf{1}_{(t<\tau_0^-)}, t \geq 0)$ every time it is about to die; indeed, it degenerates to the latter process when $\mathfrak{p} = 0$. In this respect the ricocheted process is related to the forthcoming work of [6]. This paper highlights the interesting phenomenon as to whether such resurrected processes will eventually continuously absorb at the origin or not.

As alluded to above, ricocheted processes were considered in [4]. In particular, it was shown in [4, Proposition 11] that the characteristic exponent of the Lévy process associated to Y^* via the Lamperti transform takes the shape

$$\Psi^*(\theta) = \frac{\Gamma(\alpha - \mathrm{i}\theta)\Gamma(1+\mathrm{i}\theta)}{\pi}\left[\sin(\pi(\alpha\hat{\rho} - \mathrm{i}\theta)) - \mathfrak{p}\sin(\pi\alpha\hat{\rho})\right], \qquad \theta \in \mathbb{R}. \tag{4.2}$$

Moreover, by setting

$$\sigma := \frac{1}{2} - \alpha\hat{\rho} \in \left(-\frac{1}{2}, \frac{1}{2}\right) \text{ and } b := \frac{1}{\pi}\arccos(\mathfrak{p}\cos(\pi\sigma)) \in \left[|\sigma|, \frac{1}{2}\right],$$

we may also write

$$\Psi^*(\theta) = 2^\alpha \frac{\Gamma\left(\frac{1+\mathrm{i}\theta}{2}\right)\Gamma\left(\frac{2+\mathrm{i}\theta}{2}\right)}{\Gamma\left(\frac{\sigma+b+\mathrm{i}\theta}{2}\right)\Gamma\left(\frac{\sigma-b+\mathrm{i}\theta+2}{2}\right)} \times \frac{\Gamma\left(\frac{\alpha-\mathrm{i}\theta}{2}\right)\Gamma\left(\frac{1+\alpha-\mathrm{i}\theta}{2}\right)}{\Gamma\left(\frac{b-\sigma-\mathrm{i}\theta}{2}\right)\Gamma\left(\frac{2-\sigma-b-\mathrm{i}\theta}{2}\right)}, \qquad \theta \in \mathbb{R}. \tag{4.3}$$

With this in hand we now see an apparent relation with Corollary 2.1.

Proposition 4.1. *Provided* $\alpha \in [1-\sigma-b, b-\sigma+1]$, *equivalently* $\mathfrak{p}\sin(\pi\alpha\hat{\rho}) \le \sin(\pi\alpha\rho)$, *then the two factors either side of* $\times$ *in* (4.3) *identify the Wiener–Hopf factorisation. That is to say, the exponent in* (4.3) *belongs to the class of double hypergeometric Lévy processes.*

There are a number of remarks that are worth making at this point. First, $b \in \{\sigma, -\sigma\}$ occurs if and only if $\mathfrak{p} = 1$. In that case we see that ξ^* has infinite lifetime; moreover, by computing $\Psi^{*\prime}(0)$, we see that Y^* is absorbed at the origin in finite time, oscillates between 0 and ∞, or drifts to ∞, according as $\sigma < 0$, $\sigma = 0$, or $\sigma > 0$ (equivalently $\alpha\hat{\rho} > 1/2$, $\alpha\hat{\rho} = 1/2$, or $\alpha\hat{\rho} < 1/2$), which resonates with the work on resurrected Lévy processes in [6].

Second, when $\mathfrak{p} = 0$, and hence $b = \frac{1}{2}$, we recover the Wiener–Hopf factorisation of the Lamperti stable Lévy process corresponding to killing the stable process on hitting $(-\infty, 0)$ (in this case the condition $\alpha \in [1-\sigma-b, b-\sigma+1]$ is met).

Third, we can easily verify that the condition $\alpha \in [1-\sigma-b, b-\sigma+1]$ is a necessary and sufficient condition needed to ensure the 'interlacing' property of Theorem 2.1. Despite the exponent Ψ^* falling outside of the double hypergeometric class when the aforementioned condition fails, we nonetheless conjecture that (4.3) is in fact the correct factorisation for the whole of the parameter regime. By way of example, taking, e.g., $\alpha = 21/20$, $\rho = 5/21$, and $\mathfrak{p} = 0.9$, then the condition $\alpha \in [1-\sigma-b, b-\sigma+1]$ fails, and indeed numerically it is seen that the second factor in (4.3) does not correspond to a Pick function; still, an inspection of the first few derivatives suggests that it nevertheless corresponds to a Bernstein function.

Finally, we note that, in the spirit of [5] and [4], two more characteristic exponents of Lévy processes appear which are obtained by a simple Esscher transform once we note that $\mathrm{i}(b+\sigma)$ and $-\mathrm{i}(b-\sigma)$ are roots of Ψ^*. Indeed, again provided $\alpha \in [1-\sigma-b, b-\sigma+1]$, we also have that $\Psi^*(z+\mathrm{i}(b+\sigma))$ and $\Psi^*(z-\mathrm{i}(b-\sigma))$ fall into the double hypergeometric class.

Now let T_0^* be the first hitting time of 0 by the process Y^*, and let ζ^* be the lifetime of ξ^*. By the Lamperti transform we may express

$$T_0^* = \int_0^{\zeta^*} \mathrm{e}^{\alpha\xi_s^*}\,\mathrm{d}s = 2^{-\alpha}\int_0^{2^\alpha\zeta^*} \mathrm{e}^{-(-2\xi^*_{2^{-\alpha}u})/(2/\alpha)}\,\mathrm{d}u,$$

so that Proposition 2.2 (coupled with Remark 2.1) yields the Mellin transform of T_0^* provided $0 < b - \sigma < \alpha$. We leave making the eventual expression explicit to the interested reader. We do, however, give the following detail.

Corollary 4.1. *Assume* $\alpha \in [1 - \sigma - b, b - \sigma + 1]$. *Let, respectively,* $\overline{Y^*}_{T_0^*-} := \sup\{Y_s^* : s \in [0, T_0^*)\}$ *and* $\underline{Y}^*_{T_0^*-} := \inf\{Y_s^* : s \in [0, T_0^*)\}$ *be the overall supremum and the overall infimum of* Y^* *before absorption at zero. Correspondingly introduce* $\overline{\xi^*}_{\zeta^*-}$ *and* $\underline{\xi}^*_{\zeta^*-}$ *in the obvious way. We have the following assertions:*

(i) If $\mathfrak{p} = 1$ *and* $\alpha\hat{\rho} \leq \frac{1}{2}$, *then, almost surely (a.s.),* $T_0^* = \zeta^* = \overline{Y^*}_{T_0^*-} = \overline{\xi^*}_{\zeta^*-} = \infty$. *Otherwise, the quantities* T_0^* *and* $\overline{Y^*}_{T_0^*-} = \exp(\overline{\xi^*}_{\zeta^*-})$ *are a.s. finite, and the Laplace transform of the law of* $\log(\overline{Y^*}_{T_0^*-}/Y_0) = \overline{\xi^*}_{\zeta^*-} - \xi_0^*$ *(this law also being that of* $\log(Y^*_{T_0^*-}/\underline{Y^*}_{T_0^*-}) = \xi^*_{\zeta-} - \underline{\xi}^*_{\zeta^*-}$ *when* $\mathfrak{p} < 1$*) is given by the map*

$$[0, \infty) \ni z \mapsto \frac{\sqrt{\pi}\,\Gamma(\alpha)2^{1-\alpha}}{\Gamma\left(\frac{b-\sigma}{2}\right)\Gamma\left(\frac{2-\sigma-b}{2}\right)} \frac{\Gamma\left(\frac{b-\sigma+z}{2}\right)\Gamma\left(\frac{2-\sigma-b+z}{2}\right)}{\Gamma\left(\frac{\alpha+z}{2}\right)\Gamma\left(\frac{1+\alpha+z}{2}\right)}.$$

(ii) If $\mathfrak{p} = 1$ *and* $\alpha\hat{\rho} \geq \frac{1}{2}$, *then* $\zeta^* = \infty$ *and* $\underline{Y}^*_{T_0^*-} = 0$. *Otherwise,* $\underline{Y}^*_{T_0^*-} = \exp(\underline{\xi}^*_{\zeta^*-})$ *is a.s. strictly positive, and the Laplace transform of the law of* $\log(Y_0/\underline{Y}^*_{T_0^*-}) = \xi_0^* - \underline{\xi}^*_{\zeta^*-}$ *(this law also being that of* $\log(\overline{Y^*}_{T_0^*-}/Y^*_{T_0^*-}) = \overline{\xi^*}_{\zeta^*-} - \xi^*_{\zeta^*-}$ *when* $\mathfrak{p} < 1$*) is given by the map*

$$[0, \infty) \ni z \mapsto \frac{\sqrt{\pi}}{\Gamma\left(\frac{\sigma+b}{2}\right)\Gamma\left(\frac{\sigma-b+2}{2}\right)} \frac{\Gamma\left(\frac{\sigma+b+z}{2}\right)\Gamma\left(\frac{\sigma-b+z+2}{2}\right)}{\Gamma\left(\frac{1+z}{2}\right)\Gamma\left(\frac{2+z}{2}\right)}.$$

Proof. The various (a.s.; we have in some places omitted this qualification for brevity) equalities of random variables follow from the Lamperti transformation. The equalities in law are part of the statement of the Wiener–Hopf factorisation of ξ^*. To see (i), let η^* be the lifetime of the increasing ladder heights subordinator H^{+*} of ξ^* and denote by Φ^{+*} the Laplace exponent of H^{+*}. Then $\eta^* \sim \mathrm{Exp}(\Phi^{+*}(0))$ and $H^{+*}_{\eta^*-} = \overline{\xi^*}_{\zeta^*-}$. When $\Phi^{+*}(0) = 0$, i.e. $b = \sigma$, this renders $\zeta^* = \overline{\xi^*}_{\zeta^*-} = \infty$ a.s., hence, by the Lamperti transform, a.s. $T_0^* = \overline{\xi^*}_{\zeta^*-} = \infty$. Otherwise, when $\Phi^{+*}(0) > 0$, it then follows from the statements surrounding the Wiener–Hopf factorisation of Lévy processes that $\mathbf{E}[\mathrm{e}^{-\theta H^{+*}_{\zeta^*-}}] = \frac{\Phi^{+*}(0)}{\Phi^{+*}(\theta)}$ for $\theta \in [0, \infty)$. The argument for (ii) is similar. □

We can also push the notion of a ricocheted stable process into the realms of rssMp, but before doing so we present another interesting example of a pssMp due to Alex Watson [31], $Y^\natural = (Y_t^\natural, t \geq 0)$, associated to the symmetric stable process X, and whose corresponding Lévy process in the Lamperti transform also lies in the double hypergeometric class. Indeed, fix a $\mathfrak{q} \in [0, 1)$, take $\rho = 1/2$ and let τ_0^- be as before. The stochastic dynamics of $Y^\natural$ are as follows. From its point of issue in $(0, \infty)$, $Y^\natural$ evolves as X until its first passage into $(-\infty, 0)$. At that time an independent coin is flipped with probability $\mathfrak{q}$ of heads. If heads is thrown, then the process $Y^\natural$ is immediately transported to X_{σ_0}, where $\sigma_0 = \inf\{s \geq \tau_0 : X_s > 0\}$ (i.e., it is 'glued' to its next positive position, the negative part being thus 'censored away'). If tails is thrown, then it is sent to 0 and the process is killed. In the event $Y^\natural$ is glued, it continues to evolve as an independent copy of X, flipping a new coin on first pass into $(-\infty, 0)$, and so on. Had we allowed $\mathfrak{q} = 1$, then this case would correspond to the censored stable process of [23].

The characteristic exponent of $Y^\natural$, denoted by $\Psi^\natural$, may then be identified as

$$\Psi^\natural(\theta) = \frac{\Gamma(\frac{\alpha}{2} - \mathrm{i}\theta)\Gamma(\alpha - \mathrm{i}\theta)}{\Gamma(-\gamma + \frac{\alpha}{2} - \mathrm{i}\theta)\Gamma(\gamma + \frac{\alpha}{2} - \mathrm{i}\theta)} \times \frac{\Gamma(1 - \frac{\alpha}{2} + \mathrm{i}\theta)\Gamma(1 + \mathrm{i}\theta)}{\Gamma(1 - \gamma - \frac{\alpha}{2} + \mathrm{i}\theta)\Gamma(1 + \gamma - \frac{\alpha}{2} + \mathrm{i}\theta)}, \quad \theta \in \mathbb{R}, \tag{4.4}$$

where $\gamma := \frac{1}{\pi} \arcsin(\sqrt{\mathfrak{q}} \sin(\pi \frac{\alpha}{2}))$; see the comments below the proof of Proposition 4.2. By noting that $\gamma < \frac{\alpha}{2} \wedge (1 - \frac{\alpha}{2})$, we can check that the two factors either side of the sign $\times$ identify the Wiener–Hopf factors and that $Y^\natural$ belongs to the double hypergeometric class.

As indicated above, we can also introduce a real-valued version, X^*, of the ricocheted stable process. Indeed, we can define the process X^* similarly to Y^* but with some slight differences. On crossing the origin from $(0, \infty)$ to $(-\infty, 0)$, independently with probability $\mathfrak{p}$ it is ricocheted and with probability $1 - \mathfrak{p}$, rather than being killed, it continues to evolve with the dynamics of a stable process. Additionally, as the state space of X^* is $\mathbb{R}$, when passing from $(-\infty, 0)$ to $(0, \infty)$, the same path adjustment is made on the independent flip of a coin, albeit with a different probability $\hat{\mathfrak{p}}$. Note, in the special case that $\mathfrak{p} = \hat{\mathfrak{p}} = 0$, we have that X^* is equal in law to nothing more than the underlying stable process X from which its paths are derived. The reader should also bear in mind that we do not assume any a priori relation between $\mathfrak{p}$ and $\hat{\mathfrak{p}}$: we stress this because we use the notation $\hat{\rho}$ for $1 - \rho$, so the reader might be misled into thinking that $\hat{\mathfrak{p}} = 1 - \mathfrak{p}$, which is *not* being assumed.

It is worth spending a little bit of time to address the question as to why the processes X^*, $Y^\natural$, and Y^* are indeed self-similar Markov processes (ssMp). As alluded to earlier, the fact that Y^* is a ssMp comes from the reasoning of [4]. The explanation given there was via the use of the associated infinitesimal generators, which, in principle, needs a little care with the rigorous association of the generator to the appropriate semigroup of Y^*. We argue here that there is a relatively straightforward pathwise justification that comes directly out of the Lamperti transform for Y^*.

Proposition 4.2. *Y^*, $Y^\natural$, and X^* are all ssMp with self-similarity index α.*

Proof. Our strategy will be simply to identify the pathwise definition of Y^*, $Y^\natural$, and X^* directly with a Lamperti/Lamperti–Kiu decomposition. Once this has been done, the statement follows immediately from the bijection onto the class of pssMp/rssMp in the Lamperti/Lamperti–Kiu transform.

Let us start with the case of Y^*. From [5] it is known that $(X_t \mathbf{1}_{(t<\tau_0^-)}, t \geq 0)$ is a pssMp; moreover, the Lévy process that underlies its Lamperti transform, say $\xi^\dagger$, has an explicit form for its characteristic exponent (indeed it belongs to the hypergeometric Lévy processes). As a Lévy process it falls into the category which are killed at an independent and exponentially distributed time with a strictly positive rate, say $q^\dagger$. Suppose we write $\Psi^\dagger$ for the characteristic exponent of $\xi^\dagger$; consider the process ξ^0 whose characteristic exponent is $\Psi^\dagger - q^\dagger$. That is to say, ξ^0 is the Lévy process whose dynamics are those of $\xi^\dagger$, albeit with the exponential killing removed.

Next, define a compound Poisson process ξ° with arrival rate $q^\dagger \mathfrak{p}$ and jump distribution which is equal in distribution to the random variable $\log(|X_{\tau_0^-}|/X_{\tau_0^- -})$. Note, straightforward scaling arguments (of both the stopping time τ_0^- and X) can be used to show that the distribution of $\log(|X_{\tau_0^-}|/X_{\tau_0^- -})$ does not depend on the point of issue of X; see, for example, [20, Chapter 13]. The characteristic exponent of this compound Poisson process is given by $q^\dagger \mathfrak{p}(1 - \mathbb{E}_1[(|X_{\tau_0^-}|/X_{\tau_0^- -})^{\mathrm{i}\theta}])$.

Finally, we build the Lévy process ξ to have characteristic exponent given by

$$\Psi^*(\theta) = \Psi^\dagger(\theta) - q^\dagger + q^\dagger \mathfrak{p}(1 - \mathbb{E}_1[(|X_{\tau_0^-}|/X_{\tau_0^- -})^{\mathrm{i}\theta}]) + q^\dagger(1-\mathfrak{p}). \tag{4.5}$$

In other words, the independent sum of ξ^0, the Lévy process $\xi^\dagger$ without killing, and the compound Poisson process described in the previous paragraph with the resulting stochastic process killed at an independent and exponentially distributed time with rate $q^\dagger(1-\mathfrak{p})$.

Now consider the Lamperti transform of ξ. By splitting the resulting path over the events corresponding to the arrival of points in the compound Poisson processes and the killing time, it is straightforward to see that the stochastic evolution of (3.1) matches that of Y^* and hence they are equal in law. In other words, Y^* is a positive self-similar Markov process.

In the above description, instead of killing at rate $q^\dagger(1-\mathfrak{p})$, we could also see this as the rate at which switching of an independent Markov chain occurs from $+1$ to -1. Accordingly, to build up a MAP which corresponds to the process X^*, we need to define

$$G^*_{1,-1}(\theta) := \mathbb{E}_1[(|X_{\tau_0^-}|/X_{\tau_0^- -})^{\mathrm{i}\theta}] = \frac{\Gamma(\alpha - \mathrm{i}\theta)\Gamma(1+\mathrm{i}\theta)}{\Gamma(\alpha)}, \qquad \theta \in \mathbb{R},$$

where the second equality comes from [8, Corollary 11]. Note that, although we have seen the Fourier transform of $\log(|X_{\tau_0^-}|/X_{\tau_0^- -})$ before, this is also the change in the radial exponent ξ when X^* passes from $(0, \infty)$ to $(-\infty, 0)$.

By replacing the role of ρ by $\hat\rho$ in the definition of $\Psi^\dagger$, $q^\dagger$, $\mathfrak{p}$ (henceforth written $\hat\Psi^\dagger$, $\hat q^\dagger$, $\hat{\mathfrak{p}}$), we can similarly describe the part of the MAP that corresponds to X^* until its first passage from $(-\infty, 0)$ to $(0, \infty)$ when issued from a negative value.

Thanks to the piecewise evolution of X^*, considered at each crossing of the origin, we can thus conclude that it is a rssMp with underlying MAP whose matrix exponent is given by

$$\begin{aligned}
&\boldsymbol{\Psi}^*(\theta) \\
&= -\begin{bmatrix} \Psi^\dagger(\theta) - q^\dagger + q^\dagger\mathfrak{p}\big(1 - \mathbb{E}_1\big[\big(|X_{\tau_0^-}|/X_{\tau_0^- -}\big)^{\mathrm{i}\theta}\big]\big) & 0 \\ 0 & \hat\Psi^\dagger(\theta) - \hat q^\dagger + \hat q^\dagger\hat{\mathfrak{p}}\big(1 - \hat{\mathbb{E}}_1\big[\big(|X_{\tau_0^-}|/X_{\tau_0^- -}\big)^{\mathrm{i}\theta}\big]\big) \end{bmatrix} \\
&\quad + \begin{bmatrix} -q^\dagger(1-\mathfrak{p}) & q^\dagger(1-\mathfrak{p})\mathbb{E}_1\big[\big(|X_{\tau_0^-}|/X_{\tau_0^- -}\big)^{\mathrm{i}\theta}\big] \\ \hat q^\dagger(1-\hat{\mathfrak{p}})\hat{\mathbb{E}}_1\big[\big(|X_{\tau_0^-}|/X_{\tau_0^- -}\big)^{\mathrm{i}\theta}\big] & -\hat q^\dagger(1-\hat{\mathfrak{p}}) \end{bmatrix} \\
&= \begin{bmatrix} -\Psi^\dagger(\theta) + q^\dagger\mathfrak{p}\mathbb{E}_1\big[\big(|X_{\tau_0^-}|/X_{\tau_0^- -}\big)^{\mathrm{i}\theta}\big] & q^\dagger(1-\mathfrak{p})\mathbb{E}_1\big[\big(|X_{\tau_0^-}|/X_{\tau_0^- -}\big)^{\mathrm{i}\theta}\big] \\ \hat q^\dagger(1-\hat{\mathfrak{p}})\hat{\mathbb{E}}_1\big[\big(|X_{\tau_0^-}|/X_{\tau_0^- -}\big)^{\mathrm{i}\theta}\big] & -\hat\Psi^\dagger(\theta) + \hat q^\dagger\hat{\mathfrak{p}}\hat{\mathbb{E}}_1\big[\big(|X_{\tau_0^-}|/X_{\tau_0^- -}\big)^{\mathrm{i}\theta}\big] \end{bmatrix}, \qquad \theta \in \mathbb{R},
\end{aligned}$$

where $\hat{\mathbb{P}}$ is the law of $-X$.

Finally, we consider the case of $Y^\natural$ and recall that $\rho = \frac{1}{2}$. The analysis is the same as in the case of Y^* except that the compound Poisson process ξ° there gets replaced with a different compound Poisson process $\xi^\bullet$. From the path analysis of $Y^\natural$ the jumps of $\xi^\bullet$

have the distribution of $\log(X_{\sigma_0}/X_{\tau_0^- -})$, while its rate is $q^\dagger \mathfrak{q}$. The characteristic function of $\log(X_{\sigma_0}/X_{\tau_0^- -})$ follows from the analysis in [23, Proposition 4.2]:

$$\mathbb{E}_1\left[\left(\frac{X_{\sigma_0}}{X_{\tau_0^- -}}\right)^{\mathrm{i}\theta}\right] = \frac{\Gamma(1-\frac{\alpha}{2}+\mathrm{i}\theta)\Gamma(\frac{\alpha}{2}-\mathrm{i}\theta)\Gamma(1+\mathrm{i}\theta)\Gamma(\alpha-\mathrm{i}\theta)}{\Gamma(\frac{\alpha}{2})\Gamma(1-\frac{\alpha}{2})\Gamma(\alpha)}, \qquad \theta\in\mathbb{R}.$$

This completes the proof. □

From the above proof we can recover the result of [4]. Indeed, from [5, Corollary 1],

$$\Psi^\dagger(\theta) = \frac{\Gamma(\alpha-\mathrm{i}\theta)\Gamma(1+\mathrm{i}\theta)}{\Gamma(\alpha\hat\rho-\mathrm{i}\theta)\Gamma(1-\alpha\hat\rho+\mathrm{i}\theta)}, \qquad \theta\in\mathbb{R},$$

and in particular $q^\dagger = \Psi^\dagger(0) = \Gamma(\alpha)/(\Gamma(\alpha\hat\rho)\Gamma(1-\alpha\hat\rho))$. It now follows from (4.5) that

$$\begin{aligned}\Psi^*(\theta) &= \Psi^\dagger(\theta) - q^\dagger \mathfrak{p}\mathbb{E}_1[(|X_{\tau_0^-}|/X_{\tau_0^- -})^{\mathrm{i}\theta}] \\ &= \frac{\Gamma(\alpha-\mathrm{i}\theta)\Gamma(1+\mathrm{i}\theta)}{\Gamma(\alpha\hat\rho-\mathrm{i}\theta)\Gamma(1-\alpha\hat\rho+\mathrm{i}\theta)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat\rho)\Gamma(1-\alpha\hat\rho)}\mathfrak{p}\frac{\Gamma(\alpha-\mathrm{i}\theta)\Gamma(1+\mathrm{i}\theta)}{\Gamma(\alpha)} \\ &= \frac{\Gamma(\alpha-\mathrm{i}\theta)\Gamma(1+\mathrm{i}\theta)}{\pi}\left[\sin(\pi(\alpha\hat\rho-\mathrm{i}\theta)) - \mathfrak{p}\sin(\pi\alpha\hat\rho)\right], \qquad \theta\in\mathbb{R},\end{aligned}$$

which agrees with (4.2). Likewise, we can deduce the explicit form of $\Psi^\natural$ reported in (4.4).

In a similar spirit, we can proceed to compute the matrix exponent $\boldsymbol{\Psi}^*$ to obtain

$$\boldsymbol{\Psi}^*(\theta) = \frac{\Gamma(\alpha-\mathrm{i}\theta)\Gamma(1+\mathrm{i}\theta)}{\pi}\begin{bmatrix}\mathfrak{p}\sin(\pi\alpha\hat\rho)-\sin(\pi(\alpha\hat\rho-\mathrm{i}\theta)) & (1-\mathfrak{p})\sin(\pi\alpha\hat\rho) \\ (1-\hat{\mathfrak{p}})\sin(\pi\alpha\rho) & \hat{\mathfrak{p}}\sin(\pi\alpha\rho)-\sin(\pi(\alpha\rho-\mathrm{i}\theta))\end{bmatrix}$$

for $\theta\in\mathbb{R}$. For $\mathfrak{p}=\hat{\mathfrak{p}}=0$ this reduces to the matrix exponent of a stable process killed on hitting 0, as discussed in [21]. We conclude this discussion with the following corollary which identifies $\boldsymbol{\Psi}$ in a form that is similar to the double hypergeometric analogue.

Corollary 4.2. *For* $\theta\in\mathbb{R}$,

$$\boldsymbol{\Psi}^*(\theta) = -2^\alpha\begin{bmatrix}\frac{\Gamma\left(\frac{1+\mathrm{i}\theta}{2}\right)\Gamma\left(\frac{2+\mathrm{i}\theta}{2}\right)}{\Gamma\left(\frac{\sigma+b+\mathrm{i}\theta}{2}\right)\Gamma\left(\frac{\sigma-b+\mathrm{i}\theta+2}{2}\right)}\frac{\Gamma\left(\frac{\alpha-\mathrm{i}\theta}{2}\right)\Gamma\left(\frac{1+\alpha-\mathrm{i}\theta}{2}\right)}{\Gamma\left(\frac{b-\sigma-\mathrm{i}\theta}{2}\right)\Gamma\left(\frac{2-\sigma-b-\mathrm{i}\theta}{2}\right)} & -\frac{\Gamma\left(\frac{1+\mathrm{i}\theta}{2}\right)\Gamma\left(\frac{2+\mathrm{i}\theta}{2}\right)}{\Gamma\left(\frac{\sigma+b}{2}\right)\Gamma\left(\frac{\sigma-b+2}{2}\right)}\frac{\Gamma\left(\frac{\alpha-\mathrm{i}\theta}{2}\right)\Gamma\left(\frac{1+\alpha-\mathrm{i}\theta}{2}\right)}{\Gamma\left(\frac{b-\sigma}{2}\right)\Gamma\left(\frac{2-\sigma-b}{2}\right)} \\ -\frac{\Gamma\left(\frac{1+\mathrm{i}\theta}{2}\right)\Gamma\left(\frac{2+\mathrm{i}\theta}{2}\right)}{\Gamma\left(\frac{\hat\sigma+\hat b}{2}\right)\Gamma\left(\frac{\hat\sigma-\hat b+2}{2}\right)}\frac{\Gamma\left(\frac{\alpha-\mathrm{i}\theta}{2}\right)\Gamma\left(\frac{1+\alpha-\mathrm{i}\theta}{2}\right)}{\Gamma\left(\frac{\hat b-\hat\sigma}{2}\right)\Gamma\left(\frac{2-\hat\sigma-\hat b}{2}\right)} & \frac{\Gamma\left(\frac{1+\mathrm{i}\theta}{2}\right)\Gamma\left(\frac{2+\mathrm{i}\theta}{2}\right)}{\Gamma\left(\frac{\hat\sigma+\hat b+\mathrm{i}\theta}{2}\right)\Gamma\left(\frac{\hat\sigma-\hat b+\mathrm{i}\theta+2}{2}\right)}\frac{\Gamma\left(\frac{\alpha-\mathrm{i}\theta}{2}\right)\Gamma\left(\frac{1+\alpha-\mathrm{i}\theta}{2}\right)}{\Gamma\left(\frac{\hat b-\hat\sigma-\mathrm{i}\theta}{2}\right)\Gamma\left(\frac{2-\hat\sigma-\hat b-\mathrm{i}\theta}{2}\right)}\end{bmatrix},$$

where, as before, $\sigma = -\alpha\hat\rho + 1/2$ *and* $b := \arccos(\mathfrak{p}\cos(\pi\sigma))/\pi$, *and* $\hat\sigma$ *and* $\hat b$ *have the obvious meaning.*

We note that the condition $\{\mathfrak{p}, \hat{\mathfrak{p}}\}\subset[0, 1)$ ensures that J is irreducible. If $\mathfrak{p}=1$, then, albeit possibly only after the first crossing of the origin (in the case that X^* is issued from a negative value), we are basically in the case of the pssMp Y^*. An analogous statement can be made for $\hat{\mathfrak{p}}=1$.

For real θ in some neighbourhood of 0, the explicit expression for $\boldsymbol{\Psi}^*(-\mathrm{i}\theta)$ can be used to determine its Perron–Frobenius eigenvalue $\chi(\theta)$ [1, Paragraph XI.2C]. From the aforesaid reference, it is known that $\chi(0)=0$ and that, when it exists, $\chi'(0)$ dictates the long-term behaviour of (ξ, J), the underlying MAP of X^*, in the sense that a.s. $\lim_{t\to\infty}\xi_t/t=\chi'(0)$ [1, Corollary XI.2.8]. Moreover, ξ drifts to ∞, oscillates, or drifts to $-\infty$, according to whether $\chi'(0)$ is valued >0, $=0$, or <0 [1, Proposition XI.2.10]. By the Lamperti–Kiu transform, this corresponds to $|X^*|$ drifting to ∞, oscillating between 0 and ∞, and hitting 0 continuously, respectively.

An explicit computation yields

$$\chi'(0)=\Gamma(\alpha)\frac{(1-\hat{\mathfrak{p}})\sin(\pi\alpha\rho)\cos(\pi\alpha\hat{\rho})+(1-\mathfrak{p})\sin(\pi\alpha\hat{\rho})\cos(\pi\alpha\rho)}{(1-\hat{\mathfrak{p}})\sin(\pi\alpha\rho)+(1-\mathfrak{p})\sin(\pi\alpha\hat{\rho})}.$$

From this, we note that when $\mathfrak{p}=\hat{\mathfrak{p}}$ (in particular when both are equal to 0 and $X^*=(X_t\mathbf{1}_{(t<\tau_0^-)}, t\geq 0))$, we obtain $\chi'(0)(\sin(\pi\alpha\rho)+\sin(\pi\alpha\hat{\rho}))=\Gamma(\alpha)\sin(\pi\alpha)$ (which is consistent with the polarity of 0 for X when $\alpha\in(0, 1]$ and the fact that X hits 0 when $\alpha\in(1, 2)$). Furthermore, whether or not $\mathfrak{p}=\hat{\mathfrak{p}}$, $\chi'(0)>0$ whenever $\{\alpha\rho, \alpha\hat{\rho}\}\subset(0, \frac{1}{2})$, in particular whenever $\alpha<\frac{1}{2}$, but in general the sign of $\chi'(0)$ will depend on α, ρ, $\mathfrak{p}$, $\hat{\mathfrak{p}}$ in a non-trivial 'balancing' way. For instance, if $\hat{\rho}\alpha\in(\frac{1}{2}, 1)$, $\hat{\mathfrak{p}}$ is sufficiently close to 0, while $\mathfrak{p}$ is sufficiently close to 1, then X^* will hit 0 continuously, even though one may also have $\alpha<1$, in which case $(X_t\mathbf{1}_{(t<\tau_0^-)}, t\geq 0)$ will not be absorbed continuously at the origin. More generally, we have the following proposition.

Proposition 4.3. *X^* hits zero continuously if and only if*

$$(1-\hat{\mathfrak{p}})\sin(\pi\alpha\rho)\cos(\pi\alpha\hat{\rho})+(1-\mathfrak{p})\sin(\pi\alpha\hat{\rho})\cos(\pi\alpha\rho)<0,$$

equivalently $\sin(\pi\alpha)<\hat{\mathfrak{p}}\sin(\pi\alpha\rho)\cos(\pi\alpha\hat{\rho})+\mathfrak{p}\sin(\pi\alpha\hat{\rho})\cos(\pi\alpha\rho)$*; for this condition to prevail it is necessary (but not sufficient) that* $\alpha\hat{\rho}>\frac{1}{2}$ *or* $\alpha\rho>\frac{1}{2}$.

Proof. The denominator of $\chi'(0)$ is always strictly positive because $0<\alpha\rho<1$ and $0<\alpha\hat{\rho}<1$ (see, for instance, [21, p. 3] for the identification of the admissible pairs (α, ρ) under our standing assumptions). Therefore, the condition for X^* to hit 0 continuously, i.e., $\chi'(0)<0$, becomes the stated one, with its equivalent form following upon using the addition formula for the sine. For the final claim we note that $\cos(\pi\alpha\hat{\rho})$ and $\cos(\pi\alpha\rho)$ are both non-negative unless $\alpha\hat{\rho}>\frac{1}{2}$ or $\alpha\rho>\frac{1}{2}$. □

5. Proofs of main results

5.1 Proof of Theorem 2.1

Proof. Let $\alpha<\beta$ and $\gamma<\delta$ be strictly positive real numbers, and assume that $\{\alpha-\beta, \gamma-\delta, \gamma-\alpha, \delta-\alpha, \gamma-\beta, \delta-\beta\}\cap\mathbb{Z}=\emptyset$, i.e., assume $\gamma+\mathbb{Z}_{\leq 0}$, $\delta+\mathbb{Z}_{\leq 0}$, $\alpha+\mathbb{Z}_{\leq 0}$, and $\beta+\mathbb{Z}_{\leq 0}=\emptyset$ are pairwise disjoint. Then we claim that $\{\gamma, \delta\}+\mathbb{Z}_{\leq 0}$ interlaces with $\{\alpha, \beta\}+\mathbb{Z}_{\leq 0}$ if and only if either

(I) there is a $k\in\mathbb{N}_0$ with $\beta-k-1<\gamma<\alpha<\delta-k<\beta-k$, or

(II) there is a $k\in\mathbb{N}$ with $\delta-k<\alpha<\gamma<\beta-k<\delta-k+1$.

The conditions are clearly sufficient. For necessity, assume the interlacing property. Clearly we must have $\alpha < \delta < \beta$ and then either (i) $\gamma < \alpha < \delta < \beta$ or else (ii) $\alpha < \gamma < \delta < \beta$. Assume (i) first. Then $\beta - 1 < \delta$. If even $\beta - 1 < \alpha$, then necessarily $\beta - 1 < \gamma$, and we are done. Otherwise, $\gamma < \alpha < \beta - 1 < \delta < \beta$. But then we must have $\gamma < \alpha < \delta - 1 < \beta - 1 < \delta < \beta$. An inductive argument gives the required result. Now assume (ii). We must have $\alpha < \gamma < \beta - 1 < \delta < \beta$. If $\delta - 1 < \alpha$ then we are done. Otherwise, $\alpha < \delta - 1$ and necessarily $\alpha < \gamma < \beta - 2 < \delta - 1 < \beta - 1 < \delta < \beta$. An inductive argument again gives the result. Clearly (I) and (II) are exclusive.

Assume further that the interlacing property holds. The above characterisation implies that $-\alpha - \beta < -\gamma - \delta < -\alpha - \beta + 1$, i.e., $\gamma + \delta < \alpha + \beta < \gamma + \delta + 1$.

Next, via Euler's infinite product formula for the gamma function, we may write, for $z \in \mathbb{C}$ where the expression is defined,

$$\Phi(z) := \frac{\Gamma(z+\alpha)\Gamma(z+\beta)}{\Gamma(z+\gamma)\Gamma(z+\delta)} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)\Gamma(\delta)} \prod_{n=0}^{\infty} \frac{\left(1 - \frac{z}{-n-\gamma}\right)\left(1 - \frac{z}{-n-\delta}\right)}{\left(1 - \frac{z}{-n-\alpha}\right)\left(1 - \frac{z}{-n-\beta}\right)}.$$

Then Φ is a real meromorphic function, and it follows from [24, Theorem 27.2.1] and from the interlacing condition together with $\{\alpha, \beta, \gamma, \delta\} \subset (0, \infty)$ that it maps the open upper complex half-plane into itself, i.e., it is a Pick function, and hence a non-constant completely monotone Bernstein function. It follows furthermore, see [24, Problem 27.2.1], that Φ admits the representation, for some $\{\omega_1, \omega_2\} \subset \mathbb{Z} \cup \{-\infty, \infty\}$, $\omega_1 \leq \omega_2$, and at least for z in the open upper complex half-plane,

$$\Phi(z) = az + b + \frac{d}{z} + \sum\nolimits_{n=\omega_1}^{\omega_2} c_n \left[\frac{1}{a_n - z} - \frac{1}{a_n}\right], \tag{5.1}$$

where $\{a, -d\} \subset [0, \infty)$, $b \in \mathbb{R}$, all the c_n are non-negative, the sequence a_n (where $n \in \{\omega_1, \omega_1 + 1, \ldots, \omega_2\}$) consists of non-zero real numbers and is strictly increasing, and finally $\sum_{n=\omega_1}^{\omega_2} c_n/a_n^2 < \infty$. Then the a_n and 0 must exhaust the poles of Φ and the c_n must be the respective negatives of the residua of Φ at these poles, which can easily be computed: for $k \in \mathbb{N}_0$,

$$\operatorname{Res}(\Phi, -\alpha - k) = (-1)^k \frac{\Gamma(\beta - \alpha - k)}{k!\Gamma(\gamma - \alpha - k)\Gamma(\delta - \alpha - k)},$$

$$\operatorname{Res}(\Phi, -\beta - k) = (-1)^k \frac{\Gamma(\alpha - \beta - k)}{k!\Gamma(\gamma - \beta - k)\Gamma(\delta - \beta - k)}.$$

The coefficient d equals 0 because Φ has no pole at 0 (as $\{\alpha, \beta\} \subset (0, \infty)$). By continuity of Φ and dominated convergence for the right-hand side, the equality (5.1) extends to $[0, \infty)$. Then, by dominated convergence and by Stirling's formula for the gamma function, using $\alpha + \beta < \gamma + \delta + 1$, we see that $a = \lim_{z\to\infty} \Phi(z)/z = 0$ (limit over $(0, \infty)$).

On the other hand, from the definition of the hypergeometric function ${}_2F_1$, for $s \in (0, \infty)$,

$$f(s) = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{\Gamma(\beta - \alpha - k)}{k!\Gamma(\gamma - \alpha - k)\Gamma(\delta - \alpha - k)} \mathrm{e}^{-(\alpha+k)s}$$
$$+ \sum_{k=0}^{\infty} (-1)^{k+1} \frac{\Gamma(\alpha - \beta - k)}{k!\Gamma(\gamma - \beta - k)\Gamma(\delta - \beta - k)} \mathrm{e}^{-(\beta+k)s}.$$

A direct computation using Tonelli–Fubini (note that we know a priori from the preceding that $\sum_{n=\omega_1}^{\omega_2} c_n/a_n^2 < \infty$) then reveals that, for $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geq 0$,

$$\begin{aligned} H(z) &:= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)\Gamma(\delta)} + \int_0^\infty (1 - \mathrm{e}^{-zs}) f(s)\,\mathrm{d}s \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)\Gamma(\delta)} + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{\Gamma(\beta-\alpha-k)}{k!\Gamma(\gamma-\alpha-k)\Gamma(\delta-\alpha-k)} \left(\frac{1}{\alpha+k} - \frac{1}{\alpha+k+z}\right) \\ &\quad + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{\Gamma(\alpha-\beta-k)}{k!\Gamma(\gamma-\beta-k)\Gamma(\delta-\beta-k)} \left(\frac{1}{\beta+k} - \frac{1}{\beta+k+z}\right). \end{aligned} \tag{5.2}$$

Comparing (5.1) and (5.2), we see that certainly $\Phi = H$ on $[0, \infty)$ and hence, by analyticity and continuity, on the whole of the closed right complex half-plane, which identifies the subordinator corresponding to Φ as being pure-jump with Lévy density f. Incidentally, because all of the c_n must be non-negative, we also obtain that the two series appearing in the above representation of f are each of positive terms only.

The assumption $\{\alpha, \beta, \gamma, \delta\} \subset (0, \infty)$ ensures that $\Phi(0) > 0$, i.e., the subordinator corresponding to Φ is killed. By monotone convergence Φ has infinite or finite activity according as $\int_0^\infty f(s)\,\mathrm{d}s = \lim_{z\to\infty} \Phi(z) = \infty$ or $< \infty$, and Stirling's formula for the gamma function together with $\gamma + \delta < \alpha + \beta$ guarantees that it is the former that is taking place here.

Further, as a complete Bernstein function, the potential measure associated to Φ has a density with respect to Lebesgue measure; cf. [27, Remark 11.6] and [28, Lemma 2.3]. In order to check that (2.2) gives the potential density, write it out using the definition of ${}_2F_1$: for $x \in (0, \infty)$,

$$\begin{aligned} u(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\delta-\gamma-k)}{k!\Gamma(\alpha-\gamma-k)\Gamma(\beta-\gamma-k)} \mathrm{e}^{-(\gamma+k)x} \\ &\quad + \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\gamma-\delta-k)}{k!\Gamma(\alpha-\delta-k)\Gamma(\beta-\delta-k)} \mathrm{e}^{-(\delta+k)x}. \end{aligned}$$

Note that the two series consist of positive terms owing to the characterisations (I) and (II) and the properties of the sign of the gamma function on the real line. By Tonelli's theorem we then compute the Laplace transform $\hat{u}$ of (2.2),

$$\begin{aligned} \hat{u}(z) &= \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\delta-\gamma-k)}{k!\Gamma(\alpha-\gamma-k)\Gamma(\beta-\gamma-k)} \frac{1}{\gamma+k+z} \\ &\quad + \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\gamma-\delta-k)}{k!\Gamma(\alpha-\delta-k)\Gamma(\beta-\delta-k)} \frac{1}{\delta+k+z} \end{aligned}$$

($z \in \mathbb{C}$, $\mathrm{Re}(z) \geq 0$), and check that it coincides with $1/\Phi$.

To this end, let $z \in \mathbb{C}$, $\mathrm{Re}(z) \geq 0$; we verify that $\hat{u}(z) = \frac{1}{\Phi(z)}$. But Φ being a complete Bernstein function implies (see [27, Proposition 7.1]) that $\frac{\mathrm{id}_{(0,\infty)}}{\Phi}$ is itself a complete Bernstein function. Then [27, Remark 6.4] guarantees that we can write

$$\frac{1}{\Phi(z)} = \frac{\Gamma(\gamma+z)\Gamma(\delta+z)}{\Gamma(\alpha+z)\Gamma(\beta+z)} = \frac{a}{z} + b + \int_{(0,\infty)} \frac{1}{z+t} \eta(\mathrm{d}t), \qquad z \in (0, \infty),$$

where $\{a, b\} \subset [0, \infty)$ and η is a measure on $\mathcal{B}_{(0,\infty)}$ such that $\int (1+t)^{-1}\eta(\mathrm{d}t) < \infty$. Clearly we must have $b = 0$ because $\lim_{z\to\infty} 1/\Phi(z) = 0$. The measure η is supported by $\{\delta, \gamma\} + \mathbb{N}_0$, which follows by appealing to, e.g., [27, Corollary 6.3]. Moreover, $a = 0$ because $\lim_{z\downarrow 0} 1/\Phi(z) < \infty$. By analytic extension, the masses of η are identified by identifying the residues of $\Phi(z)$, which concludes the argument.

When, *ceteris paribus*, the interlacing condition fails, it follows from its representation as an infinite product given above, and from the characterisation of Pick maps of [24, Theorem 27.2.1], that the function $\Phi(z)$ is not Pick, hence also not a non-constant complete Bernstein function.

The final assertion of the theorem follows by recalling that complete Bernstein functions are closed under pointwise limits [27, Corollary 7.6(ii)]. □

Let us make some remarks concerning the above proof. First, to establish the complete Bernstein property of (1.3) under the 'interlacing conditions' we could have relied directly on the results of [16] (see Lemma 1 and the discussion following its proof therein). However, we wanted to establish the precise, not just sufficient, conditions under which the 'non-constant complete Bernstein' property obtains, which is why we have provided above a direct and detailed study of the Pick property of (1.3). Besides, this has the advantage of keeping our paper more self-contained.

Secondly, the expression (2.1) for the Lévy density of course does not just 'fall from the sky'; rather, it is obtained by twice differentiating the inverse Laplace transform of

$$z \mapsto \frac{1}{z^2}\frac{\Gamma(\alpha+z)\Gamma(\beta+z)}{\Gamma(\gamma+z)\Gamma(\delta+z)}$$

(cf. [27, Eq. (3.4)]), and the latter in turn is obtained (at least on a pro forma level) by the Heaviside residue expansion [13, Theorem 10.7(c)]. A similar remark pertains to (2.2): we know [27, Eq. (5.20)] that the Laplace transform of U is $1/\Phi$, so that at least on a pro forma level we can Laplace invert via the Heaviside expansion.

Finally, we recover from the proof of Theorem 2.1 the non-obvious identities

$$\begin{aligned}\frac{\Gamma(\alpha+z)\Gamma(\beta+z)}{\Gamma(\gamma+z)\Gamma(\delta+z)} = {} & \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)\Gamma(\delta)} \\ & + \sum_{k=0}^{\infty}(-1)^{k+1}\frac{\Gamma(\beta-\alpha-k)}{k!\Gamma(\gamma-\alpha-k)\Gamma(\delta-\alpha-k)}\left(\frac{1}{\alpha+k}-\frac{1}{\alpha+k+z}\right) \\ & + \sum_{k=0}^{\infty}(-1)^{k+1}\frac{\Gamma(\alpha-\beta-k)}{k!\Gamma(\gamma-\beta-k)\Gamma(\delta-\beta-k)}\left(\frac{1}{\beta+k}-\frac{1}{\beta+k+z}\right), \\ & z \in \mathbb{C}\backslash(\{-\alpha, -\beta\} + \mathbb{Z}_{\le 0}), \\ & \sum_{k=0}^{\infty}(-1)^{k}\frac{\Gamma(\delta-\gamma-k)}{k!\Gamma(\alpha-\gamma-k)\Gamma(\beta-\gamma-k)}\frac{1}{\gamma+k+z} \\ & + \sum_{k=0}^{\infty}(-1)^{k}\frac{\Gamma(\gamma-\delta-k)}{k!\Gamma(\alpha-\delta-k)\Gamma(\beta-\delta-k)}\frac{1}{\delta+k+z} \\ = {} & \frac{\Gamma(\gamma+z)\Gamma(\delta+z)}{\Gamma(\alpha+z)\Gamma(\beta+z)}, \qquad z \in \mathbb{C}\backslash(\{-\delta, -\gamma\} + \mathbb{Z}_{\le 0}).\end{aligned}$$

5.2. Proof of Proposition 2.1

Proof.

(i). The meromorphic property can be seen from [16, Theorem 1(v)], simply using the representation of the real meromorphic function $\psi := (z \mapsto -\Psi(-\mathrm{i}z))$ as an infinite product that was essentially procured already in the proof of Theorem 2.1 (via Euler's infinite product formula for the gamma function). Then, according to [16, p. 1105], this means a priori that the Lévy measure Π of ξ admits a density π of the form

$$\frac{\pi(\mathrm{d}x)}{\mathrm{d}x} = \mathbf{1}_{(0,\infty)}(x) \sum_{n\in\mathbb{N}} a_n \rho_n \mathrm{e}^{-\rho_n x} + \mathbf{1}_{(-\infty,0)}(x) \sum_{n\in\mathbb{N}} \hat{a}_n \hat{\rho}_n \mathrm{e}^{\hat{\rho}_n x}, \qquad x \in \mathbb{R},$$

where all the coefficients are non-negative and the sequences $(\rho_n)_{n\in\mathbb{N}}$ and $(\hat{\rho}_n)_{n\in\mathbb{N}}$ are strictly positive and strictly increasing to ∞. Furthermore, from [16, Eq. (8)], we have that, for some $\sigma^2 \in [0, \infty)$ and $\mu \in \mathbb{R}$,

$$-\frac{\Gamma(\alpha - z)\Gamma(\beta - z)}{\Gamma(\gamma - z)\Gamma(\delta - z)} \frac{\Gamma(\hat{\alpha} + z)\Gamma(\hat{\beta} + z)}{\Gamma(\hat{\gamma} + z)\Gamma(\hat{\delta} + z)} = \psi(z)$$
$$= \Psi(0) + \frac{1}{2}\sigma^2 z^2 + \mu z + z^2 \sum_{n\in\mathbb{N}} \left[\frac{a_n}{\rho_n(\rho_n - z)} + \frac{\hat{a}_n}{\hat{\rho}_n(\hat{\rho}_n + z)} \right].$$

Dividing in the previous display by z^2, and considering the residua, we find that $-\frac{a_n}{\rho_n} = \mathrm{Res}(\psi(z)/z^2; z = \rho_n)$, i.e.

$$a_n \rho_n = \mathrm{Res}(-\psi; \rho_n) = -\mathrm{B}(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}; \rho_n)\mathrm{Res}(\mathrm{B}(\alpha, \beta, \gamma, \delta; \cdot), -\rho_n);$$

similarly that $\frac{\hat{a}_n}{\hat{\rho}_n} = \mathrm{Res}(\psi(z)/z^2; z = -\hat{\rho}_n)$, i.e.

$$\hat{a}_n \hat{\rho}_n = -\mathrm{B}(\alpha, \beta, \gamma, \delta; \hat{\rho}_n)\mathrm{Res}(\mathrm{B}(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}; \cdot), -\hat{\rho}_n);$$

and that the ρ_n, $n \in \mathbb{N}$, must run over $\{\alpha, \beta\} + \mathbb{N}_0$, while the $\hat{\rho}_n$, $n \in \mathbb{N}$, must run over $\{\hat{\alpha}, \hat{\beta}\} + \mathbb{N}_0$. Computing the residua and simplifying yields the expression for the Lévy density.

(ii) By the asymptotic properties of the gamma function [11, Formula 8.328.1] we see that, for $\theta \in \mathbb{R}$, $|\Psi(\theta)| \sim |\theta|^{\alpha+\beta+\hat{\alpha}+\hat{\beta}-\gamma-\delta-\hat{\gamma}-\hat{\delta}}$ as $|\theta| \to \infty$. Now the claim follows using [2, Proposition I.2(i)], since we know a priori that $\alpha + \beta \le \gamma + \delta + 1$ and $\hat{\alpha} + \hat{\beta} \le \hat{\gamma} + \hat{\delta} + 1$.

(iii). This follows from inspecting whether or not the Laplace exponents of the increasing and decreasing ladder heights subordinators of ξ (which are known from Corollary 2.1) vanish at 0. □

5.3. Proof of Proposition 2.2

As alluded to above, we will use the verification argument of [19, Proposition 2]. The idea there is to verify that the right-hand side of (2.3) has certain analytical properties so that it matches the unique solution of the functional equation that the left-hand side of (2.3) must necessarily solve (cf. [25]).

Proof. The Laplace exponent $\psi_c(z) := \log \mathbf{E}[\exp(z\xi_1/c)]$ is given by the map $z \mapsto \psi(z/c)$, where ψ is the Laplace exponent of ξ. The conditions on the parameters ensure that ξ/c either drifts to ∞ or is killed (see Proposition 2.1(iii)), and that it satisfies the so-called Cramér condition, $\psi_c(-\hat{\gamma}c) = \psi(-\hat{\gamma}) = 0$ and ψ_c is finite on $(-\hat{\alpha}c, 0)$, which contains $-\hat{\gamma}c$. Furthermore, letting $\mathcal{M}(s)$ be the right-hand side of (2.3), we see immediately from the properties of the double gamma function that $\mathcal{M}$ is analytic and free of zeros in the strip $\mathrm{Re}(s) \in (0, 1+\hat{\gamma}c)$, and $\mathcal{M}(1) = 1$ and $\mathcal{M}(s+1) = -s\mathcal{M}(s)/\psi_c(-s)$ for $s \in (0, \hat{\gamma}c)$.

It remains to check that $\lim_{|y|\to\infty} |\mathcal{M}(x+\mathrm{i}y)|^{-1} \mathrm{e}^{-2\pi|y|} = 0$ uniformly in $x \in (0, \hat{\gamma}c+1)$. Lemma 1 in [19] implies that, as $|y| \to \infty$,

$$\ln \left| \frac{G(p+\mathrm{i}y;c)}{G(q+\mathrm{i}y;c)} \right| = \frac{p-q}{c} \mathrm{Re}[(q+\mathrm{i}y)\ln(q+\mathrm{i}y)] + o(|y|) = -\pi \frac{p-q}{2c}|y| + o(|y|)$$

uniformly in bounded real p and q. Note that this was observed in [4, p. 44], albeit there without the uniformity, which is needed for our purposes. At the same time, by Stirling's asymptotic formula for the gamma function [11, Formula 8.327], as $|y| \to \infty$, we have $\ln|\Gamma(x+\mathrm{i}y)| = \mathrm{Re}[(x+\mathrm{i}y)(\ln(x+\mathrm{i}y)-1)] + o(|y|) = -\frac{\pi}{2}|y| + o(|y|)$ uniformly for x bounded in $\mathbb{R}$.

Therefore, it follows that, uniformly for $x \in (0, \hat{\gamma}c+1)$, as $|y| \to \infty$, we have

$$\ln|\mathcal{M}(x+\mathrm{i}y)| = -\frac{\pi}{2}\left(1 + \frac{1}{2}(\gamma+\delta-\alpha-\beta+\hat{\alpha}+\hat{\beta}-\hat{\gamma}-\hat{\delta})\right)|y| + o(|y|).$$

Because $\gamma+\delta \le \alpha+\beta$ and $\hat{\alpha}+\hat{\beta} \le \hat{\gamma}+\hat{\delta}+1$ (see Theorem 2.1), which gives us that $1 + (\gamma+\delta-\alpha-\beta+\hat{\alpha}+\hat{\beta}-\hat{\gamma}-\hat{\delta})/2 < 4$, the proof is now complete. □

Acknowledgements

MV acknowledges financial support from the Slovenian Research Agency (research programmes nos. P1-0222 & P1-0402). JCP acknowledges support from CONACyT-MEXICO. AEK acknowledges financial support from the EPSRC (EP/P009220/1 & EP/L002442/1). JCP and MV are grateful for the kind hospitality of the University of Bath, where both were on a sabbatical while this research was conducted, with JCP holding a David Parkin Visiting Professorship at the same university. We thank Alexey Kuznetsov for providing some very useful insight into Pick functions, and an anonymous referee for his/her careful review of our paper.

References

[1] Asmussen, S. (2003). *Applied Probability and Queues*. Applications of Mathematics: Stochastic Modelling and Applied Probability. Springer, New York.
[2] Bertoin, J. (1996). *Lévy Processes*. *Cambridge Tracts in Mathematics*. Cambridge University Press.
[3] Bertoin, J. and Yor, M. (2001). On subordinators, self-similar Markov processes and some factorizations of the exponential variable. *Electron. Commun. Prob.* **6**, 95–106.
[4] Budd, T. (2018). The peeling process on random planar maps coupled to an $O(n)$ loop model. arXiv:1809.02012v1.
[5] Caballero, M. E. and Chaumont, L. (2006). Conditioned stable Lévy processes and the Lamperti representation. *J. Appl. Prob.* **43**, 967–983.
[6] Caballero, M. E., Chaumont, L. and Rivero, V. (2019). Resurrected Lévy processes. Preprint.
[7] Caballero, M. E., Pardo, J. C. and Pérez, J. L. (2010). On Lamperti stable processes. *Prob. Math. Statist.* **30**, 1–28.
[8] Chaumont, L., Pantí, H. and Rivero, V. (2013). The Lamperti representation of real-valued self-similar Markov processes. *Bernoulli*, **19**, 2494–2523.
[9] Chybiryakov, O. The Lamperti correspondence extended to Lévy processes and semi-stable Markov processes in locally compact groups. *Stoch. Process. Appl.* **116**, 857–872.
[10] Dereich, S., Döring, L. and Kyprianou, A. E. (2017). Real self-similar processes started from the origin. *Ann. Prob.* **45**, 1952–2003.

[11] Gradshteyn, I. S. and Ryzhik, I. M. (2007). *Table of Integrals, Series, and Products*. Elsevier, Amsterdam.
[12] Graversen, S. E. and Vuolle-Apiala, J. (1986). α-self-similar Markov processes. *Prob. Theory Relat. Fields* **71**, 149–158.
[13] Henrici, P. (1991). *Applied and Computational Complex Analysis*, Vol. 2. Wiley, Chichester.
[14] Horton, E. L. and Kyprianou, A. E. (2016). More on hypergeometric Lévy processes. *Adv. Appl. Prob.* **48**, 153–158.
[15] Kiu, S. C. (1980). Semistable Markov processes in $\mathbf{R}^n$. *Stoch. Process. Appl.* **10**, 183–191.
[16] Kuznetsov, A., Kyprianou, A. E. and Pardo, J. C. (2012). Meromorphic Lévy processes and their fluctuation identities. *Ann. Appl. Prob.* **22**, 1101–1135.
[17] Kuznetsov, A., Kyprianou, A. E., Pardo, J. C. and van Schaik, K. (2011). A Wiener–Hopf Monte Carlo simulation technique for Lévy processes. *Ann. Appl. Prob.* **21**, 2171–2190.
[18] Kuznetsov, A., Kyprianou, A. E., Pardo, J. C. and Watson, A. R. (2014). The hitting time of zero for a stable process. *Electron. J. Prob.* **19**, 1–26.
[19] Kuznetsov, A. and Pardo, J. C. (2013). Fluctuations of stable processes and exponential functionals of hypergeometric Lévy processes. *Acta Appl. Math.* **123**, 113–139.
[20] Kyprianou, A. E. (2014). *Fluctuations of Lévy Processes with Applications: Introductory Lectures*. Springer, Berlin.
[21] Kyprianou, A. E. (2016). Deep factorisation of the stable process. *Electron. J. Prob.* **21**, 1–28.
[22] Kyprianou, A. E., Pardo, J. C. and Watson, A. R. (2014). The extended hypergeometric class of Lévy processes. *J. Appl. Prob.* **51**, 391–408.
[23] Kyprianou, A. E., Pardo, J. C. and Watson, A. R. (2014). Hitting distributions of α-stable processes via path censoring and self-similarity. *Ann. Prob.* **42**, 398–430.
[24] Levin, B. Y. (1996). *Lectures on Entire Functions*. Translations of Mathematical Monographs. American Mathematical Society, Providence, RI.
[25] Maulik, K. and Zwart, B. (2006). Tail asymptotics for exponential functionals of Lévy processes. *Stoch. Process. Appl.* **116**, 156–177.
[26] Rogers, L. C. G. (1983). Wiener–Hopf factorization of diffusions and Lévy processes. *Proc. London Math. Soc.* **47**, 177–191.
[27] Schilling, R. L., Song, R. and Vondraček, Z. (2010). *Bernstein Functions: Theory and Applications*. De Gruyter, Berlin.
[28] Song, R. and Vondraček, Z. (2010). Some remarks on special subordinators. *Rocky Mt. J. Math.* **40**, 321–337.
[29] Vigon, V. (2002). *Simplifiez vos Lévy en titillant la factorisation de Wiener–Hopf*. Éditions universitaires européennes, Saarbrücken.
[30] Vuolle-Apiala, J. and Graversen, S. E. (1986). Duality theory for self-similar processes. *Ann. Inst. H. Poincaré Prob. Statist.* **22**, 323–332.
[31] Watson, A. R. (2019). Private commnication.