OPTIMAL STOPPING PROBLEMS DRIVEN BY LÉVY PROCESSES AND PASTING PRINCIPLES

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Optimal Stopping Problems Driven by Lévy Processes and Pasting Principles
With 31 Figures

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(met een samenvatting in het Nederlands)

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This thesis is dedicated to:
Mama and Papa in Padang
Ibu and Bapak in Jakarta
my beloved wife and daughter

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Summary

Solving optimal stopping problems driven by Lévy processes has been a challenging task and has found many applications in modern theory of mathematical finance. For example situations in which optimal stopping problems typically arise include the problems of finding the arbitrage-free price of the *American put (call) option* and determining an optimal *bankruptcy level* in the problem of endogenous default.

The main concern in pricing the American put (call) option lies in finding the critical value of the stock price process below (above) which the option is exercised. In the case of endogenous default, the problem mainly deals with finding an optimal bankruptcy level of a firm which keeps a constant profile of debt and chooses its default level endogenously, to maximize the equity value. In the context of the theory of optimal stopping, the arbitrage-free price of the American put (call) option and the equity value of the defaultable firm correspond to the value function of an optimal stopping problem while the critical value of the stock price process and the optimal bankruptcy level correspond to the optimal stopping boundary.

In general, optimal stopping problems are two-dimensional in the sense that they consist of finding the value function and the optimal boundary simultaneously; that is to say that the value function can be seen as a function of an unknown stopping boundary. Thus, from an analytical point of view, solving the problem is difficult.

A major technique that has been widely used in the theory of optimal stopping problems driven by diffusion processes is the *free boundary* formulation for the value function and the optimal boundary. The free boundary formulation consists primarily of a partial differential equation and (among other boundary conditions) the *continuous* and *smooth pasting* conditions used to determine the unknown boundary and specify the value function. The first condition requires the value function to be continuous at the boundary while the second condition imposes a C^1 smoothness of the value function at the boundary. Depending on the nature of the problem and the sample paths of the Lévy process, the smooth pasting condition may break down. As will be shown in this thesis, this phenomenon can happen to be the case when the Lévy process has paths of bounded variation. As a result, for this type of Lévy processes, the continuous pasting condition appears to be the only criterion for choosing the boundary. Thus, a better understanding of the appropriate choice of pasting conditions to determine the boundary can play an important role in the theory.

Much of this thesis is concerned with solving optimal stopping problems driven by Lévy processes in a general setting. The aim is to propose a framework by which semi-explicit solutions can be obtained. Using such solutions, we give sufficient and necessary conditions for the continuous and smooth pasting conditions to occur in the considered problem. In this thesis we give examples of different cases.

For finite expiration date, we focus on the American put option problem where the evolution of the stock price is driven by a bounded variation Lévy process. The problem is solved by using a change of variable formula with local time on curves for bounded variation Lévy processes. Combining this with Itô-Doob-Meyer decomposition of the value process of the American put option problem into martingale and potential processes, we show that the optimal stopping boundary can be characterized as a solution to a nonlinear integral equation. Taking account of the continuous pasting condition, we show using the change of variable formula that such integral equation admits, under some conditions, a unique solution for the optimal boundary. By the uniqueness of such solution, we show that the value function of the American put option problem and the optimal stopping boundary represent the unique pair solution to a free boundary problem of parabolic integro-differential type.

In the case of infinite maturity, we give an optimal solution to a perpetual optimal stopping problem for a general class of payoff functions under Lévy processes. The solution is obtained by reducing the stopping problem into an averaging problem. Using solution to the latter problem, we obtain using the *Wiener-Hopf* factorization a *fluctuation identity* of Lévy processes. This fluctuation identity relates the solution of the averaging problem with the expected value of discounted payoff function up to a first passage time. Based on the identity, we show that if the solution to the averaging problem has a certain monotonicity property then an optimal solution to the stopping problem can be described in terms of such a monotone function, and the boundary is given by a level at which the function changes its sign. Using such solution, we are able to show that the smooth pasting condition is satisfied if and only if the optimal stopping boundary is regular for interior of the stopping region for the Lévy process. A number of problems are studied in detail, in particular for polynomial payoff and the arbitrage-free pricing of the American put and call options.

For the problem of endogenous default, we show that within a particular class of models, the issue of choosing an optimal bankruptcy level can be dealt with analytically and numerically when the underlying source of randomness for the value of the firm's asset is replaced by a general *Lévy process with no positive jumps.* By working with the latter process, we bring to light a new phenomenon, namely that, depending on the nature of the small jumps, the optimal bankruptcy level may be determined by a continuous pasting condition as opposed to the usual smooth pasting condition. Moreover, we are able to prove the optimality of the bankruptcy level according to the appropriate choice of pasting conditions.

Most of the main results presented in this thesis are verified by means of numerical examples for Lévy processes having one-sided jumps.

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Glossary of Notations

$\mathbb{R}_+$, $\mathbb{R}$, $\mathbb{C}$	positive, real, and complex numbers
$\mathfrak{Im}(z)$	imaginary part of a complex number z
$\mathfrak{Re}(z)$	real part of a complex number z
$\mathbf{1}_A(x)$	indicator function of a set A
$Q_n(x)$	Appell polynomial of order n
$a \wedge b$	minimum of two real numbers a and b
X	real valued Lévy process
X_{t-}	left limit of X at time t
$\widehat{X}$	dual process $(= -X)$
$\Delta, \Delta X$	jump process of X
Ω	space right-continuous path having limits to the left
$\mathcal{F}_t$	$\mathbb{P}-$ complete sigma-field generated by $(X_s, s \leq t)$
H, $\widehat{H}$	ascending, descending ladder height processes
$\mathbb{P}, \mathbb{P}_0$	law of the Lévy process started at 0
$\mathbb{P}_x$	law of the Lévy process started at $x \in \mathbb{R}$
$\mathbb{P}^\nu$	Esscher transform defined by $\frac{d\mathbb{P}^\nu}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \frac{e^{\nu X_t}}{\mathbb{E}(e^{\nu X_t})}$ for $t \geq 0$
$\mathbb{E}$, $\mathbb{E}_x$	expectation operators associated with $\mathbb{P}$, $\mathbb{P}_x$
$\mathbb{E}^\nu$, $\mathbb{E}^\nu_x$	expectation operators associated with $\mathbb{P}^\nu$, $\mathbb{P}^\nu_x$
d	drift coefficient of a bounded variation Lévy process
δ_x	Dirac unit mass at a point x
σ^-_b, τ^-_b	first exit time of X below a boundary b
σ^+_b, τ^+_b	first exit time of X above a boundary b
Π	Lévy measure of the jump process ΔX
$\overline{\Pi}^-$, $\overline{\Pi}^+$	lower, upper tails of the Lévy measure
$\nu(dx, dt)$	Poisson random measure
$\mathcal{L}_X$	infinitesimal generator of X
$\mathbf{e}_q$	exponential time of parameter q, independent of X

$\underline{X}_t$, $\overline{X}_t$	infimum, supremum processes
$\Psi(\lambda)$	characteristic exponent of a Lévy process
$\kappa(\lambda)$	Laplace exponent of a Lévy process
$\Phi(\alpha)$	the largest root of the equation $\kappa(p) = \alpha$
$\kappa(\alpha,\beta)$,$\widehat{\kappa}(\alpha,\beta)$	Laplace exponents of ascending, descending ladder process
$\Psi_q^{(-)}(\lambda)$, $\Psi_q^{(+)}(\lambda)$	Fourier transforms of the law of $\underline{X}_{\mathbf{e}_q}$, $\overline{X}_{\mathbf{e}_q}$
$\kappa_q^{(-)}(\lambda)$, $\kappa_q^{(+)}(\lambda)$	Laplace transforms of the law of $\underline{X}_{\mathbf{e}_q}$, $\overline{X}_{\mathbf{e}_q}$
$W^{(q)}(x)$	scale function under measure $\mathbb{P}$
$W_\nu^{(q)}(x)$	scale function under measure $\mathbb{P}^\nu$
$G(x)$	payoff function of an optimal stopping problem (OSP)
$V(x)$, $V(t,x)$	optimal value function of a perpetual, finite maturity OSP
$V_y(x)$, $V(x;y)$	candidate solution of a perpetual OSP
$\mathscr{P}_G^{(q)}(x)$	a function for which we have for a given $q \geq 0$ and payoff G that $\mathbb{E}\big(\mathscr{P}_G^{(q)}\big(x + \underline{X}_{\mathbf{e}_q}\big)\big) = G(x)$
$\mathscr{C}_G^{(q)}(x)$	a function for which we have for a given $q \geq 0$ and payoff G that $\mathbb{E}\big(\mathscr{C}_G^{(q)}\big(x + \overline{X}_{\mathbf{e}_q}\big)\big) = G(x)$
$\mathscr{R}$	class of sufficiently regular functions
L^b	local time-space on a curve boundary b

Chapter 1

Introduction and Preliminaries

1.1 Motivation

The American put option problem

The valuation of contingent claims has been a widely known topic in the theory of modern finance. Typical claims such as call and put options have been playing significant role not only in the theory but also in the real financial markets. A put (call) option is the "right" but not the obligation to sell (buy) a certain asset at a specified price until or at a predetermined maturity date in the future. If the option specifies that the option holder may exercise this right only at the given future date, the claim is termed *European*.

The pricing of European puts and calls on stocks has an interesting history, dating back to the work of Bachelier [9]. In 1900 Bachelier was the first to use a linear Brownian motion to model the movement of stock price fluctuations. The theory reaches a milestone with the celebrated papers of Black and Scholes [18] and Merton [83] in which the principles of *hedging* and *arbitrage-free pricing* were introduced for the first time. These idea were formalized and extended further by Harrison and Kreps [56] and Harrison and Pliska [57] by applying the fundamental concepts of stochastic integrals and the Girsanov theorem in stochastic calculus. Based on the important principle of hedging, Black and Scholes [18] derived the now famous formula for the value of the European call option, which bears their name and which was extended by Merton [83] in a variety of very significant ways. For this foundational work, Robert Merton and Myron Scholes were awarded the 1997 Nobel Prize in economics.

It is worth noting that most of the traded options, however, are of *American style* (or in the sequel, *American options*)-that is, the holder has the right to exercise an option at any instant before the option's expiry. It is the added feature of early exercise which makes the American options more interesting and complex to evaluate. According to the theory of modern finance[1], the arbitrage-free price of the American

[1] See for instance Karatzas and Shreve [66] and Myneni [90] for extensive review of the theory

put option with strike price K coincides with the value function V of an optimal stopping problem with *payoff function* $G(x) = (K - x)^+$. That is to say that the *arbitrage-free* price of the American put option is given by

$$V(t,x) \;=\; \sup_{0\le\tau\le t} \mathbb{E}\Big(e^{-r\tau}\big(K - S_\tau(x)\big)^+\Big), \tag{1.1.1}$$

for all $(t,x) \in [0,T] \times \mathbb{R}_+$, where T is the maturity of the option and τ is a stopping time of the stock price process S the evolution of which is given by exponential of a linear Brownian motion

$$S_t(x) \;=\; xe^{(r+\omega)t+\sigma B_t}, \tag{1.1.2}$$

taken under a chosen martingale measure $\mathbb{P}$ (with associated expectation operator $\mathbb{E}$) under which $S_0 = x$. The parameter ω is chosen to be $-\frac{1}{2}\sigma^2$ so that the discounted stock price process $e^{-rt}S_t(x)$ is $\mathbb{P}$-*martingale*, implying that

$$\mathbb{E}\Big(e^{-rt}S_t(x)\Big) \;=\; x.$$

Although the American put option problem was treated as an optimal stopping problem, a financial justification using hedging arguments was given only later by Bensoussan [12] and Karatzas [64], [65]. The optimal stopping time in the American put option problem (1.1.1) is the first time when the stock price process S goes below a time-dependent boundary b. When the maturity time T of the option is finite, the problem (1.1.1) is essentially two-dimensional in the sense that it consists of finding the value function V and the optimal stopping boundary b simultaneously; that is to say that the value function can be seen as a function of the unknown stopping boundary. Therefore, from an analytical point of view, solving the problem is difficult.

The first and one of the most penetrating mathematical analysis of the problem (1.1.1) was due to McKean [82]. There the problem was transformed into a *free boundary problem* for the value function V and the boundary b. Solving the free boundary problem, McKean obtained the American option price explicitly in terms of the boundary. McKean's work was taken further by van Moerbeke [86]. Motivated by the physical problem of *the condition of heat balance* (i.e., the law of conservation of energy), van Moerbeke [86] introduced a so-called the *smooth pasting* condition to determine the boundary and specify the value function. This condition dictates that the value and the payoff functions must join smoothly at the boundary.

The derivation of the smooth pasting condition for diffusion processes are given by Grigelionis and Shiryaev [55], Shiryaev [113], Chernoff [30], McKean [82] and Myneni [90] using Taylor approximation of the value function around the boundary and by Bather [11] and van Moerbeke [86] using Taylor expansion of the payoff function around the boundary plus the assumption that the boundary is *regular*[2] for the interior of the stopping region for the underlying process. Since the value function is not

and methods of pricing American type options for diffusion processes.

[2]Starting at the boundary, the underlying process makes an immediate visit to the interior points of the stopping region. See Definition 2.1.3 in Chapter 3.

known a priori, the approach of Bather [11] and van Moerbeke [86] is more satisfactory than the others.

As an alternative to the Taylor expansion method, Peskir [98] introduced a probabilistic approach to prove the smooth pasting condition. The main approach of the proof is based on a *change of variable formula with local time-space* on curves which he derived recently in [97]. This formula extends further the Itô-Tanaka formula for convex functions (see for instance Revusz and Yor [106]). Using the change of variable formula and the free boundary problem, Peskir [98] derived the smooth pasting condition. (See also Peskir and Shiryaev [99] for more discussion on recent development of local time-space calculus in the theory of optimal stopping.)

Based on the free boundary problem formulation of the optimal stopping problem (1.1.1), with continuous and smooth pasting conditions in place, and combining with the Itô-Doob-Meyer decomposition of the value function of the problem (1.1.1) into martingale and potential processes, van Moerbeke [86], Myneni [90], El Karoui and Karatzas [45], Jacka [62], Carr et al. [23], and later Peskir [98] showed that the optimal stopping boundary can be characterized as a solution to a nonlinear integral equation. Such an equation was already obtained earlier by Friedman [51] in 1959 for a one-dimensional free boundary problem of ice melting. This nonlinear integral equation for the optimal boundary is known as the *Riesz decomposition* for the value function of the problem (1.1.1) and has a clear economical meaning to the *early exercise premium* representation of the value function. We refer among others to Kim [67], Myneni [90] and Carr et al. [23] and the literature therein for details.

The existence and local uniqueness of a solution to the nonlinear integral equation for the boundary was proved by Friedman [51] and van Moerbeke [86] using the fixed point theorem (contraction principle) first for a small time interval and extending it to any interval of time using induction arguments. The result of applying the fixed point theorem is that the nonlinear equation involves continuous differentiability of the curve boundary, a condition that is needed to be proved a priori, and results in a long computation and strong condition imposed on the boundary. In contrast to the fixed point method, Jacka [62] and later Peskir [98] introduced a probabilistic approach to prove the existence and uniqueness of a solution to the nonlinear integral equation. The key ingredient of the proof is based on the smooth pasting condition and the Itô-Doob-Meyer decomposition of the value function of the optimal stopping problem (1.1.1). (Note that the Itô-Doob-Meyer decomposition underlies the basic principle of the theory of optimal stopping developed earlier by Snell [115], Dynkin [39] and Dynkin and Yushkevich [41].) However, the incorporation of the smooth pasting condition in the proof was made clear by Peskir [98] using his change of variable formula.

Alternative modelling for underlying processes

Until now we have discussed exponential of a linear Brownian motion as the continuous time model for the evolution of the stock price process (1.1.2). In recent years, there has been a lot of interest in looking for alternative models for the evolution of the stock price process which gives a better fit to the real data. Empirical study of financial data reveals the fact that the distribution of the log-return of stock price exhibits features which cannot be captured by the normal distribution such as heavy tails and asymmetry. For the purpose of replicating more effectively these features, there has been a general shift in the literature to modelling with exponential *Lévy process* as an alternative to exponential of a linear Brownian motion.

A Lévy process is a stochastic process with stationary independent increments whose paths are right-continuous and have left limits. Most recent examples of Lévy processes used in modelling the evolution of the stock price process we refer among others to the normal inverse Gaussian model of Bandorff-Nielsen [10], the hyperbolic model of Eberlein and Keller [42], the variance gamma model of Madan and Seneta [80], the CGMY model of Carr et al. [24], and tempered stable process first introduced by Koponen [68] and extended further by Boyarchenko and Levendorski [21].

Working with a Lévy process leads to many intriguing mathematical issues which need to be resolved to completely settle the problem of valuing American options. In a market where the underlying dynamics for the stock price process is driven by the exponential of a linear Brownian motion, as discussed before, the valuation is transformed into a free boundary problem. The critical value (the stopping boundary) of the stock price process is determined by imposing continuous and smooth pasting conditions as optimality criterion for choosing the stopping boundary. However, by allowing jumps in the sample paths of the underlying dynamics of the stock price process, the smooth pasting *may* break down at the stopping boundary as the stock price process may jump over the boundary. As a result, the *continuous pasting* condition is perhaps the only criterion for determining the stopping boundary.

When maturity T is infinite and the underlying is a general Markov process, the optimal stopping problem (1.1.1) could be solved without necessarily being transformed into a free boundary problem and using the smooth pasting condition. The solution can be obtained using probabilistic approach. This approach was first introduced by Darling et al. [33] for random walks and was extended further using similar arguments in [33] to continuous time among others by Mordecki [87], Asmussen et al. [6], and Alili and Kyprianou [3]. Taking the result of Mordecki [87], it was shown recently by Alili and Kyprianou [3] that the existence of the smooth pasting condition for the problem (1.1.1) is determined by the *regularity* of the sample paths of the underlying process; for the problem considered there the smooth pasting occurs if and only if 0 is regular for the lower half-line $(-\infty, 0)$ for the process itself.

However, the solution to a perpetual optimal stopping problem with a more general payoff function was not discussed by the aforementioned authors. This problem was addressed by Boyarchenko and Levendorski [21]. There they considered the problem of solving perpetual optimal stopping for payoff functions with exponential growth. Their approach is much more sophisticated in which potential theory of Lévy processes and the theory of pseudo-differential operators are heavily used to solve the problem. See for instance [20] for recent work in this direction. Working under a particular class of Lévy processes with stable like characteristic exponent, Boyarchenko and Levendorski [21] gave an integral test for the smooth pasting condition to occur.

1.2 The main contribution of this thesis

This section outlines the main contribution of this thesis to the theory of optimal stopping problems driven by Lévy processes. The aim is to propose a framework by which semi-explicit solutions can be obtained. The solutions are given for both finite and infinite maturity and are obtained without using the continuous and smooth pasting conditions. Using the semi-explicit solutions in the problems we consider, we give sufficient and necessary conditions for the pasting conditions to be fulfilled.

The thesis consists of seven self-contained chapters. The content of the chapters is outlined in what follows.

Chapter 1 This chapter overviews some past and recent developments in the theory of optimal stopping and outlines some points that have not been discussed in the literature. The missing gaps in the theory are explained in this chapter and constitute the main source of motivation of the writing of this thesis.

Chapter 2 This chapter provides a brief introduction to Lévy processes and the Wiener-Hopf factorization formula which underlies the fluctuation theory of Lévy processes and forms one of the two main principles for solving an infinite horizon optimal stopping problem under Lévy processes. We also discuss in this chapter some important classes of Lévy processes for which the two factors of the Wiener-Hopf factorization have explicit expressions. Among theses classes, we use spectrally negative Lévy processes for the numerical computation performed in the last four chapters.

Chapter 3 In this chapter we establish a change of variable formula for 'ripped' time-space functions of Lévy processes of bounded variation at the cost of an additional integral with respect to local time-space in the formula. Roughly speaking, by a ripped function, we mean here a time-space function which is $C^{1,1}$ on either side of a time dependent barrier and which may exhibit a discontinuity along the barrier itself. Such functions have appeared in the theory of optimal stopping problems for Markov processes of bounded variation (cf. Peskir and Shiryaev ([95], [96]), Chan ([26], [27]), Avram et al. [7]. This result complements the recent work of Föllmer et al. [50], Eisenbaum ([43], [44]) and Peskir ([97], [98]) and Elworthy et al. [46] in which generalized versions of Itô's formula were established with local time-space. Using the change of variable formula, we address the finite maturity American put option problem where the evolution of the stock price process is driven by a bounded variation Lévy process. Combining this with Itô-Doob-Meyer decomposition of the value process of the American put option problem into martingale and potential processes, we show that the optimal stopping boundary can be characterized as a solution to a nonlinear integral equation. Taking account of the continuous pasting condition, we show using the change of variable formula that such integral equation admits, under some conditions, an unique solution for the optimal boundary. By the uniqueness of such solution, we show that the value function of the American put option problem and the optimal stopping boundary represent an unique pair solution to a free boundary problem of parabolic integro-differential type.

Chapter 4 This chapter discusses a relatively new optimal stopping problem where the payoff is an integer power function. This problem was first introduced by Novikov and Shiryaev [91] for random walks based on other similar examples given by Darling et al [33]. We give the analogue of their results when the random walks are replaced by Lévy processes. The main ingredient of solving this problem is central to using Appell polynomials and fluctuation theory of Lévy processes.

Chapter 5 In this chapter, we generalize the recent work of Boyarchenko and Levendorski [21] on a perpetual optimal stopping problem under Lévy processes. Unlike their approach, we do not appeal to the theory of pseudo-differential operators to solve the problem. We work with a more general class of Lévy processes and we allow for a more general class of payoffs. The solution is obtained by reducing the problem into an averaging problem from which we obtain, using the *Wiener-Hopf* factorization, a *fluctuation identity* for overshoots of Lévy processes. This fluctuation identity relates the solution of the averaging problem with the expected value of discounted payoff function up to a first passage time and is the key element in obtaining the value function and the optimal boundary of the stopping problem. Using our approach, we are able to verify the smooth pasting condition analytically and to reproduce the special results of those discussed among others by Darling et al. [33], Mordecki [87], Boyarchenko and Levendorskii [21], Alili and Kyprianou [3], Novikov and Shiryaev [91], and Kyprianou and Surya [73] (also presented in Chapter 4). Furthermore, assuming that the moment generating function of the underlying Lévy process exists on an open set containing zero, we obtain a lower and upper bounds for the arbitrage-free price of the finite maturity American put option in terms of the value function of the perpetual American put option problem.

Chapter 6 In this chapter we consider an endogenous bankruptcy problem. This problem is closely related to a perpetual type optimal stopping problem which primarily deals with finding an optimal bankruptcy level V_B of a firm which keeps a constant level of its debt and chooses its bankruptcy level endogenously so that the value of its equity is maximized. The firm declares bankruptcy when the value of its asset goes below the level V_B. This problem has been investigated by Leland and Leland and Toft in a sequence of their papers in [77] and [76], respectively. The work of Leland and Toft was extended further from diffusion to a Lévy process which is the independent sum of a linear Brownian motion and a compound Poisson process with one-sided exponential jumps by Hilberink and Rogers [58]. As it was suggested by Leland and Toft [76] and later by Hilberink and Rogers [58] that, subject to the limited liability constraint[3] of the equity value, the smooth pasting condition is used for optimality criterion for choosing the bankruptcy level V_B. In other recent work, Chen and Kou [29] generalized the works of Leland and Toft [76] and Hilberink and Rogers [58] by adding a two-sided exponential jumps compound Poisson process to a linear Brownian motion. They succeeded in *proving* that the optimal bankruptcy level is obtained by using the smooth pasting condition for the case considered there.

The main purpose of this chapter is threefold. Firstly to revisit the previous works of Leland and Toft [76] and Hilberink and Rogers [58] and show that the issue of choosing an optimal endogenous bankruptcy level can be dealt with analytically and numerically when the underlying source of randomness for the value of the firm's asset is replaced by a general Lévy process with no positive jumps. Secondly, by working with the latter class of Lévy processes we bring to light a new phenomenon, namely that, depending on the nature of the small jumps, the optimal default level may be determined by a principle of *continuous pasting* as opposed to the usual *smooth pasting*. Thirdly, we are able to *prove* the optimality of the default level according to the appropriate choice of pasting. This improves on the results of Hilberink and Rogers [58] who were only able to give a numerical justification for the case of smooth pasting. Our calculations are greatly eased by the recent perspective on fluctuation theory of spectrally negative Lévy processes in which many new identities are expressed in terms of the so called *scale functions*. To finish this chapter, we study analytically and numerically the behaviour of the term structure of credit spreads for very short maturity bonds when we allow the firm's assets to be driven by a general Lévy process with no positive jumps. The study reveals the fact that the credit spreads have strictly positive values, a feature typically observed in the financial market.

Chapter 7 In this chapter we discuss a robust numerical method to numerically produce the q-scale function $\{W^{(q)}(x) : q \geq 0, x \in \mathbb{R}\}$ of a general spectrally negative Lévy process $(X, \mathbb{P})$. The method is based on the *Esscher transform* of measure $\mathbb{P}^{\nu}$ under which X is taken and the scale function is determined. This change of measure makes it possible for the scale function to be bounded and hence makes numerical computation easier, fast and stable. Working under the new measure $\mathbb{P}^{\nu}$ and using the method of Abate and Whitt [1] and Choudhury et al. [31], we give a fast stable numerical algorithm for the computation. The algorithm has been extensively used to give numerical verification of the main results presented in this thesis.

[3]Equity must worth non-negative for all values V of the firm's asset bigger than equal to the bankruptcy level V_B.

1.3 Publication details

The material presented in this thesis has resulted in the following research papers.

(i) Kyprianou, A. E. and Surya, B. A. A note on the change of variable formula with local time-space for bounded variation Lévy processes. To appear in *Séminaire de Probabilité XL*, Lecture Notes in Mathematics, Springer-Verlag.

(ii) Kyprianou, A. E. and Surya, B. A. On the Novikov-Shiryaev optimal stopping problems in continuous time. Appeared in *Electronic Communications in Probability*, Vol. **10** (2005), 146-154.

(iii) Kyprianou, A. E. and Surya, B. A. Principles of smooth and continuous fit in the determination of endogenous bankruptcy levels. To appear in *Finance and Stochastics*, Springer-Verlag.

(iv) Surya, B. A. An approach for solving perpetual optimal stopping problems driven by Lévy processes. To appear in *Stochastics*.

(v) Surya, B. A. Evaluating scale functions of spectrally negative Lévy processes. Submitted for publication to *Journal of Applied Probability*.

Chapter 2

A Brief Introduction to Lévy Processes

In this chapter, we present a brief introduction to Lévy processes which underlie the main object of interest of the optimal stopping problems considered in this thesis.

We refer among others to Applebaum [5], Bertoin [13], Kyprianou [69], Protter [105], and Sato [111] for a detailed account on Lévy processes.

2.1 Introduction

Definition 2.1.1 (Lévy process) Let $\mathbb{P}$ be a probability measure on probability space $(\Omega, \mathcal{F})$. A process $X = (X_t, t \geq 0)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Lévy process if the paths of X are right continuous with left limits $\mathbb{P}$-almost surely, for every $s, t \geq 0$, the increment $X_{t+s} - X_t$ is independent of the process $(X_u, 0 \leq u \leq t)$ and has the same law as X_s. In particular, $\mathbb{P}(X_0 = 0) = 1$.

From now on, the law of the Lévy process started at $x \in \mathbb{R}$ will be denoted by $\mathbb{P}_x$. For convenience we write $\mathbb{P} = \mathbb{P}_0$ and we shall write $\mathbb{E}_x$ for the expectation operator associated with $\mathbb{P}_x$ and in the special case that $x = 0$ we write $\mathbb{E}$.

The characteristic exponent of X is given by the well known *Lévy-Khintchine formula* which shall be given by the following theorem (see for instance Theorem 1 in Chapter I of Bertoin [13]).

Theorem 2.1.2 (Lévy-Khintchine formula) *Suppose that $\mu \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure concentrated on $\mathbb{R}\backslash\{0\}$ such that $\int_{-\infty}^{\infty}(1 \wedge y^2)\Pi(dy) < \infty$. From this triple define for each $\theta \in \mathbb{R}$ a continuous function $\Psi : \mathbb{R} \to \mathbb{C}$ given by*

$$\Psi(\theta) \; = \; i\mu\theta + \frac{\sigma^2}{2}\theta^2 + \int_{-\infty}^{\infty} \big(1 - e^{i\theta y} + i\theta y 1_{\{|y| \leq 1\}}\big)\Pi(dy). \tag{2.1.1}$$

Then there exists a unique probability measure $\mathbb{P}$ on Ω under which $(X_t, t \geq 0)$ is a Lévy process with characteristic exponent Ψ, i.e.,

$$\mathbb{E}\big(e^{i\theta X_t}\big) \; = \; e^{-t\Psi(\theta)} \quad \text{for } \theta \in \mathbb{R} \text{ and } t \geq 0. \tag{2.1.2}$$

Moreover, the jump process of X, namely $\Delta X = (\Delta X_t, t \geq 0)$, is a Poisson point process with characteristic measure Π.

The measure Π is called the *Lévy measure* and the parameter σ the *Gaussian coefficient*. The Lévy-Khintchine formula has a simpler expression when the sample paths of the Lévy process have *bounded variation* on every compact time interval almost surely. For short, we will then say that the Lévy process has bounded variation. Specifically, a Lévy process has bounded variation if and only if $\sigma = 0$ and the Lévy measure Π satisfies the integral test $\int_{-\infty}^{\infty}(1 \wedge |y|)\Pi(dy) < \infty$. In that case, the mapping $\lambda \mapsto \int_{-\infty}^{\infty} \lambda x \mathbf{1}_{\{|y|<1\}}\Pi(dy)$ is a well-defined linear function and the characteristic exponent Ψ can be re-expressed for each $\theta \in \mathbb{R}$ as

$$\Psi(\theta) \;=\; -i\mathrm{d}\theta + \int_{-\infty}^{\infty} \big(1 - e^{i\theta y}\big)\Pi(dy), \tag{2.1.3}$$

for some $\mathrm{d} \in \mathbb{R}$ which is known as the *drift coefficient*. Moreover, if $\Delta = (\Delta_t, t \geq 0)$ is a Poisson point process with characteristic measure Π, then the process

$$X_t \;=\; \mathrm{d}t + \sum_{0 \leq s \leq t} \Delta_s, \quad \text{for } t \geq 0,$$

is a Lévy process of bounded variation (recall that the series is absolutely convergent almost surely if and only if the Lévy measure Π satisfies $\int_{-\infty}^{\infty}(1 \wedge |y|)\Pi(dy) < \infty$, see for instance Chapter I of Bertoin [13]) with characteristic exponent Ψ (2.1.3). It is clear that a compound Poisson process has bounded variation and, conversely, a Lévy process with bounded variation is a compound Poisson process if and only if its drift coefficient d is null and its Lévy measure Π has finite mass.

For every rapidly decreasing function[1] f, one can use the Lévy-Khintchine formula to get the infinitesimal generator $\mathcal{L}_X$ of the Lévy process X defined by

$$\begin{aligned}\mathcal{L}_X f(x) \;&:=\; \lim_{t \downarrow 0} t^{-1}\big(\mathbb{E}_x f(X_t) - f(x)\big)\\ &=\; \mu \frac{df}{dx}(x) + \frac{1}{2}\sigma^2 \frac{d^2 f}{dx^2}(x) + \int_{-\infty}^{\infty} \Big(f(x+y) - f(x) - y\frac{df}{dx}(x)1_{\{|y|\leq 1\}}\Big)\Pi(dy).\end{aligned}$$

We refer to Section 4.1 in Skorohod [114] for the details of the calculations.

To finish this section, let us introduce the notion of regularity of a point for an open or closed set $\mathcal{O}$ for a Lévy process. This notion becomes relevant to the discussions on the smooth and continuous pasting conditions later in this thesis.

Definition 2.1.3 (Regularity of a point for a Lévy process) For a Lévy process X, the point $x \in \mathbb{R}$ is said to be regular (respectively, irregular) for an open or closed set $\mathcal{O}$ if

$$\mathbb{P}_x\big(\tau_{\mathcal{O}} = 0\big) = 1 \;\text{ (respectively, 0)},$$

[1] We say a function $f(x)$ is *rapidly decreasing* if there are constants M_N such that $|f(x)| \leq M_N|x|^{-N}$ as $x \to \infty$ for $N = 1, 2, 3, \cdots$. See for example Gel'fand and Shilov [52] for more details.

thanks to Blumenthal's zero-one law, where the stopping time

$$\tau_{\mathcal{O}} = \inf\{t > 0 : X_t \in \mathcal{O}\}.$$

Intuitively speaking, x is regular for $\mathcal{O}$ if, when starting from x, the Lévy process hits $\mathcal{O}$ immediately.

2.2 The Wiener-Hopf factorization

In this section we discuss the fundamental notion of the Wiener-Hopf factorization in fluctuation theory of Lévy processes. The results presented in this section will be used later in Chapters 4, 5 and 6. What we shall say in this section mainly refers to the Chapters VI and 6 of the books of Bertoin [13] and Kyprianou [69], respectively.

To begin with, let us introduce a so-called the *supremum* and *infimum* processes

$$\overline{X}_t \;=\; \sup_{0\le s\le t} X_s \quad \text{and} \quad \underline{X}_t \;=\; \inf_{0\le s\le t} X_s.$$

We see that $\overline{X}$ and $-\underline{X}$ are two nonnegative increasing right-continuous processes, which are adapted to the filtration $\mathcal{F}_t = \sigma(X_s, s \le t)$. It is well known (see Proposition 1 in Chapter VI of Bertoin [13]) that the *reflected process at supremum* $\overline{X} - X$ is a Markov process in the filtration $\mathcal{F}_t$. (Note that $X - \underline{X}$, *the reflected process at infimum*, can also be viewed as the dual process $-X$ reflected at its supremum.)

We denote by $L = (L(t), t \ge 0)$ a local time of the reflected process $\overline{X} - X$ at zero and by $L^{-1}(t) = \inf\{s > 0 : L(s) > t\}$ its right-continuous inverse also known as the (ascending) *ladder time process*. Note that the range of the inverse local time L^{-1} corresponds to the set of real times at which new maxima occur. Recalling from Chapter IV of Bertoin [13] and Chapter 6 of Kyprianou [69], the support of the Stieltjes measure dL_t coincides with the closure of the zero set of the reflected process.

Next, let us introduce a *ascending ladder height process* H, using the inverse local time to time-change the supremum process. The process H is defined by

$$H(t) \;=\; \overline{X}_{L^{-1}(t)} \quad \text{if } L^{-1}(t) < \infty, \quad H(t) = \infty \text{ otherwise.} \tag{2.2.1}$$

The pair (L^{-1}, H) is known as the *ascending ladder process*. Analogously, the process $(\widehat{L}^{-1}, \widehat{H})$ constructed from the dual process $-X$ is called the *descending ladder process*. The law of the ladder process is characterized by the bivariate Laplace exponent κ and $\widehat{\kappa}$ defined by

$$e^{-\kappa(\alpha,\beta)} \;=\; \mathbb{E}\Big(e^{-\alpha L^{-1}(1) - \beta H(1)}\Big) \quad \text{and} \quad e^{-\widehat{\kappa}(\alpha,\beta)} \;=\; \mathbb{E}\Big(e^{-\alpha \widehat{L}^{-1}(1) - \beta \widehat{H}(1)}\Big)$$

for $\alpha, \beta \ge 0$. It is therefore important to evaluate explicitly the quantities κ and $\widehat{\kappa}$ as they play fundamental role in the fluctuation theory of Lévy processes.

The random variables of interest in fluctuation theory are the following. Let $\mathbf{e}_q$ be an independent exponentially distributed random time with parameter $q \ge 0$. We shall work with the convention that when $q = 0$, the random variable $\mathbf{e}_q$ is understood to be equal to ∞ with probability one. Next, we define by

$$\overline{G}_{\mathbf{e}_q} = \sup\{t < \mathbf{e}_q : X_t = \overline{X}_t\} \quad \text{and} \quad \underline{G}_{\mathbf{e}_q} = \sup\{t < \mathbf{e}_q : X_t = \underline{X}_t\}$$

the last zero of the reflected processes before the exponential random time $\mathbf{e}_q$. According to Proposition VI.4 in Bertoin [13], it is known that, if X is not a compound Poisson process, $\overline{G}_{\mathbf{e}_q}$ is actually the unique instant time t in the random interval $[0, \mathbf{e}_q]$ such that $X_t = \overline{X}_{\mathbf{e}_q}$ or $X_{t-} = \overline{X}_{\mathbf{e}_q}$.

We move on now to introducing the fluctuation identity which provides many results concerning the distributional decomposition of the excursions of any Lévy process when sampled at an independent exponentially distributed random time.

Theorem 2.2.1 (The Wiener-Hopf factorization) [2] *Suppose that X is any Lévy process other than compound Poisson process.*

(i) *The pairs $(\overline{G}_{\mathbf{e}_q}, \overline{X}_{\mathbf{e}_q})$ and $(\mathbf{e}_q - \overline{G}_{\mathbf{e}_q}, \overline{X}_{\mathbf{e}_q} - X_{\mathbf{e}_q})$ are independent and infinitely divisible, yielding the factorization*

$$\mathbb{E}\Big(e^{i\vartheta \mathbf{e}_q + i\theta X_{\mathbf{e}_q}}\Big) = \frac{q}{q - i\vartheta + \Psi(\theta)} = \Psi_q^{(-)}(\vartheta,\theta).\Psi_q^{(+)}(\vartheta,\theta), \tag{2.2.2}$$

where $\vartheta, \theta \in \mathbb{R}$,

$$\Psi_q^{(-)}(\vartheta,\theta) = \mathbb{E}\Big(e^{i\vartheta \underline{G}_{\mathbf{e}_q} + i\theta \underline{X}_{\mathbf{e}_q}}\Big) \quad \textit{and} \quad \Psi_q^{(+)}(\vartheta,\theta) = \mathbb{E}\Big(e^{i\vartheta \overline{G}_{\mathbf{e}_q} + i\theta \overline{X}_{\mathbf{e}_q}}\Big).$$

The pair $\Psi_q^{(-)}(\vartheta,\theta)$ and $\Psi_q^{(+)}(\vartheta,\theta)$ are called the Wiener-Hopf factors.

(ii) *The Wiener-Hopf factors may themselves be identified in terms of the analytically extended Laplace exponent $\kappa(\alpha,\beta)$ and $\widehat{\kappa}(\alpha,\beta)$ via the Laplace transforms,*

$$\mathbb{E}\Big(e^{-\alpha \underline{G}_{\mathbf{e}_q} + \beta \underline{X}_{\mathbf{e}_q}}\Big) = \frac{\widehat{\kappa}(q,0)}{\widehat{\kappa}(q+\alpha,\beta)} \quad \textit{and} \quad \mathbb{E}\Big(e^{-\alpha \overline{G}_{\mathbf{e}_q} - \beta \overline{X}_{\mathbf{e}_q}}\Big) = \frac{\kappa(q,0)}{\kappa(q+\alpha,\beta)}$$

for every complex numbers α, β having positive real part.

(iii) *The Laplace exponent $\kappa(\alpha,\beta)$ and $\widehat{\kappa}(\alpha,\beta)$ may also be identified in terms of the law of X in the following way,*

$$\kappa(\alpha,\beta) = k \exp\Big(\int_0^\infty \int_{(0,\infty)} \big(e^{-t} - e^{-\alpha t - \beta x}\big) t^{-1} \mathbb{P}(X_t \in dx) dt\Big) \tag{2.2.3}$$

and

$$\widehat{\kappa}(\alpha,\beta) = \widehat{k} \exp\Big(\int_0^\infty \int_{(-\infty,0)} \big(e^{-t} - e^{-\alpha t + \beta x}\big) t^{-1} \mathbb{P}(X_t \in dx) dt\Big), \tag{2.2.4}$$

where $\alpha, \beta \in \mathbb{R}$ and k and $\widehat{k}$ are strictly positive constants.

[2] We refer to Theorem 6.16 of Chapter 6 of Kyprianou [69].

(iv) *By setting $\vartheta = 0$ and taking limits as q tends to zero in (2.2.2), we obtain*

$$k'\Psi(\theta) \;=\; \kappa(0, -i\theta)\widehat{\kappa}(0, i\theta)$$

for some constants $k' > 0$ (which may be taken equal to unity by a suitable normalization of local time).

We conclude this section with an important result of Theorem 2.2.1. Recall that Ψ is the characteristic exponent of X so that, for $q > 0$, $q/(q+\Psi(\theta))$ is the characteristic function of $X_{\mathbf{e}_q}$. Theorem 2.2.1 yields the following remarkable fluctuation identity:

$$\mathbb{E}\big(e^{i\theta X_{\mathbf{e}_q}}\big) \;=\; \frac{q}{q+\Psi(\theta)} \;=\; \Psi_q^{(-)}(\theta)\Psi_q^{(+)}(\theta), \tag{2.2.5}$$

for $q > 0$, where $\Psi_q^{(-)}(\theta)$ and $\Psi_q^{(+)}(\theta)$ are respectively the characteristic function of the random variable $\underline{X}_{\mathbf{e}_q}$ and $X_{\mathbf{e}_q} - \underline{X}_{\mathbf{e}_q}$ defined by

$$\Psi_q^{(-)}(\theta) \;=\; \mathbb{E}\big(e^{i\theta \underline{X}_{\mathbf{e}_q}}\big), \tag{2.2.6}$$

and

$$\Psi_q^{(+)}(\theta) \;=\; \mathbb{E}\big(e^{i\theta (X_{\mathbf{e}_q} - \underline{X}_{\mathbf{e}_q})}\big) \;=\; \mathbb{E}\big(e^{i\theta \overline{X}_{\mathbf{e}_q}}\big). \tag{2.2.7}$$

Notice that $\Psi_q^{(-)}(\theta)$ (respectively, $\Psi_q^{(+)}(\theta)$) admits the analytic continuation into the lower half-plane $\mathfrak{Im}(\theta) < 0$ (respectively, upper half-plane $\mathfrak{Im}(\theta) > 0$), and does not vanish there. We refer among others to Applebaum [5], Bertoin [13], Kyprianou [69], and Sato [111] for more details.

2.3 Some important classes of Lévy processes

In the next section, we outline some important class of Lévy processes for which the two factors $\Psi_q^{(-)}(\lambda)$ and $\Psi_q^{(+)}(\lambda)$ of the Wiener-Hopf factorization formula (2.2.5) have explicit expressions. These class of Lévy processes can be found among others in Bertoin [13], [15], Kyprianou [69], Mordecki [87], [88], [89], and Asmussen et al. [6].

Working under these classes of Lévy processes, numerical computation for the problem discussed in Chapters 4, 5 and 6 can be performed quite easily.

2.3.1 Lévy processes with no positive jumps

This class of processes has a great interest from a practical point of view, because they are processes for which fluctuation theory takes the nicest form and can be developed explicitly to its full extent. The degenerate case when X is either the negative of a subordinator or a deterministic drift has no interest and will not be discussed throughout. What we shall say here is based on Chapter VII in Bertoin [13].

Due to the absence of the positive jumps, the characteristic function $\theta \mapsto \mathbb{E}(e^{i\theta X_t})$ $(\theta \in \mathbb{R})$ can be extended to define an analytic function in the complex lower half-plane

$(\mathfrak{Im}(\theta) \leq 0)$. Because of the fact that the Lévy measure vanishes on the positive half-line, the Lévy-Khintchine formula shows that the characteristic exponent $\Psi(\theta)$ is well defined and analytic on $(\mathfrak{Im}(\theta) \leq 0)$. Hence, it is therefore sensible to define

$$\kappa(\theta) \;=\; -\Psi(-i\theta) \;=\; -\mu\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,0)} \big(e^{\theta y} - 1 - \theta y \mathbf{1}_{\{y>-1\}}\big)\Pi(dy),$$

and, hence, we see that the identity $\mathbb{E}(\exp\{\theta X_t\}) = \exp\{t\kappa(\theta)\}$ holds whenever $\mathfrak{Re}(\theta) \geq 0$. The function $\kappa : [0,\infty) \to (-\infty,\infty)$ also called as the *Laplace exponent* of X is zero at the origin and is strictly convex with $\lim_{\theta\uparrow\infty}\kappa(\theta) = \infty$. Next we denote by $\Phi(\alpha)$ the largest solution of the equation

$$\kappa(p) \;=\; \alpha \quad \text{for all } \alpha \geq 0.$$

Note that due to the convexity of κ, there exists at most two solutions for a given α and precisely one root when $\alpha > 0$. A special feature of spectrally negative Lévy processes is that $\overline{X}_{\mathbf{e}_q}$ is known to have exponential law with parameter $\Phi(q)$, namely

$$\mathbb{P}\big(\overline{X}_{\mathbf{e}_q} \in dx\big) \;=\; \Phi(q)e^{-\Phi(q)x}dx, \tag{2.3.1}$$

whose Laplace transform is given by

$$\kappa_q^+(\lambda) \;=\; \mathbb{E}\big(e^{-\lambda\overline{X}_{\mathbf{e}_q}}\big) \;=\; \int_0^\infty e^{-\lambda x}\mathbb{P}\big(\overline{X}_{\mathbf{e}_q} \in dx\big) \;=\; \frac{\Phi(q)}{\lambda + \Phi(q)}, \tag{2.3.2}$$

for all $\lambda \geq 0$ which in turn by the Wiener-Hopf factorization (2.2.5) yields

$$\kappa_q^-(\lambda) \;=\; \mathbb{E}\big(e^{\lambda\underline{X}_{\mathbf{e}_q}}\big) \;=\; \int_0^\infty e^{-\lambda x}\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} \in dx\big) \;=\; \frac{q}{\Phi(q)}\frac{(\lambda - \Phi(q))}{(\kappa(\lambda) - q)}. \tag{2.3.3}$$

In principle, there is no difficulty to invert the above equation numerically. However, by introducing a special class of functions known as scale functions, the definition of which is given below, the measure $\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} \in dx\big)$ can be recovered theoretically in terms of such functions.

Definition 2.3.1 (q-Scale function) For a given spectrally negative Lévy process X with Laplace exponent κ, there exists for every $q \geq 0$ a right-continuous function $W^{(q)} : [0,\infty) \to [0,\infty)$, called the q-scale function, with Laplace transform given by

$$\int_0^\infty e^{-\lambda x}W^{(q)}(x)dx \;=\; \frac{1}{\kappa(\lambda) - q}, \quad \text{for } \lambda > \Phi(q), \tag{2.3.4}$$

where $\Phi(q)$ was defined previously. We shall write for short $W^{(0)} = W$.

Following the definition of $W^{(q)}$ introduced above and by applying Laplace inversion method to (2.3.3), the measure of the random variable $-\underline{X}_{\mathbf{e}_q}$ is given by

$$\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} \in dx\big) \;=\; \frac{q}{\Phi(q)}dW^{(q)}(x) - qW^{(q)}(x)dx. \tag{2.3.5}$$

If the Lévy measure Π has no atoms, it is known that the q-scale function $W^{(q)}$ is at least C^1 smooth, see for instance Lambert [75] and Chan and Kyprianou [28] for more details. For some spectrally negative Lévy processes, the q-scale functions $W^{(q)}$ are available in explicit form. In general, numerical methods are required to compute $W^{(q)}$ from the equation (2.3.4). Further discussion on the property of scale functions and their numerical computation are given in Chapters 6 and 7, respectively.

2.3.2 Lévy processes with mixed exponential jumps

Consider now a Lévy process with Lévy measure Π given by

$$\Pi(dy) \;=\; \mathbf{1}_{(y>0)}\pi(dy) + \mathbf{1}_{(y<0)}\lambda \sum_{k=1}^{n} a_k \alpha_k e^{\alpha_k y} dy, \tag{2.3.6}$$

where π is an arbitrary Lévy measure on $(0, \infty)$, $0 < \alpha_1 < ... < \alpha_n$, $a_k > 0$, for $k = 1, ..., n$ and $\sum_{k=1}^{n} a_k = 1$. The magnitude of the negative jumps of X is mixed exponentially distributed, with parameter α_k chosen with probability a_k, and they occur at the times of a Poisson process with rate λ. Simple computations give

$$\Psi(\theta) \;=\; i\eta\theta - \frac{\sigma^2}{2}\theta^2 + \int_0^{\infty} \big(e^{i\theta y} - 1 - i\theta h(y)\big)\pi(dy) - \lambda \sum_{k=1}^{n} a_k \frac{i\theta}{\alpha_k + i\theta}, \tag{2.3.7}$$

where $h(y) = y\mathbf{1}_{\{0<y<1\}}$ and η is given by

$$\eta \;=\; \mu + \lambda \sum_{k=1}^{n} \frac{a_k}{\alpha_k}\big(1 - (1 + \alpha_k)e^{-\alpha_k}\big).$$

Considered as a function with complex domain, the characteristic exponent $iq \mapsto \Psi(q)$ in (2.3.7) can be extended analytically to the complex strip $\{z = p+iq : p \in (-\alpha_1, 0]\}$ and, for $-\alpha_1 < p \leq 0$, we have the Laplace exponent of X defined by

$$\kappa(p) \;=\; \eta p + \frac{\sigma^2}{2}p^2 + \int_0^{\infty} \big(e^{py} - 1 - ph(y)\big)\pi(dy) - \lambda \sum_{k=1}^{n} a_k \frac{p}{\alpha_k + p}. \tag{2.3.8}$$

Due to the convexity of $\kappa(p)$ on $(-\alpha_1, 0]$, it follows when $\sigma > 0$ that under the condition

$$\kappa'(0-) \;=\; \lim_{p\to 0-} \frac{1}{p}\kappa(p) \;=\; \eta + \int_1^{\infty} y\pi(dy) - \lambda \sum_{k=1}^{n} \frac{a_k}{\alpha_k} \;>\; 0, \tag{2.3.9}$$

(where the integral can take the value ∞), the equation $\kappa(p) = 0$ has $n + 1$ negative roots $-p_j$, $j = 1, \cdots, n+1$, that satisfy

$$0 < p_1 < \alpha_1 < p_2 < ... < p_n < \alpha_n < p_{n+1}. \tag{2.3.10}$$

Furthermore, observe that when $\gamma > 0$, the *Cramèr-Lundberg equation*

$$\kappa(p) \;=\; \gamma, \quad \text{for } \gamma > 0 \tag{2.3.11}$$

has always $n+1$ negative roots $\{-p_j, j = 1, ..., n+1\}$ satisfying (2.3.10).

Denote by $\mathbf{e}_\gamma$ an exponential random variable with parameter $\gamma \geq 0$, independent of X, and for $\gamma = 0$, it is understood that $\mathbf{e}_\gamma = \infty$ with probability 1. Assuming that the condition (2.3.9) holds when $\gamma = 0$, Mordecki [87], [88] shows that

$$\Psi_\gamma^{(-)}(s) \;=\; \sum_{j=1}^{n+1} A_j \frac{p_j}{s + p_j},$$

where $-p_1, ..., -p_{n+1}$ are the negative roots of the equation (2.3.11) and the constants $A_1, ..., A_{n+1}$ are given by

$$A_j \;=\; \frac{\prod_{k=1}^{n}(1 - p_j/\alpha_k)}{\prod_{k=1,k\neq j}^{n+1}(1 - p_j/p_k)}, \quad \text{for } j = 1, ..., n+1.$$

By applying Laplace inversion to $\Psi_\gamma^{(-)}(s)$, Mordecki [87], [88] shows that

$$\mathbb{P}\big(-\underline{X}_{\mathbf{e}_\gamma} \in dx\big) \;=\; \sum_{j=1}^{n+1} A_j p_j e^{-p_j x} dx, \quad x \geq 0.$$

2.3.3 Lévy processes with jumps of phase-type

This class of Lévy processes includes and generalizes exponential jumps Lévy processes detailed earlier. A phase-type Lévy process is constructed by independent sum of a spectrally positive process with a compound Poisson process having negative phase-type jumps. We refer to Mordecki [89] and Asmussen et al. [6] for more details.

A distribution F on $(0, \infty)$ is said to be *phase-type* if it is the distribution of the absorption time in a finite state continuous time Markov process $\{J_t : t \geq 0\}$ with one state Δ absorbing and the remaining ones $1, ..., m$ transient. The parameters of this system are m, the restriction $\mathbf{T}$ of the full intensity matrix to the m transient states and the initial probability (row) vector $\mathbf{a} = (a_1, ..., a_m)$, where $a_i = \mathbb{P}(J_0 = i)$. For any $i = 1, ..., m$, let t_i be the intensity of the transition $i \to \Delta$ and write $\mathbf{t}$ for the column vector of intensities. It follows that F has a density given by $f(x) = \mathbf{a}e^{\mathbf{T}x}\mathbf{t}$ and its Laplace transform is given for $s > 0$ by $\widehat{F}(s) = \int_0^\infty e^{-sx} f(x)dx = \mathbf{a}(s\mathbf{I} - \mathbf{T})^{-1}\mathbf{t}$ which can be analytically extended to the complex plane except at a finite number of poles (the eigen values of $\mathbf{T}$). The phase-type Lévy process has the representation

$$X_t \;=\; X_t^+ - \sum_{j=1}^{N(t)} U_j, \quad t \geq 0,$$

where $\{X_t^{(+)} : t \geq 0\}$ is a spectrally positive Lévy process, $\{N_t : t \geq 0\}$ is a Poisson process with rate λ and $\{U_j : j \geq 1\}$ are i.i.d random variables with a common distribution F; all of the objects mentioned above are mutually independent.

The corresponding Lévy-Khintchine exponent, Ψ, can be analytically extended to the complex plane $\{z \in \mathbb{C} : \mathfrak{Re}(z) \leq 0\}$ with the exception of a finite number of poles (the eigenvalues of $\mathbf{T}$). Define, for each $\alpha > 0$, the finite set of roots, with negative real part, of the Cramèr-Lundberg equation $\Psi(p_i) = \alpha$, i.e.,

$$\mathcal{I}_\alpha \;=\; \{p_i : \Psi(p_i) = \alpha, \mathfrak{Re}(p_i) \;<\; 0\},$$

where multiple roots are counted individually. Next, define, for each $\alpha > 0$, a second set of roots with negative real part

$$\mathcal{J}_\alpha \;=\; \{q_i : \frac{\alpha}{\alpha - \Psi(q_i)} = 0, \mathfrak{Re}(q_i) \;<\; 0\},$$

again taking into account of multiplicity, Mordecki [89] and Asmussen et al. [6] show that

$$\Psi_\alpha^{(-)}(s) \;=\; \frac{\prod_{j\in\mathcal{J}_\alpha}(s - q_j)}{\prod_{j\in\mathcal{J}_\alpha}(-q_j)} \frac{\prod_{j\in\mathcal{I}_\alpha}(-p_j)}{\prod_{j\in\mathcal{I}_\alpha}(s - p_j)},$$

which can be analytically extended to the whole complex plane $\mathbb{C}$ except for the poles at $\{p_j \in \mathcal{I}_\alpha\}$. Applying Laplace inversion to $\Psi_\alpha^{(-)}(s)$, it was shown in [89] and [6] that

$$\mathbb{P}\big(-\underline{X}_{\mathbf{e}_\alpha} \in dx\big) \;=\; \sum_{j=1}^{n}\sum_{k=1}^{m(j)} A_{j,k}\frac{(-p_j x)^{k-1}}{(k-1)!}e^{p_j x}dx, \quad x \geq 0,$$

where $m(j)$ is the multiplicity of root p_j, n is the number of distinct roots and

$$A_{j,k} \;=\; \frac{1}{(m-k)!}\frac{d^{m-k}}{ds^{m-k}}\Big(\frac{\Psi_\alpha^{(-)}(s)(s-p_j)^m}{(-p_j)^k}\Big)\Big|_{s=p_j}.$$

Chapter 3

A Change of Variable Formula with Local Time-Space for Bounded Variation Lévy Processes with Application to Solving the American Put Option Problem[1]

Abstract

We establish a change of variable formula with local time-space for 'ripped' functions of Lévy processes of bounded variation. Our results complement the recent work of Föllmer et al. [50], Eisenbaum ([43], [44]), Peskir ([97], [98]) and Elworthy et al. [46] in which generalized versions of Itô's formula were established with local time-space. The result is applied to solving the American put option problem driven by bounded variation Lévy processes.

3.1 Lévy processes of bounded variation and local time-space

In this chapter we shall establish a change of variable formula for 'ripped' time-space functions of Lévy processes of bounded variation at the cost of an additional integral with respect to local time-space in the formula. Roughly speaking, by a ripped function, we mean here a time-space function which is $C^{1,1}$ on either side of a time-dependent barrier and which may exhibit a discontinuity along the barrier itself. Such functions have appeared in the theory of optimal stopping problems for Markov processes of bounded variation (cf. Peskir and Shiryaev ([95], [96]), Chan ([26], [27]), Avram et al. [7]). Our starting point is to give a brief review of the relevant features of Lévy processes of bounded variation and what is meant by local time-space for these processes.

Suppose that $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is a filtered probability space with filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ satisfying the usual conditions of right continuity and completion. In this text,

[1]This chapter is the extended version of: Kyprianou, A. E. and Surya, B. A. A note on the change of variable formula with local time-space for bounded variation Lévy processes. To appear in *Séminaire de Probabilité XL*, Lecture Notes in Mathematics, Springer-Verlag.

we take as our definition of a Lévy process on $(\Omega, \mathcal{F}, \mathbb{F}, P)$, the strong Markov, $\mathbb{F}$-adapted process $X = \{X(t) : t \geq 0\}$ with paths that are right continuous with left limits (càdlàg) having the properties that $P(X(0) = 0) = 1$ and for each $0 \leq s \leq t$, the increment $X(t) - X(s)$ is independent of $\mathcal{F}_s = \sigma(X_u, u \leq s)$ and has the same distribution as $X(t-s)$. On each finite time interval, X has paths of bounded variation (or just X has bounded variation for short) if and only if for each $t \geq 0$,

$$X(t) \;=\; \mathrm{d}t + \sum_{0<s\leq t} \Delta_s, \tag{3.1.1}$$

where $\mathrm{d} \in \mathbb{R}$ and $\{(s, \Delta_s) : s \geq 0\}$ is a Poisson point process on $[0, \infty) \times (\mathbb{R}\backslash\{0\})$ with (time-space) intensity measure $dt \times \Pi\,(dx)$ satisfying

$$\int_{-\infty}^{\infty} (1 \wedge |x|)\,\Pi\,(dx) \;<\; \infty.$$

Note that the later integrability condition is necessary and sufficient for the convergence of $\sum_{0<s\leq t} |\Delta_s|$. The process X is further a compound Poisson process with drift if and only if $\Pi\,(\mathbb{R}\backslash\{0\}) < \infty$.

For any such Lévy process we say that 0 *is irregular for itself* if

$$P(T = 0) \;=\; 0$$

where T is the first visit of X to the origin,

$$T \;=\; \inf\big\{t > 0 : X_t = 0\big\}$$

with the usual definition $\inf \emptyset = \infty$ being understood in the present context as corresponding to the case that Y never visits the origin over the time interval $(0, \infty)$. Standard theory allows us to deduce that T is a stopping time. With the exception of a compound Poisson process, 0 is always irregular for itself within the class of Lévy processes of bounded variation. Further, again excluding the case of a compound Poisson process, we have that

$$P(T < \infty) > 0 \Longleftrightarrow \mathrm{d} \;\neq\; 0. \tag{3.1.2}$$

We refer to Bertoin [14] for a much deeper account of regularity properties Lévy processes. For the purpose of this text we need to extend the idea of irregularity for points to irregularity of time-space curves.

Definition 3.1.1 Given a Lévy process X with finite variation, a measurable time-space curve $b : [0, \infty) \to \mathbb{R}$ is said to be *irregular for itself for* X if for all $\infty > T \geq s \geq 0$,

$$P_{(s,b(s))}(\#\{t \in (s, T] : X(t) = b(t)\} < \infty) = 1,$$

and $t \in \{s \geq 0 : X(s) = b(s)\}$ if and only if $\lim_{s\uparrow t} |X(s) - b(s)| = 0$.

A curve b which is irregular for itself for X allows for the construction of the almost surely finite counting measure

$$L^b : \mathcal{B}[0,\infty) \to \mathbb{N}$$

defined by

$$L^b[0,t] \;=\; 1 + \sum_{0<s\leq t} \delta_{(X(s)=b(s))}(s) \tag{3.1.3}$$

where δ_x is the Dirac unit mass at point x. Further, $L^b[0,\infty]$ is almost surely 1 if and only if $\mathrm{d} = 0$. We call the right continuous process

$$L^b \;=\; \{L^b_t := L[0,t] : t \geq 0\}$$

local time-space for the curve b. Our choice of terminology here is motivated by Peskir [97] who gave the name *local time-space* for an analogous object defined for continuous semi-martingales.

There seems to be little known about local times of Lévy processes of bounded variation (see however Fitzsimmons and Port [49]) and hence a full classification of all such curves b which are irregular for themselves for X remains an open question. The definition as given is not empty however as we shall now show with the following simple examples.

Example 3.1.2 Suppose simply that $b(t) = x$ for all $t \geq 0$ and some $x \in \mathbb{R}$ and that X is not a compound Poisson process. In this case, the local time process is nothing more than the number of visits to x plus one which is a similar definition to the one given in Fitzsimmons and Port [49]. As can be deduced from the above introduction to Lévy processes of bounded variation, if $\mathrm{d} = 0$ then $L_t = 1$ for all $t > 0$. If on the other hand $\mathrm{d} \neq 0$ then since X has the property that $\{0\}$ is irregular for itself for X then the number of times X hits x in each finite time interval is almost surely finite. Further, X hits x by either creeping upwards over it or creeping downwards below according to the respective sign of d. (Creeping both upwards and downwards is not possible for Lévy processes which do not posses a Gaussian component). Creeping upwards above x occurs at first passage time T if and only if $\lim_{s\uparrow T} X(s) = x$. Since the same statement is true of downward creeping and X may only creep in at most one direction, it follows with the help of the Strong Markov Property that $t \in \{s > 0 : X(s) = x\}$ if and only if $\lim_{s\uparrow t} |X(s) - x| = 0$.

Example 3.1.3 More generally, if $\Pi(\mathbb{R}\backslash\{0\}) = \infty$ then an argument similar to the above shows that if b satisfying $b(0+) = b(0)$ and $|b'(0+)| < \infty$, belongs to the class $C^1(0,\infty)$, then it is also irregular for itself for X. One needs to take advantage in this case of the fact that b has locally linear behaviour. Furthermore, one sees that points t for which $b'(t) = \mathrm{d}$ cannot be hit. We have excluded $\Pi(\mathbb{R}\backslash\{0\}) < \infty$ in order to avoid simple pathological examples such as the case of the compound Poisson process and $b(t) = 0$ for all $t > 0$.

3.2 A generalization of the change of variable formula

In this section we state our result. The idea is to take the change of variable formula and to weaken the assumption on the class of functions to which it applies. For clarity, let us first state the Itô's change of variable formula in the special form that it takes for bounded variation Lévy processes. The proof can be found in a standard text book on semimartingales, see for instance Revuz and Yor [106], Protter [105], and Jacod and Shiryaev [63] (Theorem I.4.57).

Theorem 3.2.1 (Itô's Change of Variable Formula) *Suppose that the time-space function $f \in C^{1,1}([0,\infty)\times\mathbb{R})$. Then for any Lévy process X of bounded variation of the form (3.1.1), it holds almost surely that*

$$f(t,X(t)) - f(0,X(0)) = \int_0^t \frac{\partial f}{\partial t}(s,X(s-))ds + \mathrm{d}\int_0^t \frac{\partial f}{\partial x}(s,X(s-))ds + \sum_{0<s\le t}\Big\{f(s,X(s)) - f(s,X(s-))\Big\}. \tag{3.2.1}$$

Remark 3.2.2 By inspection of the proof of the change of variable formula it is also clear that if for some random time T, $X_t \in D$ for all $t < T$ where D is an open set, then the change of variable formula as given above still holds on the event $\{t \le T\}$ for functions $f \in C^{1,1}([0,\infty), D)$.

The generalization we are interested in consists of weakening the class $C^{1,1}([0,\infty)\times\mathbb{R})$ in the Change of Variable formula to the following class.

Definition 3.2.3 Suppose that $b : [0,\infty) \to \mathbb{R}$ is a measurable function. A function f is said to be $C^{1,1}([0,\infty)\times\mathbb{R})$ *ripped along* b if

$$f(t,x) = \begin{cases} f^{(1)}(t,x) & x > b(t),\ t \ge 0 \\ f^{(2)}(t,x) & x < b(t),\ t \ge 0 \end{cases} \tag{3.2.2}$$

where $f^{(1)}$ and $f^{(2)}$ each belong to the class $C^{1,1}([0,\infty)\times\mathbb{R})$.

We shall prove the following theorem.

Theorem 3.2.4 *Suppose that b is a measurable function which is irregular for itself for X and f is $C^{1,1}([0,\infty)\times\mathbb{R})$ ripped along b. Then for any Lévy process of bounded variation, X, it holds almost surely that*

$$\begin{aligned} f(t,X(t)) - f(0,X(0+)) = &\int_0^t \frac{\partial f}{\partial t}(s,X(s-))ds + \mathrm{d}\int_0^t \frac{\partial f}{\partial x}(s,X(s-))ds \\ &+ \sum_{0<s\le t}\Big\{f(s,X(s)) - f(s,X(s-))\Big\} \qquad (3.2.3)\\ &+ \int_0^t \big\{f(s,X(s+)) - f(s,X(s-))\big\}dL^b_s, \end{aligned}$$

where dL_s^b refers to integration with respect to $s \mapsto L_s^b$.

Note, the term $f(0, X(0+))$ is deliberate in place of $f(0, X(0))$ as, in the case that $X(0) = b(0)$, it is possible that the process $f(\cdot, X(\cdot))$ starts with a jump.

This result complements the recent results of Peskir [97] which concern an extension of Itô's formula for continuous semi-martingales. Peskir accommodates for the case that the time-space function, f, to which Itô's formula is applied has a disruption in its smoothness along a *continuous* space time barrier of *bounded variation*. In particular, on either side of the barrier, the function is equal to a $C^{1,2}(\mathbb{R} \times [0, \infty))$ time-space function but, unlike the case here, it is assumed that there is continuity in f across the barrier. The formula that Peskir obtained has an additional integral with respect to the semi-martingale local time at zero of the distance of the underlying semi-martingale from the boundary (this is again a semi-martingale) which he calls *local time-space*. As mentioned above, we have chosen for obvious reasons to refer to the integrator in the additional term obtained in Theorem 3.2.4 as local time-space also. Peskir's results build further on those of Föllmer et al. [50] and Eisenbaum [43] for Brownian motion and in this sense our results now bring the discussion into the particular and somewhat simpler class of bounded variation semimartingales that we study here. Eisenbaum [44], Elworthy et al. [46] and Peskir [97] all have further results for general and special types of semi-martingales. However, the present study is currently the only one which considers discontinuous functions and hence the necessity to introduce a local time-space as a counting measure rather than an occupation density at zero of the semimartingale $X - b$ as one normally sees. Note in the case at hand, the semimartingale definition of local time at zero of $X - b$ is in fact identically zero (cf. Protter [105]). Other definitions of local time-space may be possible in order to work with more general classes of curves than those given in Definition 1 and hence the current presentation merely scratches the surface of the problem considered.

3.3 Proofs and main calculations

Proof of Theorem 3.2.4

Proof The essence of the proof is based around a telescopic sum which we shall now describe. Define the inverse local time process $\tau = \{\tau_t : t \geq 0\}$ where

$$\tau_t \;=\; \inf\left\{s > 0 : L_s^b > t\right\}$$

for each $t \geq 0$. Note the second strict inequality in the definition ensures that τ is a càdlàg process and since $L_0^b = 1$ by definition, it follows that $\tau_0 = 0$. The process τ is nothing more than a step function which increases on the integers $k = 1, 2, 3, ...$ by an amount corresponding to the length of the excursion of X from b whose right end point corresponds to the k-th crossing of b by X. Note that even when $X_0 \neq b(0)$ we count the section of the path of X until it first meets b as an (incomplete) excursion.

The increment in $\{f(s, X(s)) : s \geq 0\}$ between $s = 0+$ and $s = t$ can be seen as the accumulation of the increments incurred by X crossing the boundary b, the excursions of X from b and the final increment between the last time of contact of X with b and time t. We have

$$\begin{aligned} f(t, X(t)) - f(0, X(0+)) &= \int_0^t \big\{f(s, X(s+)) - f(s, X(s-))\big\} dL_s^b \\ &+ \sum_{s \leq L_t^b} \big\{f(\tau_s, X_{\tau_s}) - f(\tau_{s-}, X_{\tau_{s-}})\big\} \mathbf{1}_{\{|\Delta\tau_s|>0\}} \quad (3.3.1) \\ &+ \big\{f(t, X(t)) - f(\tau_{L_t^b}, X_{\tau_{L_t^b}+})\big\}. \end{aligned}$$

The proof is then completed once we know that the increments in the curly brackets of the second and third term on the right hand side of (3.3.1) observe the same development as the change of variable formula. Indeed, taking into account of the Strong Markov Property, it would suffice to prove that under the given assumptions on f we have that for all $t \in (0, \infty]$

$$\begin{aligned} & f(t \wedge \eta, X(t \wedge \eta)) - f(0, X(0+)) \\ &= \int_0^{t\wedge\eta} \frac{\partial f}{\partial t}(s, X(s-)) ds + \mathrm{d} \int_0^{t\wedge\eta} \frac{\partial f}{\partial x}(s, X(s-)) ds \\ &+ \sum_{0<s\leq t\wedge\eta} \big\{f(s, X(s)) - f(s, X(s-))\big\}. \quad (3.3.2) \end{aligned}$$

Note that η is the first strictly positive time that $X - b = 0$, that is to say that

$$\eta = \inf\big\{t > 0 : X_t - b(t) = 0\big\}.$$

The statement in (3.3.2) is intuitively appealing since up to the stopping time η the process X does not intersect with the boundary b and hence the discontinuity in f should not appear in a development of the function $f(\cdot, X(\cdot))$. The result is proved in the lemma below and thus concludes the proof of the main result.

Lemma 3.3.1 *Under the assumptions of Theorem 3.2.4, the identity (3.3.2) holds for all* $t \in (0, \infty]$.

Proof First fix some $\kappa > 0$, define

$$\sigma_{\kappa,0} = \inf\big\{t \geq 0 : |X(t) - b(t)| > \kappa\big\}.$$

and $\Omega_\kappa = \{\omega \in \Omega : \sigma_{\kappa,0} < \eta\}$. Next define for each $j \geq 1$ the stopping times

$$\sigma_{\kappa,j} = \inf\Big\{t > \sigma_{\kappa,j-1} : |X(t) - b(t)| < \frac{1}{2}\big|X(\sigma_{\kappa,j-1}) - b(\sigma_{\kappa,j-1})\big|\Big\}$$

where we again work with the usual definition $\inf\emptyset = \infty$. On the set $\Omega_\kappa \cap \{\eta < \infty\}$ we have that

$$\limsup_{j\uparrow\infty} \big|X(\sigma_{\kappa,j}) - b(\sigma_{\kappa,j})\big| \leq \lim_{j\uparrow\infty} \left(\frac{1}{2}\right)^j \big|X_{\sigma_{\kappa,0}}\big| = 0$$

and hence by the definition of irregularity of b for itself for X,

$$\lim_{j\uparrow\infty}\sigma_{\kappa,j} \;=\; \eta \tag{3.3.3}$$

where the limit is interpreted to be infinite on the set $\{\eta=\infty\}$. It is also clear that, since X has right continuous paths,

$$\lim_{\kappa\downarrow 0} P\left(\Omega_\kappa\right) \;=\; 1. \tag{3.3.4}$$

Over the time interval $[\sigma_{\kappa,j-1},\sigma_{\kappa,j})$ the process X does not enter a tube of positive, $\mathcal{F}_{\sigma_{\kappa,j-1}}$-measurable radius around the curve b, we may appeal to then standard Change of Variable Formula to deduce that on Ω_κ

$$\begin{aligned}
&f\left(\sigma_{\kappa,j}\wedge t, X_{\sigma_{\kappa,j}\wedge t}\right)-f\left(\sigma_{\kappa,j-1}\wedge t, X_{\sigma_{\kappa,j-1}\wedge t}\right)\\
&=\int_{\sigma_{\kappa,j-1}\wedge t}^{\sigma_{\kappa,j}\wedge t}\frac{\partial f}{\partial t}(s,X(s-))ds+\mathrm{d}\int_{\sigma_{\kappa,j-1}\wedge t}^{\sigma_{\kappa,j}\wedge t}\frac{\partial f}{\partial x}(s,X(s-))ds\\
&\quad+\sum_{\sigma_{\kappa,j-1}\wedge t<s\le\sigma_{\kappa,j}\wedge t}\big\{f(s,X(s))-f(s,X(s-))\big\}.
\end{aligned}$$

Hence on Ω_κ we have

$$\begin{aligned}
&f(\eta\wedge t,X(\eta\wedge t))-f(\sigma_{\kappa,0},X(\sigma_{\kappa,0}))\\
&=\sum_{j\ge1}\left\{f\left(\sigma_{\kappa,j}\wedge t,X(\sigma_{\kappa,j}\wedge t)\right)-f\left(\sigma_{\kappa,j-1}\wedge t,X(\sigma_{\kappa,j-1}\wedge t)\right)\right\}\\
&=\sum_{j\ge1}\int_0^{\eta\wedge t}\Big\{\frac{\partial f}{\partial t}(s,X(s-))+\mathrm{d}\frac{\partial f}{\partial x}(s,X(s-))\Big\}\mathbf{1}_{(\sigma_{\kappa,j-1}\wedge t<s\le\sigma_{\kappa,j}\wedge t)}ds\\
&\quad+\sum_{j\ge1}\sum_{0<s\le\eta\wedge t}\big\{f(s,X(s))-f(s,X(s-))\big\}\mathbf{1}_{(\sigma_{\kappa,j-1}\wedge t<s\le\sigma_{\kappa,j}\wedge t)}\\
&=\int_0^{\eta\wedge t}\frac{\partial f}{\partial t}(s,X(s-))ds+\mathrm{d}\int_0^{\eta\wedge t}\frac{\partial f}{\partial x}(s,X(s-))ds\\
&\quad+\sum_{0<s\le\eta\wedge t}\big\{f(s,X(s))-f(s,X(s-))\big\},
\end{aligned}$$

where the final equality follows from (an almost sure version of) Fubini's theorem which in turn appeals to the assumption that the limits of f, $\partial f/\partial t$ and $\partial f/\partial x$ all exist and are finite when approaching any point on the curve b. In particular, to deal with the final term in the second equality, note that an almost sure uniform bound of the form

$$\big|f(s,X(s))-f(s,X(s-))\big|\le C\big|\triangle X(s)\big|$$

holds (for random C) because of the assumptions on $\partial f/\partial x$ and hence the double sum converges (as X is a process of bounded variation). Since κ may be chosen arbitrarily small, (3.3.4) shows that (3.3.2) is true almost surely on Ω. □

3.4 Application to solving the American put option problem

We present in this section a connection between the change of variable formula developed in the previous section with the problem of finding the arbitrage-free price of the finite maturity American put option. The option confers the right to sell a unit of stock at any time up to a finite time horizon T at a strike price K.

We assume that the stock pays no dividends during the lifetime of the option and the evolution of the stock price process $S_t = e^{X_t}$ is driven under a chosen martingale measure $\mathbb{P}_x$ by a bounded variation Lévy process of the form

$$X_t \;=\; x + (r+\omega)t + J_t, \tag{3.4.1}$$

where $(J_t, t \geq 0)$ is a pure jump Lévy process of bounded variation defined as $J_t = \sum_{0<s\leq t} \Delta_s$ and $\{(s,\Delta_s) : s \geq 0\}$ is a Poisson point process on $[0,\infty) \times (\mathbb{R}\backslash\{0\})$ with (time-space) intensity measure $dt \times \Pi\,(dx)$ (see the expression (3.1.1)). Throughout the remaining of this section, we assume that the Lévy measure Π satisfies

$$\int_{-\infty}^{\infty} \big(e^{|y|} - 1\big)\Pi(dy) \;<\; \infty, \tag{3.4.2}$$

and the rate $(r+\omega)$ in (3.4.1) is assumed to be strictly positive. Furthermore, we assume that the discounted process $(e^{-rt}S_t, t \geq 0)$ is $\mathbb{P}_x$-martingale, implying that

$$\mathbb{E}_x\big(e^{-rt}S_t\big) \;=\; e^x.$$

This condition implies that the parameter ω in (3.4.1) is given by

$$\omega \;=\; -\int_{-\infty}^{\infty} \big(e^y - 1\big)\Pi(dy), \tag{3.4.3}$$

which is well-defined due to the integral test (3.4.2). Note that under the martingale condition (3.4.3) and the integral test (3.4.2), it can be shown using the formula (3.2.1) that the stock price process S_t fulfills the *arbitrage-free* condition

$$\mathbb{E}_x\Big(dS_t - rS_{t-}dt\Big) \;=\; 0.$$

The problem of interest in this section is to characterize the arbitrage-free price of the finite maturity American put option

$$V(t,x) \;=\; \sup_{0\leq\tau\leq t} \mathbb{E}_x\Big(e^{-r\tau}\big(K - e^{X_\tau}\big)^+\Big) \tag{3.4.4}$$

for all $(t,x) \in [0,T] \times \mathbb{R}_+$ where τ is a Markov stopping time of X.

Adapting arguments of Peskir [98], we derive using the change of variable formula (3.2.3) a nonlinear integral equation for optimal stopping boundary of the problem (3.4.4) within a bounded variation Lévy process and show that the optimal value function V is continuous across the boundary. Taking into account the continuous pasting condition, we give a proof similar to Jacka [62] and Peskir [98] for the uniqueness of the nonlinear integral equation and show that the value function V solves uniquely a free boundary problem of parabolic integro-differential type [2].

The results are given by the following two theorems.

Theorem 3.4.1 (Free boundary problem) *Assume that the Lévy measure* Π *of* X *(3.4.1) satisfies the integrability condition (3.4.2) and* $b : [0,T] \to (-\infty, \log(K)]$ *is a curved boundary which is irregular for itself for* X*. Suppose that* (U, b) *is a solution pair, with* $U \in C^{1,1}([0,T] \times \mathbb{R})$ *ripped along the curved boundary* b*, to the problem:*

$$\Big(-\frac{\partial}{\partial t} + \mathcal{L}_X - r\Big)U(t,x) = 0 \quad \textit{for } (t,x) \in \mathcal{C} \tag{3.4.5}$$

$$U(0,x) = (K - e^x)^+ \quad \textit{for all } x \in \mathbb{R}, \tag{3.4.6}$$

$$U(t,x) = (K - e^x)^+ \quad \textit{for } x = b(t) \ \textit{(continuous fit)}, \tag{3.4.7}$$

$$U(t,x) > (K - e^x)^+ \quad \textit{for } (t,x) \in \mathcal{C} \tag{3.4.8}$$

$$U(t,x) = (K - e^x)^+ \quad \textit{for } (t,x) \in \mathcal{D} \tag{3.4.9}$$

where the continuation region $\mathcal{C}$ *and the stopping region* $\mathcal{S} = \overline{\mathcal{D}}$ *are defined by*

$$\mathcal{C} = \big\{(t,x) \in [0,T] \times \mathbb{R} \,\big|\, x > b(t)\big\} \tag{3.4.10}$$

and

$$\mathcal{D} = \big\{(t,x) \in [0,T] \times \mathbb{R} \,\big|\, x < b(t)\big\}, \tag{3.4.11}$$

and the infinitesimal generator $\mathcal{L}_X$ *of* X *(3.4.1) is defined by*

$$\mathcal{L}_X U(t,x) = (r+\omega)\frac{\partial U}{\partial x}(t,x) + \int_{-\infty}^{\infty} \Big(U(t,x+y) - U(t,x)\Big)\Pi(dy). \tag{3.4.12}$$

Then, the curved boundary b *solves for all* $t \in (0,T]$ *the nonlinear integral equation*[3]

$$K - e^{b(t)} = e^{-rt}\mathbb{E}_{b(t)}\big(K - e^{X_t}\big)^+ + rK\int_0^t e^{-ru}\mathbb{P}_{b(t)}\big(X_{u-} \le b(t-u)\big)du. \tag{3.4.13}$$

Theorem 3.4.2 (Uniqueness) *If the value function* V *of the problem (3.4.4) solves the free boundary problem (3.4.5)-(3.4.12) and the optimal stopping time is the first passage time* τ_b^- *of* X *below a curved boundary* b *solving the integral equation (3.4.13), then* (V, b) *represents the unique pair solution to the problem (3.4.5)-(3.4.12).*

[2]For jump-diffusion processes, the uniqueness of a free boundary problem of parabolic integro-differential type associated to the optimal stopping problem (3.4.4) was discussed in Pham [100].

[3]For exponential of a linear Brownian motion $S_t(x) = x\exp\big(\sigma B_t + (r - 1/2\sigma^2)t\big)$, it was shown in Kim [67], El Karoui and Karatzas [45], Jacka [62], Myneni [90], Carr et al. [23], and Peskir [98] that the optimal boundary $h(t)$ of the stopping problem $V(t,x) = \sup_{0\le\tau\le t}\mathbb{E}\big(e^{-r\tau}\big(K - S_\tau(x)\big)^+\big)$ solves a nonlinear integral equation of the similar form:

$$K - h(t) = e^{-rt}\mathbb{E}\big(K - S_t(h(t))\big)^+ + rK\int_0^t e^{-ru}\mathbb{P}\big(S_u(h(t)) \le h(t-u)\big)du.$$

3.4.1 Proof and main calculations of Theorem 3.4.1

To start with, let us remind ourself that the curved boundary b is irregular for itself for the Lévy process (3.4.1) and T is a finite maturity time. Next, let us consider for a fixed $t \in (0, T]$ a function $f : [0, t] \times \mathbb{R} \to \mathbb{R}_+$ defined by

$$f(u, x) \;=\; e^{-ru} U(t - u, x),$$

where U is a $C^{1,1}$ function, ripped along the curved boundary b, that solves the free boundary problem (3.4.5)-(3.4.12). Observe that the functions f and U have the same number of continuous derivatives w.r.t t and x. Since U is ripped along b, we have by the construction that f is also a $C^{1,1}$ function that is ripped along b. Therefore, we can now apply the change of variable formula (3.2.3) to get

$$\begin{aligned} f(s, X_s) \;&=\; f(0, X_0) + \int_0^s \frac{\partial f}{\partial u}(u, X_{u-})du + (r+\omega)\int_0^s \frac{\partial f}{\partial x}(u, X_{u-})du \\ &\quad + \sum_{0<u\leq s} \Big\{ f(u, X_{u-} + \Delta X_u) - f(u, X_{u-}) \Big\} \qquad (3.4.14) \\ &\quad + \int_0^s \big(f(u, X(u+)) - f(u, X(u-)) \big) dL_u^b(X). \end{aligned}$$

Using the chain rule for partial differentiation, we see that

$$\frac{\partial f}{\partial u} \;=\; -\Big(re^{-ru} U + e^{-ru}\frac{\partial U}{\partial t}\Big) \quad \text{and} \quad \frac{\partial f}{\partial x} \;=\; e^{-rt}\frac{\partial U}{\partial x}.$$

Inserting these expressions in (3.4.14), we obtain

$$\begin{aligned} &e^{-rs} U(t - s, X_s) \\ &\quad = U(t, x) + \int_0^s e^{-ru}\Big(-rU - \frac{\partial U}{\partial t} + (r+\omega)\frac{\partial U}{\partial x}\Big)(t-u, X_{u-})du \\ &\qquad + \sum_{0<u\leq s} e^{-ru}\Big\{ U(t-u, X_{u-} + \Delta X_u) - U(t-u, X_{u-}) \Big\} \qquad (3.4.15) \\ &\qquad + \int_0^s e^{-ru}\big\{ U(t-u, X_{u+}) - U(t-u, X_{u-}) \big\} dL_u^b(X). \end{aligned}$$

Notice that the sum in the foregoing expression converges in absolute value due to the assumptions on U and the fact that X has path of bounded variation.

On recalling the fact that for a Borel set $\Lambda \subset \mathbb{R}$, with $0 \notin \Lambda$, we have for every (bounded) measurable function h that

$$\int_0^t \int_\Lambda h(u, y)\nu(dy, du) \;=\; \sum_{0<u\leq t} h(u, \Delta X_u)\mathbf{1}_\Lambda(\Delta X_u), \qquad (3.4.16)$$

where $\nu(dy, du)$ is a Poisson random measure, with the space-time compensator $\Pi(dy) \times du$, defined by

$$\nu(dy, du) \;=\; \sum_{s\geq 0} \mathbf{1}_{\{\Delta X_s \neq 0\}} \delta_{(\Delta X_s, s)}(dy, du),$$

where δ_x denotes the Dirac measure at point x, see for instance Proposition 1.16 in Chapter II of Jacod and Shiryaev [63] for details. By adding and subtracting

$$\int_0^s \int_{\mathbb{R}} e^{-ru}\Big(U(t-u, X_{u-}+y) - U(t-u, X_{u-})\Big)\Pi(dy)du$$

in the equation (3.4.15) we finally obtain

$$\begin{aligned} &e^{-rs}U(t-s, X_s) \\ =&U(t,x) + \int_0^s e^{-ru}\Big(-\frac{\partial}{\partial t} + \mathcal{L}_X - r\Big)U(t-u, X_{u-})du + \mathcal{M}_s \\ &+ \int_0^s e^{-ru}\big\{U(t-u, X_{u+}) - U(t-u, X_{u-})\big\}dL_u^b(X), \end{aligned} \tag{3.4.17}$$

where $\mathcal{L}_X$ is the infinitesimal generator of the Lévy process (3.4.1) defined earlier in (3.4.12) and the stochastic process $(\mathcal{M}_s)_{0\le s\le t}$ is a $\mathbb{P}_x$-(local) martingale defined by

$$\begin{aligned} \mathcal{M}_s \ =& \int_0^s \int_{\mathbb{R}} e^{-ru}\Big(U(t-u, X_{u-}+y) - U(t-u, X_{u-})\Big)\nu(dy, du) \\ &- \int_0^s \int_{\mathbb{R}} e^{-ru}\Big(U(t-u, X_{u-}+y) - U(t-u, X_{u-})\Big)\Pi(dy)du. \end{aligned} \tag{3.4.18}$$

Using the fact that U is $C^{1,1}$ ripped at b and is continuous at b (see (3.4.7)), we see that there exists for all $u \in [0, s]$ a positive constant C_s such that

$$\big|U(u, x+y) - U(u, x)\big| \ \le\ C_s\big|y\big| \quad \text{for all } x, y \in \mathbb{R}.$$

Note that the constant C_s depends on the interval of time over which u is considered, i.e., the estimate has a uniform constant C_s for all u in the interval $[0, s]$. Hence, in view of the integrability condition (3.4.2), we see that the last double integral in (3.4.18) converges in absolute value, and therefore we have that

$$\begin{aligned} \mathbb{E}\big|\mathcal{M}_s\big| \ \le& \ \mathbb{E}\int_0^s \int_{\mathbb{R}} e^{-ru}\Big|U(t-u, X_{u-}+y) - U(t-u, X_{u-})\Big|\nu(dy, du) \\ &+ \ \mathbb{E}\int_0^s \int_{\mathbb{R}} e^{-ru}\Big|U(t-u, X_{u-}+y) - U(t-u, X_{u-})\Big|\Pi(dy)du \\ \le& \ \frac{2C_s}{r}\big(1-e^{-rs}\big)\int_{\mathbb{R}} \big|y\big|\Pi(dy). \end{aligned} \tag{3.4.19}$$

Moreover, by recalling that U is a $C^{1,1}$ function and is continuous along the curved boundary b and that the Lévy process X has paths of bounded variation, we can apply the *compensation formula* (see for instance Theorem 4.4 in Kyprianou [69]) to have that

$$\begin{aligned} &\mathbb{E}\Big(\int_0^s \int_{\mathbb{R}} e^{-ru}\Big(U(t-u, X_{u-}+y) - U(t-u, X_{u-})\Big)\nu(dy, du)\Big) \\ &\quad = \ \mathbb{E}\Big(\int_0^s \int_{\mathbb{R}} e^{-ru}\Big(U(t-u, X_{u-}+y) - U(t-u, X_{u-})\Big)\Pi(dy)du\Big). \end{aligned} \tag{3.4.20}$$

Hence, in view of (3.4.19) it follows that the process $(\mathcal{M}_s)_{0\le s\le t}$ is an L_1-integrable $\mathbb{P}_x$-martingale and hence vanishes after taking expectation under the measure $\mathbb{P}_x$.

On noticing the fact that $(-\frac{\partial}{\partial t}+\mathcal{L}_X-r)U(t,x)=0$ for all $(t,x)\in\mathcal{C}$ and

$$\Big(-\frac{\partial}{\partial t}+\mathcal{L}_X-r\Big)(K-e^x) \;=\; -rK,$$

we have following (3.4.17) that

$$\begin{aligned} e^{-rs}U(t-s,X_s) \;=\; & U(t,x)-rK\int_0^s e^{-ru}\mathbf{1}_{(X_{u-}\le b(t-u))}du+\mathcal{M}_s \\ & +\int_0^s e^{-ru}\big\{U(t-u,X_{u+})-U(t-u,X_{u-})\big\}dL_u^b(X), \end{aligned} \tag{3.4.21}$$

holds $\mathbb{P}_x$ almost surely. Note that we have used in (3.4.21) the fact that the curved boundary b is bounded from above by $\log(K)$. Since $U(0,x)=(K-e^x)^+$ for all $x\in\mathbb{R}$ we have after inserting $s=t$ in (3.4.21) and taking expectation under $\mathbb{P}_x$ that

$$\begin{aligned} U(t,x) =& e^{-rt}\mathbb{E}_x\Big(K-e^{X_t}\Big)^+ + rK\int_0^t e^{-ru}\mathbb{P}_x\big(X_{u-}\le b(t-u)\big)du \\ & -\mathbb{E}_x\Big(\int_0^t e^{-ru}\big\{U(t-u,X_{u+})-U(t-u,X_{u-})\big\}dL_u^b(X)\Big) \end{aligned} \tag{3.4.22}$$

for all $(t,x)\in[0,T)\times\mathbb{R}$. On recalling that $U(t,x)=(K-e^x)^+$ for $(t,x)\in\overline{\mathcal{D}}$, we see that

$$\begin{aligned} (K-e^x)^+ =& e^{-rt}\mathbb{E}_x\Big(K-e^{X_t}\Big)^+ + rK\int_0^t e^{-ru}\mathbb{P}_x\big(X_{u-}\le b(t-u)\big)du \\ & -\mathbb{E}_x\Big(\int_0^t e^{-ru}\big\{U(t-u,X_{u+})-U(t-u,X_{u-})\big\}dL_u^b(X)\Big). \end{aligned} \tag{3.4.23}$$

Since U is a $C^{1,1}$ function ripped along the curved boundary b and is continuous at b (see (3.4.7)), the second expectation on the right hand side of (3.4.23) vanishes. Hence, we deduce that b must solve the integral equation

$$(K-e^x)^+ = e^{-rt}\mathbb{E}_x\Big(K-e^{X_t}\Big)^+ + rK\int_0^t e^{-ru}\mathbb{P}_x\big(X_{u-}\le b(t-u)\big)du,$$

for all $x\le b(t)$ and all $t\in(0,T]$. By inserting $x=b(t)$ in the foregoing equation, we come to rest at a free-boundary equation that the boundary b has to solve:

$$\big(K-e^{b(t)}\big)^+ \;=e^{-rt}\mathbb{E}_{b(t)}\Big(K-e^{X_t}\Big)^+ + rK\int_0^t e^{-ru}\mathbb{P}_{b(t)}\big(X_{u-}\le b(t-u)\big)du.$$

Thus, the claim that the curved boundary b solves the nonlinear integral equation (3.4.13) is then established. □

3.4.2 Proof and main calculations of Theorem 3.4.2

To start with let us denote by τ_h^- the first exit time of X below a curved boundary h defined by

$$\tau_h^- \ = \ \inf\{u > 0 : X_u \le h(t-u)\} \wedge t. \tag{3.4.24}$$

Suppose that (W, c) is a solution pair, with $W \in C^{1,1}([0,T]\times\mathbb{R})$ ripped along a curved boundary c, to the free boundary problem (3.4.5)-(3.4.12). By applying the formula (3.2.3) to the function W subject to the pasting condition (3.4.7), we have following the similar calculations as before that

$$e^{-rt}W(0, X_t) \ = \ W(t,x) - rK\int_0^t e^{-ru}\mathbf{1}_{(X_{u-}\le c(t-u))}du + \mathcal{M}_t \tag{3.4.25}$$

where $\mathcal{M}$ is in principle a $\mathbb{P}_x$ (local) martingale process, but in view of (3.4.19) and (3.4.20) one can argue that it is $\mathbb{P}_x$-martingale. Recall that $W(0,x) = (K - e^x)^+$ for all $x \in \mathbb{R}_+$ and $W(t,x) = (K - e^x)^+$ for all $x \le c(t)$. Following these two facts, one can deduce following the same arguments as before that the curved boundary c solves the integral equation (3.4.13). Moreover, by replacing t with stopping time τ_c^- in the expression (3.4.25) and taking expectation under $\mathbb{P}_x$, we have for all $(t,x) \in [0,T]\times\mathbb{R}$ that

$$W(t,x) \ = \ \mathbb{E}_x\Big(e^{-r\tau_c^-}(K - e^{X_{\tau_c^-}})^+\Big). \tag{3.4.26}$$

Since the value function V of the problem (3.4.4) is assumed to solve the free boundary problem (3.4.5)-(3.4.12) and the optimal stopping time is the first exit of X below a curved boundary b solving (3.4.13), we have that

$$V(t,x) \ = \ \sup_{0\le\tau\le t}\mathbb{E}_x\Big(e^{-r\tau}(K - e^{X_\tau})^+\Big) \ = \mathbb{E}_x\Big(e^{-r\tau_b^-}(K - e^{X_{\tau_b^-}})^+\Big). \tag{3.4.27}$$

Following (3.4.26) and (3.4.27), we see for every $t > 0$ and $x \in \mathbb{R}$ that

$$V(t,x) \ \ge \ W(t,x). \tag{3.4.28}$$

This inequality implies that

$$c(t) \ \ge \ b(t) \quad \text{for all } t \in [0,T]. \tag{3.4.29}$$

Suppose that there exists $t \in (0,T)$ such that $c(t) > b(t)$. Next, let us take for a given $t \in (0,T)$ a point $x \in (b(t), c(t))$. By replacing s and t with the stopping time τ_b^- in (3.4.25) and (3.4.21), respectively, we obtain after taking expectation under the measure $\mathbb{P}_x$ that

$$\mathbb{E}_x\Big(e^{-r\tau_b^-}(K - e^{X_{\tau_b^-}})^+\Big) \ = \ W(t,x) - rK\mathbb{E}_x\Big(\int_0^{\tau_b^-} e^{-ru}\mathbf{1}_{(X_{u-}\le c(t-u))}du\Big),$$

and

$$\mathbb{E}_x\Big(e^{-r\tau_b^-}(K - e^{X_{\tau_b^-}})^+\Big) \ = \ V(t,x).$$

On remarking that $V(t,x) \geq W(t,x)$ for all $(t,x) \in [0,T] \times \mathbb{R}$ (see equation (3.4.28)), we deduce from the two foregoing equations that

$$\mathbb{E}_x\Big(\int_0^{\tau_b^-} e^{-ru} \mathbf{1}_{(X_{u-} \leq c(t-u))} du \Big) \;\leq\; 0,$$

which can not be true. Hence, in absence of the existence of such a point x, it follows that

$$b(t) \;=\; c(t) \quad \text{for all } t \in [0,T].$$

As a result, having shown that the integral equation (3.4.13) admits a unique solution for the optimal stopping boundary of the problem (3.4.4), we deduce following the two expressions (3.4.26) and (3.4.27) that

$$W(t,x) \;=\; \mathbb{E}_x\Big(e^{-r\tau_b^-} (K - e^{X_{\tau_b^-}})^+ \Big) \;=\; V(t,x)$$

for all $(t,x) \in [0,T] \times \mathbb{R}$. Thus, the claim that the integral equation (3.4.13) and the free boundary problem (3.4.5)-(3.4.12) admit a unique solution is then proved. □

Remark 3.4.3 In fact using the change of variable formula (3.2.3), it can be shown in the similar calculations as before that a solution to the free boundary problem (3.4.5)-(3.4.12) coincides with the value function V of the optimal stopping problem (3.4.4), and the optimal stopping time of (3.4.4) is the first exit time τ_b^- of X below a curved boundary b that solves the integral equation (3.4.13).

3.5 Concluding remarks

To summarize, we have seen using the free boundary problem (3.4.5)-(3.4.11) that the change of variable formula (3.2.3) with local time-space on a irregular curved boundary b, developed earlier in Section 2, has been able to deliver three important things. Firstly, we show using the formula that the smallest superharmonic majorant property of the value function V (3.4.4) is simplified to the analytic condition of continuous pasting at the stopping boundary b. By imposing the continuous pasting condition, we derive using the change of variable formula a nonlinear integral equation for b and show that b is the optimal stopping boundary for the problem (3.4.4). Secondly, given the continuous pasting condition, we show using (3.2.3) that such a nonlinear integral equation admits a unique solution. Thirdly, as a final result of the use of the formula (3.2.3), we see that it is possible to show that (V,b) is the unique solution pair to the free boundary problem.

Chapter 4

On the Novikov-Shiryaev Optimal Stopping Problems in Continuous Time[1]

Abstract

Novikov and Shiryaev [91] give explicit solutions to a class of optimal stopping problems for random walks based on other similar examples given in Darling et al. [33]. We give the analogue of their results when the random walks are replaced by Lévy processes. Further we show that the solutions show no contradiction with the conjecture given in Alili and Kyprianou [3] that there is smooth pasting at the optimal boundary if and only if the boundary of the stopping region is irregular for the interior of the stopping region.

4.1 Introduction

Let $X = \{X_t; t \geq 0\}$ be a Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ satisfying the usual conditions, see Chapter 2 for more details. Consider for a given Lévy process X an optimal stopping problem of the form

$$V(x) = \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau} G(X_\tau) \mathbf{1}_{(\tau < \infty)}\Big), \tag{4.1.1}$$

where $q \geq 0$ and $\mathcal{T}_{[0,\infty]}$ is the family of stopping times with respect to the filtration $\mathcal{F}_t = \sigma(X_s, s \leq t)$. The purpose of this chapter is to characterize the solution to the problem (4.1.1) for the choices of gain functions

$$G(x) = (x^+)^n, \quad \text{for } n = 1, 2, 3, \ldots$$

under the hypothesis that

$$\text{either } q > 0 \text{ or } q = 0 \text{ and } \limsup_{t \uparrow \infty} X_t < \infty. \tag{H}$$

[1]This chapter is the extended version of: Kyprianou, A.E. and Surya, B. A. On the Novikov-Shiryaev optimal stopping problems in continuous time. *Elect. Comm. Probab.*, **10** (2005), 146-154.

Note that when $q = 0$ and $\limsup_{t\uparrow\infty} X_t < \infty$ it is clear that it is never optimal to stop in (4.1.1) for the given choices of gain function G.

This chapter thus verifies that the results of Novikov and Shiryaev [91] for random walks carry over into the context of the Lévy process as predicted by the aforementioned authors. Novikov and Shiryaev [91] write:

"The results of this paper can be generalized to the case of stochastic processes with continuous time parameter (that is for Lévy processes instead of random walk). This generalization can be done by passage of limit from discrete time case (similarly to the techniques used in Mordecki [87] for pricing American options) or by use of the technique of pseudo-differential operators (described e.g. in the monograph Boyarchenko and Levendorskii [20] in the context of Lévy processes)".

We appeal to neither of the two methods referred to by Novikov and Shiryaev however. Instead we work with fluctuation theory of Lévy processes which is essentially the direct analogue of the random walks counterpart used in Novikov and Shiryaev [91]. In this sense our proofs are loyal to those of the latter. Minor additional features of our proofs are that we also allow for discounting as well avoiding the need to modify the gain function in order to obtain the solution. Truncation techniques are also avoided as much as possible. Undoubtedly however, the link with Appell polynomials as laid out by Novikov and Shiryaev remains the driving force of the solution. In addition we show that the solutions show no contradiction with the conjecture given in Alili and Kyprianou [3] that there is smooth pasting at the optimal boundary if and only if the boundary of the stopping region is regular for the interior of the stopping region.

4.2 Main results

In order to state the main results we need to introduce one of the tools identified by Novikov and Shiryaev to be instrumental in solving the optimal stopping problems at hand.

Definition 4.2.1 (Appell Polynomials) Suppose that Y is a non-negative random variable with n-th cummulant given by $\kappa_n \in (0, \infty]$ for $n = 1, 2, ...$ Then define the Appell polynomials iteratively as follows. Take $Q_0(x) = 1$ and assuming that $\kappa_n < \infty$ (equivalently Y has an n-th moment) given $Q_{n-1}(x)$ we define $Q_n(x)$ via

$$\frac{d}{dx}Q_n(x) \;=\; nQ_{n-1}(x). \tag{4.2.1}$$

This defines Q_n up to a constant. To pin this constant down we insist that $\mathbb{E}(Q_n(Y)) = 0$. The first three Appell polynomials are given for example by

$$\begin{aligned} Q_0(x) \;=\; 1,\;\; Q_1(x) \;=\; x - \kappa_1,\;\; Q_2(x) \;=\; (x-\kappa_1)^2 - \kappa_2 \\ Q_3(x) \;= (x-\kappa_1)^3 - 3\kappa_2(x-\kappa_1) - \kappa_3, \end{aligned} \tag{4.2.2}$$

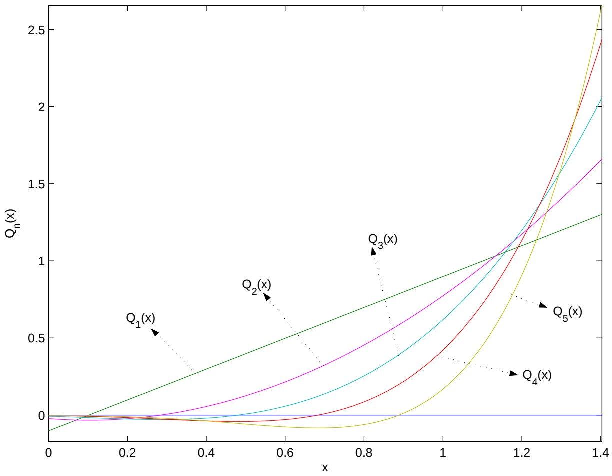


Figure 4.1: Plots of the first five Appell polynomials $Q_n(x)$, $n = 1, 2, \ldots, 5$, generated by upward jumps compound Poisson process having drift $\mathrm{d} = -0.1$.

under the assumption that $\kappa_3 < \infty$ (see Figure 4.1 above for the plots of $Q_n(x)$, $n = 1, 2, ..5$). We refer to Schoutens [112] for further details of Appell polynomials.

In the following theorem, we shall work with the Appell polynomials generated by the random variable $Y = \overline{X}_{\mathbf{e}_q}$ where for each $t \in [0, \infty]$, $\overline{X}_t = \sup_{s \in [0,t]} X_s$ and $\mathbf{e}_q$ is an exponentially distributed random variable which is independent of the Lévy process X. We shall work with the convention that when $q = 0$, the variable $\mathbf{e}_q$ is understood to be equal to ∞ with probability 1.

Theorem 4.2.2 *Let $\zeta = \mathbf{1}_{(q=0)}$. Fix $n \in \{1, 2, ...\}$. Suppose that the assumption (H) holds as well as*

$$\int_{(1,\infty)} x^{n+\zeta} \Pi(dx) \; < \; \infty.$$

Then $Q_n(x)$ has finite coefficients and there exists $x_n^\star \in [0, \infty)$ being the largest root of the equation $Q_n(x) = 0$. Let

$$\tau_n^\star \; = \; \inf\{t \geq 0 : X_t \geq x_n^\star\}.$$

Then $\tau_n^\star$ is an optimal strategy to (4.1.1) with $G(x) = (x^+)^n$. Further,

$$V_n(x) \; = \; \mathbb{E}_x\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q} \geq x_n^\star)}\Big).$$

Theorem 4.2.3 *For each $n = 1, 2, ...$ the solution V_n to the optimal stopping problem in the previous theorem is continuous and has the property that*

$$\frac{d}{dx} V_n(x_n^\star -) \; = \; \frac{d}{dx} V_n(x_n^\star +) - \frac{d}{dx} Q_n(x_n^\star) \mathbb{P}(\overline{X}_{\mathbf{e}_q} = 0).$$

Hence there is smooth pasting at $x_n^\star$ if and only if 0 *is regular for* $(0,\infty)$ *for* X.

Remark 4.2.4 (Regularity of 0 for $(0,\infty)$ for Lévy processes) Suppose that X is any Lévy process other than a compound Poisson process. The theory of Lévy processes offers us the opportunity to specify when regularity of 0 for $(0,\infty)$ for X occurs in terms of the triple (a,σ,Π) appearing the Lévy-Khintchine exponent (2.1.1). When X has bounded variation it will be more convenient to write (2.1.1) in the form

$$\Psi(\theta) \;=\; -i\mathrm{d}\theta + \int_{-\infty}^{\infty}(1-e^{i\theta x})\Pi(dx) \qquad (4.2.3)$$

where $\mathrm{d}\in\mathbb{R}$ is known as the *drift coefficient*. We have that the point 0 is regular for $(0,\infty)$ for X (i.e., $\mathbb{P}(\overline{X}_{\mathbf{e}_q}=0)=0$) if and only if one of the following three conditions are fulfilled.
(i) $\int_{(-1,1)}|x|\Pi(dx)=\infty$ (so that X has unbounded variation).
(ii) $\int_{(-1,1)}|x|\Pi(dx)<\infty$ (so that X has bounded variation) and in the representation (4.2.3) we have $\mathrm{d}>0$.
(iii) $\int_{(-1,1)}|x|\Pi(dx)<\infty$ (so that X has bounded variation) and in the representation (4.2.3) we have $\mathrm{d}=0$ and further

$$\int_{(0,1)}\frac{x}{\int_{(0,x)}\Pi(-\infty,-y)dy}\Pi(dx) \;=\; \infty.$$

The latter conclusions being collectively due to Rogozin [108], Shtatland [116] and Bertoin [16].

Intuitively, the conditions $(i)-(iii)$ can be explained as follows. In case (i) when $\sigma>0$ regularity follows as a consequence of the presence of Brownian motion whose behavior on the small time scale always dominates the path of the Lévy process. If on the other hand $\sigma=0$, the condition $\int_{(-1,1)}|x|\Pi(dx)=\infty$ causes small jumps to have behavior on the small time scale close to Brownian motion. The case (ii) says that when the Poisson point process of jumps fulfills the condition $\int_{(-1,1)}|x|\Pi(dx)<\infty$, over small time scales, the sum of the jumps grows sublinearly in time almost surely. Therefore if the drift is present, this dominates the initial movement of the path. In case (iii) when there is no dominant drift, the integral test may be thought of as a statement about what the 'relative weight' of the small positive jumps compared to the small negative jumps needs to be in order for regularity to occur.

4.3 Preliminary lemmas

We need some preliminary results given in the following series of lemmas. All have previously been dealt with in Novikov and Shiryaev [91] for the case of random walks. For some of these lemmas we include slightly more direct proofs which work equally well for random walks (for example avoiding the use of truncation methods).

Lemma 4.3.1 (Moments of the supremum) *Let $\zeta = \mathbf{1}_{(q=0)}$. Fix $n > 0$ and $q \geq 0$. Suppose that the Lévy process X has jump measure Π satisfying*

$$\int_{(1,\infty)} x^{n+\zeta}\Pi(dx) \ < \ \infty. \tag{4.3.1}$$

Then $\mathbb{E}\big((X_1^+)^{n+\zeta}\big) < \infty$. Suppose further that (H) holds. Then $\mathbb{E}\big(\overline{X}_{\mathbf{e}_q}^n\big) < \infty$.

Although the analogue of this lemma is well known for random walks, it seems that one cannot find so easily the equivalent statement for Lévy processes in the existing literature; in particular the final statement of the lemma. None the less the proof can be extracted from a number of well known facts concerning Lévy processes.

Proof The fact that $\mathbb{E}\big((X_1^+)^{n+\zeta}\big) < \infty$ follows from the integral condition (4.3.1) can be seen by combining the result of Theorem 25.3 and Proposition 25.4 of Sato [111]. The remaining statement follows when $q \geq 0$ by Theorem 25.18 of the same book.

To see this let us denote by X^K the Lévy process with the same characteristic as X except that the Lévy measure Π is replaced by the truncated one Π^K defined as

$$\Pi^K(dx) \ = \ \Pi(dx)\mathbf{1}_{(x>-K)} + \delta_{-K}(dx)\Pi(-\infty,-K).$$

In other words, the paths of the Lévy process X^K are an adjustment of the paths of X in that all negative jumps of magnitude K or greater are replaced by a negative jump of precisely magnitude K. To establish our claim, first suppose that $q > 0$. The Wiener-Hopf factorization (see Theorem 2.2.1 in Chapter 2) gives us

$$\mathbb{E}\Big(e^{-i\theta\overline{X}^K_{\mathbf{e}_q}}\Big) \ = \ \mathbb{E}\Big(e^{i\theta X^K_{\mathbf{e}_q}}\Big) \times \frac{\widehat{\kappa}^K(q,i\theta)}{\widehat{\kappa}^K(q,0)} \tag{4.3.2}$$

where $\mathbf{e}_q$ is an independent and exponentially distributed random variable with mean $1/q$ and $\widehat{\kappa}^K$ is the Laplace-Fourier exponent of the bivariate descending ladder process $\widehat{H}$ (see Section 2.2). Note that the descending ladder height process of $\widehat{H}$ cannot have jumps of size greater than K as X^K cannot jump downwards by more than K. Hence the Lévy measure of the descending ladder height process of X^K has bounded support which with the help of Theorem 25.3 and Proposition 25.4 of Sato [111] imply that all moments of the aforementioned process exist. Since $\mathbb{E}\big((X_1^+)^n\big) < \infty$, it implies that $\mathbb{E}(|X_t^K|^n) < \infty$ for all $t > 0$. The latter implies that the right-hand side of (4.3.2) has a Maclaurin expansion up to order n. Specifically this means that $\mathbb{E}\big((\overline{X}^K_{\mathbf{e}_q})^n\big) < \infty$. Due to the truncation, we finally have $\overline{X}_{\mathbf{e}_q} < \overline{X}^K_{\mathbf{e}_q}$ and hence $\mathbb{E}(\overline{X}^n_{\mathbf{e}_q}) < \infty$.

Now suppose that $\limsup_{t\uparrow\infty} X_t < \infty$ and $q = 0$ so that $\overline{X}_{\mathbf{e}_q} = \overline{X}_\infty$. In the absence of the killing constant, we assume[2] that $\mathbb{E}\big((X_1^+)^{n+1}\big) < \infty$. This condition follows from the integral condition (4.3.1) combining the result of Theorem 25.3 and

[2] This is a sufficient condition used in [91] to prove that $\mathbb{E}(\overline{X}^n_\infty) < \infty$ for random walk. In our case, this condition is required in case $\Psi^K(\theta)$ and $\widehat{\kappa}^K(0,i\theta)$ has a factor θ cancelling in (4.3.3).

Proposition 25.4 of Sato [111]. As before, we appeal to the Wiener-Hopf factorization for X^K in the form (up to a multiplicative constant)

$$\kappa^K(0,-i\theta) \;=\; \frac{\Psi^K(\theta)}{\widehat{\kappa}^K(0,i\theta)} \tag{4.3.3}$$

where κ^K and Ψ^K are obviously defined. The same reasoning in the previous paragraphs shows that the Maclaurin expansion on the right-hand side above exists up to order n and hence the same is true for the left-hand side. We make the truncation level K large enough so that it is still the case that $\lim_{t\to\infty} X_t^K = -\infty$. This is possible by choosing K sufficiently large so that $\mathbb{E}(X_1^K) < 0$.

We now have that $\widehat{\kappa}^K(0,0) = 0$ and that $\widehat{\kappa}^K(0,i\theta)$ has an infinite Maclaurin expansion. The assumption $\mathbb{E}\big((X_1^+)^{n+1}\big) < \infty$ implies that $\Psi^K(\theta)$ has Maclaurin expansion up to order $n+1$ and as a matter of fact $\Psi^K(0) = 0$. It now follows that the ratio $\Psi^K(\theta)/\widehat{\kappa}^K(0,i\theta)$ has a Maclaurin expansion up to order n. Since $\kappa^K(0,-i\theta)$ is the cumulative generating function of the ascending ladder height process of X^K it follows that the aforementioned process has finite nth moments. Since $\overline{X}_\infty^K$ is equal in law to the ascending ladder height process of X^K stopped at an independent and exponentially distributed random time, we have that $\mathbb{E}\big((\overline{X}_\infty^K)^n\big) < \infty$. Finally we have $\mathbb{E}\big((\overline{X}_\infty)^n\big) < \infty$ since $\overline{X}_\infty \le \overline{X}_\infty^K$, which can be shown similar to above. □

Lemma 4.3.2 (Mean value property) *Fix $n \in \{1,2,...\}$. Suppose that Y is a non-negative random variable satisfying $\mathbb{E}(Y^n) < \infty$. Then if Q_n is the n-th Appell polynomial generated by Y then we have that*

$$\mathbb{E}\Big(Q_n(x+Y)\Big) \;=\; x^n \quad \textit{for all } x \in \mathbb{R}.$$

Proof As remarked in Novikov and Shiryaev [91], this result can be obtained by truncation of the variable Y. However, it can also be derived from the definition of Q_n given in (4.2.1). Indeed note the result is trivially true for $n = 1$. Next suppose the result is true for Q_{n-1}. Then using dominated convergence we have from (4.2.1)

$$\frac{d}{dx}\mathbb{E}\Big(Q_n(x+Y)\Big) = \mathbb{E}\left(\frac{d}{dx}Q_n(x+Y)\right) = n\mathbb{E}\Big(Q_{n-1}(x+Y)\Big) = nx^{n-1}.$$

Solving together with the requirement that $\mathbb{E}(Q_n(Y)) = 0$ we have the result. □

Lemma 4.3.3 (Fluctuation identity) *Let $\zeta = \mathbf{1}_{(q=0)}$. Fix $n \in \{1,2,...\}$ and suppose that*

$$\int_{(1,\infty)} x^{n+\zeta}\Pi(dx) \;<\; \infty,$$

and that (H) holds. Define $\tau_a^+ = \inf\{t \ge 0 : X_t > a\}$. Then for all $a > 0$ and $x \in \mathbb{R}$

$$\mathbb{E}_x\Big(e^{-q\tau_a^+}X_{\tau_a^+}^n \mathbf{1}_{(\tau_a^+<\infty)}\Big) = \mathbb{E}_x\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}>a)}\Big).$$

Proof Note that on the event $\{\tau_a^+ < \mathbf{e}_q\}$ we have that $\overline{X}_{\mathbf{e}_q} = X_{\tau_a^+} + S$ where S is independent of $\mathcal{F}_{\tau_a^+}$ and has the same distribution as $\overline{X}_{\mathbf{e}_q}$. It follows that

$$\mathbb{E}_x\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}>a)}\Big|\mathcal{F}_{\tau_a^+}\Big) = \mathbf{1}_{(\tau_a^+<\mathbf{e}_q)}h(X_{\tau_a^+})$$

where $h(x) = \mathbb{E}_x(Q_n(\overline{X}_{\mathbf{e}_q}))$. From Lemma 4.3.2 with $Y = \overline{X}_{\mathbf{e}_q}$ one also has that $h(x) = x^n$. We see then by taking expectations again in the previous calculation that

$$\mathbb{E}_x\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}>a)}\Big) = \mathbb{E}_x\Big(e^{-q\tau_a^+}X^n_{\tau_a^+}\mathbf{1}_{(\tau_a^+<\infty)}\Big)$$

as required. □

Lemma 4.3.4 (Largest positive root) *Let $\zeta = \mathbf{1}_{(q=0)}$. Fix $n \in \{1, 2, ...\}$ and suppose that*

$$\int_{(1,\infty)} x^{n+\zeta}\Pi(dx) \ < \ \infty.$$

Suppose that (H) holds and Q_n is generated by $\overline{X}_{\mathbf{e}_q}$. Then Q_n has a unique positive root x_n^* such that $Q_n(x)$ is negative on $[0, x_n^*)$ and positive and increasing on $[x_n^*, \infty)$.

Proof The proof follows proof of the same statement given for random walks in Novikov and Shiryaev [91] with minor modifications. (It is important to note that in following their proof, it is not necessary to make an approximation of the Lévy process by a random walk). Notice that the statement of the lemma is straightforward for $n = 1$. The proof for $n > 1$ is done using induction arguments.

The first step is to show that $Q_n(0) \leq 0$. To start with let us denote by

$$\tau_a^+ \ = \ \inf\{t \geq 0 : X_t \geq a\}$$

the first time X goes above a level a and

$$\gamma(a, n) \ = \ \mathbb{E}\Big(e^{-q\tau_a^+}X^n_{\tau_a^+}\mathbf{1}_{\tau_a^+<\infty}\Big).$$

Note that $\gamma(a, n) \geq 0$ for all $a \geq 0$ and $n = 1, 2, ...$ On the other hand, we see that

$$\begin{aligned}\gamma(a,n) \ &= \ \mathbb{E}\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}\geq a)}\Big)\\ &= \ -\mathbb{E}\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}<a)}\Big)\\ &= \ -\mathbb{P}\big(\overline{X}_{\mathbf{e}_q} < a\big)Q_n(0)\\ &\quad +\mathbb{E}\Big((Q_n(0) - Q_n(\overline{X}_{\mathbf{e}_q}))\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}<a)}\Big)\end{aligned}$$

where the first equality follows from applying Lemma 4.3.3 while the second equality follows from using Lemma 4.3.2. Following the definition

$$Q_n(x) \ = \ Q_n(0) + n\int_0^x Q_{n-1}(y)dy \qquad (4.3.4)$$

for all $x \geq 0$ we have the estimate

$$\left|\mathbb{E}\Big((Q_n(0) - Q_n(\overline{X}_{\mathbf{e}_q}))\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}<a)}\Big)\right| \leq na \sup_{y\in[0,a]}\Big|Q_{n-1}(y)\Big|\mathbb{P}\big(\overline{X}_{\mathbf{e}_q} < a\big),$$

which tends to zero as $a \downarrow 0$. Thus, we then deduce that

$$0 \leq \gamma(a,n) \leq -\mathbb{P}\big(\overline{X}_{\mathbf{e}_q} < a\big)\big[Q_n(0) + o(a)\big]$$

as a approaches zero. This implies that $Q_n(0) \leq 0$. Under the induction hypothesis for Q_{n-1}, we see from (4.3.4) together with the fact that $Q_n(0) \leq 0$ that Q_n is negative and decreasing on the interval $[0, x^\star_{n-1})$. The point $x^\star_{n-1}$ corresponds to the minimum of Q_n thanks to the positivity and the monotonicity of $Q_{n-1}(x)$ for $x > x^\star_{n-1}$. In particular, $Q_n(x)$ tends to infinity from its minimum point and hence there must be a unique strictly positive root of the equation $Q_n(x) = 0$. Thus, our claim that Q_n has a unique positive root is then established. □

4.4 Proofs of theorems

Proof of Theorem 4.2.2

Proof In light of the Novikov-Shiryaev optimal stopping problems and their solutions, we verify that the analogue of their solution, namely the one proposed in Theorem 4.2.2, is also a solution for (4.1.1) for $G(x) = (x^+)^n$, $n = 1, 2, ...$

To this end, fix $n \in \{1, 2, ...\}$ and define

$$V_n(x) = \mathbb{E}_x\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}>x^*_n)}\Big).$$

First note from Lemma 4.3.3 that

$$V_n(x) = \mathbb{E}_x\Big(e^{-q\tau^*_n}(X^+_{\tau^*_n})^n\mathbf{1}_{(\tau^*_n<\infty)}\Big)$$

and hence the pairs (V_n, τ^*_n) are a candidate pair to solve the problem (4.1.1).

Secondly we prove that $V_n(x) \geq (x^+)^n$ for all $x \in \mathbb{R}$. Note that this statement is obvious for $x \in (-\infty, 0] \cup [x^*_n, \infty)$ just from the definition of V_n. Otherwise when $x \in (0, x^*_n)$ we have, using the mean value property in Lemma 4.3.2 that

$$\begin{aligned} V_n(x) &= \mathbb{E}_x\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}>x^*_n)}\Big) \\ &= x^n - \mathbb{E}_x\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}\leq x^*_n)}\Big) \\ &\geq (x^+)^n \end{aligned}$$

where the final inequality follows from Lemma 4.3.4 and specifically the fact that $Q_n(x) \leq 0$ on $(0, x^*_n]$. Note in particular, embedded in this argument is the statement that $V_n(x-) = (x^+)^n$ at $x = x^*_n$.

Thirdly, we have $\mathbb{P}_x$ almost surely that $Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}>x_n^*)} \geq 0$. Using the latter together with the fact that, on the event that $\{\mathbf{e}_q > t\}$ we have $\overline{X}_{\mathbf{e}_q}$ is equal in distribution to $X_t + I$ where I is independent of $\mathcal{F}_t$ and equal in distribution to $\overline{X}_{\mathbf{e}_q}$, it follows that

$$\begin{aligned} V_n(x) &\geq \mathbb{E}_x\Big(\mathbf{1}_{(\mathbf{e}_q>t)}Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}>x_n^*)}\Big) \\ &= \mathbb{E}_x\Big(\mathbf{1}_{(\mathbf{e}_q>t)}\mathbb{E}_x\Big(Q_n(X_t+\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(X_t+\overline{X}_{\mathbf{e}_q}>x_n^*)}\Big|\mathcal{F}_t\Big)\Big) \\ &= \mathbb{E}_x\Big(e^{-qt}V_n(X_t)\Big). \end{aligned}$$

From this inequality together with the Markov property, it is easily shown that $\{e^{-qt}V_n(X_t) : t \geq 0\}$ is a supermartingale.

Finally we put these three facts together as follows to complete the proof. From the supermartingale property and the lower bound on V_n it follows that

$$V_n(x) \geq \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}V_n(X_\tau)\mathbf{1}_{(\tau<\infty)}\Big) \geq \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}(X_\tau^+)^n\mathbf{1}_{(\tau<\infty)}\Big). \quad (4.4.1)$$

On the other hand, rather trivially, we have

$$\sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}(X_\tau^+)^n\mathbf{1}_{(\tau<\infty)}\Big) \geq \mathbb{E}_x\Big(e^{-q\tau_n^*}(X_{\tau_n^*}^+)^n\mathbf{1}_{(\tau_n^*<\infty)}\Big) = V_n(x). \quad (4.4.2)$$

and the proof of the theorem follows. □

Proof of Theorem 4.2.3

Proof It has already been noted in the previous proof that there is continuity of V_n at the point x_n^*. To establish when there is a smooth pasting at this point, we calculate as follows. For $x < x_n^*$

$$\begin{aligned} \frac{V_n(x_n^*)-V(x)}{x_n^*-x} &= \frac{(x_n^*)^n-x^n}{x_n^*-x} + \frac{\mathbb{E}_x(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}\leq x_n^*)})}{x_n^*-x} \\ &= \frac{(x_n^*)^n-x^n}{x_n^*-x} + \frac{\mathbb{E}_x((Q_n(\overline{X}_{\mathbf{e}_q})-Q_n(x_n^*))\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}\leq x_n^*)})}{x_n^*-x} \end{aligned}$$

where the final equality follows because $Q_n(x_n^*) = 0$. Clearly

$$\lim_{x\uparrow x_n^*}\frac{(x_n^*)^n-x^n}{x_n^*-x} = \frac{dV_n}{dx}(x_n^*+).$$

However,

$$\begin{aligned} \frac{\mathbb{E}_x((Q_n(\overline{X}_{\mathbf{e}_q})-Q_n(x_n^*))\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}\leq x_n^*)})}{x_n^*-x} =& \frac{\mathbb{E}_x((Q_n(\overline{X}_{\mathbf{e}_q})-Q_n(x))\mathbf{1}_{(x<\overline{X}_{\mathbf{e}_q}\leq x_n^*)})}{x_n^*-x} \\ &- \frac{\mathbb{E}_x((Q_n(x_n^*)-Q_n(x))\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}\leq x_n^*)})}{x_n^*-x}, \end{aligned} \quad (4.4.3)$$

where in the first term on the right hand side we may restrict the expectation to $\{x < \overline{X}_{\mathbf{e}_q} \leq x_n^*\}$ as the atom of $\overline{X}_{\mathbf{e}_q}$ at zero gives zero mass to the expectation. Denote by A_x and B_x the two expressions on the right hand side of equation (4.4.3). We have that

$$\lim_{x \uparrow x_n^*} B_x = -\frac{dQ_n(x_n^*)}{dx}\mathbb{P}(\overline{X}_{\mathbf{e}_q} = 0).$$

Integration by parts also gives

$$\begin{aligned} A_x &= \int_{(0,x_n^*-x]} \frac{Q_n(x+y) - Q_n(x)}{x_n^* - x}\mathbb{P}(\overline{X}_{\mathbf{e}_q} \in dy) \\ &= \frac{Q_n(x_n^*) - Q_n(x)}{x_n^* - x}\mathbb{P}(\overline{X}_{\mathbf{e}_q} \in (0, x_n^* - x]) \\ &\quad - \frac{1}{x_n^* - x}\int_0^{x_n^*-x} \mathbb{P}(\overline{X}_{\mathbf{e}_q} \in (0,y])\frac{dQ_n}{dx}(x+y)dy. \end{aligned}$$

Hence it follows that

$$\lim_{x \uparrow x_n^*} A_x = 0.$$

In conclusion we have that

$$\lim_{x \uparrow x_n^*} \frac{V_n(x_n^*) - V(x)}{x_n^* - x} = \frac{dV_n}{dx}(x_n^*+) - \frac{dQ_n(x_n^*)}{dx}\mathbb{P}(\overline{X}_{\mathbf{e}_q} = 0)$$

which concludes the proof. □

4.5 Numerical examples

This section discusses some numerical examples of the results presented in Section 2.

For this numerical purposes, we consider two cases. Firstly, we choose X to be a spectrally negative Lévy process of bounded variation. Necessarily, it takes the form of a linear drift minus a subordinator. We take the drift d to be at rate 0.1 and the subordinator to be a compound Poisson process with exponentially distributed jumps; that is to say that X has Laplace exponent

$$\kappa(\lambda) \;=\; \mathrm{d}\lambda + \int_{-\infty}^{0} ace^{cx}(e^{\lambda x} - 1)dx \;=\; \mathrm{d}\lambda - \frac{a\lambda}{c+\lambda}. \tag{4.5.1}$$

As explained in Section 2.3 of Chapter 2, it is known that the moment generating function $\Psi_q^{(+)}(\lambda)$ of the random variable $\overline{X}_{\mathbf{e}_q}$ is given for $q \geq 0$ and $\mathfrak{Re}(\lambda) \geq 0$ by

$$\Psi_q^{(+)}(\lambda) \;=\; \int_0^\infty e^{-\lambda x}\mathbb{P}(\overline{X}_{\mathbf{e}_q} \in dx) \;=\; \frac{\Phi(q)}{\lambda + \Phi(q)}.$$

Following this Laplace transform, we deduce using Tauberian theorem that

$$\mathbb{P}(\overline{X}_{\mathbf{e}_q} = 0) \;=\; \lim_{\lambda \uparrow \infty} \frac{\Phi(q)}{\lambda + \Phi(q)} = 0, \tag{4.5.2}$$

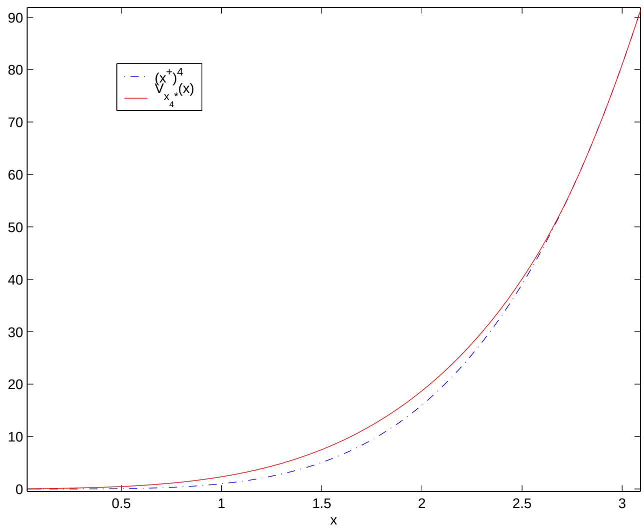


Figure 4.2: The shape of the value function of an optimal stopping problem with payoff function $G(x) = (x^+)^4$ driven by downward jumps compound Poisson process with drift $\mathrm{d} = 0.1$. The optimal stopping boundary is given by $x_4^\star = 2.7789$.

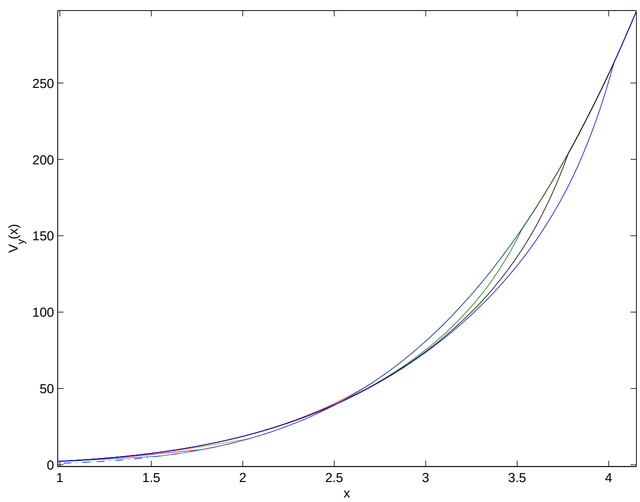


Figure 4.3: The shape of a candidate solution $V_y(x)$ for different values of boundary y of an optimal stopping problem with payoff function $G(x) = (x^+)^4$ driven by downward jumps compound Poisson process with drift $\mathrm{d} = 0.1$. The optimal stopping boundary is given by $x_4^\star = 2.7789$.

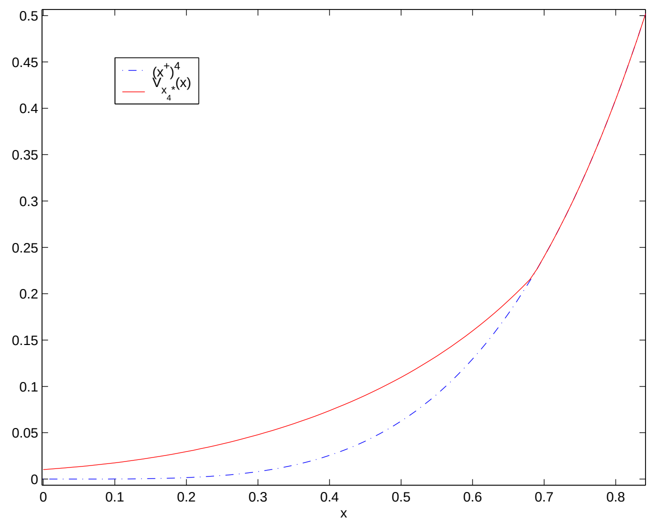


Figure 4.4: The shape of the value function of an optimal stopping problem with payoff function $G(x) = (x^+)^4$ driven by upward jumps compound Poisson process with drift $\mathrm{d} = -0.1$. The optimal stopping boundary is given by $x_4^\star = 0.6832$.

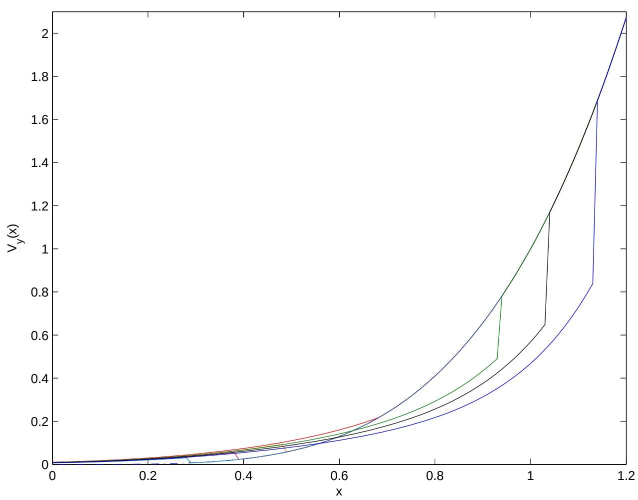


Figure 4.5: The shape of a candidate solution $V_y(x)$ for different values of boundary y of an optimal stopping problem with payoff function $G(x) = (x^+)^4$ driven by upward jumps compound Poisson process with drift $\mathrm{d} = -0.1$. The optimal stopping boundary is given by $x_4^\star = 0.6832$.

from which it follows that 0 is regular for the upper half-line $(0,\infty)$ for X.

Secondly, we consider the dual process $\widehat{X} = -X$ of the former spectrally negative compound Poisson process X. We denote by $\overline{\widehat{X}}_{\mathbf{e}_q} = \sup_{0\le s\le \mathbf{e}_q} \widehat{X}_s$ the running supremum of the dual process $\widehat{X}$ up to random time $\mathbf{e}_q$. Since $\overline{\widehat{X}}_{\mathbf{e}_q} = -\underline{X}_{\mathbf{e}_q}$, by duality arguments, it is known (see for instance Section 5 of Bingham [17]) that the moment generating function $\Psi_q^{(+)}(\lambda)$ of the random variable $\overline{\widehat{X}}_{\mathbf{e}_q}$ is given by

$$\Psi_q^{(+)}(\lambda) \;=\; \int_0^\infty e^{-\lambda x}\mathbb{P}(\overline{\widehat{X}}_{\mathbf{e}_q} \in dx) \;=\; \frac{q}{\Phi(q)}\Big(\frac{\lambda-\Phi(q)}{\kappa(\lambda)-q}\Big),$$

for $q>0$ and $\mathfrak{Re}(\lambda)\ge 0$, where $\Phi(a)$ is the largest root γ of the equation $\kappa(\gamma)=a$. (See also Section 2.3 of Chapter 2 for more details.) For numerical inversion of Laplace transform, we refer to Chapter 7 for further discussions. Applying the same arguments as before, we deduce from the foregoing expression that

$$\mathbb{P}(\overline{\widehat{X}}_{\mathbf{e}_q}=0) \;=\; \lim_{\lambda\uparrow\infty}\frac{q}{\Phi(q)}\Big(\frac{\lambda-\Phi(q)}{\kappa(\lambda)-q}\Big) \;=\; \frac{q}{\Phi(q)}\Big(\frac{1}{\mathrm{d}}\Big) \;>\; 0. \tag{4.5.3}$$

The expression (4.5.3) tells us that 0 is irregular for $(0,\infty)$ for the dual process $\widehat{X}$.

For all computations, we set $q=0.075$, $n=4$, $c=9$ and $a=0.5$. The numerical computation is carried out using MATLAB6.5.

For each of these two Lévy processes, we consider the function

$$V_y(x) \;=\; (x^+)^n - \mathbb{E}_x\Big(Q_4(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}\le y)}\Big) \tag{4.5.4}$$

as a candidate solution to the optimal stopping problem (4.1.1). By varying the values of the boundary y, we present in Figures 4.2-4.5 the plots of the function $x\mapsto V_y(x)$ for $x\ge 0$ with steps $dx=0.01$. From these plots, we notice in all respects that all curves V_y are upper bounded by that of associated with the optimal boundary $y=x_4^\star$ (the largest root of the equation $Q_4(x)=0$). This majorant property of $V_{x_4^\star}$ can be explained using (4.4.1)-(4.4.2) and the result of Lemma 4.3.3 as follows

$$\begin{aligned}
V_{x_4^\star}(x) \;&=\; \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}(X_\tau^+)^n\mathbf{1}_{(\tau<\infty)}\Big) \quad \{\text{by (4.4.1) and (4.4.2)}\}\\
&\ge\; \mathbb{E}_x\Big(e^{-q\tau_y^+}(X_{\tau_y^+}^+)^n\mathbf{1}_{(\tau_y^+<\infty)}\Big)\\
&=\; \mathbb{E}_x\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}>y)}\Big) \qquad \{\text{by Lemma 4.3.3}\}\\
&=\; V_y(x).
\end{aligned}$$

For the spectrally negative compound Poisson process X (4.5.1), we observe from Figures 4.2-4.3 that the continuous pasting condition $V_y(x)=(x^+)^4$ holds at point $x=y$, for any $y\ge 0$. This is because by evaluating the candidate solution (4.5.4) at the point $x=y$, we see that

$$V_y(x) \;=\; (x^+)^4 - \mathbb{P}(\overline{X}_{\mathbf{e}_q}=0)Q_4(x) \quad \text{at } x=y. \tag{4.5.5}$$

Hence, since $\mathbb{P}(\overline{X}_{\mathbf{e}_q} = 0) = 0$ for this process (see the expression (4.5.2)) we obtain following (4.5.5) that $V_y(x) = (x^+)^4$ at the point $x = y$ of any stopping boundary $y \geq 0$. In addition, we observe also from Figures 4.2 and 4.3 that the smooth pasting condition $\frac{d}{dx}V_y(x) = \frac{d}{dx}(x^+)^4$ only holds at the point $x = y$ of the optimal boundary $y = x_4^\star$. In view of Theorem[3] 4.2.3, this observation is obvious following the fact that $\mathbb{P}(\overline{X}_{\mathbf{e}_q} = 0) = 0$ for this process. Thus, our claim in Theorem 4.2.3 is then verified.

In contrast to the first two plots, we observe from Figures 4.4-4.5 that the candidate solution $V_y(x)$ for the dual process $\widehat{X}$ satisfies the continuous pasting condition $V_y(x) = (x^+)^4$ only at the point $x = y$ of the optimal stopping boundary $y = x_4^\star$ (as displayed in Figure 4.4) and exhibit negative jumps of magnitude $\mathbb{P}(\overline{\widehat{X}}_{\mathbf{e}_q} = 0)Q_4(y)$ when $y \neq x_4^\star$ (as Figure 4.5 shows). This phenomenon is well understood following the equation (4.5.5). Moreover, taking account of the fact that $\mathbb{P}(\overline{\widehat{X}}_{\mathbf{e}_q} = 0) > 0$ for the dual process $\widehat{X}$ (see expression (4.5.3)), it is clear following Theorem 4.2.3 that the smooth pasting condition $\frac{d}{dx}V_y(x) = \frac{d}{dx}(x^+)^4$ does not hold at the point $x = y$ of the optimal stopping boundary $y = x_4^\star$ as illustrated in Figure 4.4.

To summarize, we have seen that all the numerical results obtained in this section are found to be consistent with the main results of Section 2.

4.6 Concluding remarks

(i) As in Alili and Kyprianou [3] one can argue that the occurrence of continuous pasting for irregularity and smooth pasting for regularity appear as a matter of principle. The way to see this is to consider the candidate solutions (V_y, τ_y^+) where $\tau_y^+ = \inf\{t \geq 0 : X_t > y\}$ and $V_y(x) = \mathbb{E}_x\big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q} > y)}\big)$. By varying the value of y in $(0, \infty)$ one will find that, when there is irregularity, in general there is a discontinuity of V_y at y (as illustrated in Figure 4.5) and otherwise when there is regularity, there is always continuity at y (as Figure 4.5 displays). In both cases, let $\mathcal{C}$ be the class of $y > 0$ for which V_y is lower bounded by the gain and is superharmonic (it composes with X to make a supermartingale when discounted at rate q). When there is irregularity, the choice of $y = x_n^*$ is the unique point in $\mathcal{C}$ for which the discontinuity at y is closed and hence the function V_y turns out to be pointwise minimal. When there is regularity, the minimal curve indexed in $\mathcal{C}$ will occur by adjusting y so that the gradients either side of y match which again turns out to be the unique value $y = x_n^*$.

(ii) From arguments presented in Novikov and Shiryaev [91] together with the supporting arguments given in this chapter, it is now clear how to handle the gain function $G(x) = 1 - e^{x^+}$ for Lévy processes instead of random walks as well as how to handle the pasting principles at the optimal boundary.

[3] See also Theorem 5.4.3 in Chapter 5.

Chapter 5

An Approach for Solving Perpetual Optimal Stopping Problems Driven by Lévy Processes[1]

Abstract

In this chapter, we propose an approach for solving perpetual optimal stopping problems for a general class of payoff functions under Lévy processes. This approach was inspired by the work of Boyarchenko and Levendorski [21]. In contrast to [21], our approach does not appeal to a free boundary problem associated to the optimal stopping problem nor to the theory of pseudodifferential operators to solve the problem. Instead, we introduce an averaging problem from which we obtain, using the Wiener-Hopf factorization, a fluctuation identity for first passage of Lévy processes. This identity constitutes the main principle in solving the optimal stopping problem. If a solution to the averaging problem can be found and has certain monotonicity properties, we show using the fluctuation identity that an optimal solution to the optimal stopping problem can be written in terms of such a monotone function.

Using the optimal solution, we give sufficient and necessary condition for the C^1 smooth pasting condition to occur in the considered problem. Our conclusion over the smooth pasting condition extends further the recent result of Alili and Kyprianou [3] into a more general payoff function.

Furthermore, assuming that the moment generating function of the Lévy process exists on an open set containing zero, we give an estimate for the value function of the finite maturity American put option problem in terms of the value function of the perpetual American put option problem.

5.1 Introduction and problem formulation

Let $X = \{X_t; t \geq 0\}$ be a Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ satisfying the usual conditions, see Chapter 2 for more details. For

[1]This chapter is the extended version of: Surya, B. A. An approach for solving perpetual optimal stopping problems driven by Lévy processes. To appear in *Stochastics*.

a given Lévy process X, we consider the following optimal stopping problem which consists of finding the *optimal value function*

$$V(x) \ = \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x \Big(e^{-q\tau} G(X_\tau) \mathbf{1}_{(\tau \, < \, \infty)} \Big), \tag{5.1.1}$$

where G is a measurable function, and the supremum is taken over the class $\mathcal{T}_{[0,\infty]}$ of Markov stopping times taking values in $[0, \infty]$ with respect to the filtration $\mathcal{F}_t = \sigma(X_s, s \leq t)$. We say that a stopping time $\tau^\star$ is optimal if

$$V(x) \ = \ \mathbb{E}_x \Big(e^{-q\tau^\star} G(X_{\tau^\star}) \mathbf{1}_{(\tau^\star \, < \, \infty)} \Big). \tag{5.1.2}$$

From general results for the optimal stopping problem of diffusion processes (see for instance Shiryaev [113]), it is well-known that if the stopping time

$$\tau^\star \ = \ \inf\{t > 0 : X_t \in \mathcal{S}\}, \tag{5.1.3}$$

where $\mathcal{S}$ is the *stopping region* in which the optimal value function V is equal to the payoff G, is finite $\mathbb{P}_x$ a.s. for any $x \in \mathbb{R}$ then under very general assumptions on the payoff G it is optimal in the class $\mathcal{T}_{[0,\infty]}$ of Markov stopping times. Thus finding such a stopping time $\tau^\star$ completely determines the value function V in (5.1.1). For diffusion processes, it was shown in Shiryaev and Grigelionis [55], van Moerbeke [86], and Shiryaev [113] that the boundary $\partial\mathcal{S}$ of $\mathcal{S}$ is determined by using the *smooth pasting principle*, and solving the optimal stopping problem (5.1.1) is then reduced to solving a corresponding Stefan's free boundary problem. However, when the sample paths of X are not continuous, this smooth pasting principle may break down. This observation over the breakdown of the smooth pasting was studied by Peskir and Shiryaev in [96] for the problem of sequential testing for compound Poisson processes, by Boyarchenko and Levendorskii in [21], Hirsa and Madan [59], Matache et al. [81], Almendral and Oosterlee [4], and Alili and Kyprianou [3] for the problem of pricing the American put option under a Lévy process, and Kyprianou and Surya [73] for an American call-type optimal stopping problem with integer power function of Lévy processes. (Note that in [21], [59], [81], and [4] the solution to the problem (5.1.1) was obtained by solving the free boundary problem without imposing the smooth pasting condition). Observe that even though the stopping time (5.1.3) characterizes the value function V, it presents only qualitative features of the solution to the problem (5.1.1), but it does not present an effective way of finding the value function or for constructing the optimal stopping boundary explicitly.

In this chapter we propose an effective approach for solving the problem (5.1.1) in a general setting. This approach was inspired by the work of Boyarchenko and Levendorski [21] on perpetual American put-type optimal stopping problem for payoff function with exponential growth under stable like Lévy processes. In contrast to [21], our approach does not appeal to a free boundary problem associated to the problem (5.1.1) nor to the theory of pseudodifferential operators to solve the problem. Instead, we introduce an averaging problem from which we obtain, using the Wiener-Hopf factorization, a fluctuation identity for first passage of Lévy processes. This fluctuation identity represents the main principle in obtaining an optimal solution to the problem (5.1.1). This identity gives a generic link to some known identities which have been used to solve the problem (5.1.1) for special payoff functions. See for instance Darling et al. [33], Mordecki [87], Asmussen et al. [6], Alili and Kyprianou [3], Novikov and Shiryaev [91], and Kyprianou and Surya [73]. If a solution to the averaging problem can be found and has certain monotonicity properties, we show using the fluctuation identity that an optimal solution to the problem (5.1.1) can be written explicitly in terms of such monotone function.

Using our approach, we are able to reproduce the special results of those discussed among others by Darling et al. [33], Mordecki [87], Boyarchenko and Levendorskii [21], Alili and Kyprianou [3], Novikov and Shiryaev [91], and Kyprianou and Surya [73]. Using the optimal solution to the problem (5.1.1), we show that the C^1 smooth pasting condition exists if and only if the optimal stopping boundary is *regular* for the interior points of the stopping region for the Lévy process. But, in the case when the optimal boundary is *irregular* for the interior points of the stoping region for the Lévy process, we replace the principle of smooth pasting by a principle of *continuous pasting* in determining the optimal boundary. Our observation on the smooth pasting principle extends further the recent work of Alili and Kyprianou [3] and Kyprianou and Surya [73], into a more general payoff function.

In addition, assuming that the moment generating function of a Lévy process exists on an open set containing zero, we obtain an estimate for the value function of the finite maturity American put option problem in terms of the value function of the perpetual American put option problem. The estimate allows us to have a quick access of information about an estimate of what the arbitrage-free price $V(t,x)$ of the finite maturity American put option would be at time t given the initial value x of the stock price process.

The outline of this chapter is as follows. Building upon the Wiener-Hopf factorization introduced in Chapter 2, we present in Section 2 an averaging problem and fluctuation identity for first passage of Lévy processes which form the main principle in obtaining an optimal solution to the problem (5.1.1) in a general setting. The result is presented in Section 3. In Section 4, we discuss sufficient and necessary conditions for the C^1 smooth pasting to occur in the considered problem. In Section 5, we use our approach to reproduce the results of the aforementioned authors. Section 6 presents details of derivation of the main results of Sections 2-4. We exemplify in Section 7 the main results by means of numerical examples for pricing the perpetual American put and call options driven by tempered stable Lévy processes with downward jumps. The estimate for the the arbitrage-free price of the finite maturity American put option is given in Section 8. Finally, Section 9 concludes this chapter.

5.2 Preliminary results

Before establishing our general solution to the problem (5.1.1), we present in this section our main tool needed to obtain the main results.

5.2.1 An averaging problem

Suppose that X is a Lévy process with the assumption that

$$\text{either } q > 0 \text{ or } \Big(\; q = 0 \text{ and } \mathbb{P}\big(\liminf_{t\to\infty} X_t > -\infty\big) \;=\; 1\Big). \tag{H1}$$

The problem consists of finding a function $\mathscr{P}_G^{(q)}$ such that for a given continuous function G and $q \geq 0$, we have for every $x \in \mathbb{R}$ that

$$\mathbb{E}\Big(\mathscr{P}_G^{(q)}\big(x + \underline{X}_{\mathbf{e}_q}\big)\Big) \;=\; G(x). \tag{5.2.1}$$

In general, this problem may or may not have a solution in the class $\mathcal{C}_b(\mathbb{R})$ of bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$, and the solution of which may not necessarily be unique.

However, there are examples for which the problem (5.2.1) is solved. We give below some solutions to this problem for exponential, linear combination of exponential, polynomials, and sufficiently regular functions.

In the sequel below let us remind ourself that $\underline{X}_{\mathbf{e}_q}$ is a negative valued random variable and therefore for a positive $\theta \in \mathbb{R}$ we see that the Wiener-Hopf factor $\Psi_q^{(-)}(-i\theta) = \mathbb{E}\big(e^{\theta \underline{X}_{\mathbf{e}_q}}\big)$ (see Section 2.2 of Chapter 2) is finite and, in particular, for $q = 0$ it is strictly positive due to the assumption (H1) imposed on the Lévy process.

Example 5.2.1 (Exponential) Suppose that X is a Lévy process with property $(H1)$. Define, for a given $\theta \geq 0$, a function $G(x) = e^{\theta x}$. Then, a solution to the problem (5.2.1) is given for every $x \in \mathbb{R}$ by

$$\mathscr{P}_G^{(q)}(x) \;=\; \frac{e^{\theta x}}{\Psi_q^{(-)}(-i\theta)}. \tag{5.2.2}$$

It is clear to see for each $q \geq 0$ and every $x \in \mathbb{R}$ that $\mathbb{E}\big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\big) = G(x)$.

Example 5.2.2 (Linear combination of exponential) Suppose that X is a Lévy process with property $(H1)$. Define, for a given $m = 1, 2, \cdots$, a function $G(x) = \sum_{j=1}^m c_j e^{\theta_j x}$ with $\theta_j \geq 0$. Then, a solution to the problem (5.2.1) is given for each $q \geq 0$ and every $x \in \mathbb{R}$ by

$$\mathscr{P}_G^{(q)}(x) \;=\; \sum_{j=1}^m c_j \frac{e^{\theta_j x}}{\Psi_q^{(-)}(-i\theta_j)}, \quad \text{with } \theta_j \geq 0. \tag{5.2.3}$$

It is clear to see for each $q \geq 0$ and every $x \in \mathbb{R}$ that $\mathbb{E}\big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\big) = G(x)$.

Example 5.2.3 (Polynomials) Suppose that X is a Lévy process with property $(H1)$. Define, for a given $n = 1, 2, \cdots$, a function $G(x) = x^n$.

Let us consider the *Esscher transform*

$$\frac{e^{\theta x}}{\mathbb{E}\big(e^{\theta \underline{X}_{\mathbf{e}_q}}\big)} \;=\; \sum_{n=0}^{\infty} \frac{\theta^n}{n!} Q_n(x), \quad \text{with } \theta \;\geq\; 0. \tag{5.2.4}$$

In literature, $Q_n(x)$, $n = 1, 2, \cdots$, are known as the *Appell polynomials* generated by the random variable $\underline{X}_{\mathbf{e}_q}$. We refer to Schoutens [112] for more details. For the polynomials x^n, $n = 1, 2, \cdots$, a solution to the problem (5.2.1) is given by

$$\mathscr{P}_G^{(q)}(x) \;=\; Q_n(x). \tag{5.2.5}$$

Following the Esscher transform (5.2.4), it is not difficult to check that for each $q \geq 0$ and every $x \in \mathbb{R}$ we have $\mathbb{E}\big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\big) = G(x)$.

Example 5.2.4 (Sufficiently regular function) Denote by $\mathscr{R}$ a subset of L_1- integrable functions $h : \mathbb{R} \to \mathbb{R}$ within which the *Fourier transform* $\widehat{h}$ of h, defined by

$$\widehat{h}(\lambda) \;=\; \int_{-\infty}^{\infty} e^{-i\lambda x} h(x) dx, \tag{5.2.6}$$

satisfies the integrability condition

$$\int_{-\infty}^{\infty} \big(1 + |\lambda|^3\big) \big|\widehat{h}(\lambda)\big| d\lambda \;<\; \infty. \tag{5.2.7}$$

From the fact that the function h and its Fourier transform $\widehat{h}$ are L_1-integrable, every function in $\mathscr{R}$ can be decomposed into the *Fourier integral* representation

$$h(x) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} \widehat{h}(\lambda) d\lambda. \tag{5.2.8}$$

We refer to Rudin [109] for more details of discussion. We call throughout $\mathscr{R}$ as the set of *sufficiently regular functions*. It is clear that the set $\mathscr{R}$ belongs to the class $\mathcal{C}_b^3$ of continuously differentiable functions bounded with its derivatives f^j with $j = 1, 2, 3$ and contains the *Schwartz class*[2] $\mathcal{S}(\mathbb{R})$ of *rapidly decreasing* functions and the class $\mathcal{C}_0^\infty$ of infinitely differentiable functions which tend to zero at infinity.

[2]If h is in the *Schwartz class* $\mathcal{S}(\mathbb{R})$ of *rapidly decreasing* functions, then using integration by parts it can be checked straightforwardly from (5.2.6) that the function $\widehat{h}$ admits the estimate $|\widehat{h}(\lambda)| \leq C\big((1 + |\lambda|)^{-N}\big)$, for $C > 0$, as $\lambda \longrightarrow \infty$ for any integer $N = 1, 2, 3....$ This is the reason that the class $\mathcal{S}(\mathbb{R})$ is useful in studying Fourier transform since $\widehat{h} \in \mathcal{S}(\mathbb{R})$ whenever $h \in \mathcal{S}(\mathbb{R})$. We refer to Hörmander [61] for more details on general theory of Fourier integral operators.

Lemma 5.2.5 *Suppose that the Wiener-Hopf factor* $\Psi_q^{(-)}(\lambda)$ *(2.2.6) is nowhere zero. For a function* $G \in \mathscr{R}$ *and a fixed* $q > 0$*, the problem (5.2.1) has a unique solution within the class of* C_b^1 *given for each* $q > 0$ *and every* $x \in \mathbb{R}$ *by*

$$\mathscr{P}_G^{(q)}(x) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}d\lambda. \tag{5.2.9}$$

The proof of this claim can be found in Section 6.

5.2.2 Fluctuation identity for first passage of Lévy processes

We present in this section a fluctuation identity for the first passage above or below a certain level of a Lévy process. To a fixed level $y \in \mathbb{R}$ we associate the first strict passage time τ_y^- (resp. τ_y^+) below (resp. above) y defined by

$$\tau_y^- \;=\; \inf\{t > 0 : X_t \;<\; y\} \quad \text{and} \quad \tau_y^+ \;=\; \inf\{t > 0 : X_t \;>\; y\}. \tag{5.2.10}$$

The identity is stated as follows.

Lemma 5.2.6 *Let* X *be a Lévy process under the hypothesis* $(H1)$*. Suppose that* $\mathscr{P}_G^{(q)}$ *solves the problem (5.2.1). Then for every* $x, y \in \mathbb{R}$ *such that* $x \geq y$*, we have*

$$\mathbb{E}_x\Big(e^{-q\tau_y^-}G(X_{\tau_y^-})\mathbf{1}_{(\tau_y^- \,<\, \infty)}\Big) \;=\; \mathbb{E}_x\Big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q})\mathbf{1}_{(\underline{X}_{\mathbf{e}_q}<y)}\Big). \tag{5.2.11}$$

Proof The proof is mainly based on the fact that conditionally on $\mathcal{F}_{\tau_y^-}$ and on the event $\{\mathbf{e}_q > \tau_y^-\}$, $\underline{X}_{\mathbf{e}_q} - X_{\tau_y^-}$ is independent of $\mathcal{F}_{\tau_y^-}$, and has the same distribution as $\underline{X}_{\mathbf{e}_q}$, thanks to the lack of memory property of exponential random variable $\mathbf{e}_q$ and the stationary independent increment of X. Combined with the fact that the function $\mathscr{P}_G^{(q)}$ solves the problem (5.2.1), we see that

$$\begin{aligned}
\mathbb{E}_x\Big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q})\mathbf{1}_{(\underline{X}_{\mathbf{e}_q}<y)}\Big) &= \mathbb{E}_x\Big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q})\mathbf{1}_{(\mathbf{e}_q>\tau_y^-)}\Big)\\
&= \mathbb{E}_x\Big(\mathbb{E}\big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q})\mathbf{1}_{(\mathbf{e}_q>\tau_y^-)}\big|\mathcal{F}_{\tau_y^-}\big)\Big)\\
&= \mathbb{E}_x\Big(\mathbf{1}_{(\mathbf{e}_q>\tau_y^-)}\mathbb{E}\big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q})\big|\mathcal{F}_{\tau_y^-}\big)\Big)\\
&= \mathbb{E}_x\Big(\mathbf{1}_{(\mathbf{e}_q>\tau_y^-)}\mathbb{E}\big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q} - X_{\tau_y^-} + X_{\tau_y^-})\big|\mathcal{F}_{\tau_y^-}\big)\Big)\\
&= \mathbb{E}_x\Big(\mathbf{1}_{(\mathbf{e}_q>\tau_y^-)}\mathbb{E}_{X_{\tau_y^-}}\big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q} - X_{\tau_y^-})\big|\mathcal{F}_{\tau_y^-}\big)\Big)\\
&= \mathbb{E}_x\Big(\mathbf{1}_{(\mathbf{e}_q>\tau_y^-)}\mathbb{E}_{X_{\tau_y^-}}\big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q})\big)\Big)\\
&= \mathbb{E}_x\Big(\mathbf{1}_{(\mathbf{e}_q>\tau_y^-)}G(X_{\tau_y^-})\Big)\\
&= \mathbb{E}_x\Big(e^{-q\tau_y^-}G(X_{\tau_y^-})\Big),
\end{aligned}$$

which indeed establishes our claim. □

Example 5.2.7 Now let us consider, for a given $\theta \geq 0$, the function $G(x) = e^{\theta x}$. Then using (5.2.2), we deduce from the foregoing theorem that

$$\mathbb{E}_x\Big(e^{-q\tau_y^- + \theta X_{\tau_y^-}} \mathbf{1}_{(\tau_y^- < \infty)}\Big) \;=\; \frac{\mathbb{E}_x\left(e^{\theta \underline{X}_{\mathbf{e}_q}} \mathbf{1}_{(\underline{X}_{\mathbf{e}_q} < y)}\right)}{\mathbb{E}\big(e^{\theta \underline{X}_{\mathbf{e}_q}}\big)}. \tag{5.2.12}$$

This identity can be analytically extended to $\theta \in \mathbb{C}$ with $\mathfrak{Re}(\theta) > 0$. This particular fluctuation identity goes back to the work of Darling et al. [33] for random walks, and has been extended to continuous-time, among others, by Alili and Kyprianou [3], Asmussen et al. [6], and Mordecki [87], and was used to solve the optimal stopping problem (5.1.1) with payoff function $G(x) = (K - e^x)^+$.

By replacing X with its dual $\widehat{X} = -X$ and y with $-y$, the problem of first exit above a level y for X can be transformed into the problem of first exit of $\widehat{X}$ below a level $-y$. The following result is the dual form of Lemma 5.2.6.

Corollary 5.2.8 *Let X be a Lévy process with the assumption that*

$$\textit{either } q > 0 \textit{ or } \Big(\; q = 0 \textit{ and } \mathbb{P}\big(\limsup_{t\to\infty} X_t < \infty\big) \;=\; 1\Big). \tag{H2}$$

Suppose that for a given continuous function G and $q \geq 0$, $\mathscr{C}_G^{(q)}$ solves the problem

$$\mathbb{E}\Big(\mathscr{C}_G^{(q)}\big(x + \overline{X}_{\mathbf{e}_q}\big)\Big) \;=\; G(x), \quad \textit{for every } x \in \mathbb{R}. \tag{5.2.13}$$

Then for every $x, y \in \mathbb{R}$ such that $x \leq y$, we have

$$\mathbb{E}_x\Big(e^{-q\tau_y^+} G(X_{\tau_y^+}) \mathbf{1}_{(\tau_y^+ < \infty)}\Big) \;=\; \mathbb{E}_x\Big(\mathscr{C}_G^{(q)}(\overline{X}_{\mathbf{e}_q}) \mathbf{1}_{(\overline{X}_{\mathbf{e}_q} > y)}\Big). \tag{5.2.14}$$

Apart from exponential, linear combination of exponential, polynomials, and sufficiently regular functions, we provide here another example of solution to the problem (5.2.13).

Example 5.2.9 (Appell functions with index $\nu < 0$ and $\nu > 0$) It was shown recently by Novikov and Shiryaev [92] that it is possible to construct Appell functions $Q_\nu(x)$ associated with the random variable $\overline{X}_{\mathbf{e}_q}$ with index $\nu < 0$ and $\nu > 0$. What we shall say below is based on [92]. The construction is based on the *Esscher-Mellin transform*

$$\mathscr{C}_G^{(q)}(x;\nu) \;=\; \frac{1}{\Gamma(-\nu)} \int_0^\infty \lambda^{-\nu-1} \frac{e^{-\lambda x}}{\mathbb{E}\big(e^{-\lambda \overline{X}_{\mathbf{e}_q}}\big)} d\lambda, \quad \text{for } \nu < 0. \tag{5.2.15}$$

Following this, we see for $x > 0$ and $\nu < 0$ that

$$\frac{d\mathscr{C}_G^{(q)}}{dx}(x;\nu) \;=\; \frac{\nu}{\Gamma(1-\nu)} \int_0^\infty \lambda^{-\nu} \frac{e^{-\lambda x}}{\mathbb{E}\big(e^{-\lambda \overline{X}_{\mathbf{e}_q}}\big)} d\lambda \;=\; \nu \mathscr{C}_G^{(q)}(x;\nu-1), \tag{5.2.16}$$

and (5.2.15) solves the problem (5.2.13) for $G(x) = x^{\nu}$. That is to say

$$\mathbb{E}\Big(\mathscr{C}_G^{(q)}(x+\overline{X}_{\mathbf{e}_q};\nu)\Big) = \frac{1}{\Gamma(-\nu)}\int_0^{\infty}\lambda^{-\nu-1}\frac{\mathbb{E}\big(e^{-\lambda(x+\overline{X}_{\mathbf{e}_q})}\big)}{\mathbb{E}\big(e^{-\lambda\overline{X}_{\mathbf{e}_q}}\big)}d\lambda \\ = x^{\nu}, \quad \text{for } x>0. \tag{5.2.17}$$

An analytical continuation of the function $\nu \mapsto \mathscr{C}_G^{(q)}(x;\nu)$, $\nu < 0$, to the region $\nu > 0$ can be constructed with the help of (5.2.16), see Novikov and Shiryaev [92] for more details. Thus, we see that the Appell function $\mathscr{C}_G^{(q)}(x;\nu)$ is a solution to the problem (5.2.13) for $G(x) = x^{\nu}$.

Example 5.2.10 Let $\zeta = \mathbf{1}_{(q=0)}$. Assume that the assumption $(H2)$ holds and that the Lévy measure Π satisfies the integrability condition

$$\int_{(1,\infty)} x^{n+\zeta}\Pi(dx) < \infty.$$

For a fixed $n = 1, 2, \cdots$, let us consider a function $G(x) = x^n$. Then using (5.2.15), we deduce for every $x \in \mathbb{R}$ and $n = 1, 2, \cdots$ that

$$\mathbb{E}_x\Big(e^{-q\tau_y^+}X^n_{\tau_y^+}\mathbf{1}_{(\tau_y^+<\infty)}\Big) = \mathbb{E}_x\Big(Q_n(\overline{X}_{\mathbf{e}_q})\mathbf{1}_{(\overline{X}_{\mathbf{e}_q}>y)}\Big),$$

where Q_n, $n = 1, 2, \cdots$, are now the Appell polynomials generated by the random variable $\overline{X}_{\mathbf{e}_q}$. This particular fluctuation identity goes back to the work of Darling et al. [33] for random walks, and was used recently by Novikov and Shiryaev [91] and Kyprianou and Surya [73] to solve the optimal stopping problem (5.1.1) with integer power function $G(x) = (x^+)^n$ of random walks and Lévy processes, respectively.

If the function $\mathscr{P}_G^{(q)}$ (resp. $\mathscr{C}_G^{(q)}$), solving the problem (5.2.1) (resp. (5.2.13)), has a certain monotonicity property, using the fluctuation identity (5.2.11) (resp. (5.2.14)), we will show in the next section that the optimal solution to the problem (5.1.1), with payoff function G, can be written in terms of the function $\mathscr{P}_G^{(q)}$ (resp. $\mathscr{C}_G^{(q)}$).

5.3 General results on optimal stopping problems

In this section, we present a general solution to the perpetual optimal stopping problem (5.1.1). The solution is expressed in terms of the function $\mathscr{P}_G^{(q)}$ (resp. $\mathscr{C}_G^{(q)}$) that solves the problem (5.2.1) (resp. (5.2.13)).

5.3.1 American put-type optimal stopping problems

A general solution to the problem (5.1.1) is given by the following theorem.

Theorem 5.3.1 (General solution) *Suppose that $\mathscr{P}_G^{(q)}$ is a continuous function that solves the problem (5.2.1), and there exists $\widehat{x} \in \mathbb{R}$ such that $\mathscr{P}_G^{(q)}(\widehat{x}) = 0$, $\mathscr{P}_G^{(q)}(x)$ is non-increasing for $x < \widehat{x}$, and $\mathscr{P}_G^{(q)}(x) \leq 0$ for $x > \widehat{x}$, under the assumption that $(H1)$ holds. Denote by $x^\star$ the smallest root of the equation*

$$\mathscr{P}_G^{(q)}(x) \;=\; 0. \tag{5.3.1}$$

Then the optimal solution to the problem (5.1.1), with payoff G, is given by

$$V_{x^\star}(x) \;=\; \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\underline{X}_{\mathbf{e}_q} < x^\star\}}\Big), \tag{5.3.2}$$

for every $x \in \mathbb{R}$ while the optimal stopping time is given by

$$\tau_{x^\star}^- \;=\; \inf\{t > 0 : X_t \;<\; x^\star\}. \tag{5.3.3}$$

That is to say that

$$V_{x^\star}(x) \;=\; \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}G(X_\tau)\mathbf{1}_{(\tau\,<\,\infty)}\Big) \;=\; \mathbb{E}_x\Big(e^{-q\tau_{x^\star}^-}G(X_{\tau_{x^\star}^-})\mathbf{1}_{(\tau_{x^\star}^-\,<\,\infty)}\Big).$$

The result of the previous theorem still holds true while the payoff G is replaced by the function $\widetilde{G}(x) = \max\{G(x), 0\}$. The following lemma establishes this claim.

Lemma 5.3.2 *Let $\widetilde{V}(x)$ be the value function of the problem (5.1.1) under the payoff function $\widetilde{G}(x) = \max\{G(x), 0\}$. Then for all $x \in \mathbb{R}$, we see that $V(x) \;=\; \widetilde{V}(x)$.*

To obtain the main result in Theorem 5.3.1, the following lemma is needed.

Lemma 5.3.3 (Candidate solution) *Suppose that $\mathscr{P}_G^{(q)}$ is a continuous function that solves the problem (5.2.1) and fulfills the requirements of Theorem 5.3.1. Define for every $y \in \mathbb{R}$ and $q \geq 0$ a candidate solution to the problem (5.1.1) as*

$$V_y(x) \;\triangleq\; \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\underline{X}_{\mathbf{e}_q} < y\}}\Big). \tag{5.3.4}$$

Let $x^\star$ be the smallest root of the equation (5.3.1). Then it holds true that

(i) *for all $x, y \in \mathbb{R}$ such that $x < y$, we have*

$$V_y(x) \;=\; G(x);$$

(ii) *for any $x \in \mathbb{R}$, we have*

$$V_{x^\star}(x) \;\geq\; G(x);$$

(iii) *and $\{e^{-qt}V_{x^\star}(X_t), t \geq 0\}$ is a $\mathbb{P}_x$-supermartingale.*

As a result of the optimality of the function $V_{x^\star}$, we have the following result.

Proposition 5.3.4 *For every $x, y \in \mathbb{R}$, it is then true that*

$$V_{x^\star}(x) \;\geq\; V_y(x); \tag{5.3.5}$$

and if $y < x^\star$, then there exists x such that

$$V_y(x) \;<\; G(x). \tag{5.3.6}$$

5.3.2 American call-type optimal stopping problems

By replacing X with its dual $\widehat{X} = -X$ and y with $-y$, the problem of first exit above a level y for X can be transformed into the problem of first exit of $\widehat{X}$ below a level $-y$. Thus, the results below are the obvious dual forms with respect to those achieved previously for the American put-type optimal stopping problem.

Theorem 5.3.5 (General solution) *Suppose that $\mathscr{C}_G^{(q)}$ is a continuous function that solves the problem (5.2.13), and there exists $\widehat{x} \in \mathbb{R}$ such that $\mathscr{C}_G^{(q)}(\widehat{x}) = 0$, $\mathscr{C}_G^{(q)}(x)$ is non-decreasing for $x > \widehat{x}$, and $\mathscr{C}_G^{(q)}(x) \leq 0$ for $x < \widehat{x}$, under the assumption that $(H2)$ holds. Denote by $x^\star$ the largest root of the equation*

$$\mathscr{C}_G^{(q)}(x) \;=\; 0. \tag{5.3.7}$$

Then the optimal solution to the problem (5.1.1), with payoff G, is given by

$$V_{x^\star}(x) \;=\; \mathbb{E}\Big(\mathscr{C}_G^{(q)}(x + \overline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\overline{X}_{\mathbf{e}_q} > x^\star\}}\Big), \tag{5.3.8}$$

for every $x \in \mathbb{R}$ while the optimal stopping time is given by

$$\tau_{x^\star}^+ \;=\; \inf\{t > 0 : X_t \; > x^\star\}. \tag{5.3.9}$$

That is to say that

$$V_{x^\star}(x) \;=\; \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}G(X_\tau)\mathbf{1}_{(\tau\,<\,\infty)}\Big) \;=\; \mathbb{E}_x\Big(e^{-q\tau_{x^\star}^+}G(X_{\tau_{x^\star}^+})\mathbf{1}_{(\tau_{x^\star}^+\,<\,\infty)}\Big).$$

It can be shown in the similar way as before that the result of the previous theorem still holds true when the payoff G is replaced by $\widetilde{G}(x) = \max\{G(x), 0\}$.

5.4 The continuous and smooth pasting principles

In this section, we discuss the behaviour of the candidate solution (5.3.4) of an American put-type optimal stopping problem at a stopping boundary $y \in \mathbb{R}$. The behaviour of the candidate solution $V_y(x) = \mathbb{E}\big(\mathscr{C}_G^{(q)}(x + \overline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\overline{X}_{\mathbf{e}_q} > y\}}\big)$ of an American call-type optimal stopping problem can be obtained similarly.

Firstly, assume that the solution is evaluated at the optimal stopping boundary $x^\star$ (5.3.1). Then we have the following results.

Theorem 5.4.1 *Suppose that the functions G and $\mathscr{P}_G^{(q)}$ are continuously differentiable. Then, the optimal value function (5.3.2) of the problem (5.1.1) is continuous at the optimal stopping boundary $x^\star$, i.e.,*

$$V_{x^\star}(x^\star) \;=\; G(x^\star),$$

and has the property that

$$\frac{dV_{x^\star}}{dx}(x^\star) \;=\; \frac{dG}{dx}(x^\star) - \mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} = 0\big)\frac{d\mathscr{P}_G^{(q)}}{dx}(x^\star). \tag{5.4.1}$$

Hence there is C^1 smooth pasting at $x^\star$ if and only if

$$\mathbb{P}(-\underline{X}_{\mathbf{e}_q} = 0) \;=\; 0,$$

that is when 0 is regular for the lower half-line $(-\infty, 0)$ for the Lévy process X.

The theory of Lévy processes offers the opportunity to specify when regularity of 0 for the lower half-line $(-\infty, 0)$ (resp. for the upper half-line $(0, \infty)$) for the Lévy process X occurs in terms of the triplet (μ, σ, Π) of the Lévy-Khintchine exponent (2.1.1). When X has bounded variation, it will be more convenient to write (2.1.1) in the form

$$\Psi(\theta) \;=\; -i\mathrm{d}\theta + \int_{-\infty}^{\infty}\big(1 - e^{i\theta x}\big)\Pi(dx).$$

Theorem 5.4.2 (Regularity of half-line for Lévy processes) *Suppose that X is any Lévy process other than a compound Poisson process. Denote the upper and lower tails $\overline{\Pi}^{\pm}$ of the Lévy measure Π by*

$$\overline{\Pi}^{+}(x) = \Pi((x,\infty)), \quad \text{and} \quad \overline{\Pi}^{-}(x) = \Pi((-\infty,x)).$$

We have that 0 is regular for $(-\infty, 0)$ (respectively, for $(0, \infty)$) for X if and only if one of the following conditions[3] is satisfied:

(i) *X has bounded variation with $\mathrm{d} < 0$ (respectively, with $\mathrm{d} > 0$).*

(ii) *X has bounded variation, $\mathrm{d} = 0$, and the Lévy measure Π satisfies*

$$\int_{-1}^{0-} \frac{|x|\Pi(dx)}{\int_0^{|x|}\overline{\Pi}^{+}(y)dy} \;=\; \infty, \quad \left(\textit{respectively,} \int_0^1 \frac{x\Pi(dx)}{\int_0^x \overline{\Pi}^{-}(-y)dy} \;=\; \infty\right)$$

(iii) *X has unbounded variation.*

On the other hand, when the candidate solution (5.3.4) to the problem (5.1.1) is evaluated at a stopping boundary $y \in \mathbb{R}$ other than the optimal stopping boundary $x^\star$ (5.3.1), we have the following results.

[3] See for instance [16], [3], [73], and the literature therein for more details on regularity of half-line for Lévy processes.

Theorem 5.4.3 *Suppose that the functions G and $\mathscr{P}_G^{(q)}$ are continuously differentiable. Let us denote by $P_q^{(-)}(x) := \mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} \leq x\big)$ the distribution function of the random variable $-\underline{X}_{\mathbf{e}_q}$. Consider the candidate solution (5.3.4) of the problem (5.1.1)*

$$V_y(x) \;=\; \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x+\underline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\underline{X}_{\mathbf{e}_q}<y\}}\Big).$$

If the limit

$$p_q^{(-)}(x) \;\triangleq \lim_{h\downarrow 0}\frac{1}{h}\Big(P_q^{(-)}(x+h)-P_q^{(-)}(x)\Big) \tag{5.4.2}$$

exists for every $x\in\mathbb{R}$, then we see at $x=y$ that

$$V_y(y) \;=\; G(y)-\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q}=0\big)\mathscr{P}_G^{(q)}(y), \tag{5.4.3}$$

and the derivative at $x=y$ of the function $V_y(x)$ is given by

$$\frac{dV_y}{dx}(y) \;=\; \frac{dG}{dx}(y)-\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q}=0\big)\frac{d\mathscr{P}_G^{(q)}}{dx}(y)-p_q^{(-)}(0)\mathscr{P}_G^{(q)}(y). \tag{5.4.4}$$

Hence, when $y\neq x^\star$ we see that there is discontinuity for the candidate solution $V_y(x)$ and its derivative at $x=y$ when X is a Lévy process for which 0 is irregular for $(-\infty,0)$ for X. In the regular case, there is only discontinuity for the derivative. Additionally, if $|p_q^{(-)}(0)|=\infty$, then there is an infinite gradient of the candidate solution $V_y(x)$ at the point $x=y$.

Below are examples of Lévy processes for which $p_q^{(-)}(0)=\infty$.

Lemma 5.4.4 *Suppose that the Lévy measure Π of X has no atoms and $\Pi(0,\infty)=0$ so that the Laplace exponent of X exists and is given by*

$$\kappa(\lambda) \;=\; -\Psi(-i\lambda) \;=\; \mathrm{d}\lambda+\frac{1}{2}\sigma^2\lambda^2+\int_{(-\infty,0)}\big(e^{\lambda x}-1-\lambda x\mathbf{1}_{\{x>-1\}}\big)\Pi(dx).$$

For every $q\geq 0$ the value of the limit (5.4.2) at zero is given by

$$p_q^{(-)}(0)=\begin{cases}\frac{2q}{\sigma^2\Phi(q)}, & \textit{when } X \textit{ has unbounded variation and } \sigma\neq 0,\\ \infty, & \textit{when } X \textit{ has unbounded variation with } \sigma=0,\\ \infty, & \textit{when } X \textit{ has bounded variation and } \Pi(-\infty,0)=\infty,\\ \frac{q(\Pi(-\infty,0)+q)}{\mathrm{d}^2\Phi(q)}-\frac{q}{\mathrm{d}}, & \textit{when } X \textit{ has bounded variation and } \Pi(-\infty,0)<\infty,\end{cases}$$

where $\Phi(q)=\sup\{\lambda:\kappa(\lambda)=q\}$ is the largest root of the equation $\kappa(x)-q=0$.

Example 5.4.5 (Regular Lévy process of exponential type [21]) Suppose that X is a regular Lévy process of exponential type[4] with the Wiener-Hopf factor

$$\Psi_q^{(-)}(\lambda) \;=\; C(\alpha+i\lambda)^{-\beta}+\widehat{f}(\lambda), \tag{5.4.5}$$

where α and C are positive constants, $\beta\in(0,1]$, and $\widehat{f}(\lambda)=\mathcal{O}((1+|\lambda|)^{-s})$ as $\lambda\to\infty$ for some $s>1$. Hence, f, the inverse Fourier transform of $\widehat{f}$, is a bounded continuous function. Since it is known that

$$\int_{-\infty}^{\infty} e^{-i\lambda x}x^{\nu-1}e^{-\alpha x}\mathbf{1}_{(0,\infty)}(x)dx \;=\; \Gamma(\nu)(\alpha+i\lambda)^{-\nu}, \quad \text{for } \nu>0,$$

following the Wiener-Hopf factorization (2.2.6), discussed in Section 2.2 of Chapter 2, we deduce from equation (5.4.5) that the function

$$p_q^{(-)}(x) \;=\; C\Gamma(\beta)^{-1}x^{\beta-1}e^{-\alpha x}\mathbf{1}_{(0,\infty)}(x)+f(x),$$

is continuous on $(0,+\infty)$ and is unbounded as $x\to 0$. On noticing the fact that $p_q^{(-)}$ represents the density of the distribution function $\mathbb{P}(-\underline{X}_{\mathbf{e}_q}\le x)$, it is clear following Theorem 5.4.3 that there exists an infinite gradient for the candidate solution $V_y(x)$ at a stopping boundary y, unless $y=x^\star$ or $\beta=1$.

5.5 Consistency with existing literature

In this section, we use our approach to reproduce the special results of those discussed, among others, by Darling et al. [33], Mordecki [87], Boyarchenko and Levendorskii [21], Novikov and Shiryaev [91], [92], and Kyprianou and Surya [73].

Example 5.5.1 (Option with a relatively general payoff) In [21], Boyarchenko and Levendorskii considered perpetual optimal stopping problems of American put option type with a relatively general payoff function G of the form

$$G(x) \;=\; \sum_{j=1}^{m} c_j e^{\theta_j x}, \quad \text{with } \theta_j\ge 0, \tag{5.5.1}$$

for a class of regular Lévy processes of exponential type which includes normal inverse Gaussian, hyperbolic processes, tempered stable processes, and Variance Gamma processes. Using the result (5.2.3) of Section 2.1, we see that the function

$$\mathscr{P}_G^{(q)}(x) \;=\; \sum_{j=1}^{m} c_j\Psi_q^{(-)}(-i\theta_j)^{-1}e^{\theta_j x}$$

[4] This is a process of pure jumps whose characteristic exponent is given for $c>0$, $\kappa_-<0<\kappa_+$, $\nu\in(0,2]$ and $\nu_1<\nu$ by $\Psi(\lambda)=-i\mu\lambda+\phi(\lambda)$ where $\phi(\lambda)=c|\lambda|^\nu+\mathcal{O}(|\lambda|^{\nu_1})$ as $\lambda\to\infty$ in the strip $\mathfrak{Im}(\lambda)\in[\kappa_-,\kappa_+]$. This type of Lévy process was considered by Boyarchenko and Levendorskii [21]. They showed in [21] that under some regularity conditions imposed on $\phi(\lambda)$ the Wiener-Hopf factor $\Psi_q^{(-)}(\lambda)$ (2.2.6) is of the form (5.4.5).

solves for a given $q \geq 0$ and the payoff function G the averaging problem (5.2.1). Denote by $x^\star$ the root of the equation $\mathscr{P}_G^{(q)}(x) = 0$. Suppose that the measure $\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} \in dx\big)$ is absolutely continuous w.r.t the Lebesque measure dx. (This assumption was used in Theorem 4.6 of [21] to prove the optimality of the stopping time $\tau_{x^\star}^-$).

By applying Fourier transform, with $\mathfrak{Im}(\lambda) = \sigma$, for some $\sigma > 0$, to the optimal value function (5.3.2), we come to rest at the following expression

$$\int_{-\infty}^{\infty} e^{-i\lambda x} V_{x^\star}(x)dx \;=\; \widehat{w}(\lambda)\Psi_q^{(-)}(\lambda), \tag{5.5.2}$$

where $\widehat{w}(\lambda)$ is the Fourier transform, with $\mathfrak{Im}(\lambda) = \sigma$, for some $\sigma > 0$, of the function $x \mapsto \mathscr{P}_G^{(q)}(x)\mathbf{1}_{(-\infty,x^\star)}(x)$ defined by

$$\widehat{w}(\lambda) \;=\; \int_{-\infty}^{\infty} e^{-i\lambda x}\mathscr{P}_G^{(q)}(x)\mathbf{1}_{(-\infty,x^\star)}(x)dx.$$

The result in (5.5.2) was given by Boyarchenko and Levendorskii [21] (see Section 4.2 of [21]). They obtained the result by reducing the problem (5.1.1) using potential theory of Lévy processes and Dynkin's formula to a free boundary problem and solving the latter using the standard theory of pseudodifferential operators.

Example 5.5.2 (Perpetual American put option) Let us consider an optimal stopping problem (5.1.1) with the payoff function $G(x) = K - e^x$ under the hypothesis $(H1)$. Applying the result in (5.2.2), we see that $\mathscr{P}_G^{(q)}(x) = K - \Psi_q^{(-)}(-i)^{-1}e^x$ solves the averaging problem (5.2.1) for a given $q \geq 0$ and the payoff function G.

According to Theorem 5.3.1, the optimal stopping boundary $y = x^\star$ is determined as the smallest root of the equation

$$0 \;=\; \mathscr{P}_G^{(q)}(x) \;=\; K - \Psi_q^{(-)}(-i)^{-1}e^x.$$

That is to say that $e^{x^\star} \;=\; K\Psi_q^{(-)}(-i) \;=\; K\mathbb{E}\big(e^{\underline{X}_{\mathbf{e}_q}}\big)$. The rational price can be calculated using the explicit pricing formula (5.3.2) and is given by

$$\begin{aligned} V_{x^\star}(x) &= \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\underline{X}_{\mathbf{e}_q} < x^\star\}}\Big) \\ &= \mathbb{E}\Big(\big(K - \Psi_q^{(-)}(-i)^{-1}e^{(x+\underline{X}_{\mathbf{e}_q})}\big)\mathbf{1}_{\{x+\underline{X}_{\mathbf{e}_q} < x^\star\}}\Big) \\ &= \frac{\mathbb{E}\big(K\mathbb{E}\big(e^{\underline{X}_{\mathbf{e}_q}}\big) - e^{(x+\underline{X}_{\mathbf{e}_q})}\big)^+}{\mathbb{E}\big(e^{\underline{X}_{\mathbf{e}_q}}\big)}. \end{aligned} \tag{5.5.3}$$

This expression for the rational price was given by Mordecki [87].

Example 5.5.3 (Perpetual American call option) Let us consider the optimal stopping problem (5.1.1) with the payoff function $G(x) = e^x - K$ under the hypothesis $(H2)$. Similar to (5.2.2), it is clear that $\mathscr{C}_G^{(q)}(x) \;=\; \Psi_q^{(+)}(-i)^{-1}e^x - K$ solves the averaging problem (5.2.13).

According to Theorem 5.3.5, the optimal stopping boundary $y = x^\star$ is determined as the largest root of the equation

$$\mathscr{C}_G^{(q)}(x) \; = \; \Psi_q^{(+)}(-i)^{-1} e^x - K = 0.$$

That is to say that $e^{x^\star} = K\Psi_q^{(+)}(-i) = K\mathbb{E}\big(e^{\overline{X}_{\mathbf{e}_q}}\big)$. The rational price can be calculated using the explicit pricing formula (5.3.8) and is given by

$$\begin{aligned} V_{x^\star}(x) &= \mathbb{E}\Big(\mathscr{C}_G^{(q)}(x+\overline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\overline{X}_{\mathbf{e}_q}>x^\star\}}\Big) \\ &= \mathbb{E}\Big(\big(\Psi_q^{(+)}(-i)^{-1}e^{x+\overline{X}_{\mathbf{e}_q}} - K\big)\mathbf{1}_{\{x+\overline{X}_{\mathbf{e}_q}>x^\star\}}\Big) \\ &= \frac{\mathbb{E}\big(e^{(x+\overline{X}_{\mathbf{e}_q})} - K\mathbb{E}\big(e^{\overline{X}_{\mathbf{e}_q}}\big)\big)^+}{\mathbb{E}\big(e^{\overline{X}_{\mathbf{e}_q}}\big)}. \end{aligned} \tag{5.5.4}$$

This solution was given by Darling et al [33] for random walks and by Mordecki in [87] for continuous time.

Example 5.5.4 (Option with integer power function) This is a special type of optimal stopping problem where the payoff is an integer power function $G(x) = (x^+)^n$, $n = 1, 2, ...$, of the underlying process. For random walks, this problem was introduced by Novikov and Shiryaev [91] and was extended to continuous time by Kyprianou and Surya [73]. Similar to (5.2.15), it is clear for $\nu = n = 1, 2, ...$ that

$$\mathscr{C}_G^{(q)}(x) \; = \; Q_\nu(x).$$

According to Theorem 5.3.5, the optimal boundary $y = x^\star$ is determined as the largest root of the equation $Q_n(x) = 0$ and the optimal value function is given by

$$\begin{aligned} V_{x^\star}(x) &= \mathbb{E}\Big(\mathscr{C}_G^{(q)}(x+\overline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\overline{X}_{\mathbf{e}_q}>x^\star\}}\Big) \\ &= \mathbb{E}\Big(Q_n(x+\overline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\overline{X}_{\mathbf{e}_q}>x^\star\}}\Big). \end{aligned}$$

This result is equal to the one given by Novikov and Shiryaev [91] for discrete time and to the one in Kyprianou and Surya [73] for continuous time.

In particular, for $n = 1$, we have $Q_1(x) \; = \; x - \mathbb{E}\big(\overline{X}_{\mathbf{e}_q}\big)$ and the optimal boundary is given by $x^\star = \mathbb{E}\big(\overline{X}_{\mathbf{e}_q}\big)$. The optimal value function is given by

$$\begin{aligned} V_{x^\star}(x) &= \mathbb{E}\Big(\big(x+\overline{X}_{\mathbf{e}_q} - \mathbb{E}\big(\overline{X}_{\mathbf{e}_q}\big)\big)\mathbf{1}_{\{x+\overline{X}_{\mathbf{e}_q}>x^\star\}}\Big) \\ &= \mathbb{E}\Big(x+\overline{X}_{\mathbf{e}_q} - \mathbb{E}\big(\overline{X}_{\mathbf{e}_q}\big)\Big)^+. \end{aligned}$$

This result was also given by Darling et al [33] for payoff function $G(x) = x^+$ under random walks.

Remark 5.5.5 It was shown recently by Novikov and Shiryaev [92] that it is possible to extend the result of Darling et al. [33] to the function x^{ν}, for $\nu < 0$ and $\nu > 0$, as a payoff function of random walks and Lévy processes. Our results presented in Section 4 show no contradiction with their results for the case of Lévy processes.

Thus, we have seen that our results in Theorems 5.3.1 and 5.3.5 are consistent with those provided in the above mentioned literature.

5.6 Proofs and main calculations

Proof of Lemma 5.2.5

Proving uniqueness of the solution

To see that (5.2.9) is the solution to the problem (5.2.1), let us first show that the integral in (5.2.9) exists. Due to the regularity assumption (5.2.7) imposed on the payoff function G, using the Wiener-Hopf factorization (2.2.5), we see for $q > 0$ that

$$\begin{aligned} \int_{-\infty}^{\infty} \Big| \frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)} \Big| d\lambda &= \int_{-\infty}^{\infty} \Big| \frac{\Psi_q^{(+)}(\lambda)\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)\Psi_q^{(+)}(\lambda)} \Big| d\lambda \\ &= q^{-1} \int_{-\infty}^{\infty} \big|\big(q + \Psi(\lambda)\big)\big| \big|\Psi_q^{(+)}(\lambda)\big| \big|\widehat{G}(\lambda)\big| d\lambda \\ &\le q^{-1} \int_{-\infty}^{\infty} \big|\big(q + \Psi(\lambda)\big)\big| \big|\widehat{G}(\lambda)\big| d\lambda \\ &\le \int_{-\infty}^{\infty} \big|\widehat{G}(\lambda)\big| d\lambda + q^{-1} \int_{-\infty}^{\infty} \big|\Psi(\lambda)\big| \big|\widehat{G}(\lambda)\big| d\lambda. \end{aligned} \tag{5.6.1}$$

In view of (5.2.7), it is clear that the first integral in (5.6.1) is finite. To see that the second integral is finite, we need to take account on the fact that

$$\big| e^{i\lambda x} - 1 - i\lambda x \mathbf{1}_{\{|x|\le 1\}} \big| \;\le\; \frac{1}{2}|\lambda|^2 |x|^2 \mathbf{1}_{\{|x|\le 1\}} + 2\mathbf{1}_{\{|x|>1\}},$$

which, following (2.1.1), implies that

$$\big|\Psi(\lambda)\big| \le \mu|\lambda| + \frac{1}{2}|\lambda|^2 \Big(\sigma^2 + \int_{\{|y|\le 1\}} |y|^2 \Pi(dy) \Big) + 2 \int_{\{|y|>1\}} \Pi(dy),$$

where the Lévy measure Π satisfies the integrability condition

$$\int_{-\infty}^{\infty} (1 \wedge |y|^2) \Pi(dy) \;<\; \infty.$$

On observing that

$$\begin{aligned} \int_{-\infty}^{\infty} |\lambda| \big|\widehat{G}(\lambda)\big| d\lambda &= \int_{\{|\lambda|\le 1\}} |\lambda| \big|\widehat{G}(\lambda)\big| d\lambda + \int_{\{|\lambda|>1\}} |\lambda| \big|\widehat{G}(\lambda)\big| d\lambda \\ &\le \int_{-\infty}^{\infty} \big|\widehat{G}(\lambda)\big| d\lambda + \int_{-\infty}^{\infty} |\lambda|^3 \big|\widehat{G}(\lambda)\big| d\lambda \;<\; \infty, \end{aligned}$$

and similarly

$$\int_{-\infty}^{\infty} |\lambda|^2 \big|\widehat{G}(\lambda)\big| d\lambda \;<\; \infty, \tag{5.6.2}$$

we see that the integral in (5.2.9) is convergent in absolute value.

We move now to showing that the function (5.2.9) is the solution to the averaging problem (5.2.1).

Taking account of the fact that every sufficiently regular function in $\mathscr{R}$ can be decomposed as the Fourier integral representation (5.2.8), and the Wiener-Hopf factor $\Psi_q^{(-)}(\lambda)$ is nowhere zero, we see for every $x \in \mathbb{R}$ that

$$\begin{aligned} G(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\widehat{G}(\lambda)d\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\mathbb{E}\big(e^{i\lambda \underline{X}_{\mathbf{e}_q}}\big)\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}d\lambda \\ &= \mathbb{E}\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda(x+\underline{X}_{\mathbf{e}_q})}\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}d\lambda\Big) \\ &= \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x+\underline{X}_{\mathbf{e}_q})\Big), \end{aligned}$$

where the third equality was obtained by applying (in view of the integrability conditions (5.6.1)-(5.6.2)) Fubini's theorem.

Proving boundedness and continuous differentiability

Following (5.6.1)-(5.6.2), it is clear that the function $\mathscr{P}_G^{(q)}(x)$ is bounded in $\mathbb{R}$. To see that the function $\mathscr{P}_G^{(q)}(x)$ is continuous in $\mathbb{R}$, let us take $a \in \mathbb{R}$ and $\epsilon > 0$ arbitrarily such that for the chosen $\epsilon > 0$ there exists an $R > 1$ such that

$$\int_{-\infty}^{-R}\Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big|d\lambda < \frac{\epsilon}{10} \quad \text{and} \quad \int_{R}^{\infty}\Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big|d\lambda < \frac{\epsilon}{10}. \tag{5.6.3}$$

The existence of such an R is guaranteed by the fact that the integral (5.6.1) is finite. For such an R, we see for all $x \in \mathbb{R}$ that

$$\begin{aligned} \Big|\int_{-\infty}^{-R}\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}e^{i\lambda x}d\lambda\Big| \;&\le\; \int_{-\infty}^{-R}\Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big|\Big|\cos(\lambda x)\Big|d\lambda + \int_{-\infty}^{-R}\Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big|\Big|\sin(\lambda x)\Big|d\lambda \\ &\le 2\int_{-\infty}^{-R}\Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big|d\lambda \;<\; \frac{\epsilon}{5}. \end{aligned}$$

Likewise, following the similar arguments, we see that

$$\Big|\int_{R}^{\infty}\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}e^{i\lambda x}d\lambda\Big| \;<\; \frac{\epsilon}{5}.$$

Hence, following (5.2.9), we see that

$$\begin{aligned}\Big|\mathscr{P}_G^{(q)}(x)-\mathscr{P}_G^{(q)}(a)\Big| &= \Big|\int_{-\infty}^{\infty}(\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)})e^{i\lambda x}d\lambda-\int_{-\infty}^{\infty}(\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)})e^{i\lambda a}d\lambda\Big| \\ &< \frac{4\epsilon}{5}+\Big|\int_{-R}^{R}(\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)})e^{i\lambda x}d\lambda-\int_{-R}^{R}(\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)})e^{i\lambda a}d\lambda\Big| \\ &\le \frac{4\epsilon}{5}+\int_{-R}^{R}\Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big|\Big|\cos(\lambda x)-\cos(\lambda a)\Big|d\lambda \\ &\quad+\int_{-R}^{R}\Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big|\Big|\sin(\lambda x)-\sin(\lambda a)\Big|d\lambda.\end{aligned}$$

Let us now define

$$\delta \;=\; \Big(2R^2\int_{-R}^{R}\Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big|d\lambda\Big)^{-1}\frac{\epsilon}{10}.$$

Making use of the inequalities

$$\Big|\cos(\lambda x)-\cos(\lambda a)\Big| \;\le\; 2\big(1\wedge|\lambda(x-a)|^2\big),$$

and

$$\Big|\sin(\lambda x)-\sin(\lambda a)\Big| \;\le\; 2\big(1\wedge|\lambda(x-a)|\big),$$

where $a\wedge b=\min\{a,b\}$, it is easy to see for all $x\in\mathbb{R}$, with $|x-a|<\delta$, that

$$\int_{-R}^{R}\Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big|\Big|\cos(\lambda x)-\cos(\lambda a)\Big|d\lambda \;<\; \frac{\epsilon}{10},$$

and

$$\int_{-R}^{R}\Big|\frac{\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big|\Big|\sin(\lambda x)-\sin(\lambda a)\Big|d\lambda \;<\; \frac{\epsilon}{10}. \tag{5.6.4}$$

Thus, combining the results of (5.6.3)-(5.6.4), we see for all $x\in\mathbb{R}$, with $|x-a|<\delta$, that

$$\Big|\mathscr{P}_G^{(q)}(x)-\mathscr{P}_G^{(q)}(a)\Big| \;<\; \epsilon.$$

By replacing $\widehat{G}(\lambda)$ with $\lambda\widehat{G}(\lambda)$ in (5.6.3)-(5.6.4) we see that

$$\int_{-\infty}^{\infty}\Big|\frac{\lambda\widehat{G}(\lambda)}{\Psi_q^{(-)}(\lambda)}\Big|d\lambda \;<\; \infty, \tag{5.6.5}$$

and therefore the function $\frac{d\mathscr{P}_G^{(q)}}{dx}$ is bounded in $\mathbb{R}$. Taking account of the condition (5.6.5), it can be shown following the similar steps as before that

$$\Big|\frac{d\mathscr{P}_G^{(q)}}{dx}(x)-\frac{d\mathscr{P}_G^{(q)}}{dx}(a)\Big| \;<\; \epsilon,$$

for all $x\in\mathbb{R}$ with $|x-a|<\delta$. Thus, our claim that the function $\mathscr{P}_G^{(q)}$ and its derivative $\frac{d\mathscr{P}_G^{(q)}}{dx}$ are both continuous and bounded in $\mathbb{R}$ is then proved. □

Proof of Lemma 5.3.3

To establish the results, let us recall that the function $\mathscr{P}_G^{(q)}$ solves the problem (5.2.1) and our notation for the first passage time below a level y of X is given by

$$\tau_y^- \ = \ \inf\big\{t > 0 : X_t \ < \ y\big\}.$$

Proof of (i)-(iii)

(i) Let us now consider the function (5.3.4):

$$V_y(x) \ = \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\underline{X}_{\mathbf{e}_q}<y\}}\Big). \tag{5.6.6}$$

Since $\underline{X}_{\mathbf{e}_q} \le 0$ almost surely, it is obvious from equations (5.6.6) and (5.2.1) that $V_y(x) = G(x)$ for all $x, y \in \mathbb{R}$ such that $x < y$.

(ii) Majorant property We want to show that the function $V_{x^\star}(x)$ (5.3.2) is majorant to the payoff function $G(x)$, namely $V_{x^\star}(x) \ge G(x)$ for every $x \in \mathbb{R}$.

On noticing the fact that $\mathscr{P}_G^{(q)}(x) \le 0$ for all $x \ge x^\star$, we see for each $q \ge 0$ and every $x \in \mathbb{R}$ that

$$\begin{aligned}
V_{x^\star}(x) \quad &= \quad \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\underline{X}_{\mathbf{e}_q}<x^\star\}}\Big) \\
&= \quad \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\Big) \\
&\qquad - \ \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\mathbf{1}_{\{x+\underline{X}_{\mathbf{e}_q}\ge x^\star\}}\Big) \\
&\ge \quad \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x + \underline{X}_{\mathbf{e}_q})\Big) \\
&= \quad G(x),
\end{aligned}$$

where the inequality is due to the fact that $\mathscr{P}_G^{(q)}(x) \le 0$ for all $x \ge x^\star$, while the last equality is based on the fact that $\mathscr{P}_G^{(q)}$ solves the problem (5.2.1). Thus, the claim that the function $V_{x^\star}$ is majorant to the payoff function G is then proved.

(iii) Supermartingale property Let us now show that the function $V_{x^\star}$ (5.3.2) has the supermartingale property. The proof is obtained by noticing the fact that conditionally on the event $\{\mathbf{e}_q > t\}$, the following identity

$$\underline{X}_{\mathbf{e}_q} \ = \ \underline{X}_t \wedge (\mathcal{I} + X_t)$$

holds, and conditionally on the filtration $\mathcal{F}_t$, the random variable $\mathcal{I}$ has the same distribution as $\underline{X}_{\mathbf{e}_q}$. Following this and the fact that the function $x \mapsto \mathscr{P}_G^{(q)}(x)$ is non-increasing on the interval $(-\infty, x^\star]$, we see for every $q \geq 0$ that

$$\begin{aligned} V_{x^\star}(x) &= \mathbb{E}_x\Big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q})\mathbf{1}_{\{\underline{X}_{\mathbf{e}_q}<x^\star\}}\Big) \\ &= \mathbb{E}_x\Big(\mathbb{E}\big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q})\mathbf{1}_{\{\underline{X}_{\mathbf{e}_q}<x^\star\}}\Big|\mathcal{F}_t\big)\Big) \\ &\geq \mathbb{E}_x\Big(\mathbf{1}_{\{\mathbf{e}_q>t\}}\mathbb{E}\big(\mathscr{P}_G^{(q)}(X_t+\mathcal{I})\mathbf{1}_{\{X_t+\mathcal{I}<x^\star\}}\Big|\mathcal{F}_t\big)\Big) \\ &= \mathbb{E}_x\Big(\mathbf{1}_{\{\mathbf{e}_q>t\}}\mathbb{E}_{X_t}\big(\mathscr{P}_G^{(q)}(\underline{X}_{\mathbf{e}_q})\mathbf{1}_{\{\underline{X}_{\mathbf{e}_q}<x^\star\}}\big)\Big) \\ &= \mathbb{E}_x\Big(\mathbf{1}_{\{\mathbf{e}_q>t\}}V_{x^\star}(X_t)\Big) \\ &= \mathbb{E}_x\Big(e^{-qt}V_{x^\star}(X_t)\Big). \end{aligned}$$

Thus, the supermartingale property of the process $\{e^{-qt}V_{x^\star}(X_t), t \geq 0\}$ is established.

□

Proof of Theorem 5.3.1

The proof of the theorem is mainly based on the fluctuation identity (5.2.11), the majorant and supermartingale properties of the function $V_{x^\star}$ (5.3.2), see Lemma 5.3.3. On noticing the fact that τ is arbitrary in $\mathcal{T}_{[0,\infty]}$ and the function $V_{x^\star}$ is lower bounded by the payoff G and has the supermartingale property, we see for every $x \in \mathbb{R}$ that

$$V_{x^\star}(x) \geq \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}V_{x^\star}(X_\tau)\mathbf{1}_{(\tau<\infty)}\Big) \geq \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}G(X_\tau)\mathbf{1}_{(\tau<\infty)}\Big).$$

On the other hand, rather trivially, we have for every $x \in \mathbb{R}$ that

$$\sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}G(X_\tau)\mathbf{1}_{(\tau<\infty)}\Big) \geq \mathbb{E}_x\Big(e^{-q\tau_{x^\star}^-}G(X_{\tau_{x^\star}^-})\mathbf{1}_{(\tau_{x^\star}^-<\infty)}\Big) = V_{x^\star}(x),$$

where the equality is due to the fluctuation identity (5.2.11).

Thus, all the inequalities are equalities and hence

$$V_{x^\star}(x) = \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}G(X_\tau)\mathbf{1}_{(\tau<\infty)}\Big) = \mathbb{E}_x\Big(e^{-q\tau_{x^\star}^-}G(X_{\tau_{x^\star}^-})\mathbf{1}_{(\tau_{x^\star}^-<\infty)}\Big).$$

Thus, the value function $V(x)$ of the optimal stopping problem (5.1.1) coincides for every $x \in \mathbb{R}$ with the function $V_{x^\star}(x)$, and the optimal stopping time is given by $\tau_{x^\star}^-$. Thus, the claim that the function $V_{x^\star}(x)$ (5.3.2) is the optimal solution to the problem (5.1.1) is then proved. □

Proof of Proposition 5.3.4

The proof that the candidate solution $V_y(x)$ (5.3.4) satisfies the first claim (5.3.5) follows from applying the results of Lemma 5.2.6 and Theorem 5.3.1.

The proof of the other claim (5.3.6) is established as follows. Following the assumption $(H1)$ imposed on X, we see for a fixed $y \in \mathbb{R}$ that

$$\lim_{x\to\infty} V_y(x) \;=\; \lim_{x\to\infty} \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x+\underline{X}_{\mathbf{e}_q})\mathbf{1}_{(\underline{X}_{\mathbf{e}_q}<y-x)}\Big) \;=\; 0.$$

Notice also that the function $V_{x^\star}(x)$ (5.3.2) is lower bounded for every $x \in \mathbb{R}$ by the payoff function $G(x)$, and for $y < x^\star$ we have that $0 \leq V_y(x) < V_{x^\star}(x)$ for every $x \in \mathbb{R}$. Taking into account of the fact that $V_y(x) = G(x)$ for every $x < y$, we see that there exists $x \in \mathbb{R}$ such that for each $y < x^\star$ we have $V_y(x) < G(x)$. Thus, the claim that the candidate solution $V_y(x)$ (5.3.4) satisfies the inequalities (5.3.5) and (5.3.6) is then proved. □

Proof of Lemma 5.3.2

Let $\widetilde{V}$ be the optimal value function of the problem (5.1.1) under the payoff function $\widetilde{G}(x) = \max\{G(x), 0\}$. Since $\widetilde{G}(x) \geq G(x)$ for all $x \in \mathbb{R}$, we see that

$$\widetilde{V}(x) \;=\; \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}\widetilde{G}(X_\tau)\Big) \;\geq\; \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}G(X_\tau)\Big) \;=\; V_{x^\star}(x). \tag{5.6.7}$$

However, following the positivity of the optimal value function $V_{x^\star}$ and the majorant property of $V_{x^\star}$ over the payoff G, it is straightforward to verify that $V_{x^\star}(x) \geq \widetilde{G}(x)$ for all $x \in \mathbb{R}$. Thus, from the supermartingale property of $V_{x^\star}$, we then obtain

$$V_{x^\star}(x) \;\geq\; \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}V_{x^\star}(X_\tau)\Big) \;\geq\; \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}\widetilde{G}(X_\tau)\Big) \;=\; \widetilde{V}(x).$$

Hence, combining with the inequality (5.6.7), our claim is then established. □

Proof of Theorem 5.4.3

In this section, we provide details of calculations for the proof of Theorem 5.4.3. By evaluating the candidate solution $V_y(x)$ (5.3.4) at the point $x = y$, we have

$$V_y(y) \;=\; G(y) - \mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} = 0\big)\mathscr{P}_G^{(q)}(y). \tag{5.6.8}$$

Since $\mathscr{P}_G^{(q)}(y) \neq 0$, it is clear from the foregoing relation above that $V_y(y) \neq G(y)$ whenever $\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} = 0\big) > 0$, the case where 0 is irregular for $(-\infty, 0)$ for X.

The other claim is achieved as follows. Since the function $\mathscr{P}_G^{(q)}$ solves the problem (5.2.1), the candidate solution $V_y(x)$ (5.3.4) can be rewritten as follows

$$V_y(x) \;=\; G(x) \;-\; H_y(x), \tag{5.6.9}$$

where the function $H_y(x)$ is defined for every $x \in \mathbb{R}$ by

$$H_y(x) \;=\; \mathbb{E}\Big(\mathscr{P}_G^{(q)}(x+\underline{X}_{\mathbf{e}_q})\mathbf{1}_{(x+\underline{X}_{\mathbf{e}_q}\geq y)}\Big).$$

The proof will be done once we show that

$$\frac{dH_y}{dx}(y) = \mathbb{P}\big(-\underline{X}_{\mathbf{e}_q}=0\big)\frac{d\mathscr{P}_G^{(q)}}{dx}(y) + p_q^{(-)}(0)\mathscr{P}_G^{(q)}(y).$$

After some algebra, we have

$$\begin{aligned}\frac{H_y(x)-H_y(y)}{x-y} &= \mathbb{E}\Big(\Big(\frac{\mathscr{P}_G^{(q)}(x+\underline{X}_{\mathbf{e}_q}) - \mathscr{P}_G^{(q)}(y+\underline{X}_{\mathbf{e}_q})}{x-y}\Big)\mathbf{1}_{(-\underline{X}_{\mathbf{e}_q}\le x-y)}\Big)\\ &+ \mathbb{E}\Big(\mathscr{P}_G^{(q)}(y+\underline{X}_{\mathbf{e}_q})\Big(\frac{\mathbf{1}_{(-\underline{X}_{\mathbf{e}_q}\le x-y)} - \mathbf{1}_{(-\underline{X}_{\mathbf{e}_q}=0)}}{x-y}\Big)\Big)\\ &+ \mathbb{E}\Big(\frac{\mathscr{P}_G^{(q)}(y+\underline{X}_{\mathbf{e}_q})\mathbf{1}_{(-\underline{X}_{\mathbf{e}_q}=0)} - \mathscr{P}_G^{(q)}(y)\mathbf{1}_{(-\underline{X}_{\mathbf{e}_q}=0)}}{x-y}\Big).\end{aligned}$$

On noticing the fact that the third term is zero, the above expression now reduces to

$$\begin{aligned}\frac{H_y(x)-H_y(y)}{x-y} &= \mathbb{E}\Big(\Big(\frac{\mathscr{P}_G^{(q)}(x+\underline{X}_{\mathbf{e}_q}) - \mathscr{P}_G^{(q)}(y+\underline{X}_{\mathbf{e}_q})}{x-y}\Big)\mathbf{1}_{(-\underline{X}_{\mathbf{e}_q}\le x-y)}\Big)\\ &+ \mathbb{E}\Big(\mathscr{P}_G^{(q)}(y+\underline{X}_{\mathbf{e}_q})\Big(\frac{\mathbf{1}_{(-\underline{X}_{\mathbf{e}_q}\le x-y)} - \mathbf{1}_{(-\underline{X}_{\mathbf{e}_q}=0)}}{x-y}\Big)\Big).\end{aligned} \tag{5.6.10}$$

Since the functions G and $\mathscr{P}_G^{(q)}$ are assumed to be continuously differentiable, we see using the Lebesque dominated convergence theorem that

$$\frac{H_y(x)-H_y(y)}{x-y} \longrightarrow \mathbb{P}\big(-\underline{X}_{\mathbf{e}_q}=0\big)\frac{d\mathscr{P}_G^{(q)}}{dx}(y) + p_q^{(-)}(0)\mathscr{P}_G^{(q)}(y), \tag{5.6.11}$$

as $x \downarrow y$. Thus, following (5.6.9)-(5.6.11), we see that

$$\frac{dV_y}{dx}(y) = \frac{dG}{dx}(y) - \mathbb{P}\big(-\underline{X}_{\mathbf{e}_q}=0\big)\frac{d\mathscr{P}_G^{(q)}}{dx}(y) - p_q^{(-)}(0)\mathscr{P}_G^{(q)}(y). \tag{5.6.12}$$

Hence, while $y \neq x^\star$, we see that there is discontinuity at $x = y$ for the candidate solution $V_y(x)$ and its derivative when X is a Lévy processes for which 0 is *irregular* for $(-\infty, 0)$ for X. In the regular case, there is only discontinuity for the derivative when $p_q^{(-)}(0) \neq 0$. The infinite gradient only exists if and only if $|p_q^{(-)}(0)| = \infty$. □

Proof of Theorem 5.4.1

Following (5.6.12) above, we see at $y = x^\star$ that

$$\frac{dV_{x^\star}}{dx}(x^\star) = \frac{dG}{dx}(x^\star) - \mathbb{P}\big(-\underline{X}_{\mathbf{e}_q}=0\big)\frac{d\mathscr{P}_G^{(q)}}{dx}(x^\star) - p_q^{(-)}(0)\mathscr{P}_G^{(q)}(x^\star).$$

Since the optimal stopping boundary $x^\star$ solves the equation $\mathscr{P}_G^{(q)}(x) = 0$, we see following the previous equation that

$$\frac{dV_{x^\star}}{dx}(x^\star) = \frac{dG}{dx}(x^\star) - \mathbb{P}\big(-\underline{X}_{\mathbf{e}_q}=0\big)\frac{d\mathscr{P}_G^{(q)}}{dx}(x^\star).$$

Hence, the smooth pasting condition holds if and only if $\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q}=0\big) = 0$. □

Proof of Lemma 5.4.4

From Section 2.3 of Chapter 2, we see that by applying integration by part to (2.3.3) the Stieltjes measure $d\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} \le x\big)$, associated to the function $\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} \le x\big)$, can be written in terms of the q-scale function $W^{(q)}(x)$, defined in (2.3.4), as

$$d\mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} \le x\big) \;=\; \frac{q}{\Phi(q)} dW^{(q)}(x) - qW^{(q)}(x)dx. \tag{5.6.13}$$

By defining $P_q^{(-)}(x) = \mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} \le x\big)$, we have from (5.6.13) that

$$P_q^{(-)}(x) \;=\; \mathbb{P}\big(-\underline{X}_{\mathbf{e}_q} = 0\big) + \frac{q}{\Phi(q)}\big(W^{(q)}(x) - W^{(q)}(0)\big) - q\int_0^x W^{(q)}(y)dy.$$

Therefore, following (5.4.2) we see that

$$\begin{aligned} p_q^{(-)}(0) &= \lim_{h\downarrow 0}\frac{1}{h}\big(P_q^{(-)}(h) - P_q^{(-)}(0)\big) \\ &= \frac{q}{\Phi(q)}\lim_{h\downarrow 0}\frac{1}{h}\big(W^{(q)}(h) - W^{(q)}(0)\big) - q\lim_{h\downarrow 0}\frac{1}{h}\int_0^h W^{(q)}(y)dy \\ &= \frac{q}{\Phi(q)}\frac{dW^{(q)}}{dx}(0+) - qW^{(q)}(0), \end{aligned}$$

where the last equality is due to the fact that the Lévy measure has no atoms so that the q-scale function $W^{(q)}(x)$ is differentiable (we refer to Lambert [75] and Chan and Kyprianou [28]) and, hence, a right derivative at zero of $W^{(q)}(x)$ exists. Using the result of Lemma 2 in Section 6.4 of Chapter 6, our claim is then proved. □

5.7 Numerical examples: the arbitrage-free pricing of American options under tempered stable processes with downward jumps

In this section we verify our main results in Theorems 5.3.1 and 5.3.1 for the optimal stopping problem (5.1.1) with payoffs $(K - e^x)^+$ and $(e^x - K)^+$ under tempered stable processes with no positive jumps, so its Lévy measure Π has support in $(-\infty, 0]$.

5.7.1 Tempered stable processes with downward jumps

A tempered stable process is obtained by taking a one-dimensional stable process and multiplying the Lévy measure with a decreasing exponential on each half of the real axis. This exponential softening keeps the initial stable-like behaviour whereas the large jumps become much less heavy tailed. A tempered stable process is thus a Lévy process with no Gaussian component and a Lévy measure of the form

$$\Pi(dx) \;=\; C\frac{e^{-\lambda|x|}}{|x|^{1+\alpha}}\mathbf{1}_{\{x<0\}}dx \tag{5.7.1}$$

where $C > 0$, $\lambda > 0$ and $\alpha < 2$. Unlike the case of stable processes, which can only be defined for $\alpha > 0$, in the tempered stable process there is no natural lower bound on α and the expression in (5.7.1) yields a Lévy measure for $\alpha < 2$. In fact, taking negative values of α we obtain compound Poisson models with a rich structure. We refer among others to Cont and Tankov [32] for more details. It is clear that tempered stable process is of compound Poisson type if $\alpha < 0$ and has paths of bounded variation if $\alpha < 1$. Because of exponential decay of the tails of the Lévy measure, it is then more convenient to work with tempered stable process without truncation of big jumps. To compute the Laplace exponent, we consider the case that $\alpha \neq 1$ and $\alpha \neq 0$. The integration maybe performed in the following way:

$$\begin{aligned}\int_0^\infty \big(e^{-\theta x} - 1 + \theta x\big)\frac{e^{-\lambda x}}{x^{1+\alpha}}dx &= \sum_{n=2}^{\infty}\frac{(-\theta)^n}{n!}\int_0^\infty x^{n-1-\alpha}e^{-\lambda x}dx \\ &= \sum_{n=2}^{\infty}\frac{(-\theta)^n}{n!}\lambda^{\alpha-n}\Gamma(n-\alpha) \\ &= \lambda^\alpha\Gamma(-\alpha)\Big\{\Big(1+\frac{\theta}{\lambda}\Big)^\alpha - 1 - \frac{\theta\alpha}{\lambda}\Big\}. \qquad (5.7.2)\end{aligned}$$

Note that interchanging the sum and integral and the convergence of the power series are possible if $|\theta| < \lambda$. By analytic continuation, it is clear that the expression (5.7.2) exists for all values of θ such that $\mathfrak{Re}(\theta) < \lambda$.

Performing similar calculation for the case $\alpha = 1$, we obtain

$$\int_0^\infty \big(e^{-\theta x} - 1 + \theta x\big)\frac{e^{-\lambda x}}{x^{1+\alpha}}dx = -\theta + (\lambda+\theta)\log\Big(1+\frac{\theta}{\lambda}\Big),$$

and for $\alpha = 0$, we have

$$\int_0^\infty \big(e^{-\theta x} - 1 + \theta x\big)\frac{e^{-\lambda x}}{x^{1+\alpha}}dx = \frac{\theta}{\lambda} + \log\Big(\frac{\lambda}{\lambda+\theta}\Big).$$

Following the Lévy-Khintchine formula, the Laplace exponent κ of a tempered stable process having no positive jumps is given by the following proposition.

Proposition 5.7.1 *Let X be a tempered stable process having no positive jumps. In the general case ($\alpha \neq 1$ and $\alpha \neq 0$) the Laplace exponent of X is given by*

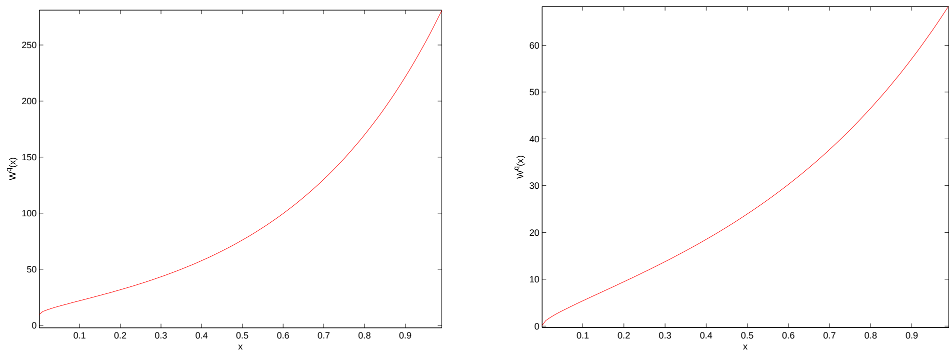
$$\kappa(\theta) = \mu\theta + \Gamma(-\alpha)\lambda^\alpha C\Big\{\Big(1+\frac{\theta}{\lambda}\Big)^\alpha - 1 - \frac{\theta\alpha}{\lambda}\Big\}. \qquad (5.7.3)$$

If $\alpha = 1$, then

$$\kappa(\theta) = (\mu - C)\theta + C(\lambda+\theta)\log\Big(1+\frac{\theta}{\lambda}\Big), \qquad (5.7.4)$$

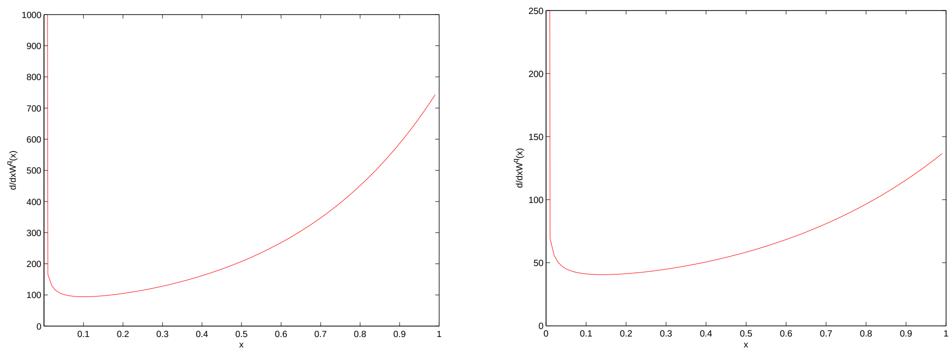
and if $\alpha = 0$, then

$$\kappa(\theta) = \mu\theta - C\Big\{-\frac{\theta}{\lambda} + \log\Big(1+\frac{\theta}{\lambda}\Big)\Big\}. \qquad (5.7.5)$$



(a) Tempered stable process with $\alpha = 0.5$. (b) Tempered stable process with $\alpha = 1.5$

Figure 5.1: Numerical plots of the scale function $W^{(q)}(x)$ for tempered stable processes of bounded and unbounded variations.



(a) Tempered stable process with $\alpha = 0.5$. (b) Tempered stable process with $\alpha = 1.5$

Figure 5.2: Numerical plots of the derivative of the scale function $W^{(q)}(x)$ for tempered stable processes of bounded and unbounded variations.

5.7.2 The rational price of perpetual American options

5.7.2.1 Perpetual American put option

Let us now consider the perpetual American put option problem

$$V(x) \;=\; \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x \Big(e^{-q\tau} \big(K - e^{X_\tau}\big)^+ \mathbf{1}_{(\tau<\infty)} \Big), \tag{5.7.6}$$

under the hypothesis that $(H1)$ holds. To get a better understanding of the problem (5.7.6) both analytically and numerically, let us consider, for a given stopping boundary $y \geq 0$, a candidate solution $V_y(x)$ to the problem (5.7.6) defined by

$$V_y(x) \;=\; \mathbb{E}\Big(\big[K - \Psi_q^{(-)}(-i)^{-1} e^{x+\underline{X}_{\mathbf{e}_q}}\big] \mathbf{1}_{\{x+\underline{X}_{\mathbf{e}_q}<y\}} \Big).$$

In the sequel below we write $V(x;y) \triangleq V_y(x)$. Using the measure (2.3.5), we can rewrite the function $V(x;y)$ explicitly in terms of the q-scale function[5] $W^{(q)}$ as

$$\begin{aligned} V(x;y) \;=\; & K - e^x + \mathbf{1}_{\{x\geq y\}}\Big(\frac{(\kappa(1)-q)}{(1-\Phi(q))}e^y - K\frac{q}{\Phi(q)}\Big)W^{(q)}(x-y) \\ &+ (\kappa(1)-q)e^x\mathbf{1}_{\{x\geq y\}}\int_0^{x-y} e^{-z}W^{(q)}(z)dz + Kq\mathbf{1}_{\{x\geq y\}}\int_0^{x-y} W^{(q)}(z)dz. \end{aligned} \tag{5.7.7}$$

Let us now define for each $q \geq 0$ a function

$$W_{\Phi(q)}(x) \;:=\; e^{-\Phi(q)x}W^{(q)}(x), \tag{5.7.8}$$

where $\Phi(q)$ is the largest root of the equation $\kappa(\theta) = q$. Due to the convexity of the Laplace exponent κ, there exists at most two solutions for a given q and precisely one root when $q > 0$. As will be shown later in Chapters 6 and 7, the scale function $W_{\Phi(q)}(x)$ is increasing and corresponds to the role of the scale function $W^{(0)}(x)$ when X is taken under the measure $\mathbb{P}^{\Phi(q)}$ defined by the *Esscher transform*

$$\frac{d\mathbb{P}^{\Phi(q)}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} \;=\; e^{\Phi(q)X_t - qt} \quad \text{for all } t \geq 0.$$

Using the transformation (5.7.8) in (5.7.7) and varying the value of the stopping boundary y in the interval $(-\infty, x]$, we obtain after some calculations that

$$\frac{dV}{dy}(x;y) \;=\; -\Big(\frac{(\kappa(1)-q)}{(1-\Phi(q))}e^y - K\frac{q}{\Phi(q)}\Big)e^{\Phi(q)(x-y)}W'_{\Phi(q)}(x-y)\mathbf{1}_{\{x\geq y\}}.$$

Next, let us define $x^\star = \log\big(K\frac{q}{\Phi(q)}\frac{(1-\Phi(q))}{(\kappa(1)-q)}\big)$, i.e., $x^\star = \log\big(K\mathbb{E}(e^{\underline{X}_{\mathbf{e}_q}})\big)$. Since the scale function $W_{\Phi(q)}(x)$ is increasing, we see from the foregoing expression that

$$\frac{dV}{dy}(x;y) \;>(<)\; 0 \;\text{ for }\; y \;<(>)\; x^\star \;\text{ and }\; \frac{dV}{dy}(x;x^\star) = 0.$$

Hence, we deduce that $y = x^\star$ is the level at which the candidate function $x \mapsto V(x;y)$ attains its maximum value. As explained previously in Section 3, it is known that the level $y = \log(K\mathbb{E}(e^{\underline{X}_{\mathbf{e}_q}}))$ corresponds to the optimal stopping boundary for the optimal stopping problem (5.7.6) and the function $V(x;x^\star)$ coincides for every $x \in \mathbb{R}$ with the value function $V(x)$ of the problem (5.5.3).

Furthermore, by evaluating (5.7.7) at the point $x = y$, we see that

$$V(y;y) \;=\; K - e^y + \Big(\frac{(\kappa(1)-q)}{(1-\Phi(q))}e^y - K\frac{q}{\Phi(q)}\Big)W^{(q)}(0), \tag{5.7.9}$$

while its derivative at $x = y$ is defined by

$$\begin{aligned} \frac{dV}{dx}(y;y) \;=\; & -e^y + \Big(\frac{(\kappa(1)-q)}{(1-\Phi(q))}e^y - K\frac{q}{\Phi(q)}\Big)\frac{dW^{(q)}}{dx}(0) \\ &+ \big((\kappa(1)-q)e^y + Kq\big)W^{(q)}(0). \end{aligned} \tag{5.7.10}$$

[5] Using excursion theory of spectrally negative Lévy processes, Avram et al [7], Pistorius [101] obtained the expression (5.7.7) as the candidate solution to the problem (5.7.6).

Thus, discontinuity or infinite gradient of the function $V(\bullet; y)$ only exists if

$$W^{(q)}(0) \neq 0 \quad \text{or} \quad \frac{dW^{(q)}}{dx}(0) = \infty,$$

respectively. The latter happens when 0 is regular for $(-\infty, 0)$ for X with zero Gaussian component and X has paths of bounded variation with $\Pi(-\infty, 0) = \infty$, see Lemma 5.4.4. Therefore, following (5.7.9) and (5.7.10), we observe that the optimal value function $V(x)$ is continuous at the point $x = x^\star$ and there exists smooth pasting if and only if $W^{(q)}(0) = 0$, the case when 0 is regular[6] for the lower half-line $(-\infty, 0)$ for X.

5.7.2.2 Perpetual American call option

Let us now consider the perpetual American call option problem

$$V(x) = \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x\Big(e^{-q\tau}\big(e^{X_\tau} - K\big)^+ \mathbf{1}_{(\tau<\infty)}\Big), \tag{5.7.11}$$

under the hypothesis that $(H2)$ holds. To obtain a better understanding of the problem (5.7.11) both analytically and numerically, let us consider, for a given stopping boundary $y \geq 0$, a candidate solution $V_y(x)$ to the problem (5.7.11) defined by

$$V_y(x) = \mathbb{E}\Big(\big(\Psi_q^{(+)}(-i)^{-1} e^{x+\overline{X}_{\mathbf{e}_q}} - K\big)\mathbf{1}_{\{x+\overline{X}_{\mathbf{e}_q}>y\}}\Big). \tag{5.7.12}$$

Again, we will write $V(x; y) \triangleq V_y(x)$. Using the measure (2.3.1), the expression for $V(x; y)$ can be simplified further as

$$\begin{aligned} V(x;y) = &\, e^x - K + K\big(1 - e^{-\Phi(q)(y-x)}\big)\mathbf{1}_{\{y\geq x\}} \\ &+ e^x\big(e^{-(\Phi(q)-1)(y-x)} - 1\big)\mathbf{1}_{\{y\geq x\}}. \end{aligned} \tag{5.7.13}$$

By varying the value of boundary y in the interval $[x, \infty)$, we obtain from (5.7.13) that

$$\frac{dV}{dy}(x; y) = -\Big((\Phi(q) - 1)e^y - K\Phi(q)\Big)e^{-\Phi(q)(y-x)}\mathbf{1}_{\{y\geq x\}}.$$

By defining $x^\star = \log\big(\frac{K\Phi(q)}{(\Phi(q)-1)}\big)$, i.e., $x^\star = \log(K\mathbb{E}(e^{\overline{X}_{\mathbf{e}_q}}))$, it is clear that

$$\frac{dV}{dy}(x;y) > (<)\ 0 \ \text{ for } \ y < (>)\ x^\star \ \text{ and } \ \frac{dV}{dy}(x;x^\star) = 0.$$

Hence, we deduce that $y = x^\star$ is the level at which the candidate function $x \mapsto V(x; y)$ attains its maximum value. As explained previously in Section 3, it is known that the level $y = \log(K\mathbb{E}(e^{\overline{X}_{\mathbf{e}_q}}))$ corresponds to the optimal boundary for the stopping

[6] By applying integration by parts and a Tauberian theorem to the Laplace transforms (2.3.3) and (2.3.4), it can be shown that $\mathbb{P}(-\underline{X}_{\mathbf{e}_q} = 0) = \frac{q}{\Phi(q)}W^{(q)}(0)$.

problem (5.7.11) and the function $V(x;x^\star)$ coincides for every $x \in \mathbb{R}$ with the value function $V(x)$ of the optimal stopping problem (5.5.4).

Furthermore, by evaluating the expression (5.7.13) at the point $x = y$, we see that

$$V(y;y) \;=\; e^y - K,$$

and the derivative at $x = y$ of the value function $V(x;y)$ is given by

$$\frac{dV}{dx}(y;y) \;= e^y \;-\; \Phi(q)\Big(\big(\frac{\Phi(q)}{(\Phi(q)-1)}\big)^{-1} e^y - K\Big).$$

On noticing the fact that $e^{x^\star} = \frac{K\Phi(q)}{(\Phi(q)-1)}$, we see that the candidate solution $V(x;y)$ obeys the smooth pasting condition at the stopping boundary $y = x^\star$. While $y \neq x^\star$, we observe that there exists discontinuity at the point $x = y$ for the derivative of the candidate solution $V(x;y)$ with no infinite gradient.

The results of numerical computation for the value functions $V(x;y)$ (5.7.7) and (5.7.13) of the perpetual American put and call option problems will be discussed in more details in the section below. In particular, for the perpetual American put option, the computation boils down to numerically produce the q-scale function $\{W^{(q)}(x) : q \geq 0, x \in \mathbb{R}_+\}$ of the Lévy process $(X,\mathbb{P})$. Further details of the computation will be elaborated in more details later in Chapter 7.

5.7.3 Numerical results

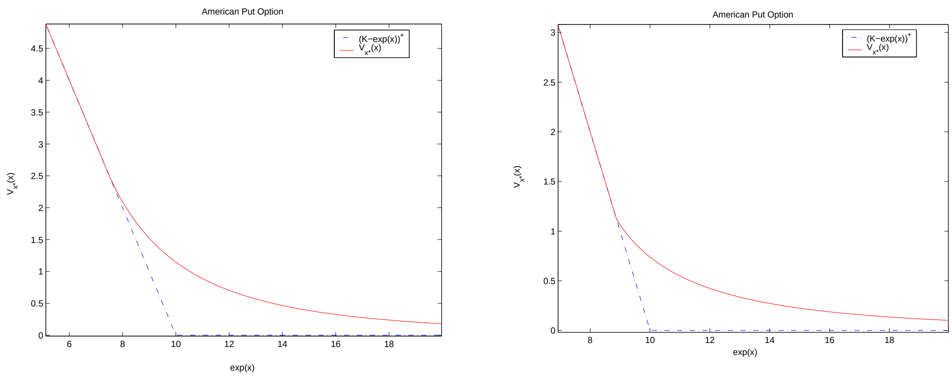
This section deals with pricing the perpetual American put and call options (5.7.6) and (5.7.11) on the stock price process S_t whose dynamics are given under a chosen martingale measure $\mathbb{P}$ by an exponential Lévy process

$$S_t(x) \;=\; xe^{X_t}. \tag{5.7.14}$$

We assume that a default-free asset exists that pays a continuous interest rate $r > 0$ and denote by δ the total payout rate of dividend. Furthermore, we assume under the measure $\mathbb{P}$ that the discounted stock price process $e^{-(r-\delta)t}S_t(x)$ is $\mathbb{P}$-martingale which implies that

$$\mathbb{E}\Big(e^{-(r-\delta)t}S_t(x)\Big) \;=\; x. \tag{5.7.15}$$

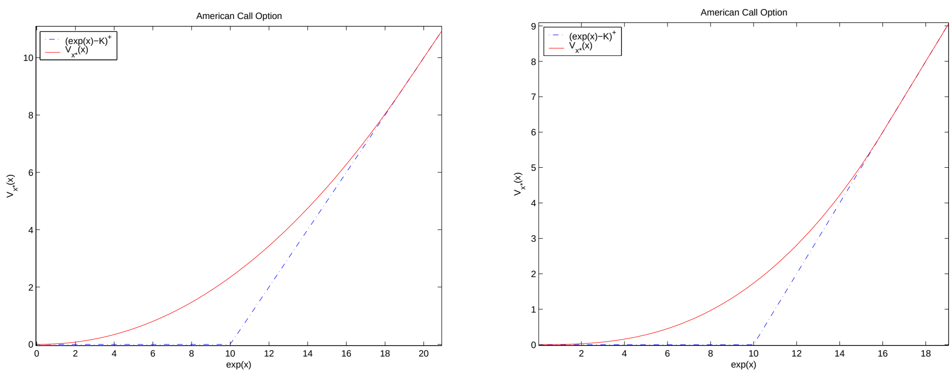
For the purpose of numerical computation, we use generalized tempered stable process for X whose Laplace exponent is given in Proposition 5.7.1. The numerical computation is carried out using MATLAB6.5. The parameter setting for interest rate r, dividend rate δ, the strike value K, and the jump rate λ are set to be 0.1, 0.07, 10, and 2.5, respectively. In the case where X has path of bounded variation we choose $\alpha = 0.5$ and the relative frequency of downward jumps C to be 0.075. In the case of unbounded variation X, we set $\alpha = 1.5$ and $C = 0.05$ and the other parameters have the same value. The drift in the Laplace exponent κ is chosen so that the martingale condition (5.7.15) is satisfied.



(a) Tempered stable process with $\alpha = 1.5$. The optimal stopping boundary $e^{x^\star} = 6.8531$.

(b) Tempered stable process with $\alpha = 0.5$. The optimal stopping boundary $e^{x^\star} = 8.8686$.

Figure 5.3: The shape of the rational price $V_{x^\star}(x)$ of the American put option.



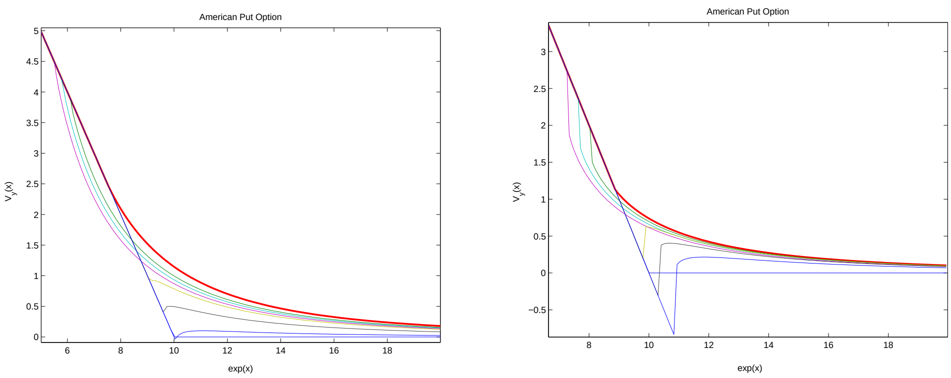
(a) Tempered stable process with $\alpha = 1.5$. The optimal stopping boundary $e^{x^\star} = 20.8455$.

(b) Tempered stable process with $\alpha = 0.5$. The optimal stopping boundary $e^{x^\star} = 16.1082$.

Figure 5.4: The shape of the rational price $V_{x^\star}(x)$ of the American call option.

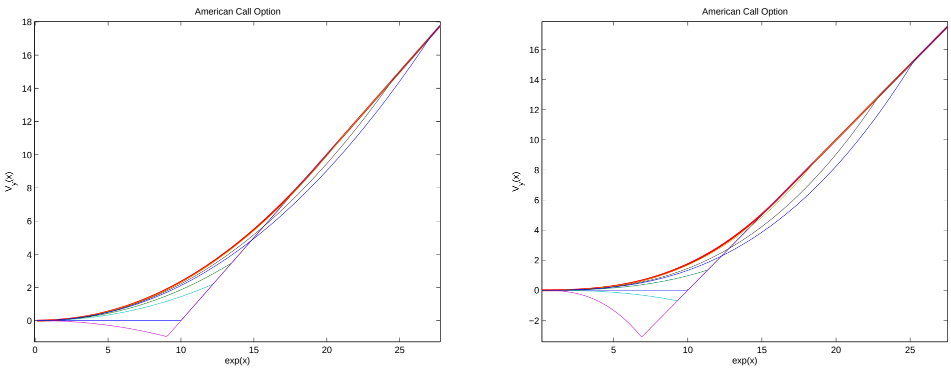
We present in Figure 5.3 plots of the value function $V_{x^\star}$ (5.5.3) of the American put option problem (5.7.6). From this figure, we observe that the value function $V_{x^\star}$ satisfies the smooth pasting condition at the optimal stopping boundary $x^\star = \log(K\mathbb{E}(e^{\underline{X}_{\mathrm{e}_q}}))$ for both Lévy processes, except for the case of $\alpha = 0.5$. Since for this case X has path of bounded variation with positive drift and hence 0 is irregular for $(-\infty, 0)$ for X, we see from figure 5.3(b) that the smooth pasting condition does not hold. All of the plots exhibit the general types of behaviour found recently by Hirsa and Madan [59], Matache et al. [81], and Almendral and Oosterlee [4].

Figure 5.4 shows plots of the value function $V_{x^\star}$ (5.5.4) of the American call option



(a) Tempered stable process with $\alpha = 1.5$. (b) Tempered stable process with $\alpha = 0.5$.

Figure 5.5: The shape of a candidate solution $V_y(x)$ of the American put option problem for different values of stopping boundary y.

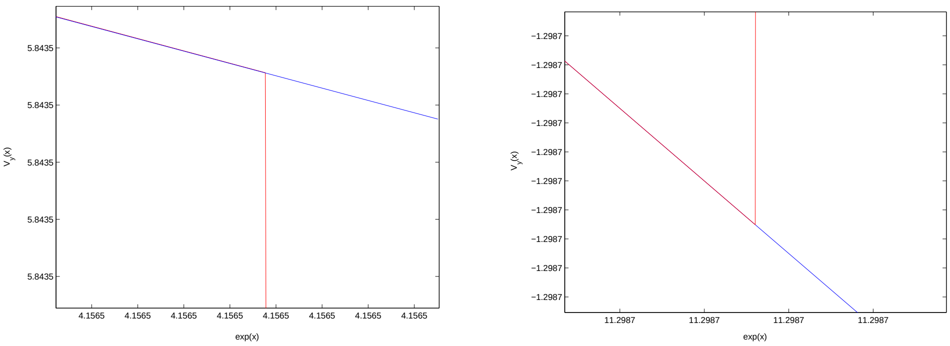


(a) Tempered stable process with $\alpha = 1.5$. (b) Tempered stable process with $\alpha = 0.5$.

Figure 5.6: The shape of a candidate solution $V_y(x)$ for different values of stopping boundary y of the American call option problem.

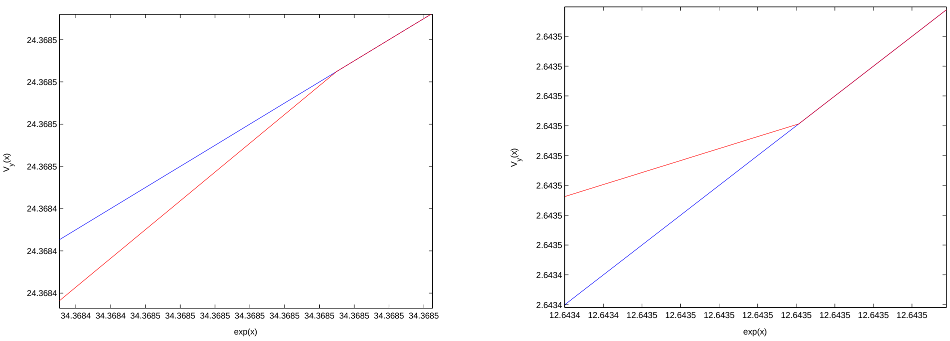
problem (5.7.11). From this figure, we observe for both Lévy processes that the value function $V_{x^\star}$ satisfies the smooth pasting condition at the optimal stopping boundary $x^\star = \log(K\mathbb{E}(e^{\overline{X}_{\mathbf{e}_q}}))$, the fact that follows from regularity of 0 for $(0,\infty)$ for both Lévy processes (see Theorem 5.4.2 for more details).

Next in Figures 5.5(a) and 5.5(b) we present plots of the candidate solution $x \mapsto V_y(x)$ (5.7.7) of the American put option problem (5.7.6) for different values of stopping boundary y. From these figures we observe that $V_y(x) = G(x)$ for every $x < y$, $V_{x^\star} \geq G(x)$ for all $x \in \mathbb{R}$, and for $y < x^\star$ we see that there exists x such that $V_y(x) < G(x)$. These are the features specified previously in Lemma 5.3.3. In particular, we observe from these figures that all of the curves $V_y(x)$ are seen to be



(a) Tempered stable process with $\alpha = 1.5$. (b) Tempered stable process with $\alpha = 1.5$.

Figure 5.7: The shape of a candidate solution $V_y(x)$ of the American put option problem at $x = y$ for $y \neq x^\star$.



(a) Tempered stable process with $\alpha = 1.5$. (b) Tempered stable process with $\alpha = 1.5$.

Figure 5.8: The shape of a candidate solution $V_y(x)$ of the American call option problem at $x = y$ for $y \neq x^\star$.

upper bounded by the curve $V_{x^\star}$ of the value function (the one associated with the stopping boundary $x^\star = \log(K\mathbb{E}(e^{\underline{X}_{\mathbf{e}_q}}))$) (see Proposition 5.3.4). Hence, the claim that $x^\star$ is the optimal stopping boundary is numerically justified.

Furthermore, In the irregular case of $\alpha = 0.5$ we notice from Figure 5.5(b) that the candidate solution $V_y(x)$ exhibits a jump of size $\left(\frac{(\kappa(1)-q)}{(1-\Phi(q))}e^y - K\frac{q}{\Phi(q)}\right)W^{(q)}(0)$ at the point $x = y$, with $y \neq x^\star$. In the regular case of $\alpha = 1.5$, we see from figures 5.5(a) that there is discontinuity only for the derivative of the candidate solution $V_y(x)$ at a stopping boundary $y \neq x^\star$; the smooth pasting only exists at the optimal stopping boundary $y = x^\star$. Moreover, since the sample path of X contains no Gaussian component, we observe from Figure 5.7 that there exists an infinite gradient at a stopping boundary $y \neq x^\star$ for the candidate solution $V_y(x)$ of the American put option problem (5.7.6). The different behaviour of the (candidate) solution of the problem (5.7.6) is the principle point of difference between the numerical results of Hirsa and Madan [59], Matache et al. [81], and Almendral and Oosterlee [4] and ours. In other respects, the results associated with the optimal boundary $y = x^\star$ are qualitatively similar. This numerical observation agrees with our claim stated previously in Theorem 5.4.3.

In the final plot, Figure 5.6 displays numerical plots of the candidate solution $x \mapsto V_y(x)$ (5.7.13) of the American call option problem (5.7.11) for various values of stopping boundary y. All of the curves are seen to be dominated by the curve $V_{x^\star}$ of the value function (the one associated with the stopping boundary $x^\star = \log(K\mathbb{E}(e^{\overline{X}_{\mathbf{e}_q}})))$. This is to say that $x^\star$ is indeed the optimal stopping boundary of the problem (5.7.11). From the plots, we also observe that $V_y(x) = G(x)$ for every $x \geq y$, $V_{x^\star} \geq G(x)$ for all $x \in \mathbb{R}$, and for $y \geq x^\star$ we see that there exists x such that $V_y(x) \leq G(x)$. These are the features specified by Lemma 5.3.3 in its dual form. In complement to the plots of the candidate solution of the American put option problem (5.7.6), we observe that there exists only discontinuity for the derivative of the curve $V_y(x)$ with no infinite gradient at the stopping boundary $y \neq x^\star$ (see Figure 5.8).

To summarize this section, we have shown that, by working with a completely general spectrally negative Lévy process, it is possible to verify both analytically and numerically the main results of Sections 3 and 4.

5.8 Connection to the finite maturity American put option problem

Let us now consider the finite maturity American put option problem

$$V(t,x) \;=\; \sup_{0\leq\tau\leq t} \mathbb{E}\Big(e^{-\alpha\tau}\big(K - S_\tau(x)\big)^+\Big) \tag{5.8.1}$$

for $\alpha \geq 0$ and all $(t,x) \in [0,T] \times \mathbb{R}_+$, where τ is a stopping time of the stock price process S whose dynamics are given under a measure $\mathbb{P}$ by

$$S_t(x) \;=\; xe^{(\alpha+\omega)t+X_t}, \tag{5.8.2}$$

where X is a Lévy process with $X_0 = 0$ under the measure $\mathbb{P}$.

We assume that the moment generating function

$$\Psi(\theta) \;=\; t^{-1}\log\mathbb{E}(e^{\theta X_t}) \text{ exists on the interval } (-\eta_1,\eta_2) \text{ with } \eta_1,\eta_2 \geq 1. \tag{H3}$$

The discount rate α is chosen so that

$$\alpha \;\geq\; (\Psi(-1) + \Psi(1)). \tag{H4}$$

Furthermore, we assume under the measure $\mathbb{P}$ that the discounted stock price process $\big(e^{-rt}S_t(x), t \geq 0\big)$ is $\mathbb{P}$-martingale, which implies that

$$\mathbb{E}\big(e^{-\alpha t}S_t(x)\big) \;=\; x.$$

The latter condition requires the parameter ω in (5.8.2) to be equal to

$$\omega \;=\; -t^{-1}\log \mathbb{E}(e^{X_t}) \;=\; -\Psi(1).$$

The problem of interest is to give an estimate for the value function V of the problem (5.8.1) in terms of the rational price of the perpetual American put option.

Remark 5.8.1 For finite maturity optimal stopping problem with payoff function $G(x)=(x^+)^n$, $n=1,2,...$ or $G(x)=1-e^{-x^+}$ for random walks, an estimate for the value function V (5.8.1) was given recently by Novikov and Shiryaev in [91].

In the sequel below it should be understood that $V(\infty,x)$ corresponds to the value function of the perpetual American put option problem (5.7.6) and will be written simply by $V(x)$. Next, let us define for a fixed level $y\in\mathbb{R}$ a first passage time under the measure $\mathbb{P}$ of the stock price process S below a level e^y:

$$\tau_y^- \;=\; \inf\{t>0 : S_t(x) \;\le\; e^y\}. \tag{5.8.3}$$

Since the moment generating function $\Psi(\theta)$ of the underlying Lévy process is assumed to exist on an open set containing zero, we have an estimate for the value function of the finite maturity American put option problem (5.8.1) in terms of the value function of the perpetual American put option problem (5.7.6). The result is given by the following theorem.

Theorem 5.8.2 *Suppose that the assumptions* $(H3)$ *and* $(H4)$ *are satisfied. Assume that* $\tau_{b^\star}^-$ *is the optimal stopping time for the perpetual optimal stopping problem associated to (5.8.1). Then for each* $x\in\mathbb{R}_+$ *and all* $t>0$ *we have the following estimate*[7]

$$\max\left\{V(x)-Ke^{-(\log(x)-b^\star)}\times e^{-(\alpha-(\Psi(1)+\Psi(-1)))t},0\right\} \;\le\; V(t,x) \;\le\; V(x). \tag{5.8.4}$$

From (5.8.4) we obtain the asymptotic value for the value function $V(t,x)$ as

$$\lim_{t\uparrow\infty} V(t,x) \;=\; V(x) \quad \text{for every } x\in\mathbb{R}.$$

and

$$\lim_{x\uparrow\infty} V(t,x) \;=\; 0 \quad \text{for every } t\ge 0.$$

The latter is quite straightforward from the equations (5.8.1) and (5.8.2), and from the fact that $\lim_{x\uparrow\infty} V(x)=0$ (see the proof of Proposition 1 and also Figure 5.5).

Proof Following (5.8.1), it is clear that $V(t,x)\ge 0$ for all $t\ge 0$ and every $x\in\mathbb{R}$. Moreover, by the nature of the increasing property of the function $t\mapsto V(t,x)$, we see that

$$V(x)-V(t,x) \;\ge\; 0 \quad \text{for all } (t,x)\in[0,T]\times\mathbb{R}_+. \tag{5.8.5}$$

[7] Note that the result in Theorem 5.8.2 can be extended to a non-negative bounded payoff function G for which the problem (5.2.1) has a solution $\mathscr{P}_G^{(q)}$ for each $q\ge 0$ given.

Furthermore, from the optimal stopping problem (5.8.1), we see for all Markov stopping time τ, taking values in $[0,t]$, that

$$\begin{aligned} V(t,x) &\geq \mathbb{E}\Big(e^{-\alpha\tau}(K-S_\tau(x))^+\mathbf{1}_{(\tau\,\leq\,t)}\Big) \\ &= \mathbb{E}\Big(e^{-\alpha\tau}(K-S_\tau(x))^+\mathbf{1}_{(\tau\,<\infty)}\Big) \\ &\quad -\mathbb{E}\Big(e^{-\alpha\tau}(K-S_\tau(x))^+\mathbf{1}_{(t<\tau<\infty)}\Big). \end{aligned} \tag{5.8.6}$$

Since the level $e^{b^\star}$ is assumed to be the optimal boundary for the perpetual counterpart of the problem (5.8.1) with the associated stopping time $\tau_{b^\star}^-$, we see that

$$V(x) \;=\; \mathbb{E}\Big(e^{-\alpha\tau_{b^\star}^-}(K-S_{\tau_{b^\star}^-}(x))^+\mathbf{1}_{(\tau_{b^\star}^-\,<\infty)}\Big).$$

Following the inequality (5.8.6), we then obtain

$$\begin{aligned} V(x)-V(t,x) &\leq \mathbb{E}\Big(e^{-\alpha\tau_{b^\star}^-}(K-S_{\tau_{b^\star}^-}(x))^+\mathbf{1}_{(t<\tau_{b^\star}^-<\infty)}\Big) \\ &\leq K\mathbb{E}\Big(\mathbf{1}_{(t<\tau_{b^\star}^-<\infty)}\Big) \\ &\leq K\mathbb{P}(t<\tau_{b^\star}^-<\infty). \end{aligned}$$

The proof is completed once we show that

$$\mathbb{P}(t<\tau_{b^\star}^-<\infty) \;\leq e^{-(\log(x)-b^\star)}\times e^{-(\alpha-(\Psi(1)+\Psi(-1)))t}. \tag{5.8.7}$$

To complete the proof, let us introduce the *Esscher transform*[8]:

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} \;=\; e^{-X_t-\Psi(-1)t} \qquad \text{for all } t\geq 0.$$

Using this Esscher transform, we see that

$$\begin{aligned} \widetilde{\mathbb{P}}(t<\tau_{b^\star}^-<\infty) &= \widetilde{\mathbb{E}}\Big(\mathbf{1}_{(t<\tau_{b^\star}^-<\infty)}\Big) \\ &= \mathbb{E}\Big(\mathbf{1}_{(t<\tau_{b^\star}^-<\infty)}e^{-X_{\tau_{b^\star}^-}-\Psi(-1)\tau_{b^\star}^-}\Big) \\ &\geq \mathbb{E}\Big(\mathbf{1}_{(t<\tau_{b^\star}^-<\infty)}e^{(\log(x)-b^\star)+(\alpha-\Psi(1))\tau_{b^\star}^--\Psi(-1)\tau_{b^\star}^-}\Big) \\ &= \mathbb{E}\Big(\mathbf{1}_{(t<\tau_{b^\star}^-<\infty)}e^{(\log(x)-b^\star)+(\alpha-(\Psi(1)+\Psi(-1)))\tau_{b^\star}^-}\Big) \\ &\geq \mathbb{E}\Big(\mathbf{1}_{(t<\tau_{b^\star}^-<\infty)}e^{(\log(x)-b^\star)+(\alpha-(\Psi(1)+\Psi(-1)))t}\Big) \\ &= e^{(\log(x)-b^\star)+(\alpha-(\Psi(1)+\Psi(-1)))t}\mathbb{E}\Big(\mathbf{1}_{(t<\tau_{b^\star}^-<\infty)}\Big) \\ &= e^{(\log(x)-b^\star)+(\alpha-(\Psi(1)+\Psi(-1)))t}\mathbb{P}(t<\tau_{b^\star}^-<\infty), \end{aligned}$$

[8] The Esscher transform is by now standard methodology in mathematical insurance, gradually however its appearance within mathematical finance is becoming more and more prominent, see for instance Gerber and Shiu [54] and the references and discussions therein.

which in turn leads to the inequality

$$\mathbb{P}(t < \tau_{b^\star}^- < \infty) \ \leq \ e^{-(\log(x)-b^\star)-(\alpha-(\Psi(1)+\Psi(-1)))t}\widetilde{\mathbb{P}}(t < \tau_{b^\star}^- < \infty).$$

Since $\widetilde{\mathbb{P}}(t < \tau_{b^\star}^- < \infty) \leq 1$, the claim that the value function $V(t,x)$ of the optimal stopping problem (5.8.1)-(5.8.2) satisfies the bounds (5.8.4) is then established. □

5.9 Conclusion and remarks

We have presented in this chapter an effective approach for solving perpetual optimal stopping problem (5.1.1) in a general setting. The approach is based on finding a solution to an averaging problem from which we obtain, using the Wiener-Hopf factorization, a fluctuation identity for first passage of Lévy processes. This fluctuation identity constitutes the main principle in obtaining an optimal solution of (5.1.1). This identity gives a generic link to some known identities used to solve the problem (5.1.1) with special payoff G, see for instance Darling et al. [33], Mordecki [87], Asmussen et al. [6], Alili and Kyprianou [3], Novikov and Shiryaev [91], and Kyprianou and Surya [73]. If a solution to the averaging problem can be found and has certain monotonicity properties, we showed that an optimal solution to the problem (5.1.1) can be written in terms of such monotone function.

Using the proposed approach, we are able to reproduce the special results of those discussed, among others, by Darling et al. [33], Mordecki [87], Boyarchenko and Levendorski [21], Alili and Kyprianou [3], Novikov and Shiryaev [91], and Kyprianou and Surya [73]. Using the optimal solution, we show that the C^1 smooth pasting condition holds if and only if the optimal stopping boundary is regular for the interior points of the stoping region for the Lévy process. Our conclusion over the smooth pasting condition extends further the recent work of Alili and Kyprianou [3] and Kyprianou and Surya [73] into a more general payoff function.

Furthermore, this conclusion shows no contradiction to the current numerical work, among others, of Hirsa and Madan [59], Matache et al.[81], and Almendral and Oosterlee [4]. Furthermore, assuming that the moment generating exists on an open set containing zero, we provided an upper and lower bounds for the value function of the finite maturity American put option problem in terms of the value function of the perpetual American put option problem.

Throughout this chapter we have assumed that the optimal stopping time belongs to a class of first passage times. This assumption boils down to computing the joint Laplace transform of the time of first exit of Lévy process below a certain level and its overshoot. It should also be possible to extend the problem (5.1.1) into the case where the class of Markov stopping times have values in finite time interval $[0,T]$. This problem amounts to solving the problem of first exit below a moving barrier for the Lévy process and solving this problem will be a challenging task both theoretically and numerically. We keep this task for possible future work.

Chapter 6

Principles of Smooth and Continuous Fit in the Determination of Endogenous Bankruptcy Levels[1]

Abstract

The purpose of this chapter is threefold. Firstly to revisit the previous works of Leland [77], Leland and Toft [76] and Hilberink and Rogers [58] on optimal capital structure and show that the issue of determining an optimal endogenous bankruptcy level can be dealt with analytically and numerically when the underlying source of randomness is replaced by that of a general spectrally negative Lévy process. Secondly, by working with the latter class of processes we bring to light a new phenomenon, namely that, depending on the nature of the small jumps, the optimal bankruptcy level may be determined by a principle of *continuous pasting* as opposed to the usual *smooth pasting*. Thirdly, we are able to *prove* the optimality of the bankruptcy level according to the appropriate choice of pasting. This improves on the results of Hilberink and Rogers [58] who were only able to give a numerical justification for the case of smooth pasting. Our calculations are greatly eased by the recent perspective on fluctuation theory of spectrally negative Lévy processes in which many new identities are expressed in terms of the so called *scale functions*.

6.1 Introduction

We consider the following model for a firm based on the earlier works of Leland [77], Leland and Toft [76] and Hilberink and Rogers [58].

The firm is assumed to be partly financed by debt, whose maturity profile is kept constant through time, by the simultaneous issue of new debt and retirement of old debt. This debt is of equal seniority, and distributes a continuous stream of coupon

[1]This chapter is the extended version of: Kyprianou, A.E. and Surya, B. A. Principles of smooth and continuous fit in the determination of endogenous bankruptcy levels. To appear in *Finance and Stochastics*, Springer-Verlag.

payment to bondholders in a fixed amount. From this the firm also receives tax benefits which are also issued as a continuous stream, providing the value of its assets is above a certain threshold, at a fixed rate. The bankruptcy level is determined endogenously by the shareholders to maximize the firm's equity value. Note that most of the authors mentioned above consider the case where the coupon is paid at a constant rate to the bondholder rather than proportional to the value of the underlying asset and the tax rebates are accordingly received at a constant rate providing the value of the underlying asset is above a certain threshold.

In this chapter we shall assume that the value of underlying assets of the firm is modelled using a general exponential spectrally negative Lévy process. This was also the case in Hilberink and Rogers [58]; however, it was necessary for them after a certain point in their calculations to work with the special case of a spectrally negative Lévy process taking the form of an independent sum of a linear Brownian motion and a compound Poisson process with negative jumps (cf. formula (3.21) on p245). As advocated by Leland and Toft [76] and Hilberink and Rogers [58], the optimal bankruptcy level should be determined by applying the smooth-pasting condition. Although for the special subclass of spectrally negative processes considered by Hilberink and Rogers [58], no rigorous proof was given to show that smooth pasting leads to the optimal choice bankruptcy level; the authors relied instead on numerical observation. By working with a *completely general spectrally negative Lévy process* here, we not only show that an analytical treatment of the optimal bankruptcy level is possible, but we are able to show that the smooth-pasting condition is not always appropriate. We give an analytical *proof of the fact* that, depending on the path regularity of the underlying Lévy process, a principle of either smooth pasting or continuous pasting should be applied accordingly as the underlying Lévy process has unbounded or bounded variation, respectively.

Among the class of spectrally negative Lévy processes, we consider the *α-stable process* with index $\alpha \in (0,1) \cup (1,2]$ for numerical examples. With the exception of the case $\alpha = 2$ which corresponds to linear Brownian motion, these are pure jump processes. Further, they have paths of *unbounded variation* when $\alpha \in (1,2]$ and paths of bounded variation when $\alpha \in (0,1)$. The numerical results for these processes give a significant differences from the jump diffusion processes considered by Hilberink and Rogers [58].

In other recent work, Chen and Kou [29] consider the same model as we do here except that the underlying source of randomness is a Lévy process which is the independent sum of a linear Brownian motion and a compound Poisson process with two-sided exponential jumps. They also succeed in *proving* that the optimal bankruptcy level is obtained by a principle of smooth pasting for the case considered there.

The chapter is organized as follows. In Section 2 we present in more mathematical terms the basic models for the evolution of the value of the firm's assets and the capital structures of the firm following Hilberink and Rogers [58]. Section 3 and 4 discuss some notions of fluctuation theory of (general) Lévy processes, including a number of identities expressed in terms of scale functions, from which we are able to give analytic expressions for the value and debt of a firm. In Section 5 we discuss the computation of the optimal endogenous bankruptcy level. The term structure of credit spreads are given in Section 6. The main conclusion of this section is that the term structure rapidly goes to zero as debt maturity approaches zero for the case where the Lévy process has no jumps and has positive value when there are jumps in the Lévy process. This observation confirms the result of Hilberink and Rogers [58] for the jump diffusion case with one-sided independent exponential jumps and Chen and Kou [29] for the jump diffusion case with two-sided independent exponential jumps. The computation of the term structure of credit spreads requires numerical inversion of a double Laplace transform. The numerical method for this is given in Section 7. In Section 8 we verify the main results of Sections 5 and 6 by means of numerical examples. Finally, Section 9 concludes this chapter.

6.2 The capital structure of the firm

Throughout this chapter we assume that Lévy processes will form the basis of the model for the value of a firm as we shall now describe. Note that with some exceptions, most of what we shall say below is fundamentally the model described in Duffie and Lando [37], Hilberink and Rogers [58], and Leland and Toft [76].

To start with, let $V(t)$ denote the value of the firm's assets at time t whose dynamics are given by an exponential Lévy process

$$V(t) \;=\; Ve^{X_t}. \tag{6.2.1}$$

We assume the existence of a default-free asset that pays a continuous interest rate $r > 0$. Further, it is assumed that under $\mathbb{P}$, the discounted value $e^{-(r-\delta)t}V(t)$ of the firm's assets is $\mathbb{P}-$ martingale, that is to say that

$$\mathbb{E}\big(e^{-(r-\delta)t}V(t)\big) = V, \tag{6.2.2}$$

where $\delta > 0$ is the total payout rate to the firm's investors (including both bond and equity holders).

The firm is assumed to be partly financed by debt, which is being constantly retired and reissued in the following way. In a time interval $(t, t+dt)$, the firm issues new debt with face value pdt, and maturity profile φ, where φ is non-negative and $\int_0^\infty \varphi(s)ds = 1$. Thus in the time the interval $(t, t+dt)$ it issues debt with face value $p\varphi(s)dtds$ maturing in the time interval $(t+s, t+s+ds)$. Therefore, at time 0 the face value of debt maturing in $(s, s+ds)$ is given by

$$\Big(\int_{-\infty}^{0} p\varphi(s-u)du\Big)ds \;=\; pF(s)ds, \tag{6.2.3}$$

where $F(s) \equiv \int_s^\infty \varphi(u)du$ is the tail of the maturity profile. Taking $s = 0$ in (6.2.3), we see that the face value of debt maturing in $(0, ds)$ is pds, the same as the face value of the newly-issued debt. Thus the face value of all debt is constant, equal to

$$P \;=\; p\int_0^\infty F(s)ds. \tag{6.2.4}$$

This is same the debt profile given in Hilberink and Rogers [58] and opposed to the paper of Leland and Toft [76] who take the Dirac delta-function at T which means that all new debt is always issued with a maturity of T. As in both of the above papers, however, we take $\varphi(t) \;=\; me^{-mt}$ for some positive m. This has the direct implication that $P = p/m$.

All debt is of equal seniority and attracts coupons of an amount ρP at time t until maturity, or until default if that occurs sooner, where $\rho > 0$. Default happens at the first time that the value of the firm's assets falls to some level V_B or lower, i.e., at

$$\sigma^-_{V_B} \;=\; \inf\{t > 0 : V(t) < V_B\}. \tag{6.2.5}$$

As we shall show later, the value of V_B can be determined endogenously for a general class of Lévy processes. At default, a fraction η of the value of the firm's asset is also assumed to be lost in reorganization.

Let us now consider a bond issued at time 0 with face value 1 and maturity t, which continuously pays a constant coupon flow at a fixed rate $\rho > 0$. Let $\frac{1}{P}$ be the fraction of the asset value $V(\sigma^-_{V_B})$ which debt of maturity t receives in the event of bankruptcy. The value of the debt with maturity t is given by

$$\begin{aligned} d(V; V_B, t) \;&= \mathbb{E}\Big(\int_0^{t\wedge\sigma^-_{V_B}} \rho e^{-rs}ds\Big) + \mathbb{E}\Big(e^{-rt} : t < \sigma^-_{V_B}\Big) \\ &\quad + \frac{1}{P}(1-\eta)\mathbb{E}\Big(e^{-r\sigma^-_{V_B}}V(\sigma^-_{V_B}) : \sigma^-_{V_B} < t\Big). \end{aligned} \tag{6.2.6}$$

The first term on the right-hand side of (6.2.6) represents the expected discounted value of all coupon payment until time t or the default time $\sigma^-_{V_B}$, whichever is sooner. The second term represents the expected discounted value of the principle repayment, if this occurs before bankruptcy, and the final term must be the net present value of what is recovered upon bankruptcy, if this happens before maturity time t. Indeed, $V(\sigma^-_{V_B})$ is the value of the firm's asset when bankruptcy occurs and $(1-\eta)V(\sigma^-_{V_B})$ is the value of the remains after bankruptcy costs are deducted. Of this, the bondholder with face value 1 gets the fraction $\frac{1}{P}$, since his debt represents this fraction of the total debt outstanding. Notice that if the process X were continuous, then $V(\sigma^-_{V_B})$ would simply be the bankruptcy level V_B; but since we allow X to have possible jumps, $V(\sigma^-_{V_B})$ can be below the bankruptcy level V_B.

Let $D(V; V_B)$ denote the total value of debt. The fraction of the firm's asset value lost in bankruptcy is η. The remaining value $(1-\eta)V(\sigma^-_{V_B})$ is distributed to debt holders so that the sum of all fractional claims $\frac{1}{P}$ for debt of all outstanding maturities equals $(1-\eta)$. We can now determine the total value at time 0 of all outstanding debt as

$$
\begin{aligned}
D(V;V_B) =& \int_0^\infty pe^{-mt} d(V;V_B,t)dt \\
=& \rho P \mathbb{E}\Big(\int_0^{\sigma^-_{V_B}} e^{-(r+m)t}dt\Big) + p\mathbb{E}\Big(\int_0^{\sigma^-_{V_B}} e^{-(r+m)t}dt\Big) \\
&+ (1-\eta)\mathbb{E}\Big(e^{-(r+m)\sigma^-_{V_B}} V(\sigma^-_{V_B})\Big) \\
=& \frac{(\rho+m)P}{r+m}\mathbb{E}\Big(1-e^{-(r+m)\sigma^-_{V_B}}\Big) + (1-\eta)\mathbb{E}\Big(e^{-(r+m)\sigma^-_{V_B}} V(\sigma^-_{V_B})\Big).
\end{aligned}
\tag{6.2.7}
$$

We assume that there is a corporate tax rate $\tau > 0$ which depends on the value of the underlying risky asset in the following way. As introduced by Leland and Toft [76] (see also Hilberink and Rogers [58]), there exists a cutoff level V_T, whose effect is that the tax rebates are 0 while $V(t) < V_T$, and are $\tau\rho P dt$ when $V(t) \geq V_T$. Under this assumption, the value of the firm at time zero becomes

$$
v(V;V_B) \;=\; V - \eta\mathbb{E}\Big(e^{-r\sigma^-_{V_B}} V(\sigma^-_{V_B})\Big) + \tau\rho P\mathbb{E}\Big(\int_0^{\sigma^-_{V_B}} e^{-rt}\mathbf{1}_{\{V(t)\geq V_T\}}dt\Big). \tag{6.2.8}
$$

In terms of (6.2.8) and (6.2.7), the value of the firm's equity is given by

$$
E(V;V_B) = v(V;V_B) - D(V;V_B). \tag{6.2.9}
$$

The expressions for the expectation in (6.2.7) and (6.2.8) cannot be written in closed form in general, although this is possible in the Brownian motion case of Leland and Toft [76]. These difficulties can be circumvented by modeling the dynamics of the firm's asset value by Lévy processes having downward jumps.

6.3 Lévy processes with no positive jumps

Now, let us return to the dynamics (6.2.1) for the value of the firm's assets. We assume throughout the remaining of this chapter that X is a real-valued Lévy process having no positive jumps, that is, its Lévy measure Π is concentrated on $(-\infty, 0)$. This class of processes has a great interest from theoretical point of view, because they are processes for which fluctuation theory can be developed to a fuller extent. As X will be chosen from this class in our financial model, we devote a little time in this section and the next to an overview of a number of relevant results from the above-mentioned fluctuation theory. Unless otherwise stated, all of what follows in this section can be extracted from the books of Bertoin [13] or Kyprianou [69].

The degenerate case when X is either the negative of a subordinator or a deterministic drift has no interest and will be excluded throughout. The Laplace exponent κ of X is given by

$$
\mathbb{E}\big(e^{\lambda X_t}\big) \;=\; e^{t\kappa(\lambda)} \quad \text{for } \lambda, t \geq 0. \tag{6.3.1}
$$

The function $\kappa : [0, \infty) \to (-\infty, \infty)$ is defined by

$$\kappa(\lambda) \;=\; -\Psi(-i\lambda) \;=\; \mu\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} \big(e^{\lambda x} - 1 - \lambda x \mathbf{1}_{\{x>-1\}}\big)\Pi(dx). \quad (6.3.2)$$

It is easily shown that κ is zero at the origin, tends to infinity at infinity and is strictly convex. We denote by $\Phi : [0, \infty) \to [0, \infty)$ the right continuous inverse of $\kappa(\lambda)$, so that

$$\Phi(\alpha) \;=\; \sup\{p > 0 : \kappa(p) = \alpha\}$$

and

$$\kappa(\Phi(\lambda)) \;=\; \lambda \quad \text{for all} \quad \lambda \geq 0.$$

Note that due to the convexity of κ, there exist at most two roots for a given α and precisely one root when $\alpha > 0$.

The class of spectrally negative Lévy processes is very rich. Among other things it allows for processes which have paths of both unbounded and bounded variations. The latter case occurs if and only if $\sigma = 0$ and

$$\int_{(-\infty,0)} |x|\Pi(dx) \;<\; \infty.$$

In that case one may rearrange (6.3.2) into the form

$$\kappa(\lambda) = \mathrm{d}\lambda - \int_{(-\infty,0)} \big(1 - e^{\lambda x}\big)\Pi(dx) \quad (6.3.3)$$

where necessarily $\mathrm{d} > 0$. This reflects the fact that a spectrally negative Lévy process of bounded variation must be the difference of a linear drift and a pure jump subordinator. If further it is assumed that $\Pi(-\infty, 0) < \infty$, then X is nothing more than the difference of a linear drift and a compound Poisson subordinator.

The path variation for a spectrally negative Lévy process also dictates how the process moves away from its initial position. It can be shown that a general Lévy process has one of four types of behaviour in this respect which we shall now describe. Let

$$\sigma_0^+ \;=\; \inf\{t > 0 : X_t \;>\; 0\} \quad \text{and} \quad \sigma_0^- \;=\; \inf\{t > 0 : X_t \;<\; 0\}.$$

Then either

(i) $\mathbb{P}(\sigma_0^+ = 0) = \mathbb{P}(\sigma_0^- = 0) = 1$,

(ii) $\mathbb{P}(\sigma_0^+ = 0) = \mathbb{P}(\sigma_0^- > 0) = 1$,

(iii) $\mathbb{P}(\sigma_0^+ > 0) = \mathbb{P}(\sigma_0^- = 0) = 1$ or

(iv) $\mathbb{P}(\sigma_0^+ > 0) = \mathbb{P}(\sigma_0^- > 0) = 1$.

Note in particular that all probabilities are either zero or one (this follows by Blumenthal's zero-one law). Case (iv) is only fulfilled by compound Poisson processes. It is well known that a spectrally negative Lévy process necessarily obeys case (i) when it has paths of unbounded variation and case (iii) when it has paths of bounded variation. To some extent, it is clear that when a spectrally negative process has a Gaussian component ($\sigma > 0$) then (i) must hold on account of the dominant behavior of the latter. If however $\sigma = 0$, then the above conclusions tell us that when $\int_{(-1,0)} |x|\Pi(dx) = \infty$, the movement of X is volatile enough that the process visits both the upper and lower half-lines immediately. If on the other hand $\int_{(-1,0)} |x|\Pi(dx) < \infty$ then, taking (6.3.3) into account, the accumulation of negative jumps in the first moments of time is not sufficient to counterbalance the upward linear motion with rate d, thus bringing X immediately into the upper half line for a strictly positive period of time.

When $\mathbb{P}(\sigma_0^+ = 0) = 1 (= 0)$ we say that 0 is regular (irregular) for $(0, \infty)$. When $\mathbb{P}(\sigma_0^- = 0) = 1 (= 0)$ we say that 0 is regular (irregular) for $(-\infty, 0)$.

6.4 Scale functions and fluctuation identities

As mentioned in the previous section, spectrally negative Lévy processes form a general class of Lévy processes that enjoy a degree of analytic tractability. The purpose of this section is to give some exposure to explicit expressions for certain fluctuation identities which will be of use when considering the problem of determining the optimal endogenous bankruptcy level for the financial model described in Section 6.2.

The starting point is the so-called scale function which features invariably in almost all known identities (see [13] and [14] for the origin of this function).

6.4.1 Scale functions

Definition 6.4.1 (Scale function) For a given spectrally negative Lévy process X with Laplace exponent κ, there exists for every $q \geq 0$ a function $W^{(q)} : \mathbb{R} \to [0, \infty)$ such that $W^{(q)}(x) = 0$ for all $x < 0$ and $W^{(q)}$ is differentiable on $[0, \infty)$, satisfying

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x) dx = \frac{1}{\kappa(\lambda) - q} \quad \text{for} \quad \lambda > \Phi(q), \tag{6.4.1}$$

where $\Phi(q)$ was defined in the previous section. We write $W^{(0)} = W$ for short.

Smoothness properties of the scale functions $W^{(q)}$ are very closely related to the roughness of the underlying paths of the associated Lévy process. The following result, found in Lambert [75] and Chan and Kyprianou [28], gives necessary and sufficient conditions for the scale function on $(0, \infty)$ to belong to $C^1(0, \infty)$.

Theorem 6.4.2 *Suppose that X is a spectrally negative Lévy process. For each $q \geq 0$,*

(i) *if X is of unbounded variation, then $W^{(q)}$ is continuously differentiable on $(0, \infty)$;*

(ii) *if X is of bounded variation, then $W^{(q)}$ is continuously differentiable on $(0, \infty)$ if and only if Π has no atoms.*

In addition the behavior of the scale function at the origin can also be established. In both lemmas below, recall that d is the drift coefficient appearing in the representation (6.3.3) of the Laplace exponent κ when X has bounded variation.

Lemma 6.4.3 *At the point zero, the value of the scale function $W^{(q)}(x)$ is determined for every $q \geq 0$ by*

$$W^{(q)}(0+) = \begin{cases} 1/\mathrm{d}, & \text{when } X \text{ has bounded variation} \\ 0, & \text{when } X \text{ has unbounded variation} \end{cases}$$

Proof From (6.4.1) we have for $q \geq 0$ that

$$\int_{[0,\infty)} \lambda e^{-\lambda x} W^{(q)}(x) dx = \frac{\lambda}{\kappa(\lambda) - q} \qquad \text{for } \lambda > \Phi(q). \tag{6.4.2}$$

When X has unbounded variation, a straightforward argument using the expression (6.3.2) shows that $\lim_{\lambda \uparrow \infty} \kappa(\lambda)/\lambda = \infty$. (In particular one can show that the integral in the expression for κ is of order λ^2). Hence by the continuity of $W^{(q)}$ it follows by taking limits as $\lambda \uparrow \infty$ in (6.4.2) that $W^{(q)}(0+) = 0$. On the other hand, when X is of bounded variation, then another straightforward argument shows that in fact $\lim_{\lambda \uparrow \infty} \kappa(\lambda)/\lambda = \mathrm{d}$. □

Lemma 6.4.4 *Following Theorem 6.4.2, we see for every $q \geq 0$ that*

$$\frac{dW^{(q)}}{dx}(0+) = \begin{cases} 2/\sigma^2, & \text{when } X \text{ has unbounded variation and } \sigma \neq 0, \\ \infty, & \text{when } X \text{ has unbounded variation with } \sigma = 0, \\ \infty, & \text{when } X \text{ has bounded variation and } \Pi(-\infty, 0) = \infty, \\ \frac{(\Pi(-\infty,0)+q)}{\mathrm{d}^2}, & \text{when } X \text{ has bounded variation and } \Pi(-\infty, 0) < \infty. \end{cases}$$

Proof Integrating (6.4.1) by parts and noting from Definition 6.4.1 and Theorem 6.4.2 that a right derivative at zero always exists, we have for each $q \geq 0$

$$\frac{dW^{(q)}}{dx}(0+) = \lim_{\lambda \uparrow \infty} \int_0^\infty \lambda e^{-\lambda x} \frac{dW^{(q)}(x)}{dx} dx = \lim_{\lambda \uparrow \infty} \frac{\lambda^2}{\kappa(\lambda) - q}.$$

In the spirit of the previous proof, it is easy to show when X has unbounded variation that $\lim_{\lambda \uparrow \infty} \kappa(\lambda)/\lambda^2 = \sigma^2/2$ (see also Proposition 2 of Section I in Bertoin [13]). This accounts for the first two cases. When X has bounded variation, a little more care is needed. Integrating again (6.4.1) by parts, taking care to note that $W^{(q)}(0+) = \mathrm{d}^{-1}$, we have

$$\begin{aligned}
&\frac{dW^{(q)}}{dx}(0+)\\
= \;& \lim_{\lambda\uparrow\infty} \frac{\lambda^2}{\mathrm{d}\lambda - \lambda\int_0^\infty e^{-\lambda x}\Pi(-\infty,-x)dx - q} - \lambda W^{(q)}(0+)\\
= \;& \lim_{\lambda\uparrow\infty} \frac{\lambda^2(1 - W^{(q)}(0+)\mathrm{d} + W^{(q)}(0+)\int_0^\infty e^{-\lambda x}\Pi(-\infty,-x)dx) + q\lambda W^{(q)}(0+)}{\mathrm{d}\lambda - \int_0^\infty \lambda e^{-\lambda x}\Pi(-\infty,-x)dx + q}\\
= \;& \lim_{\lambda\uparrow\infty} \frac{1}{\mathrm{d}}\frac{\int_0^\infty \lambda e^{-\lambda x}\Pi(-\infty,-x)dx + q}{\mathrm{d} - \int_0^\infty e^{-\lambda x}\Pi(-\infty,-x)dx}\\
= \;& \frac{\Pi(-\infty,0) + q}{\mathrm{d}^2}.
\end{aligned}$$

In particular, if $\Pi(-\infty, 0) = \infty$ then the right-hand side above is equal to ∞, and if $\Pi(-\infty, 0) < \infty$, then $dW^{(q)}(0+)/dx$ is finite and equals to $(\Pi(-\infty, 0) + q)/\mathrm{d}^2$. Thus our claim is then proved. □

It should be noted that the first of the last two lemmas is essentially not new but implicitly embedded in the literature for spectrally negative Lévy processes.

Due to the complexity of the Laplace exponent κ, the scale functions $W^{(q)}$ are not available in explicit form in general. However, it turns out that we have sufficient analytical information regarding these 'special' functions in order to achieve our main goal of establishing an optimal choice of V_B via the imposition of an appropriate pasting condition.

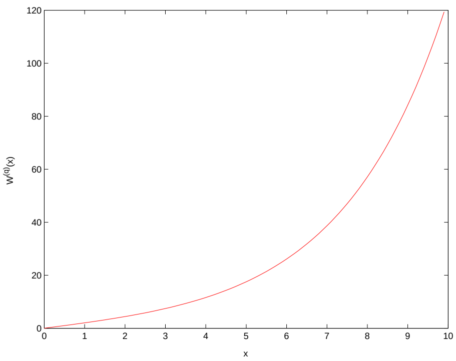
Numerical inversion of the Laplace transform (6.4.1) can always be used to compute the scale function numerically. We refer to Choudhury et al [31] for a general discussion on numerical inversion of Laplace transforms and to Surya [118] for a specific description of the case at hand (see also Chapter 7 for more details). For some spectrally negative Lévy processes, the scale functions $W^{(q)}$ are available explicitly. We consider four such examples below.

Example 6.4.5 *Standard Brownian motion.* Taking $\kappa(\lambda) = \lambda^2/2$, it is a straightforward exercise to show that the scale function is given by

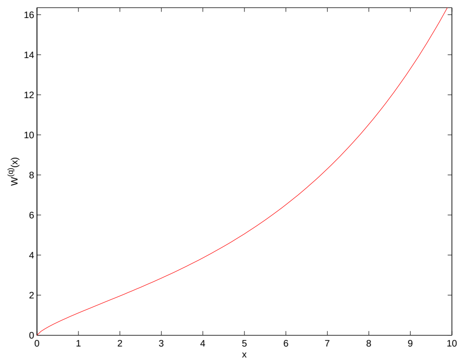
$$W^{(q)}(x) \;=\; \sqrt{\frac{2}{q}}\sinh(x\sqrt{2q}).$$

Example 6.4.6 *Spectrally negative α-stable process.* In this case X has (up to a multiplicative constant which we take as equal to 1) Laplace exponent $\kappa(\lambda) = \lambda^\alpha$ with $\alpha \in (1, 2)$. Due to [14], it is known that the scale function $W^{(q)}$ satisfies

$$\int_{[0,\infty)} e^{-\lambda x}W^{(q)}(x)dx \;=\; \frac{1}{\lambda^\alpha - q} \quad \text{for } \lambda > q^{1/\alpha},$$

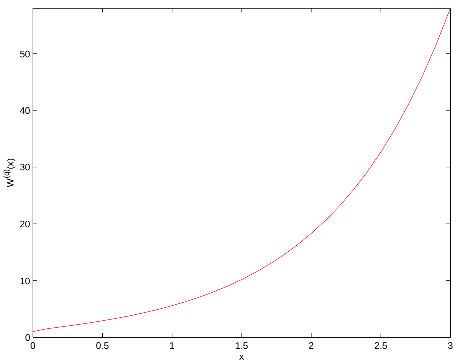


(a) X is standard Brownian motion.

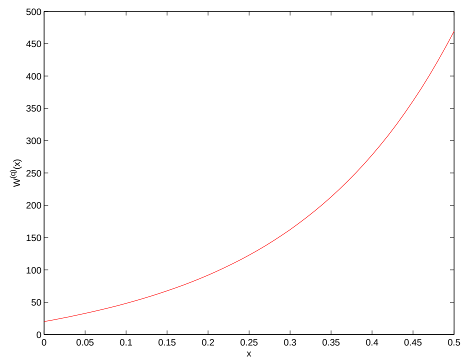


(b) X is α-stable process with $\alpha = 1.75$

Figure 6.1: The shapes of $W^{(q)}(x)$, $q = 0.075$, for unbounded variation X.



(a) X is α-stable process with $\alpha = 0.5$



(b) X is compound Poisson process with drift $\mathrm{d} = 0.05$.

Figure 6.2: The shapes of $W^{(q)}(x)$, $q = 0.075$, for bounded variation X.

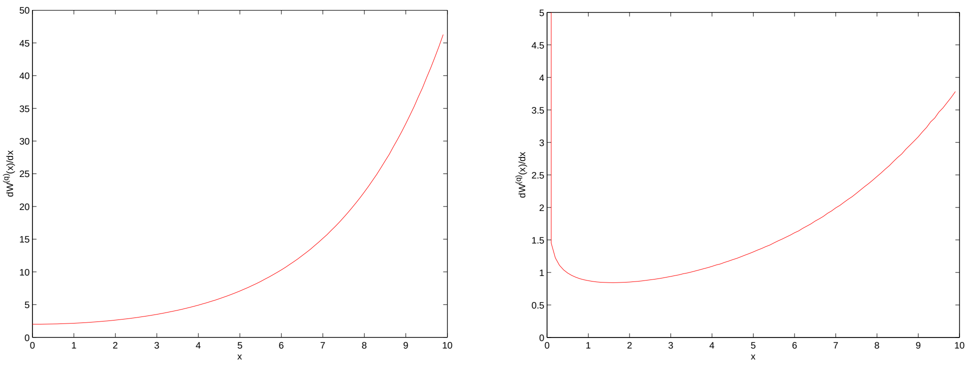
following which one can deduce that

$$W^{(q)}(x) \;=\; \alpha x^{\alpha-1} E'_{\alpha}(q x^{\alpha}) \quad \text{for } x \geq 0,$$

where $E_{\alpha}(.)$ is the Mittag-Leffler function of parameter α defined as

$$E_{\alpha}(y) \;=\; \sum_{n=0}^{\infty} \frac{y^n}{\Gamma(1+\alpha n)}, \quad y \in \mathbb{R}.$$

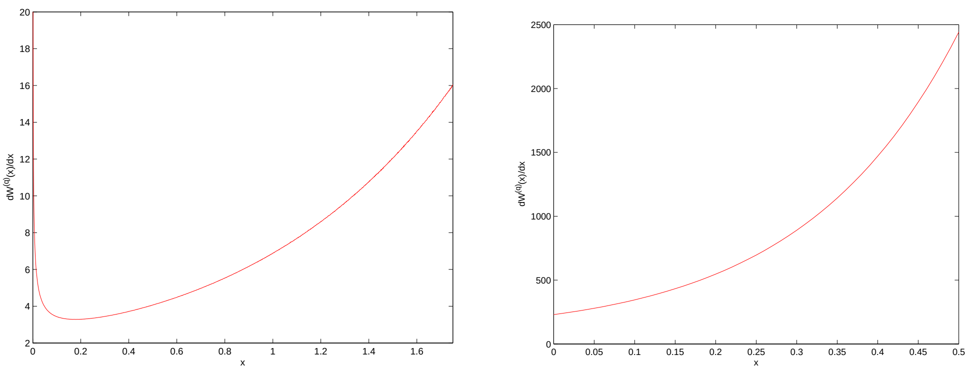
Example 6.4.7 *Spectrally negative Lévy process of bounded variation drifting to infinity.* Suppose that $X_t = \mathrm{d}t - S_t$ where $\{S_t : t \geq 0\}$ is a subordinator with Lévy measure Π having no atoms and $\mathbb{E}(X_1) > 0$ so that $\mathbb{P}(\lim_{t\uparrow\infty} X_t = \infty) = 1$. It can be



(a) X is standard Brownian motion.

(b) X is α-stable process with $\alpha = 1.75$

Figure 6.3: The shapes of $\frac{d}{dx}W^{(q)}(x)$, $q = 0.075$, for unbounded variation X.



(a) X is α-stable process with $\alpha = 0.5$

(b) X is compound Poisson process with drift $\mathrm{d} = 0.05$.

Figure 6.4: The shapes of $\frac{d}{dx}W^{(q)}(x)$, $q = 0.075$, for bounded variation X.

shown that the scale function $W(x)$ satisfies

$$\int_{[0,\infty)} e^{-\lambda x} W(x) dx = \frac{1}{\mathrm{d} - \int_{(0,\infty)} e^{-\lambda x} \Pi(x,\infty) dx},$$

from which we can deduce that

$$W(x) = \frac{1}{\mathrm{d}} \sum_{n \geq 0} \nu^{\star n}(x),$$

where $\nu^{\star n}$ denotes the nth convolution power of $\nu(x) = \mathrm{d}^{-1}\Pi(x,\infty)$ with $\nu^{\star 0}(x)$ being understood as $\delta_0(x)$.

Example 6.4.8 *Compound Poisson process with exponential jumps with parameter $\mu > 0$ and rate β.* From the previous example one may deduce further that when $\mathrm{d}\mu - \beta > 0$, the scale function is given by

$$W(x) \;=\; \frac{1}{\mathrm{d}}\Big(1 + \frac{\beta}{\mathrm{d}\mu - \beta}\big(1 - e^{-(\mu - \mathrm{d}^{-1}\beta)x}\big)\Big).$$

The scale function $W^{(q)}$ can be determined by the formula

$$W^{(q)}(x) = e^{\Phi(q)x} W_{\Phi(q)}(x) \tag{6.4.3}$$

where $W_{\Phi(q)}(x)$ plays the role of $W(x)$ when X is taken under the measure $\mathbb{P}^{\Phi(q)}$ defined by

$$\frac{d\mathbb{P}^{\Phi(q)}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = e^{\Phi(q)X_t - qt}.$$

Note that it is known that under the latter change of measure, $(X, \mathbb{P}^{\Phi(q)})$ is still a spectrally negative Lévy process whose Laplace exponent has changed to

$$\kappa_{\Phi(q)}(\lambda) \;=\; \kappa(\lambda + \Phi(q)) - \kappa(\Phi(q)).$$

Various numerical plots of the scale function $W^{(q)}$ and its derivative $\frac{d}{dx}W^{(q)}$ can be found in Figures 6.1- 6.4 for each of the above examples when $q > 0$. Note that in each case the asymptotic behaviour is that of an exponential function. This is not surprising since for $\lambda > 0$ and $V^{(q)}(x) = e^{-\Phi(q)x}W^{(q)}(x)$,

$$\int_0^\infty e^{-\lambda x} V^{(q)}(dx) \;=\; \frac{\lambda}{\kappa(\lambda + \Phi(q)) - q},$$

and hence by taking limits as $\lambda \downarrow 0$ on the right hand side to obtain $1/\kappa'(\Phi(q))$, it follows from an application of the standard Tauberian theorem that

$$W^{(q)}(x) \sim \frac{e^{\Phi(q)x}}{\kappa'(\Phi(q))} \qquad \text{as } x \to \infty.$$

6.4.2 Fluctuation identities

Recall our notation,

$$\sigma_y^- \;=\; \inf\{t > 0 : X_t < y\}, \tag{6.4.4}$$

the first time that the Lévy process X goes below a level y. Under the model described in Section 6.2, it is possible to write the equity (6.2.9) in terms of this stopping time. Via a number of fluctuation identities for spectrally negative processes this then allows us to write the equity in terms of scale functions. We devote this section to doing precisely this and we begin with quoting the necessary fluctuation identities. These come in the form of three lemmas. The first is due to Bertoin [15], the second is due to Emery [47] and the third is due to Bingham [17].

Lemma 6.4.9 *Denote by* $\mathbf{e}_q$ *an independent exponential random variable with mean* q^{-1}. *We have for every* $x, y > 0$ *and* $q > 0$ *that*

$$q^{-1}\mathbb{P}_x(X_{\mathbf{e}_q} \in dy, \mathbf{e}_q < \sigma_0^-) = \Big(e^{-\Phi(q)y}W^{(q)}(x) - \mathbf{1}_{\{x \geq y\}}W^{(q)}(x-y)\Big)dy. \quad (6.4.5)$$

Lemma 6.4.10 *For all* $q, \beta, x \geq 0$, *the joint Laplace transform under* $\mathbb{P}$ *of the stopping time* σ_{-x}^- *and its overshoot* $X_{\sigma_{-x}^-}$ *is given by*

$$\begin{aligned}\mathbb{E}_x\Big(e^{-q\sigma_0^- + \beta X_{\sigma_0^-}}\Big) &= e^{\beta x} - \frac{(\kappa(\beta) - q)}{(\beta - \Phi(q))}W^{(q)}(x) \\ &\quad - (\kappa(\beta) - q)\int_0^x e^{\beta(x-y)}W^{(q)}(y)dy.\end{aligned} \quad (6.4.6)$$

By applying Laplace transform in x in the expression (6.4.6), we then end up with the well-known identity of Pecherskii and Rogozin (see for instance Bingham [17]).

Lemma 6.4.11 *Let* X *be a spectrally negative Lévy process. For every* $q, \lambda \geq 0$ *and* $\theta \in \mathbb{C}$ *with* $\mathfrak{Re}(\theta) \geq 0$ *we have*

$$\int_0^\infty \lambda e^{-\lambda x}\mathbb{E}_x\Big(e^{-q\sigma_0^- + \theta X_{\sigma_0^-}}\Big)dx = \frac{\lambda}{\lambda - \theta}\Big(1 - \frac{\kappa_q^{(-)}(\lambda)}{\kappa_q^{(-)}(\theta)}\Big), \quad (6.4.7)$$

where $\kappa_q^{(\pm)}(\lambda)$ *are the factors of the Wiener-Hopf factorization formula defined in equations (2.3.2) and (2.3.3) of Chapter 2.*

Throughout the rest of this chapter, we define

$$\begin{aligned}\gamma(x; q, \beta) &= \mathbb{E}\Big(1 - e^{-q\sigma_{-x}^- + \beta X_{\sigma_{-x}^-}}\Big) \\ g(x; q, b) &= \mathbb{E}\Big(\int_0^{\sigma_{-x}^-} e^{-qt}\mathbf{1}_{(X_t \geq b - x)}dt\Big).\end{aligned}$$

Writing $x = \log(V/V_B)$ and reconsidering (6.2.7), the total value of the debt can be re-expressed as follows

$$\begin{aligned}D(V; V_B) &= \frac{(\rho + m)P}{m + r}\mathbb{E}_x\Big(1 - e^{-(m+r)\sigma_0^-}\Big) + (1 - \eta)V_B\mathbb{E}_x\Big(e^{-(r+m)\sigma_0^- + X_{\sigma_0^-}}\Big) \\ &= \frac{(\rho + m)P}{m + r}\gamma(x; m + r, 0) + (1 - \eta)V(1 - \gamma(x; m + r, 1)). \quad (6.4.8)\end{aligned}$$

The value of the firm (6.2.8) can be re-expressed as

$$\begin{aligned}v(V; V_B) &= V_B e^x + \tau\rho P\mathbb{E}_x\Big(\int_0^{\sigma_0^-} e^{-rt}\mathbf{1}_{\{X_t \geq b\}}dt\Big) - \eta V_B\mathbb{E}_x\Big(e^{-r\sigma_0^- + X_{\sigma_0^-}}\Big) \\ &= V(1 - \eta) + \tau\rho P g(x; r, b) + \eta V\gamma(x; r, 1), \quad (6.4.9)\end{aligned}$$

where $b = \log(\frac{V_T}{V_B})$.

Following the expression in (6.4.6) one can easily deduce an explicit expression for the function γ in terms of the scale function $W^{(q)}$.

Lemma 6.4.12 *For $x \in \mathbb{R}$, $q \geq 0$ and $\beta \geq 0$,*

$$\gamma(x;q,\beta) \;=\; \frac{(\kappa(\beta)-q)}{(\beta-\Phi(q))}e^{-\beta x}W^{(q)}(x) + (\kappa(\beta)-q)\int_0^x e^{-\beta y}W^{(q)}(y)dy.$$

Using the resolvent density (6.4.5), the expression for g, a function that appears in the expression for the value of the firm, can also be deduced explicitly in terms of the scale function $W^{(q)}$. The expression for g will be of use in the next section. The following lemma gives the expression of g.

Lemma 6.4.13 *For $x \in \mathbb{R}$, $q \geq 0$ and $b \in \mathbb{R}$,*

$$g(x;q,b) = \frac{e^{-\Phi(q)(b\vee 0)}}{\Phi(q)}W^{(q)}(x) - \int_0^{x-(b\vee 0)} W^{(q)}(y)dy \tag{6.4.10}$$

Proof Using (6.4.5) of Lemma 6.4.9, we see that

$$\begin{aligned}
\mathbb{E}_x\Big(\int_0^{\sigma_0^-} e^{-qt}\mathbf{1}_{\{X_t\geq b\}}dt\Big) &= q^{-1}\mathbb{P}_x\Big(X_{\mathbf{e}_q}\geq b, \mathbf{e}_q<\sigma_0^-\Big)\\
&= \int_{b\vee 0}^{\infty}\Big(e^{-\Phi(q)y}W^{(q)}(x) - \mathbf{1}_{\{x\geq y\}}W^{(q)}(x-y)\Big)dy\\
&= \int_{b\vee 0}^{\infty} e^{-\Phi(q)y}W^{(q)}(x)dy - \int_{b\vee 0}^{\infty}\mathbf{1}_{\{x\geq y\}}W^{(q)}(x-y)dy\\
&= \frac{e^{-\Phi(q)(b\vee 0)}}{\Phi(q)}W^{(q)}(x) - \int_0^{x-b\vee 0} W^{(q)}(y)dy,
\end{aligned}$$

where the last equality was obtained after changing variables in the integral. The required identity is then proved. □

To conclude this section, we may now write an explicit expression for the firm's equity values in terms of the scale function $W^{(q)}(x)$, namely

$$\begin{aligned}
E(V;V_B) &= V\Big(\eta\gamma(x;r,1) + (1-\eta)\gamma(x;m+r,1)\Big)\\
&\quad -\frac{(m+\rho)P}{m+r}\gamma(x;m+r,0) + \tau\rho P g(x;r,b)
\end{aligned} \tag{6.4.11}$$

where $x = \log(V/V_B)$ and $b = \log(V_T/V_B)$.

We now move on to determining an optimal bankruptcy level V_B.

6.5 Determining the bankruptcy level V_B

The expression for E in (6.4.11) gives the firm's equity value as a function of the firm's initial asset value V and the chosen bankruptcy-triggering asset level V_B. In determining the bankruptcy level V_B, the idea is to fix V and maximize E with respect to V_B subject to the limited liability constraint that the equity $E(V;V_B)$ must always be worth uniformly non-negative for $V \geq V_B$. We refer to Leland [77], Leland and Toft [76], Hilberink and Rogers [58], and the literature therein for a more detailed discussion of the underlying economics.

The main claim of this chapter is that, observing this constraint, the bankruptcy level V_B is determined in the following way.

Theorem 6.5.1 *If the spectrally negative Lévy process X has unbounded variation, so that 0 is* regular *for the lower half-line $(-\infty, 0)$, then the bankruptcy-triggering asset level V_B satisfies the condition of* smooth-pasting*; that is to say that V_B is chosen to satisfy*

$$\frac{\partial E}{\partial V}(V_B+; V_B) \; = \; 0. \tag{6.5.1}$$

However, if the spectrally negative Lévy process X has bounded variation, so that 0 is irregular *for the lower half-line $(-\infty, 0)$, then V_B satisfies the condition of continuous-pasting; that is to say that V_B is chosen to satisfy*

$$E(V_B+; V_B) \; = \; 0. \tag{6.5.2}$$

In both cases, it follows that V_B is the unique solution to the equation

$$x \; = \; \frac{\frac{(m+\rho)P}{\Phi(m+r)} - \frac{\tau\rho P}{\Phi(r)}\left(\left(\frac{x}{V_T}\right) \wedge 1\right)^{\Phi(r)}}{\left(\eta\frac{\left(r-\kappa(1)\right)}{\left(\Phi(r)-1\right)} + (1-\eta)\frac{\left(m+r-\kappa(1)\right)}{\left(\Phi(m+r)-1\right)}\right)}. \tag{6.5.3}$$

Before moving to the proof of this theorem, note that

$$f(x) = x - \frac{\frac{(m+\rho)P}{\Phi(m+r)} - \frac{\tau\rho P}{\Phi(r)}\left(\left(\frac{x}{V_T}\right) \wedge 1\right)^{\Phi(r)}}{\left(\eta\frac{\left(r-\kappa(1)\right)}{\left(\Phi(r)-1\right)} + (1-\eta)\frac{\left(m+r-\kappa(1)\right)}{\left(\Phi(m+r)-1\right)}\right)}$$

is continuous, strictly increasing in x, $f(0+) < 0$ and $f(\infty) = \infty$ so that there is a unique solution to the equation $f(x) = 0$ which we denote by $V_B^\star$. The equation (6.5.3) naturally agrees with the equation for determining the optimal V_B in Hilberink and Rogers [58] who considered the case of a linear Brownian motion plus an independent spectrally negative compound Poisson process.

In fact Hilberink and Rogers [58] show that smooth pasting leads to the equation (6.5.3) for V_B by using the Wiener-Hopf factorization where, in principle, they are working with a general spectrally negative Lévy process. However, close inspection of their calculations shows that they are implicitly assuming that X has a Gaussian component. Specifically this is because of the assumed asymptotic behaviour of their functions $\varphi(x, \lambda)$ and $\gamma(x, \theta, \lambda)$ as $x \downarrow 0$ in the text following (3.16) on p. 244. One sees that this assumed asymptotic behaviour is equivalent to the assumption that the scale function $W^{(\lambda)}$ is zero with a finite derivative at the origin which in turn implies the presence of a Gaussian component.

We also note that in Chen and Kou [29], where the underlying Lévy process takes the form of an independent sum of a linear Brownian motion and a compound Poisson process with two-sided exponential jumps, it was proved that the optimal bankruptcy level follows as a consequence of the smooth-pasting condition. Their result is consistent with the above theorem in the sense that, for the Lévy process considered there, 0 is regular for $(-\infty, 0)$ on account of the presence of the Gaussian term.

We now move to the proof of Theorem 6.5.1. Without further elaboration, we shall make use of a number of facts and notions from the theory of spectrally negative Lévy processes which are well documented in the literature. We refer to Chapter 8 of Kyprianou [69] for a recent review in which all of the used concepts are addressed.

To establish our claim in the Theorem 6.5.1, the following lemma is needed.

Lemma 6.5.2 *For each fixed $x \geq 0$, the quantity*

$$\Theta^{(q)}(x) \;:=\; W^{(q)\prime}(x) - \Phi(q)W^{(q)}(x), \tag{6.5.4}$$

is non-negative for all $x, q \geq 0$ and monotone decreasing in q.

Proof It is known that the q-resolvent measure of the descending ladder height process, $\widehat{H} = \{\widehat{H}_t : t \geq 0\}$ (see Section 2.2 of Chapter 2), of X can be identified as

$$\mathbb{E}\left(\int_0^\infty e^{-qt}\mathbf{1}_{\{\widehat{H}_t \in A\}}dt\right) = c\int_A \Theta^{(q)}(y)dy,$$

where $q \geq 0$ and $c > 0$ is a meaningless constant (determined by the normalization of local time at the minimum to generate the descending ladder height process $\widehat{H}$) and A is a Borel set in $[0, \infty)$. See for example Millar [85] and Pistorius [102]. It is immediately obvious from this relation, in particular on account of the arbitrary choice of A, that $\Theta^{(q)}(x)$ is non-negative and for each fixed $x \geq 0$ it is also monotone decreasing in q.

The fact that $\Theta^{(q)}(x) \geq 0$ for all $q, x \geq 0$ is a simple consequence of its definition as a resolvent density. Thus the claim that $\Theta^{(q)}(x) \geq 0$ is non-negative for all $x, q \geq 0$ and monotone decreasing in q is then proved. □

Proof of Theorem 6.5.1 We split the proof into two: the cases that X has paths of unbounded and bounded variation.

Firstly, we assume that X has paths of unbounded variation. Since the scale function $W^{(q)}(x)$ is continuous and equal to zero at $x = 0$, it is easy to check that the continuous pasting condition (6.5.2) is always satisfied for any bankruptcy level $V_B > 0$. We look instead at choosing $V_B^\star$ by the criterion (6.5.1).

Assume temporarily that $\sigma > 0$. By differentiating the firm's equity value E with respect to V, we see after a rather long calculation that

$$\frac{\partial E}{\partial V}(V_B+;V_B) = \frac{2}{\sigma^2 V_B}\Big(\eta\frac{(r-\kappa(1))}{(\Phi(r)-1)} + (1-\eta)\frac{(m+r-\kappa(1))}{(\Phi(m+r)-1)}\Big)f(V_B).$$

From the remarks following the statement of Theorem 6.5.1 we can now see that

$$\frac{\partial E}{\partial V}(V_B+;V_B) > (<)\, 0 \quad \text{for } V_B > (<)\ V_B^\star \quad \text{and} \quad \frac{\partial E}{\partial V}(V_B^\star+, V_B^\star) \ = \ 0.$$

In the case that $\sigma = 0$ one may similarly check with the help of the second case in the conclusion of Lemma 6.4.4 that if V_B is chosen strictly greater than $V_B^\star$ then in fact $\frac{\partial E}{\partial V}(V_B+;V_B) = \infty$ and similarly if V_B is chosen strictly less than $V_B^\star$ then $\frac{\partial E}{\partial V}(V_B+;V_B) = -\infty$. When $V_B = V_B^\star$ it conveniently turns out that $\frac{\partial E}{\partial V}(V_B+;V_B) = 0$.

Taking account of the limited liability constraint that the equity curve must be uniformly non-negative for all $V \geq V_B$, the calculations lead to the conclusion that the bankruptcy level V_B must be at least as big as $V_B^\star$, i.e., $V_B \geq V_B^\star$. We should now like to prove that $V_B^\star$ is the optimal bankruptcy level. We do this by showing that for each fixed $V > V_B^\star$, the function $V_B \mapsto E(V;V_B)$ is monotone decreasing in V_B. To this end, we note that it can be shown after some algebra that for each fixed $V > V_B^\star$ and $V_B \in [V_B^\star, V]$,

$$\begin{aligned} \frac{\partial E}{\partial V_B}(V;V_B) \quad &= \quad -\eta\frac{\big(r-\kappa(1)\big)}{\big(\Phi(r)-1\big)}\Big\{\Theta^{(r)}(x) - \Theta^{(r+m)}(x)\Big\} \\ &\quad - \frac{\tau\rho P e^{-\Phi(r)(b\vee 0)}}{\Phi(r)V_B}\Big\{\Theta^{(r)}(x) - \Theta^{(r+m)}(x)\Big\} \\ &\quad - \frac{\Theta^{(r+m)}(x)}{V_B}\Big\{\eta\frac{\big(r-\kappa(1)\big)}{\big(\Phi(r)-1\big)} + (1-\eta)\frac{\big(m+r-\kappa(1)\big)}{\big(\Phi(m+r)-1\big)}\Big\}f(V_B), \end{aligned} \tag{6.5.5}$$

where $x = \log(V/V_B)$ and the function $\Theta^{(q)}(x)$ is defined in (6.5.4). Note that in computing this derivative it is worth reminding oneself that

$$\begin{aligned} \frac{\partial \gamma}{\partial V_B}(x;q,\beta) \quad &= \quad -\frac{1}{V_B}\frac{\partial \gamma}{\partial x}(x;q,\beta) \\ &= \quad -\frac{1}{V_B}\frac{(q-\kappa(\beta))}{(\Phi(q)-\beta)}e^{-\beta x}\Theta^{(q)}(x), \end{aligned}$$

and that

$$\begin{aligned} \frac{\partial g}{\partial V_B}(x;q,b) \quad &= \quad -\frac{1}{V_B}\frac{\partial g}{\partial x}(x;q,b) - \frac{1}{V_B}\frac{\partial g}{\partial b}(x;q,b) \\ &= \quad -\frac{e^{-\Phi(q)(b\vee 0)}}{V_B\Phi(q)}\Theta^{(q)}(x), \end{aligned}$$

where special care should be taken in the derivatives of g accordingly with the sign of the value b.

Our objective now is to show that each of the three terms on the right-hand side of (6.5.5) is non-positive.

Combined with the result of Lemma 6.5.2, we see that the first two terms on the right-hand side of (6.5.5) are non-positive. The monotonicity of f also implies that the third expression is non-positive. In conclusion we see that for each fixed $V \geq V_B^\star$,

$$\frac{\partial E}{\partial V_B}(V;V_B) \ < \ 0$$

when $V_B \in [V_B^\star, V]$ thus justifying the claim that $V_B^\star$ is optimal.

Now consider the case that X has paths of bounded variation. In that case the arguments above do not apply due to the fact that, for any given choice of V_B, $0 = E(V_B-;V_B)$ is not necessarily equal to $E(V_B+, V_B)$. To see this, one can show with the help of Lemma 6.4.3 that

$$E(V_B+;V_B) = \frac{f(V_B)}{\mathrm{d}}\Big(\eta\frac{(r-\kappa(1))}{(\Phi(r)-1)} + (1-\eta)\frac{(m+r-\kappa(1))}{(\Phi(m+r)-1)}\Big).$$

The monotonicity of f in V_B now implies that

$$E(V_B+, V_B) \ > \ (<)\, 0 \quad \text{for } \ V_B \ > \ (<)\, V_B^\star \quad \text{and} \ \ E(V_B^\star+, V_B^\star) \ = \ 0. \tag{6.5.6}$$

The constraint of non-negativity of the equity curve thus implies that we must choose $V_B \geq V_B^\star$. Exactly the same analysis of the partial derivative $\frac{\partial E}{\partial V_B}(V;V_B)$ as for the unbounded variation case shows that in fact $V_B^\star$ must be optimal as $E(V;V_B)$ is decreasing in V_B for each fixed V. Thus, our claim is then established. □

6.6 The term structure of credit spreads

In this section we discuss the term structure of credit spreads. Following Hilberink and Rogers [58], we identify the credit spreads as what coupon would be required to induce an investor to lend one dollar to the firm until maturity time T. This is the interpretation that one would put on a reported credit spreads curve for a given firm.

To start the discussion, let us return to the expression (6.2.6) and compute for a fixed $t > 0$ the value $\rho^\star$ of ρ for which $d(V_0;V_B,t) = 1$. We denote by σ_y^- the time of first exit of X below a level y defined in (6.4.4). By putting $x = \log\big(V_0/V_B\big)$, we can rewrite the equation (6.2.6) as

$$\begin{aligned} f(t,x) \ &\equiv \ d(V_B e^x; V_B, t) \\ &= \ \mathbb{E}\Big(\int_0^{t\wedge\sigma_{-x}^-} \rho e^{-rs} ds\Big) + \mathbb{E}_x\Big(e^{-rt} : t < \sigma_0^-\Big) \\ &\quad + \frac{(1-\eta)}{P} V_B \mathbb{E}_x\Big(e^{-r\sigma_0^- + X_{\sigma_0^-}} : \sigma_0^- < t\Big). \end{aligned} \tag{6.6.1}$$

Taking Laplace transform in t, we have after some calculations that

$$\begin{aligned}\int_0^\infty e^{-\lambda t} f(t,x)dt &= \frac{\rho}{\lambda(\lambda+r)}\mathbb{E}\Big(1-e^{-(\lambda+r)\sigma^-_{-x}}\Big) + \frac{1}{\lambda+r}\mathbb{E}\Big(1-e^{-(\lambda+r)\sigma^-_{-x}}\Big)\\ &+ \frac{(1-\eta)V_B e^x}{\lambda P}\mathbb{E}\Big(e^{-(\lambda+r)\sigma^-_{-x}+X_{\sigma^-_{-x}}}\Big).\end{aligned}$$

If we take again Laplace transform in x, we see using (6.4.7) that

$$\begin{aligned}\widehat{f}(\lambda,\beta) &\equiv \int_0^\infty dx e^{-\beta x}\int_0^\infty e^{-\lambda t} f(t,x)dt\\ &= \frac{\rho}{\beta\lambda(\lambda+r)}\kappa^{(-)}_{\lambda+r}(\beta) + \frac{\kappa^{(-)}_{\lambda+r}(\beta)}{\beta(\lambda+r)} + \frac{(1-\eta)V_B}{\lambda P(\beta-1)}\Big(1-\frac{\kappa^{(-)}_{\lambda+r}(\beta)}{\kappa^{(-)}_{\lambda+r}(1)}\Big)\\ &= \rho\widehat{f}_1(\lambda+r,\beta)+\widehat{f}_2(\lambda+r,\beta), \qquad (6.6.2)\end{aligned}$$

where the two Laplace transforms $\widehat{f}_1$ and $\widehat{f}_2$ are defined subsequently by

$$\widehat{f}_1(\lambda+r,\beta) = \frac{1}{\beta\lambda(\lambda+r)}\kappa^{(-)}_{\lambda+r}(\beta),$$

and

$$\widehat{f}_2(\lambda+r,\beta) = \frac{\kappa^{(-)}_{\lambda+r}(\beta)}{\beta(\lambda+r)} + \frac{(1-\eta)V_B}{\lambda P(\beta-1)}\Big(1-\frac{\kappa^{(-)}_{\lambda+r}(\beta)}{\kappa^{(-)}_{\lambda+r}(1)}\Big).$$

By applying double inversion of Laplace transform, we obtain

$$f(t,x) = \mathcal{L}^{-1}_\beta\mathcal{L}^{-1}_\lambda[\widehat{f}](t,x) = \int\frac{d\lambda}{2\pi i}\int\frac{d\beta}{2\pi i}e^{t(\lambda-r)+\beta x}\widehat{f}(\lambda-r,\beta), \qquad (6.6.3)$$

from which the credit spreads is given by

$$\text{Credit spreads} = \rho^\star - r = \frac{1-\mathcal{L}^{-1}_\beta\mathcal{L}^{-1}_\lambda[\widehat{f}_2](V_Be^x;V_B,t)}{\mathcal{L}^{-1}_\beta\mathcal{L}^{-1}_\lambda[\widehat{f}_1](V_Be^x;V_B,t)} - r. \qquad (6.6.4)$$

This expression is not available in analytic form in general. Thus, numerical inversion to compute the inverse Laplace transform (6.6.3) is needed. Further technical discussion on this will be given later in Section 7.

6.6.1 Non-zero credit spreads for very short maturity bonds

This section discusses an analytical expression of the credit spreads for very short maturity bonds. It appears that credit spreads have strictly positive values.

By finding the value of $\rho=\rho^\star$ for which the right-hand side of (6.2.6) equals 1 when $t=T$ and $V(T)=V$, we find the spread $\rho^\star-r$ for borrowing with fixed maturity T. Rearrangement of (6.2.6) yields the following expression for the spreads:

$$\text{Credit spreads} = \frac{1-e^{-rT}+\mathbb{E}\Big(e^{-rT}-\frac{1-\eta}{P}V(\sigma^-_{V_B})e^{-r\sigma^-_{V_B}};\sigma^-_{V_B}\le T\Big)}{\frac{1}{r}\mathbb{E}\Big(1-e^{-r(T\wedge\sigma^-_{V_B})}\Big)} - r. \qquad (6.6.5)$$

To understand the asymptotic of the credit spreads (6.6.5) as the maturity T approaches zero, let us denote by $\nu(dx, dt)$ the Poisson random measure associated with the jumps of a Lévy process X and by $\sigma_{(-\epsilon,\epsilon)^c}$, with $\epsilon > 0$, the first entrance time of a jump of X in the set $\{\mathbb{R}\backslash(-\epsilon, \epsilon)\}$. It is known (see Proposition 2 on page 7 of Bertoin [13]) that $\sigma_{(-\epsilon,\epsilon)^c}$ is exponentially distributed with parameter $\Pi(\mathbb{R}\backslash(-\epsilon, \epsilon))$ since

$$\mathbb{P}(\sigma_{(-\epsilon,\epsilon)^c} > t) \;=\; \mathbb{P}(\nu(\{\mathbb{R}\backslash(-\epsilon, \epsilon)\} \times [0, t]) = 0) \;=\; \exp\big(- t\Pi(\mathbb{R}\backslash(-\epsilon, \epsilon))\big).$$

(See also page 143 in Kyprianou [69].) Note that this expression can be rewritten as

$$\mathbb{P}(\sigma_{(-\epsilon,\epsilon)^c} \le t) \;=\; 1 - \exp\big(- t\Pi(\mathbb{R}\backslash(-\epsilon, \epsilon))\big). \tag{6.6.6}$$

The expression in (6.6.6) tells us that if $\Pi(\mathbb{R}) = \infty$ then it becomes more and more probable to have jumps of size greater than $\epsilon > 0$ as $t \downarrow 0$. Thus the jumps have very significant influence over the initial behavior of the sample path of X, i.e., any contribution of the continuous part such as the drift and the Brownian motion to the movement of X can be ignored. Thus, by ignoring the contribution of the continuous part, a spectrally negative Lévy process could have gone below the bankruptcy level $x = \log(\frac{V_B}{V}) < 0$ only made possible by a jump. Hence, following (6.6.6) we see for a very short maturity T that

$$\mathbb{P}\big(\sigma_x^- \le T\big) \;=\; T\overline{\Pi}^-(x) + o(T) \quad \text{as} \quad T \downarrow 0,$$

where $\overline{\Pi}^-(x) = \Pi(-\infty, x)$ and $o(T)$ is the probability of having more than one jump in the very short period $[0, T]$ of time. Given that $\sigma_x^- \le T$, the law of $\log(V(\sigma_x^-))$ will be the law of a single jump conditioned to have gone below the level x, and therefore

$$\mathbb{E}\big(V(\sigma_x^-)\big|\sigma_x^- \le T\big) \;=\; \frac{1}{\overline{\Pi}^-(x)} \int_{-\infty}^{x} Ve^y\Pi(dy) \;\equiv\; \overline{V}.$$

Since the denominator of (6.6.5) is asymptotically equal to rT as $T \downarrow 0$, it is easily seen that

$$\text{Credit spreads} \to \overline{\Pi}^-(x)\Big(1 - \frac{(1-\eta)}{P}\overline{V}\Big) \tag{6.6.7}$$

as $T \downarrow 0$. Observe that when the process X is continuous, that is when $\Pi = 0$, the credit spreads go to zero as $T \downarrow 0$. Thus, as a summary, the limiting spreads have strictly positive values as $T \downarrow 0$ except when the process X is continuous. This conclusion agrees with the recent result of Hilberink and Rogers [58] and Chen and Kou [29] for jump diffusion processes and may be extended to cover a broader class of Lévy processes. To exemplify this observation over non-zero credit spreads for very short maturity bonds, we give some numerical examples in the next section.

6.7 Numerical inversion of double Laplace transform

This section discusses numerical inversion of double Laplace transforms of Abate and Whitt [1] and Choudhury et al [31] used to determine the term structure of credit spreads (6.6.4) expressed in terms of inversion of double Laplace transforms.

To begin with, let $f(t,x)$ be a complex-valued function on $\mathbb{R}_+^2$ whose double Laplace transform is given by

$$\widehat{f}(\lambda,\beta) = \int_0^\infty \int_0^\infty e^{-(\lambda t+\beta x)} f(t,x)dtdx, \tag{6.7.1}$$

which we assume to be well defined (see for example Ditkin and Prudnikov [35]). In (6.7.1), λ and β are complex variables with $\mathfrak{Re}(\lambda) > 0$ and $\mathfrak{Re}(\beta) > 0$. In this section, we discuss how to calculate $f(t,x)$ using the values of $\widehat{f}(\lambda,\beta)$.

Let F be a complex valued function on $\mathbb{R}^2$ with a well-defined *double Fourier transform*

$$\phi(\lambda,\beta) \;=\; \int_{-\infty}^\infty \int_{-\infty}^\infty e^{i(\lambda t+\beta x)} F(t,x)dtdx. \tag{6.7.2}$$

If F is a probability density function, then ϕ is known as its characteristic exponent, see for instance equation (2.1.1). Under regularity conditions, F can be recovered using the *Fourier inversion formula*

$$F(t,x) \;=\; \frac{1}{4\pi^2}\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-i(t\lambda+x\beta)} \phi(\lambda,\beta)d\lambda d\beta. \tag{6.7.3}$$

Our task is to compute the double integral numerically. The numerical approximation can be obtained using the two-dimensional *Poisson summation formula*

$$\begin{aligned}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} F\Big(t+\frac{2\pi j}{h_1}, x+\frac{2\pi k}{h_2}\Big) \\ = \frac{h_1h_2}{4\pi^2}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} e^{-i(jh_1t+kh_2x)}\phi(jh_1,kh_2).\end{aligned} \tag{6.7.4}$$

The left-hand side of (6.7.4) is constructed by *aliasing* to be a periodic function of t and x with periods h_1^{-1} and h_2^{-1}, respectively. Assuming that the series on the left in (6.7.4) converges and that this periodic function has a proper *Fourier series*, the Fourier series is given by the right side of (6.7.4).

The key point in the inversion problem is that the Fourier transform values $\phi(jh_1,kh_2)$ from (6.7.2) appear as the Fourier coefficients in (6.7.4); see equation (5.47) in Abate and Whitt [1] and Champeney [25] page 163. Note that the right-hand side of (6.7.4) can be regarded as a *trapezoidal rule* form of numerical integration applied to the inversion integral (6.7.3).

In order to control the aliasing error, we apply *exponential damping*; that is, if f is our original function of interest in the equation (6.7.1), then we replace $F(t,x)$

by the function $f(t,x)e^{-(a_1 t+a_2 x)}$ when $t,x \geq 0$ and 0 elsewhere. Then we have that $\phi(\lambda,\beta) = \widehat{f}(a_1 - i\lambda, a_2 - i\beta)$ for $\widehat{f}$ in (6.7.1), and the right-hand side of (6.7.4) can be expressed in terms of Laplace transform values. If, furthermore, we let $h_1 = \pi/(tl_1)$ and $h_2 = \pi/(xl_2)$, with $l_1, l_2 \geq 1$, and take $a_1 = A_1/(2tl_1)$ and $a_2 = A_2/(2xl_2)$, we obtain

$$\begin{aligned} f(t,x) &= \frac{\exp\big(A_1/(2l_1) + A_2/(2l_2)\big)}{4tl_1 x l_2} \\ &\quad \times \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} e^{-i(j\pi/l_1 + k\pi/l_2)} \widehat{f}\Big(\frac{A_1}{2tl_1} - \frac{ij\pi}{l_1 t}, \frac{A_2}{2xl_2} - \frac{ik\pi}{l_2 x}\Big) - \overline{e}_\infty. \end{aligned}$$

where the error $\overline{e}_\infty$ is given by

$$\overline{e}_\infty \equiv \sum_{\substack{0\leq j,k\leq\infty \\ \text{not } j=k=0}} e^{-(jA_1 + kA_2)} f\big((1+2jl_1)t, (1+2kl_2)x\big).$$

The term $\overline{e}_\infty$ can be regarded as the error term, which will not be explicitly computed. If $|f(t,x)| \leq C$ for some constant C and all t,x ($C = 1$ if $f(t,x)$ is a probability distribution), then the error can be bounded as

$$|\overline{e}_\infty| \leq \frac{C(e^{-A_1} + e^{-A_2} - e^{-(A_1+A_2)})}{(1-e^{-A_1})(1-e^{-A_2})} \approx C(e^{-A_1} + e^{-A_2}).$$

Therefore, a good approximation of the function $f(t,x)$ is given by

$$\begin{aligned} S_N(t,x) &= \frac{\exp\big(A_1/(2l_1) + A_2/(2l_2)\big)}{4tl_1 x l_2} \\ &\quad \times \sum_{j=-N}^{N} \sum_{k=-N}^{N} e^{-i(j\pi/l_1 + k\pi/l_2)} \widehat{f}\Big(\frac{A_1}{2tl_1} - \frac{ij\pi}{l_1 t}, \frac{A_2}{2xl_2} - \frac{ik\pi}{l_2 x}\Big). \end{aligned}$$

The raw value of S_N may not be a very good approximation; but by using Euler summation to smooth the values of the (nearly) alternating sums, we were able to obtain good accuracy. The approximation to $f(t,x)$ finally is given by

$$f(t,x) \doteqdot \sum_{n=0}^{M} 2^{-M} \binom{M}{n} S_{N+n}(t,x).$$

This is the formula we used in the thesis to invert numerically a double Laplace transform for the term structure of credit spreads (6.6.4).

6.8 Numerical examples

We verify the results of Sections 5 and 6 by means of numerical examples. Our main objective is to show that the bankruptcy level $V_B^\star$ is the one that maximizes the equity value $E(V;V_B)$. For our numerical examples, we pay attention to two cases. Firstly, we assume that the underlying dynamics of X is generated by α-stable processes with Laplace exponent

$$\kappa(\lambda) \;=\; K\lambda - \lambda^{\alpha}, \quad \text{and} \quad \kappa(\lambda) \;=\; K\lambda^{\alpha},$$

respectively. For the first (second) Laplace exponent, we choose $\alpha = 0.5$ ($\alpha = 1.75$). Secondly, we consider *jump diffusion processes* where the jump component of X is contributed by a compound Poisson process having independent downward jumps with exponential $\exp(c)$ distribution occurring at the times of a Poisson process with rate a, i.e., X has Laplace exponent

$$\begin{aligned}\kappa(\lambda) &= \mathrm{d}\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{-\infty}^{0} ace^{cx}(e^{\lambda x} - 1)dx \\ &= \mathrm{d}\lambda + \frac{1}{2}\sigma^2\lambda^2 - \frac{a\lambda}{c+\lambda}.\end{aligned} \tag{6.8.1}$$

This special case of spectrally negative Lévy process was considered by Hilberink and Rogers in [58]. For all computations, we fix some values of parameters: we set $r = 7.5\%$, $\delta = 7\%$, $\eta = 50\%$ and $\tau = 35\%$, $\sigma = 0.2$, $a = 0.5$, $c = 9$, which are the values used in [77],[76], and [58]. We shall also assume as in [76] and [58] that $V_T = \rho P/\delta$. The parameters in the Laplace exponent κ are chosen such that they match the martingale condition

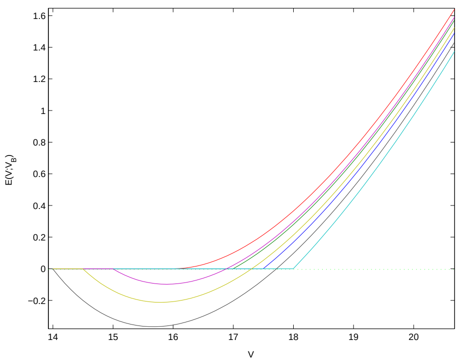
$$\mathbb{E}\Big(e^{-(r-\delta)t}V(t)\Big) \;=\; V.$$

Since our modeling for capital structures of a firm depends on the bankruptcy level V_B, we need to do the following in order to get one point on the curves (for the firm's (equity) values and debt values). Once a firm has been set up, the face value of the debt P and the coupon rate ρ are calculated for a fixed $m > 0$ in such a way that the equation (6.5.3) for the bankruptcy level V_B holds,

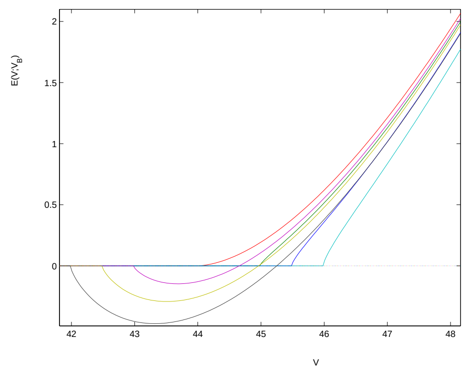
$$D(V;V_B) \;=\; P \quad \text{and} \quad L \;=\; \frac{P}{v(V;V_B)},$$

for some positive constants *leverage* L running from 5% to 95% in steps of 5%. The firm's value $v(V;V_B)$ and the total debt outstanding value $D(V;V_B)$ at time zero are defined in (6.2.8) and (6.2.7), respectively. The numerical results for the equity curves $E(V;V_B)$ are reported in Figures 6.5 and 6.6.

We present the numerical outcomes in Figures 6.5 for the case where the underlying dynamics X of the firm asset has path of unbounded variation. The first picture is for the case where X is a jump diffusion process and the other is for α-stable process with $\alpha = 1.75$. The latter process is a process of pure jumps with no Gaussian component. We see that all the curves of the equity value $E(V;V_B)$ has zero values for all $V \leq V_B$. The curves with negative (positive) gradient at $V = V_B$ correspond with bankruptcy

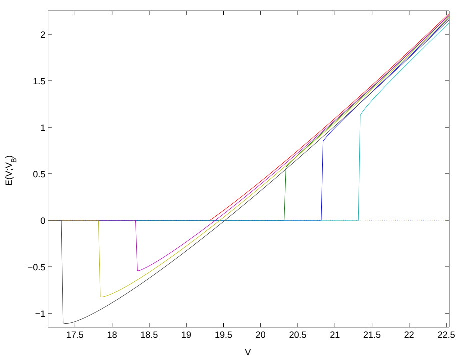


(a) X is a jump diffusion process with drift $\mathrm{d} = r - \delta - \sigma^2/2 + a/(1+c)$. The optimal bankruptcy level is $V_B^\star = 15.9964$.

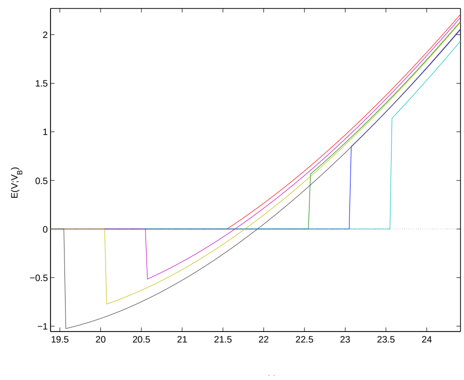


(b) X is α-stable process with Laplace exponent $\kappa(\lambda) = K\lambda^\alpha$, $\alpha = 1.75$, and $K = r - \delta$. The optimal bankruptcy level is $V_B^\star = 43.9815$.

Figure 6.5: Various shapes of the equity curves $V \mapsto E(V; V_B)$ for different values of bankruptcy level V_B for unbounded variation Lévy processes. The curve with zero gradient (smooth pasting) at $V = V_B$ (horizontal axis) corresponds to $V_B = V_B^\star$; those with negative (positive) gradient correspond to $V_B <$ $(>)$ $V_B^\star$.

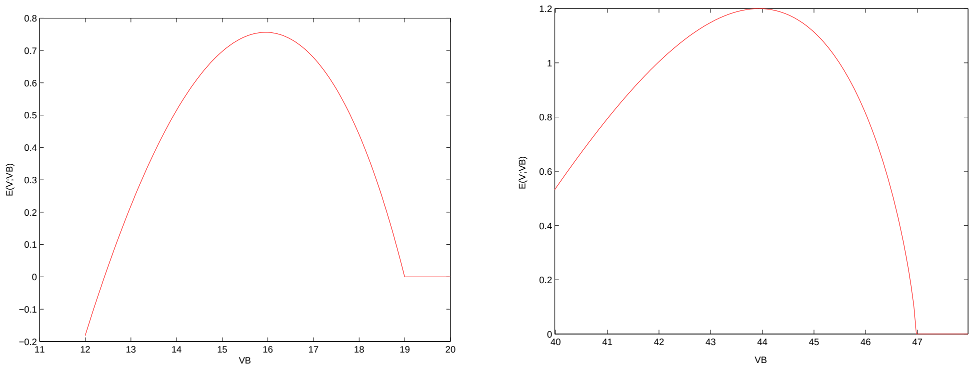


(a) X is α-stable process with Laplace exponent $\kappa(\lambda) = K\lambda - \lambda^\alpha$, $\alpha = 0.5$, and $K = 1 + r - \delta$. The optimal bankruptcy level is $V_B^\star = 19.3159$.



(b) X is compound Poisson process with drift $\mathrm{d} = r - \delta + a/(1+c)$. The optimal bankruptcy level is $V_B^\star = 21.5487$.

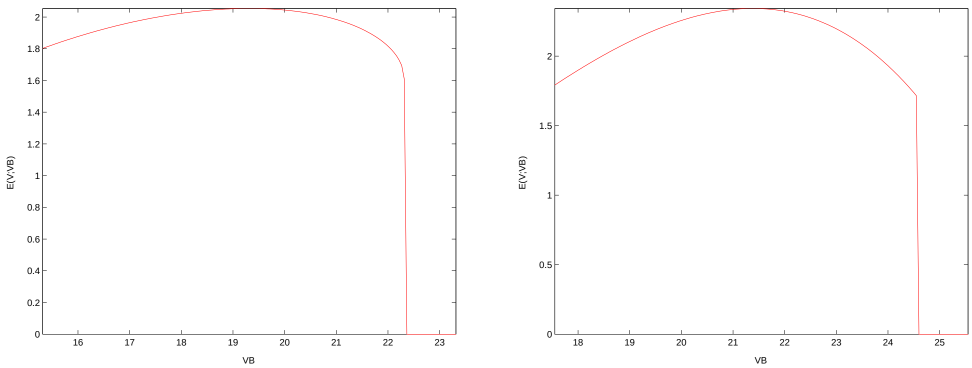
Figure 6.6: Various shapes of the equity curves $V \mapsto E(V; V_B)$ for different values of bankruptcy level V_B for bounded variation Lévy processes. The curve with zero value (continuous pasting) at $V = V_B$ (horizontal axis) corresponds to $V_B = V_B^\star$; those with negative (positive) jumps correspond to $V_B <$ $(>)$ $V_B^\star$.



(a) X is a jump diffusion process with drift $\mathrm{d} = r-\delta-\sigma^2/2+a/(1+c)$. The optimal bankruptcy level is $V_B^\star = 15.9964$.

(b) X is α-stable process with Laplace exponent $\kappa(\lambda) = K\lambda^\alpha$, $\alpha = 1.75$, and $K = r - \delta$. The optimal bankruptcy level is $V_B^\star = 43.9815$.

Figure 6.7: The shape of the equity curves $V_B \mapsto E(V; V_B)$ for a fixed initial value V of the firm's asset for unbounded variation Lévy processes. The curve achieves its maximum value at $V_B = V_B^\star$.



(a) X is α-stable process with Laplace exponent $\kappa(\lambda) = K\lambda - \lambda^\alpha$, $\alpha = 0.5$, and $K = 1 + r - \delta$. The optimal bankruptcy level is $V_B^\star = 19.3159$.

(b) X is compound Poisson process with drift $\mathrm{d} = r - \delta + a/(1 + c)$. The optimal bankruptcy level is $V_B^\star = 21.5487$.

Figure 6.8: The shape of the equity curves $V_B \mapsto E(V; V_B)$ for a fixed initial value V of the firm's asset for bounded variation Lévy processes. The curve achieves its maximum value at $V_B = V_B^\star$.

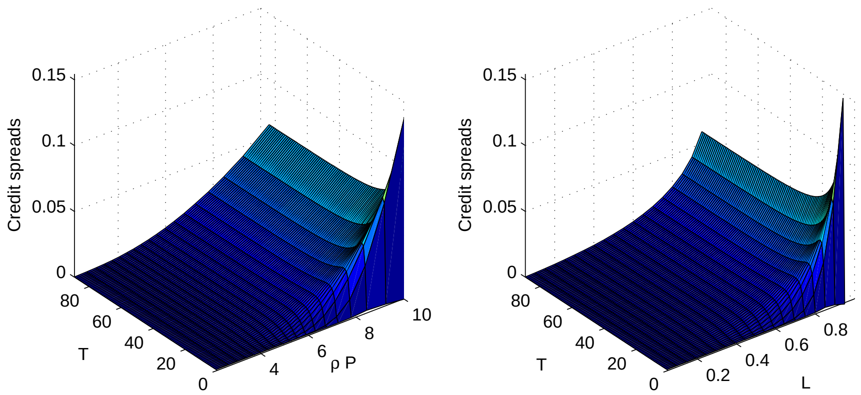


Figure 6.9: The shape of the credit spreads of a firm with debt maturity profile $m = 10$. The case where X is a pure Brownian motion. Credit spreads are zero for very short maturity bonds.

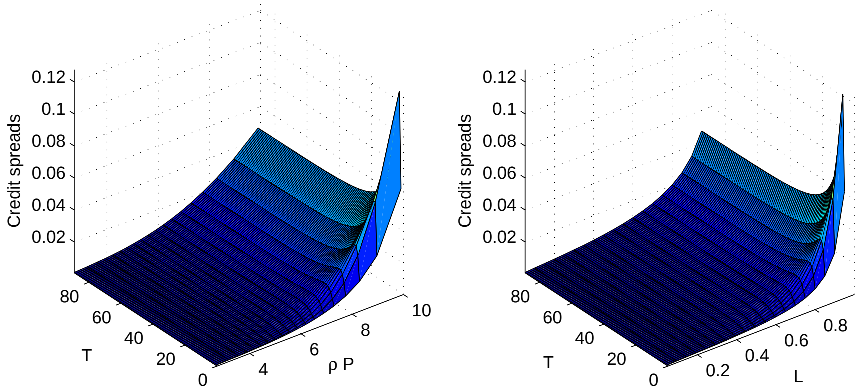


Figure 6.10: The shape of the credit spreads of a firm with debt maturity profile $m = 10$. The case where X is $\alpha-$ stable process with index $\alpha = 1.75$. Credit spreads are strictly positive for very short maturity bonds.

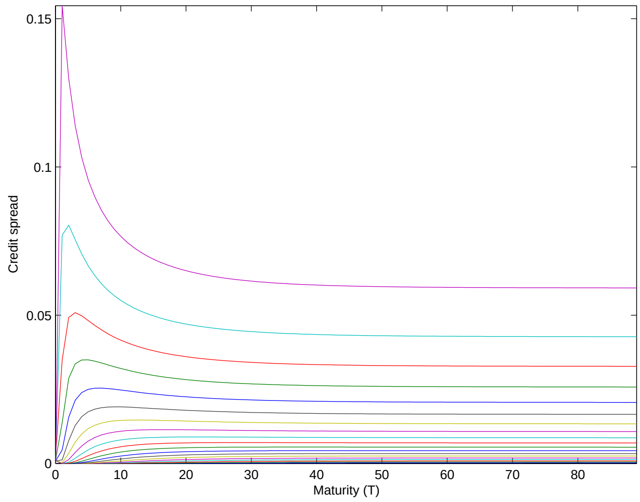


Figure 6.11: Credit spreads as a function of maturity, for different values of leverage, running from 5% to 75% in steps of 5%. The higher the leverage, the higher the spread. The case where X is a pure Brownian motion. Credit spreads are zero for very short maturity bonds.

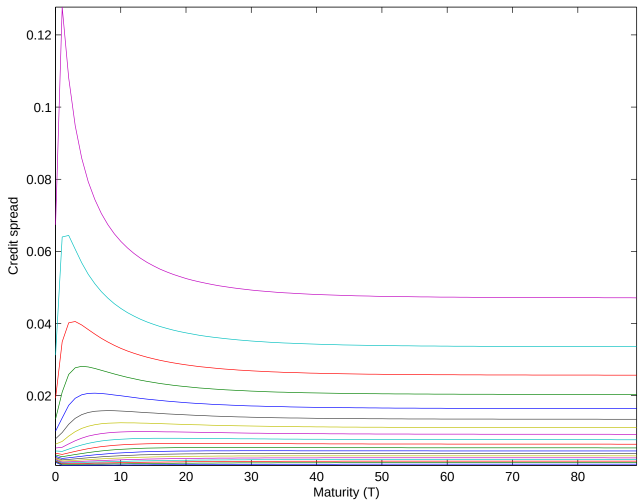


Figure 6.12: Credit spreads as a function of maturity, for different values of leverage, running from 5% to 75% in steps of 5%. The higher the leverage, the higher the spread. The case where X is $\alpha-$ stable process with index $\alpha = 1.75$. Credit spreads are strictly positive for very short maturity bonds.

level $V_B < (>) V_B^\star$. The only curve which has zero gradient (smooth pasting) at $V = V_B$ corresponds to the one with the bankruptcy level $V_B = V_B^\star$. In addition, while X has no Gaussian component we observe also that there are infinite gradient at $V = V_B$ for $V_B \neq V_B^\star$ for the equity curves.

For the case where X has paths of bounded variation, the numerical outcomes are presented in Figure 6.6. We see that all the curves of the equity value $E(V; V_B)$ have zero values for all $V < V_B$. From the picture we observe that at the bankruptcy level $V_B < (>) V_B^\star$ the equity curves $E(V; V_B)$ exhibit negative (positive) jumps. The only curve which has no jumps (continuous pasting) at $V = V_B$ corresponds to the one with the bankruptcy level $V_B = V_B^\star$.

It is seen from the two figures that the equity curve associated with $V_B = V_B^\star$ seems to dominate the other curves, even without the constraint of positive equity. This is to say that the bankruptcy level $V_B^\star$ is indeed the optimal level of bankruptcy at which the firm's equity value is maximized. This conclusion concerning the optimality of the bankruptcy level $V_B^\star$ is illustrated in Figures 6.7 and 6.8 from which we see that $V_B^\star$ is the only bankruptcy value at which, for a fixed initial value V of the firm's asset, the firm's equity value $E(V; V_B)$ is optimal. These numerical findings confirm our theoretical results given in Section 5.

The final plot, Figure 6.9-6.12, shows various shapes of the credit spreads as a function of maturity for a range of different values of leverage taken from 5% to 75% increasing in steps of 5%. Compare the continuous case, a pure Brownian motion, see Figures 6.9 and 6.11, with the other case with jumps, $\alpha-$ stable process with $\alpha = 1.75$ (see Figures 6.10 and 6.12). We notice that the credit spreads go to zero as the time to maturity T tends to zero in the pure Brownian motion case, but seem to have positive limiting values in the other case. In other respects, the numerical results obtained resemble the similar type of behavior found previously by Sarig and Warga [110], Pitts and Shelby [103], Leland [77], Leland and Toft [76], Hilberink and Rogers [58], and Chen and Kou [29].

6.9 Conclusion and remarks

We have built on the work of Leland [77], Leland and Toft [76] and Hilberink and Rogers [58] showing that one may push the model considered by these authors fully into the case that the underlying source of randomness is a spectrally negative Lévy process. We have done this by giving an analytical treatment using scale functions. This has lead to the discovery that the optimal bankruptcy level is not always achieved by a smooth pasting condition, but instead continuous pasting is sufficient according to the path regularity of the underlying Lévy process. Moreover, our justification for the pasting principles goes further than numerical observation and we give a formal proof of this fact.

Chapter 7

Evaluating Scale Functions of Spectrally Negative Lévy Processes[1]

Abstract

In this chapter we discuss a robust numerical method to compute the scale function $\{W^{(q)}(x) : q \geq 0, x \in \mathbb{R}_+\}$ of a general spectrally negative Lévy process $(X, \mathbb{P})$. The method is based on the Esscher transform of measure $\mathbb{P}^{\nu}$ under which X is taken and the scale function is determined. This change of measure makes it possible for the scale function to be bounded and hence makes numerical computation easier, fast and stable. Working under the measure $\mathbb{P}^{\nu}$ and using the method of Abate and Whitt [1] and Choudhury et al. [31], we give a fast stable numerical algorithm for the computation of $W^{(q)}(x)$ for $q \geq 0$.

7.1 Introduction

In literature, we have seen that many fluctuation identities associated with the problem of first-exit from positive half-line or finite interval of a (reflected) spectrally negative Lévy process $(X, \mathbb{P})$ can be written in terms of the so called q-scale function $\{W^{(q)}(x) : q \geq 0, x \in \mathbb{R}_+\}$. For literature review, we refer to Bertoin [14], [15], Avram et al. [7], Lambert [75], Rogers [107], and the literature therein. In connection with pricing American put and Russian options driven by spectrally negative Lévy processes, the rational price of these options appears to be some functional of this function, see for instance Avram et al. [7]. In mathematical insurance, this function appears in the problem of finding optimal dividend payments, see for instance Avram et al. [8]. In credit risk theory, the scale function plays an important role in determining an endogenous bankruptcy level V_B as well as in assessing the optimality of V_B, see for instance Kyprianou and Surya [73] (see also Chapter 5 for more details). Working under a completely general spectrally negative Lévy process, it was shown recently

[1] Submitted for publication to *Journal of Applied Probability.*

in Kyprianou and Surya [73] that not only the analytical treatment of the optimal bankruptcy level is possible, but also the smooth pasting condition used by Leland and Toft [76] and Hilberink and Rogers [58] for optimality criterion for choosing V_B can be verified both analytically and numerically. It is also worth noting that the scale function appears to be an important factor in Queuing theory (see for instance Dube et al. [36]) and the theory of continuous state branching processes (we refer to Chapter 10 in Kyprianou [69] for details).

For some spectrally negative Lévy processes, the scale function is available in explicit form. Typical examples are standard Brownian motion, α-stable processes with $\alpha \in (0, 2)$, jump diffusion process with exponential negative jumps and Compound Poisson processes. See for instance Bertoin [14], [15] and Kyprianou [69] for more details. Due to complexity of the Laplace exponent of the Lévy process, the scale function is not available in explicit form in general. An example for this is a spectrally negative tempered stable process with index less than two. Thus, numerical inversion of Laplace transform can be used to compute the scale function numerically.

Quite recently, a fast and stable numerical inversion for the scale function $W^{(0)}(x)$ of a spectrally negative Lévy process containing Brownian motion whose Laplace exponent satisfying $\kappa'(0+) > 0$ was proposed by Rogers [107]. We will come back to this point later on Section 3. For the case $q \geq 0$ and X contains no Brownian motion, the issue of how to evaluate the q-scale function $W^{(q)}(x)$ was not addressed. As will be shown later, there is a problem on the computation. The problem is that for each $q > 0$ the q-scale function $W^{(q)}(x)$ is exponentially unbounded at infinity under the measure $\mathbb{P}$ and hence numerical inversion for producing $W^{(q)}(x)$ could be unstable. We try to overcome this problem by a change of measure using the Esscher transform $\mathbb{P}^{\nu}$ under which the scale function could be bounded. Working under the measure $\mathbb{P}^{\nu}$ and using the method of Abate and Whitt [1] and Choudhury et al. [31], we give a fast stable numerical algorithm for the computation of $W^{(q)}(x)$ for $q \geq 0$.

The organization of this chapter is as follows. In Section 2, we briefly discuss spectrally negative Lévy processes and its exponential change of measure $\mathbb{P}^{\nu}$. Section 3 defines the q-scale function of a general spectrally negative Lévy process. In Section 4 we discuss numerical methods for the computation of the q-scale function. Numerical examples of the computation are given in Section 5. Finally, we provide the MATLAB program code for the computation in Section 6.

7.2 Spectrally negative Lévy processes

In this chapter, we consider a Lévy process X having the canonical decomposition

$$X_t \; = \; \mu t + \sigma B_t + J_t^{(-)},$$

where $B = \{B_t, t \geq 0\}$ is a standard Brownian motion and $J^{(-)} = \{J_t^{(-)}, t \geq 0\}$ is a non-Gaussian Lévy process, having no positive jumps, independent of B. This class of Lévy processes has a great interest from theoretical point of view, because they are processes for which fluctuation theory takes the nicest form and can be developed explicitly to its fuller extent. The degenerate case when X is either the negative of a subordinator or a deterministic drift has no interest and will not be discussed throughout. What we shall say here is based on (for the most part) Chapter VII in Bertoin [13].

The law of the Lévy process started at zero will be denoted by $\mathbb{P}$ (with the associated expectation operator $\mathbb{E}$). Since X has no positive jumps, the moment generating function $\theta \mapsto \mathbb{E}(e^{\theta X_t})$ exists for all $\theta \geq 0$ and is given by

$$\mathbb{E}(e^{\theta X_t}) \;=\; e^{t\kappa(\theta)},$$

where the function $\kappa : [0,\infty) \to (-\infty,\infty)$, also called as the Laplace exponent of X, is defined by

$$\kappa(\theta) \;=\; \mu\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,0)} \big(e^{\theta y} - 1 - \theta y \mathbf{1}_{\{y>-1\}}\big)\Pi(dy). \tag{7.2.1}$$

It is easily seen that κ is zero at the origin and is strictly convex with $\lim_{\theta\uparrow\infty}\kappa(\theta) = \infty$. Next we denote by $\Phi(\alpha)$ the largest solution of the equation

$$\kappa(p) \;=\; \alpha \quad \text{for all } \alpha \geq 0.$$

Note that due to the convexity of κ, there exists at most two roots for a given α and precisely one root when $\alpha > 0$. The asymptotic behaviour of X can be determined from the sign of $\kappa'(0+)$, the right-derivative of κ at zero. X drifts to $-\infty$, oscillates or drifts to $+\infty$ according to whether $\kappa'(0+)$ is negative, zero or positive. See for instance Kyprianou and Palmowski [70] for more details.

It is worth mentioning that under the *Esscher transform* $\mathbb{P}^\nu$ defined by

$$\frac{d\mathbb{P}^\nu}{d\mathbb{P}}\Big|_{\mathcal{F}_t} \;=\; e^{\nu X_t - \kappa(\nu)t} \quad \text{for all } \nu \geq 0, \tag{7.2.2}$$

the Lévy process $(X,\mathbb{P}^\nu)$ is still a spectrally negative Lévy process. The Laplace exponent of X under the measure $\mathbb{P}^\nu$ has changed to

$$\begin{aligned}\kappa_\nu(\theta) &= \frac{1}{t}\log\mathbb{E}^\nu\big(e^{\theta X_t}\big)\\ &= \frac{1}{t}\log\mathbb{E}\big(e^{(\theta+\nu)X_t - \kappa(\nu)t}\big)\\ &= \kappa(\theta+\nu) - \kappa(\nu).\end{aligned} \tag{7.2.3}$$

To each $\nu \geq 0$, we will denote by $\mathbb{P}^\nu_x$ the translation of $\mathbb{P}^\nu$ under which $X_0 = x$.

7.3 Scale functions

This section discusses the so called scale functions. (See Bertoin [13], [14] and [15] for the origin of this function). This function features invariably all known identities for Laplace transforms of first-exit and overshoots for spectrally negative (reflected) Lévy processes. See the aforementioned literature for more details.

Definition 7.3.1 Let $q \geq 0$ and define $\Phi_\nu(q)$ as the largest root of $\kappa_\nu(\theta) = q$.

Definition 7.3.2 (q-Scale function) For a given spectrally negative Lévy process with Laplace exponent (7.2.1), there exists for every $q \geq 0$ a right-continuous function $W^{(q)} : [0, \infty) \to [0, \infty)$, called the q-scale function, with Laplace transform given by

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x) dx \;=\; \frac{1}{\kappa(\lambda) - q} \quad \text{for} \quad \lambda > \Phi(q), \tag{7.3.1}$$

where $\Phi(q)$ was defined in the previous section. We shall write for short $W^{(0)} = W$. Furthermore, we refer to $W_\nu^{(q)}(x)$ the scale function under the measure $\mathbb{P}^\nu$.

We assume throughout the remaining of this chapter that the Lévy measure Π has no atoms. It turns out that the smoothness properties of the q-scale functions $W^{(q)}(x)$ are very closely related to the roughness of the underlying paths of the associated Lévy process. If X has paths of unbounded variation or bounded variation and the Lévy measure Π has no atoms, it is known that the q-scale function $W^{(q)}(x)$ is continuously differentiable, see for instance Lambert [75], Chan and Kyprianou [28], and the literature therein for more details.

Lemma 7.3.3 (Asymptotic behaviour) *Suppose that either ($q > 0$) or ($q = 0$ and $\kappa'(0+) > 0$). Then the scale function $\{W_{\Phi(q)}(x) : q \geq 0, x \in \mathbb{R}\}$ of X taken under the measure $\mathbb{P}^{\Phi(q)}$ is increasing and bounded from above by $1/\kappa'(\Phi(q))$. However, when X is taken under the measure $\mathbb{P}$, the q-scale function $W^{(q)}(x)$ is given by*

$$W^{(q)}(x) \;=\; e^{\Phi(q)x} W_{\Phi(q)}(x), \tag{7.3.2}$$

and hence has the asymptotic $W^{(q)}(x) \sim \frac{e^{\Phi(q)x}}{\kappa'(\Phi(q))}$ as $x \to \infty$.

Remark 7.3.4 In the case of $q = 0$ and $\kappa'(0+) = 0$, the scale function $W_{\Phi(q)}(x)$ is increasing and unbounded at infinity and hence the numerical computation may produce instability in the tail (as $x \to \infty$).

Proof From (7.3.1) and (7.2.3), it is clear to see for all $q \geq 0$ that

$$\int_0^\infty e^{-\lambda x} W_{\Phi(q)}(x) dx \;=\; \frac{1}{\kappa_{\Phi(q)}(\lambda)} \;=\; \frac{1}{\kappa(\lambda + \Phi(q)) - \kappa(\Phi(q))}, \tag{7.3.3}$$

for $\lambda > 0$. It is clear following (7.3.3) and using a standard Tauberian theorem that

$$\lim_{x \uparrow \infty} W_{\Phi(q)}(x) \;= \lim_{\lambda \downarrow 0} \frac{\lambda}{\kappa(\lambda + \Phi(q)) - \kappa(\Phi(q))} \;=\; \frac{1}{\kappa'(\Phi(q))} \;<\; \infty. \tag{7.3.4}$$

Taking into account of (7.3.4), we see using integration by parts that the Laplace-Stieltjes transform of the scale function $W_{\Phi(q)}(x)$ is given by

$$\int_0^\infty e^{-\lambda x} dW_{\Phi(q)}(x) \;=\; \frac{\lambda}{\kappa(\lambda+\Phi(q))-\kappa(\Phi(q))},$$

where $dW_{\Phi(q)}$ denotes the Stieltjes measure associated to the scale function $W_{\Phi(q)}(x)$ which gives the mass $W_{\Phi(q)}(0)$ zero value. Since the Lévy measure Π has no atoms so that $W_{\Phi(q)}$ is continuously differentiable, it is clear that the scale function $W_{\Phi(q)}(x)$ is increasing and bounded from above by $\frac{1}{\kappa'(\Phi(q))}$.

The claim that the q-scale function $W^{(q)}(x)$ is given by the expression (7.3.2) can be verified by applying Laplace transform to the both sides of (7.3.2). As a result of (7.3.2), it is clear by Tauberian theorem that the q-scale function $W^{(q)}(x)$ has the asymptotic $W^{(q)}(x) \sim \frac{e^{\Phi(q)x}}{\kappa'(\Phi(q))}$ as $x \to \infty$. Thus, our claim is then proved. □

Remark 7.3.5 Following (7.3.1), it is straightforward to check that

$$W_\nu^{(q)}(x) \;=\; e^{-\nu x} W^{(q+\kappa(\nu))}(x) \quad \text{for all } \nu \geq 0 \text{ and } q \geq -\kappa(\nu). \tag{7.3.5}$$

To verify the relation (7.3.5), apply Laplace transform to the both sides of (7.3.5).

7.4 Evaluating scale functions

In this section we discuss numerical algorithms based on a univariate version of Rogers' method [107], and the methods of Abate and Whitt [1] and Choudhury et al [31] for the computation of the q-scale function $W^{(q)}(x)$. In order to apply these algorithms, we need the result of Lemma 7.3.3 for the computation.

7.4.1 A method based on Roger's approach

In [107], Rogers gives a fast stable numerical algorithm (which is a variant of Abate and Whitt [1]) to invert the bivariate Laplace transform

$$\int_0^\infty \int_0^\infty e^{-(\lambda x + qt)} \mathbb{P}(\tau_{-x}^- \geq t) dt dx \;=\; \frac{\lambda - \Phi(q)}{(\kappa(\lambda)-q)\lambda\Phi(q)} \tag{7.4.1}$$

of the probability of first passage time $\tau_x^- = \inf\{t > 0 : X_t < -x\}$ below a level $-x \leq 0$ of a spectrally negative Lévy process having Gaussian component.

Compared to the problem (7.4.1), the inversion problem (7.3.1) is different on its own. Unlike the inversion problem (7.4.1) the problem (7.3.1) is univariate. However, the algorithm for inverting the Laplace transform (7.4.1) could be adapted to accommodate the inversion problem (7.3.1).

By identifying the fact that $\mathbb{P}(\tau_{-x}^{-} \geq t) = \mathbb{P}(-\underline{X}_t \leq x)$, we can now reduce the problem (7.4.1) into a univariate case as

$$\int_0^\infty e^{-\lambda x}\mathbb{P}(-\underline{X}_{\mathbf{e}_q} \leq x)dx \;=\; \frac{q}{\Phi(q)}\frac{1}{(\kappa(\lambda)-q)} - \frac{q}{\lambda(\kappa(\lambda)-q)}. \qquad (7.4.2)$$

Note that the expression in (7.4.2) is based on the Wiener-Hopf factorization formula (2.3.3) (see Section 3 of Chapter 2 for details) for spectrally negative Lévy processes and is obtained by applying integration by part in (2.3.3). Hence, as a result of inverting the Laplace transform (7.4.2), the q-scale function $W^{(q)}(x)$ in (7.3.1) might be deduced from the relation (7.4.2).

In the special case of $q = 0$ and $\kappa'(0+) > 0$, we see that the result of inverting (7.4.2) coincides up to multiplicative constant with that of (7.3.1). In this case, a univariate version of Rogers' algorithm might be applied to get the scale function $W^{(0)}(x)$. Due to the presence of the second term on the right-hand side of the equation (7.4.2), it is not straightforward, however, how to get $W^{(q)}(x)$ from (7.4.2).

Suppose that the problem (7.4.2) could be solved numerically using the univariate version of Rogers' algorithm to produce the distribution function $\mathbb{P}(-\underline{X}_{\mathbf{e}_q} \leq x)$. The goal is to find an expression for the q-scale functions $\{W^{(q)}(x), q \geq 0\}$ in terms of the function $\mathbb{P}(-\underline{X}_{\mathbf{e}_q} \leq x)$. To solve this problem, we can use the Esscher transform $\mathbb{P}^{\Phi(q)}$ (7.2.2) to first compute the scale function $W_{\Phi(q)}(x)$ and then use the transformation (7.3.2) to obtain $W^{(q)}(x)$ for $q \geq 0$. (Note that $W_{\Phi(q)}(x)$ plays the role of the scale function $W^{(q)}(x)$ for $q = 0$ when X is taken under the measure $\mathbb{P}^{\Phi(q)}$.) Under this measure, we obtain after some calculations that

$$W_{\Phi(q)}(x) \;=\; \frac{\Phi(q)}{q}e^{-\Phi(q)x}\mathbb{P}(-\underline{X}_{\mathbf{e}_q} \leq x) + \frac{\Phi(q)^2}{q}\int_0^x e^{-\Phi(q)y}\mathbb{P}(-\underline{X}_{\mathbf{e}_q} \leq y)dy. \quad (7.4.3)$$

To see that this is the correct expression for the scale function $W_{\Phi(q)}(x)$, take Laplace transform on both sides of the above expression and use (7.4.2) to get (7.3.3), i.e.,

$$\int_0^\infty e^{-\lambda x}W_{\Phi(q)}(x)dx \;=\; \frac{1}{(\kappa(\lambda+\Phi(q))-q)} \quad \text{for } \lambda > 0 \text{ and } q \geq 0. \qquad (7.4.4)$$

Hence, following (7.4.3) and (7.3.2) the q-scale function $W^{(q)}(x)$ is finally given by

$$W^{(q)}(x) \;=\; \frac{\Phi(q)}{q}\mathbb{P}(-\underline{X}_{\mathbf{e}_q} \leq x) + \frac{\Phi(q)^2}{q}\int_0^x e^{-\Phi(q)(y-x)}\mathbb{P}(-\underline{X}_{\mathbf{e}_q} \leq y)dy.$$

Note that the algorithm given in Rogers [107] can be used to handle the case where a spectrally negative Lévy process contains a Gaussian component ($\sigma > 0$). To deal with a more general class of spectrally negative Lévy processes, we need to modify the algorithm based on the method of Abate and Whitt [1] and Choudhury et al. [31].

7.4.2 A method based on Abate and Whitt [1] and Choudhury et al. [31]

In this section we discuss a robust numerical method based on Abate and Whitt [1] and Choudhury et al. [31] for inverting univariate Laplace transform which works with a good accuracy for bounded functions. In this thesis, we use this method to invert the Laplace transform (7.4.4) to produce the function $\{W_{\Phi(q)}(x), q \geq 0\}$ and then apply the transformation (7.3.2) to obtain the q-scale function $\{W^{(q)}(x), q \geq 0\}$.

To start with, let f be a real-valued function (not necessarily a probability density) defined on the positive real line. For such a function f, it is often convenient to work with the Laplace transform

$$\widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(x)dx, \tag{7.4.5}$$

where λ is a complex variable for which the integral exists. The standard inversion integral for the Laplace transform $\widehat{f}$ is the *Bromwich contour integral*, which can also be expressed as the integral of a real-valued function of a real variable by choosing a specific contour by any vertical line $\lambda = a_1$ such that $\widehat{f}(\lambda)$ has no singularities on or to the right of the vertical line. By applying this integral, we obtain

$$f(x) = \frac{1}{2\pi i} \int_{a_1 - i\infty}^{a_1 + i\infty} e^{\lambda x} \widehat{f}(\lambda) d\lambda = \frac{e^{a_1 x}}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} \widehat{f}(a_1 + i\lambda) d\lambda. \tag{7.4.6}$$

For some Laplace transforms $\widehat{f}$, analytic expressions for f are available explicitly, see for instance Obberhettinger [93]. When the transform can not be inverted analytically, the function f can be approximated by means of numerical approximation.

Several numerical inversion algorithms have been proposed by several authors. The fast and stable one is given by Abate and Whitt [1] and Choudhury et al. [31]. Following Abate and Whitt [1], we use the trapezoidal rule to approximate the integral in (7.4.6) and analyze the corresponding discretization error using the *Poisson summation formula*. The trapezoidal rule approximates the integral of a function g over the bounded interval $[c, d]$ by the integral of the piecewise linear function obtained by connecting the $n+1$ evenly spaced points $g(c + kh)$, $0 \leq k \leq n$ where $h = (d - c)/n$. Using the trapezoidal rule, we have

$$\int_c^d g(x)dx \approx h\Big[\frac{g(c) + g(d)}{2} + \sum_{k=1}^{n-1} g(c + kh)\Big], \tag{7.4.7}$$

see page 51 in Davis and Rabinowitz [34]. The trapezoidal rule (7.4.7) also applies for the case $c = -\infty$ and $d = \infty$ with the following modification

$$\int_{-\infty}^{\infty} g(x)dx \approx h_1 \sum_{j=-\infty}^{\infty} g(jh_1), \tag{7.4.8}$$

where h_1 is a small positive constant. Applying (7.4.8) to (7.4.6) with step size $h_1 = \pi/x$, $x > 0$, and letting $a_1 = A_1/x$ at the same time, we get

$$f(x) \approx \frac{e^{A_1}}{2x} \sum_{j=-\infty}^{\infty} (-1)^j \widehat{f}((A_1 + ij\pi)/x). \tag{7.4.9}$$

The advantage of using this simple numerical procedure of trapezoidal rule is that it tends to work well for periodic and oscillating integrands since the errors tend to cancel out and the realized errors can be substantially less than from other alternative numerical procedure such as Simpson's rule. Moreover, the Poisson summation formula can be applied to obtain a convenient representation of the discretization error associated with the trapezoidal rule. Using the Poisson summation formula, the approximation (7.4.9) can be obtained for an integrable function g by

$$\sum_{j=-\infty}^{\infty} g(x + 2\pi j/h_2) \;=\; \frac{h_2}{2\pi} \sum_{j=-\infty}^{\infty} e^{-ijh_2 x}\phi(jh_2) \tag{7.4.10}$$

where h_2 is some positive constant and $\phi(u) = \int_{-\infty}^{\infty} e^{iux} g(x)dx$, the Fourier transform of g. In order to control the aliasing[2] error, we do *exponential damping*; that is, if f is our original function of interest, then we replace $g(x)$ by the function $f(x)e^{-a_1 x}\mathbf{1}_{[0,\infty)}(x)$. Then $\phi(\lambda) = \widehat{f}(a_1 - i\lambda)$, and the right-hand side of (7.4.10) can be expressed in terms of Laplace transform values:

$$\sum_{j=0}^{\infty} e^{-a_1(x+2\pi j/h_2)} f(x + 2\pi j/h_2) \;=\; \frac{h_2}{2\pi} \sum_{j=-\infty}^{\infty} e^{-ijh_2 x}\widehat{f}(a_1 - ijh_2). \tag{7.4.11}$$

In addition, if we let $h_2 = \pi/(xl_1)$, with $l_1 \geq 1$, and $a_1 = A_1/(2xl_1)$ in (7.4.11) we obtain

$$f(x) \;=\; \frac{e^{A_1/(2l_1)}}{2xl_1} \sum_{j=-\infty}^{\infty} e^{-ij\pi/l_1}\widehat{f}\Big(\frac{A_1}{2xl_1} - \frac{ij\pi}{l_1 x}\Big) - \overline{e}_\infty, \tag{7.4.12}$$

where the error $\overline{e}_\infty$ is given by

$$\overline{e}_\infty \;=\; \sum_{j=1}^{\infty} e^{-2jA_1} f\big((2j+1)x\big).$$

Comparing (7.4.11) with (7.4.9), we conclude that $\overline{e}$ is an explicit expression for the discretization error associated with the trapezoidal rule approximation. This discretization error can easily be bounded whenever f is bounded. For example if $|f(x)| \leq C$ for some $C > 0$ and all x ($C = 1$ if $f(x)$ is a probability distribution), then we have

$$|\overline{e}_\infty| \;\leq\; \frac{Ce^{-2A_1}}{(1 - e^{-2A_1})}. \tag{7.4.13}$$

[2]Aliasing means that the new function is constructed by adding a translated version of the original function

Therefore, an approximation of the function f (7.4.6) is given by

$$S_N(x) \;=\; \frac{e^{A_1/(2l_1)}}{2xl_1}\sum_{j=-N}^{N} e^{-ij\pi/l_1}\widehat{f}\Big(\frac{A_1}{2xl_1}-\frac{ij\pi}{l_1x}\Big). \qquad (7.4.14)$$

The raw value of S_N may not be a very good approximation, but by using *Euler summation* to smooth the values of the (nearly) alternating sums, we were able to obtain good accuracy. The approximation to $f(x)$ finally is given by

$$f(x) \;\doteqdot\; \sum_{n=0}^{M} 2^{-M}\binom{M}{n} S_{N+n}(x).$$

This is the formula we used in the thesis to invert numerically a univariate Laplace transform for the scale function $W_{\Phi(q)}(x)$ (the role of the scale function $W(x)$ under the measure $\mathbb{P}^{\Phi(q)}$) with Laplace transform given by (7.3.3).

7.5 Numerical examples

For our numerical examples, we consider four different Lévy processes. Firstly, we assume that X is generated by α-stable processes with Laplace exponent

$$\kappa(\lambda) \;=\; K\lambda^{\alpha}, \quad \text{for } \alpha \in (0,2]. \qquad (7.5.1)$$

Secondly, we consider jump diffusion processes where the jump component of X is contributed by a compound Poisson process having independent downward jumps with exponential $\exp(b)$ distribution occurring at the times of a Poisson process with rate a; that is to say that X has Laplace exponent

$$\kappa(\lambda) \;=\; \mathrm{d}\lambda + \frac{\sigma^2}{2}\lambda^2 - \frac{a\lambda}{b+\lambda}. \qquad (7.5.2)$$

Thirdly, we consider X to be a one sided tempered stable process. In the general case ($\alpha \neq 1$ and $\alpha \neq 0$) the Laplace exponent of X is given by

$$\kappa(\lambda) \;= \mathrm{d}\lambda + \Gamma(-\alpha)\beta^{\alpha}C\Big\{\Big(1+\frac{\lambda}{\beta}\Big)^{\alpha} - 1 - \frac{\lambda\alpha}{\beta}\Big\}. \qquad (7.5.3)$$

(See Section 5.7.1 of Chapter 5 for further details.) Tempered stable processes were used in Section 5.7 of Chapter 5 for the computation of the arbitrage-free price of the perpetual American put and call options.

Finally, we consider the case where X is a gamma process perturbed by diffusion process. The Laplace exponent of X is described by

$$\kappa(\lambda) \;=\; \mathrm{d}\lambda + \frac{\sigma^2}{2}\lambda^2 - \psi(\lambda) \qquad (7.5.4)$$

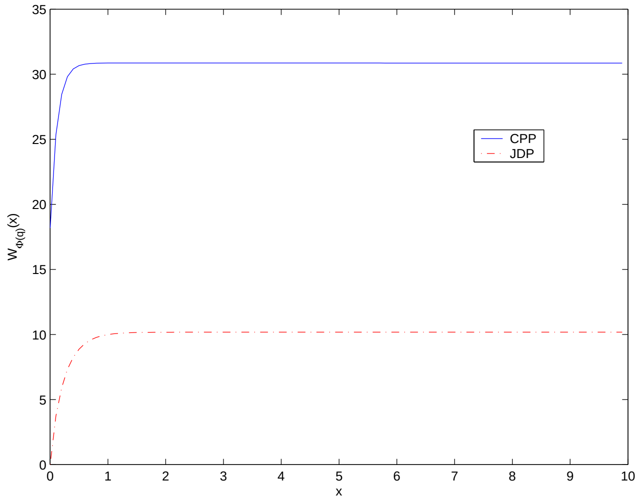


Figure 7.1: The shape of the scale function $W_{\Phi(q)}(x)$ for compound Poisson and jump diffusion processes. All the curves are bounded by $1/\kappa'(\Phi(q))$, equal to 30.8640 and 10.1787, respectively. From the plot, we notice that $W_{\Phi(q)}(0)$ is respectively positive for compound Poisson process and zero for jump diffusion process.

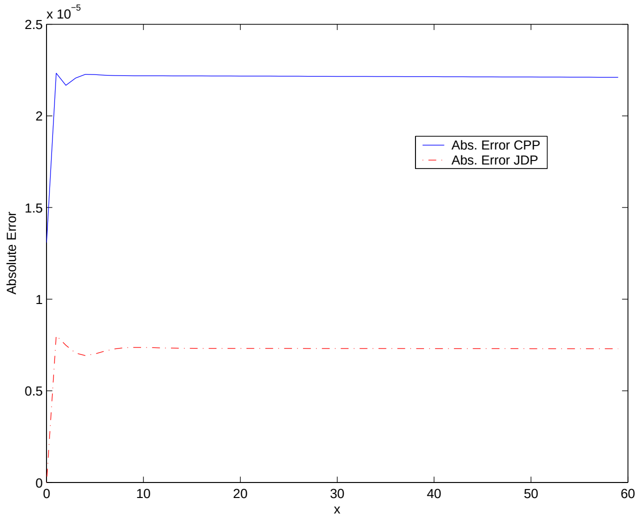


Figure 7.2: The absolute error $\left|\widehat{W}_{\Phi(q)}(x) - W_{\Phi(q)}(x)\right|$ between numerical and theoretical results for the scale function $W_{\Phi(q)}(x)$ for compound Poisson (CPP) and jump diffusion (JDP) processes. For CPP (resp. JDP) the error is bounded by 2.5664×10^{-5} (resp. 8.4639×10^{-6}). See inequality (7.5.5) for the error estimate.

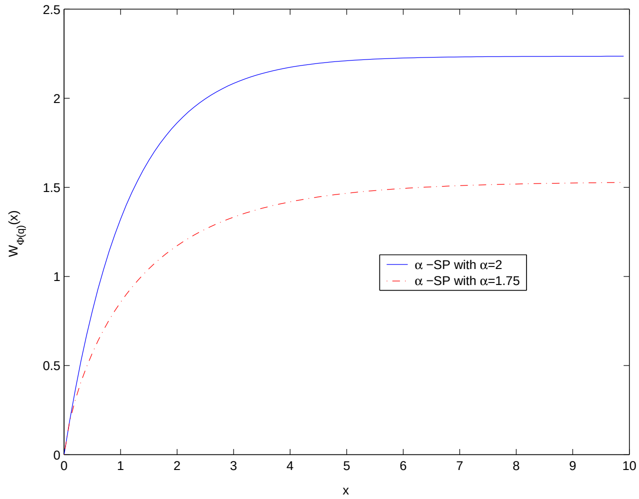


Figure 7.3: The shape of the scale function $W_{\Phi(q)}(x)$ of $\alpha-$ stable Lévy processes with indexes $\alpha = 2$ and $\alpha = 1.75$. All the curves are bounded by $1/\kappa'(\Phi(q))$, equal to 2.2361 and 1.5330, respectively. From the plot, we notice that $W_{\Phi(q)}(0)$ has zero value for both processes.

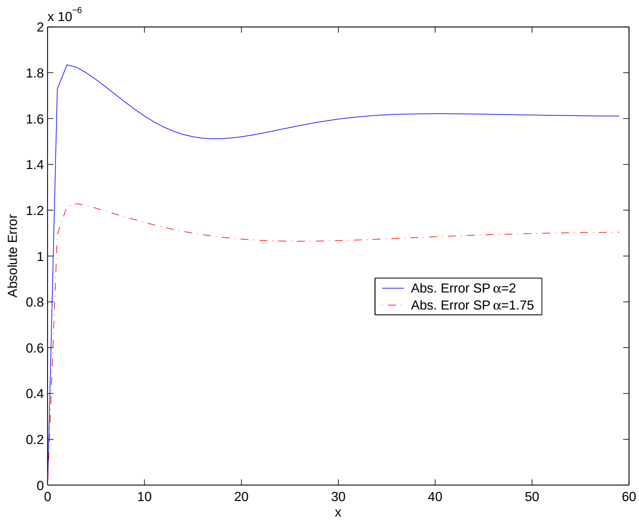


Figure 7.4: The absolute error $\left|\widehat{W}_{\Phi(q)}(x) - W_{\Phi(q)}(x)\right|$ between numerical and theoretical results for the scale function $W_{\Phi(q)}(x)$ of $\alpha-$ stable Lévy processes with indexes $\alpha = 2$ and $\alpha = 1.75$. For $\alpha = 2$ (resp. $\alpha = 1.75$) the error is bounded by 1.8594×10^{-6} (resp. 1.2747×10^{-6}). See inequality (7.5.5) for the error estimate.

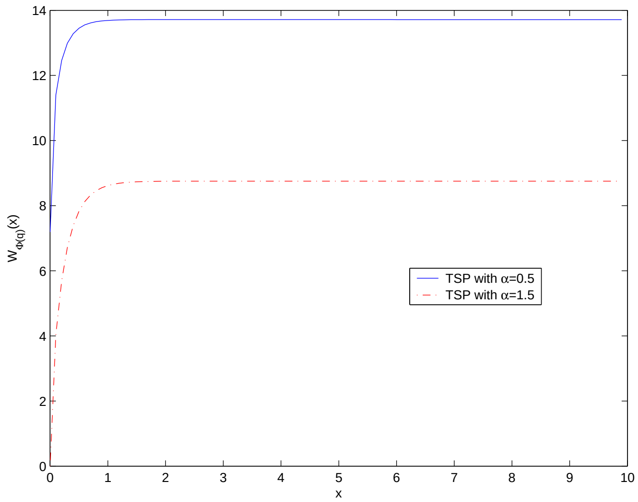


Figure 7.5: The shape of the scale function $W_{\Phi(q)}(x)$ of tempered stable Lévy processes with indexes $\alpha = 0.5$ and $\alpha = 1.5$. All the curves are bounded by $1/\kappa'(\Phi(q))$, equal to 13.7137 and 8.7532, respectively. From the plot, we notice that $W_{\Phi(q)}(0)$ is positive for $\alpha = 0.5$ and zero for $\alpha = 1.5$.

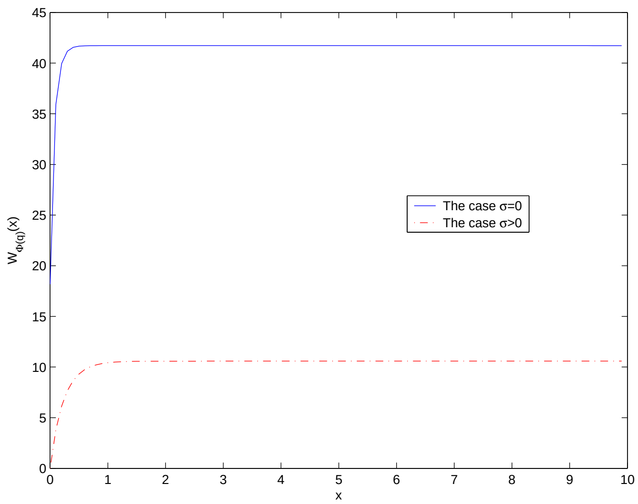


Figure 7.6: The shape of the scale function $W_{\Phi(q)}(x)$ of gamma process perturbed by diffusion process. All the curves are bounded by $1/\kappa'(\Phi(q))$, equal to 10.5823 (when $\sigma > 0$) and 41.7222 (when $\sigma = 0$), respectively. From the plot, we notice that $W_{\Phi(q)}(0)$ is positive when $\sigma = 0$ and zero when $\sigma > 0$.

where $\psi(\lambda)$ is the Laplace exponent of a gamma process S defined by

$$\psi(\lambda) \;=\; \int_0^\infty (1-e^{-\lambda x})ax^{-1}e^{-bx}dx \;=\; a\log\big(1+\frac{\lambda}{b}\big), \quad \text{for } a,b>0.$$

Note that the perturbed gamma process (7.5.4) is a slight generalization of the model studied by Dufresne and Gerber [38] and has been used in risk theory quite extensively. For related references, see for instance Dufresne and Gerber [38], Gerber [53], Gerber and Shiu [54], Yang and Zhang [119], and the literature therein.

For all computations, we fix some values of parameters: $N=11$, $M=9$, $A1=14.0$, $l1=1$; for jump diffusion and gamma processes we set $\sigma=0.2$, $\mathrm{d}=0.055$, $a=0.5$, $b=9$; for α-stable process we set $K=0.5$ for $\alpha=2$ (a standard Brownian motion) and $K=1$ for $\alpha=1.75$. In the case where X is tempered stable process with $\alpha=0.5$ (resp. $\alpha=1.5$) we choose the relative frequency of downward jumps C to be 0.075 (resp. $C=0.05$) and the jump rate β to be 2.5 (resp. $\beta=2.5$).

The numerical results of the scale function $W_{\Phi(q)}(x)$ (7.3.3) for $q=0.1$ are presented in Figures 7.1-7.6. We observe from these plots that all of the curves are increasing and bounded from above by $1/\kappa'(\Phi(q))$. These numerical outcomes support our theoretical results given previously in Lemma 7.3.3. In particular, for the case where X has path of bounded and unbounded variation we see respectively that $W_{\Phi(q)}(0)>0$ and $W_{\Phi(q)}(0)=0$, (see Section 6.4 of Chapter 6 for more discussions).

Using the explicit form of the scale function $W_{\Phi(q)}(x)$ of α-stable (7.5.1) and jump diffusion (7.5.2) processes, given in Section 6.4.1 of Chapter 6, we present in Figures 7.2 and 7.4 plots of the absolute error between the theoretical curve $W_{\Phi(q)}(x)$ and the numerical curve $\widehat{W}_{\Phi(q)}(x)$ produced by the proposed numerical method. Following (7.4.13) we observe that the absolute error is bounded by $\frac{1}{\kappa'(\Phi(q))}\frac{e^{-A_1}}{(1-e^{-A_1})}$, i.e.,

$$\big|\widehat{W}_{\Phi(q)}(x) - W_{\Phi(q)}(x)\big| \;\le\; \frac{1}{\kappa'(\Phi(q))}\frac{e^{-A_1}}{(1-e^{-A_1})}. \tag{7.5.5}$$

Since the (tuning) parameter A_1 was chosen relatively big (with $A_1=14$), we see that the absolute error is relatively small and hence our numerical method performs very good job in the computation of the scale function $W_{\Phi(q)}(x)$.

The final plot, Figures 7.5-7.6 show the shape of the scale function $W_{\Phi(q)}(x)$ for tempered stable processes (7.5.3) with indexes $\alpha=0.5$ and $\alpha=1.5$, and perturbed gamma process (7.5.4). For these stochastic processes, we notice that an explicit expression for the scale function $W_{\Phi(q)}(x)$ is not available. But nonetheless, it exhibits the important properties of the scale function as specified in Lemma 7.3.3.

7.6 MATLAB program code

We present in this section a simple MATLAB program used to compute the function $W_{\Phi(q)}(x)$ by means of numerical inversion of the Laplace transform (7.3.3). The algorithm is based on the framework of Abate and Whitt [1] and Choudhury et al. [31] described earlier. However, to handle matrix multiplication, we use the approach of Rogers [107].

```
function G=ILT(F,X,P)

% This program is for numerical inversion
% of univariate Laplace transform

% F is the Laplace transform function,
% P is a vector of parameters sitting in F

% Setting the parameters values 11, N, M and A1

N=11; M=9; A1=14.0; l1=1;

% Creating the weights to be used in the
% Euler summation of the partial sums:

mx=pascal(M+1); my=fliplr(mx); bn=diag(my)*2^(-M);
weight=ones([2*N+1 1]); head=cumsum(bn); tail=1-cumsum(bn);
tail(M+1,:)=[]; head(M+1,:)=[]; weight=[head;weight;tail];

% Setting the values of the arguments at which the
% transfor series to be evaluated:

val1=-(N+M):(N+M); val1=(i*pi*val1+A1/2)/l2; X_inv=1./X;
X_args=kron(X_inv,val1);

% Evaluating the integrand at all the points.

integrand=feval(F,X_args,P)
            .*exp(X_args*diag(kron(X,ones(1,1+2*N+2*M))));

% Preparing the matrix which will
% post-multiply the integrand

right=kron(diag(X_inv),weight)/(2*l2);
```

```
% and finally the answer is given by

G=real(integrand*right);
------------------------------------------
```

The following MATLAB m-files are needed to compute the scale function $W_{\Phi(q)}(x)$ of α-stable, jump diffusion, tempered stable, and gamma processes whose Laplace exponents are given in equations (7.5.1)-(7.5.4).

```
function g=LEXP(z,A)

% Non zero value of the parameter A is needed to compute
% the largest root of the Laplace exponent

% For Spectrally Negative (SN) alpha stable process with alpha <1
alpha=0.5; K=1; f=K.*z-z.^(alpha); g=f-A;

% For SN alpha stable processes with alpha >=1
alpha=1.75; K=1; f=K.*z.^(alpha); g=f-A;

% For SN alpha stable process with alpha =2
f=z.^2/2; g=f-A;

% For SN Jump Diffusion Processes (JDP)
sigma=0.2; a=0.5; c=9; d=0.055;
f=d.*z+0.5.*(sigma.^2).*(z.^2)-a.*z./(c+z); g=f-A;

% For SN bounded variation (BV) tempered stable process
alpha=0.5; lambda=2.5; C=0.075; d=0.0550;

% For SN unbounded variation (UBV) tempered stable process
alpha=1.5; lambda=2.5; C=0.05; d=0.0550;

f=d.*z+gamma(-alpha).*(lambda).^(alpha).*(C)...
   .*((1+z./(lambda)).^(alpha)-1-(alpha).*z./lambda); g=f-A;

% For Gamma process perturbed by diffusion process
sigma=0.2; d=0.0550; a=0.5; b=9;
f=d.*z+0.5.*(sigma.^2).*(z.^2)-a.*log(1+z./b); g=f-A;
------------------------------------------

function f=Phir(q)
```

```
% Computing the largest root of the Laplace exponent

x0=1; f=fsolve('LEXP',x0,optimset('MaxFunEvals',100),q);
------------------------------------------

function f=funcscale(lambda,q)

% The right hand side of the equation (7.4.9)

A=0; g=(LEXP(lambda+Phir(q),A)-q); f=1./g;
------------------------------------------
```

Finally, the scale function $W_{\Phi(q)}(x)$ is given by

```
function W=Scale(x,q)

% Producing the scale function

W=ILT('funcscale',x,q);
------------------------------------------
```

Having solved the problem (7.3.3), the q-scale function $W^{(q)}(x)$ is given by

$$W^{(q)}(x) \;=\; \text{exp(Phir(q)*x).*W.}$$

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Samenvatting

Het oplossen van optimale stoptijdproblemen gedreven door Lévy processen is een uitdagende opgave en kent verscheidene toepassingen in de moderne theorie van de financiering. Voorbeelden zijn het vinden van de arbitrage-vrije prijs van een Amerikaanse put (call) optie en het bepalen van een optimaal faillissementsniveau voor het probleem van endogeen faillissement.

Het belangrijkste aspect van het prijzen van de Amerikaanse put (call) optie is het vinden van de kritieke waarde van het prijsproces waaronder (waarboven) de optie uitgeoefend wordt. Voor endogeen faillissement gaat het vooral om het vinden van een optimaal faillissementsniveau van een bedrijf dat een constant schuldenprofiel heeft en dat het faillissementsniveau endogeen kiest om de *equity* waarde te maximaliseren. Binnen de context van de optimale stoptijdtheorie komen de arbitrage-vrije prijs van de Amerikaanse put (call) optie en de equity waarde overeen met de waardefunctie van een optimaal stoptijdprobleem, terwijl de kritieke waarde van het prijsproces en het optimale faillissementsniveau overeenkomen met de optimale stopgrens.

In het algemeen zijn optimale stoptijdproblemen twee-dimensionaal, omdat de waardefunctie en de optimale stopgrens tegelijk gevonden dienen te worden. Hierbij kan de waardefunctie gezien worden als functie van de te vinden optimale stopgrens. Dit is een van de redenen dat het oplossen van optimale stoptijdproblemen, vanuit een analytisch oogpunt, uitdagend is.

Een belangrijke techniek die vaak gebruikt wordt bij het oplossen van optimale stoptijdproblemen gedreven door diffusies, is de vrije-randwaarde probleem formulering van de waardefunctie en de grens. Deze formulering bestaat hoofdzakelijk uit een partiële differentiaalvergelijking en (naast andere voorwaarden) de continue en gladde verbindingvoorwaarde. Voor de eerste voorwaarde dient de waardefunctie continu te zijn op de grens, terwijl voor de tweede voorwaarde op de rand C^1- gladheid van de waardefunctie vereist is. Afhankelijk van het optimale stoptijdprobleem en van het gedrag van de paden van het Lévy proces kan het gebeuren dat de waardefunctie niet voldoet aan de gladde verbindingsvoorwaarde. In dit proefschrift zullen we laten zien dat dit fenomeen plaatsvindt als het Lévy proces van begrensde variatie is. Dit heeft tot gevolg dat voor dit type Lévy processen de continue verbindingsvoorwaarde het enige criterium blijkt te zijn volgens dewelke de grens gekozen kan worden. Een beter begrip van de juiste keus van verbindingsvoorwaarde voor het bepalen van de grens kan een belangrijke rol spelen in de theorie.

Een groot gedeelte van dit proefschrift behandelt optimale stoptijdproblemen gedreven door Lévy processen. Het doel is om semi-expliciete oplossingen te vinden voor een bepaalde klasse van optimale stoptijdproblemen. Aan de hand van de gevonden oplossingen geven we noodzakelijke en voldoende voorwaarden waaronder continue en gladde verbinding plaatsvindt. In dit proefschrift geven we voorbeelden van verschillende gevallen.

Voor eindige expiratiedatum bestuderen we de Amerikaanse put optie waarbij het prijsproces gedreven wordt door een Lévy proces van begrensde variatie. Het probleem wordt opgelost met behulp van een verandering van veranderlijke formule met lokale tijd op krommen voor Lévy processen van begrensde variatie. Gecombineerd met de Itô-Doob-Meyer decompositie van het waarde-proces van het optimale stoptijdprobleem in een martingaal en een potentiaal proces laten we zien dat de optimale stopgrens gekarakteriseerd kan worden als oplossing van een niet-lineaire integraalvergelijking. Gebruikmakend van de continue verbindingsvoorwaarde vinden we dat, onder enkele verdere voorwaarden, deze integraalvergelijking een unieke oplossing heeft. Deze uniciteit impliceert dat de waardefunctie van de Amerikaanse put optie en de bijbehorende optimale stopgrens de unieke oplossing is van een parabolisch vrije-randwaarde probleem van het integro-differentiaal type.

In het geval van oneindige expiratiedatum geven we een optimale formule voor de oplossing van optimale stoptijdproblemen voor een algemene klasse van uitbetalingsfuncties en Lévy processen. De oplossing wordt gevonden door het optimale stoptijdprobleem te reduceren tot een gemiddelde probleem. De oplossing van dit probleem leidt met behulp van de Wiener-Hopf factorisatie tot een fluctuatie-identiteit voor Lévy processen. Deze identiteit relateert de oplossing van het gemiddelde probleem aan de verwachtingswaarde van de gedisconteerde uitbetalingsfunctie tot een eerste passage tijd. Op basis van de identiteit laten we zien dat, mits de oplossing van het gemiddelde probleem een zekere monotoniciteitseigenschap heeft, een optimale oplossing van het optimale stoptijdprobleem gegeven kan worden in termen van een monotone functie en dat de grens gegeven wordt door een niveau waar deze functie van teken verandert. Aan de hand van deze oplossing kunnen we laten zien dat aan de gladde verbindingsvoorwaarde voldaan is dan en slechts dan als de rand van het stopgebied regulier is voor het inwendige van het stopgebied voor het Lévy proces. Een aantal problemen wordt in detail bestudeerd, in het bijzonder de polynomiale uitbetalingsfunctie en het vinden van de arbitrage-vrije prijzen van de Amerikaanse put en call opties.

Voor het probleem van endogeen faillissement laten we zien dat voor een specifieke klasse van modellen een optimaal faillissementsniveau gevonden kan worden wanneer de waarde van de activa van de firma gemodelleerd wordt door een Lévy proces zonder positieve sprongen. Aan de hand van dit process brengen we een nieuw fenomeen aan het licht, namelijk dat afhankelijk van het gedrag van de kleine sprongen, het optimale faillissementsniveau gekozen dient te worden door middel van de continue verbindingsvoorwaarde in plaats van de gladde verbindingsvoorwaarde, die men normaal gebruikt. Bovendien laten we zien dat de keuze van het faillissementsniveau inderdaad optimaal is overeenkomstig met de juiste keuze van verbindingsvoorwaarde.

Het merendeel van de resultaten in dit proefschrift wordt geverifieerd aan de hand van numerieke voorbeelden voor Lévy processen met eenzijdige sprongen.

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Utrecht, December 2006 Budhi Arta Surya

Curriculum Vitae

Budhi Arta Surya was born on 12 May 1974 in Padang, West Sumatra, Indonesia. He finished his high schools in Padang. He went to Bandung, West Java, to pursue his university education at the Institute of Technology Bandung in 1993. Four years later, he completed his bachelor degree with distinction (cum laude) majoring in Applied Mathematics with a final project entitled "Chapman-Kolmogorov equations and analysis of stock price fluctuations" written under supervision of Prof. dr. Sunardi Wirjosudirjo. In the following year, he was accepted a full scholarship granted by KNAW (The Royal Netherlands Academy of Arts and Sciences) to pursue his master degree in Mathematical Sciences at the University of Twente, the Netherlands, which he finished in June 2000 with a master thesis on "Estimation of stochastic volatility in the Hull-White framework using nonlinear filtering" written under supervision of Prof. dr. Arun Bagchi. In August 2000, he joined the Mathematical Research Institute (MRI) as a Master Class student in Mathematical Finance programme which he successfully completed in June 2001 with a test-problem entitled "Backward stochastic differential equations and their applications to hedging and portfolio optimization problems under transaction costs" which was written under supervision of Dr. ir. Michel Vellekoop. As a result of the Master Class Programme, he received in September 2001 a Dutch master degree (Ingenieur) in Toegepaste Wiskunde (Applied Mathematics) from the University of Twente. In February 2003, he started his PhD study under supervision of Dr. Andreas Kyprianou at the Department of Mathematics of the University of Utrecht, the Netherlands. During his PhD study he was actively involved in taking part of seminars, workshops in Lisbon, Manchester, Edinburgh, Eindhoven, Nijenrode, Lunteren, and Enschede, and giving a number of tutorial classes in Utrecht. His PhD study is resulted in this thesis.