Stochastic control for spectrally negative Lévy processes

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Abstract

Three optimal dividend models are considered for which the underlying risk process is a spectrally negative Lévy process. The first one concerns the classical dividends problem of de Finetti for which we give sufficient conditions under which the optimal strategy is of barrier type. As a consequence, we are able to extend considerably the class of processes for which the barrier strategy proves to be optimal.

The second one is a generalized version of the classical optimal dividends problem of de Finetti in which the objective function has an extra term which takes account of the ruin time of the risk process. We show that, with the exception of a small class, a barrier strategy forms an optimal strategy under the condition that the Lévy measure has a completely monotone density.

The third is an impulse control version of de Finetti's dividends problem. Here we show that when the Lévy measure has a log-convex density, then an optimal strategy is given by paying out a dividend in such a way that the reserves are reduced to a certain level c_1 whenever they are above another level c_2. Also a method to numerically find the optimal values of c_1 and c_2 is presented.

Finally, we investigate boundary crossing problems for refracted Lévy processes. The latter is a Lévy process whose dynamics change by subtracting off a fixed linear drift (of suitable size) whenever the aggregate process is above a pre-specified level. We consider in particular the case that X is spectrally negative and besides showing the existence of refracted Lévy processes, we establish a suite of identities for the case of one and two sided exit problems. We remark on a number of applications of the obtained identities to (controlled) insurance risk processes.

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Preface

As the title indicates, this thesis deals with stochastic control for spectrally negative Lévy processes. In particular, we treat a number of optimal stochastic control problems. In general a stochastic optimal control problem can be described, in a very brief and non-detailistic way, as follows. Given a 'nice' Markov process X, one is allowed to choose a control π belonging to a certain set of admissible controls. This control π changes the dynamics of the process X and we denote this controlled process by U^π. Associated to each control π is a value function v_π which is a function with respect to the initial state of X and takes the form of an expectation of a random variable which can depend on both the control π as well as on the state of the process U^π. One should think of this value function as a reward or a cost corresponding to the control. The optimal control problem then consists of finding v_* which is defined as the supremum (or infimum) of the value function amongst all possible controls, and to find an optimal control π_* (if it exists) such that this supremum (or infimum) is attained.

An important object in solving a stochastic control problem is the (infinitesimal) generator of the (time-homogeneous) Markov process X. In the case that X is a (possibly multi-dimensional) diffusion process (a Markov process which has continuous sample paths), this generator takes the form of a second order partial differential operator of elliptic type. Using Itô's formula one can then transform the stochastic problem into an analytical problem concerning these kind of operators and then use the well-established theory of elliptic partial differential equations to get results on the optimal control problem. As an example of using analytical tools to solve stochastic control problems, we mention the introductory book on stochastic control for Markov process (and in particular diffusion processes) of Fleming and Soner [19].

When the process X is a Markov process with sample paths which exhibit jumps, an integral term appears into the generator of X, making the generator a nonlocal operator for which there is not as rich a theory available as for elliptic differential operators. For these kind of processes it is therefore a lot harder to solve stochastic control problems by using analytical methods regarding the generator. We mention hereby the book of Øksendal and Sulem [48] in which control problems are considered in the case when X is a jump diffusion.

In this thesis we study a particular example of an optimal stochastic control problem which appears in the context of insurance mathematics. In this classical control problem, which was introduced by de Finetti [14], X represents the capital reserves over time of an insurance company. The insurance company is allowed to pay out part of their reserves to their beneficiaries; these payments are called dividends. The dividends payments form the control in this problem and are mathematically described by a nondecreasing, nonnegative, adapted process. The controlled process U^π is then given by the process X minus the dividend process. The company is allowed to pay out dividends up until the time the company becomes ruined, which is the first time U^π becomes strictly negative. The value function associated to a dividend strategy is defined by the expected value of the total amount of dividends paid out until ruin (discounted over time). Naturally the insurance company wants to maximize this expected value and hence wants to know what the dividend strategy is which achieves this.

A number of articles (e.g. [3, 34, 52, 57]) consider this optimal control problem in the case when X takes the form of a Brownian motion plus drift. The generator of a Brownian motion plus drift is a second order ordinary differential operator with constant coefficients and exploiting the simple form of the generator, the optimal stochastic control problem can be solved explicitly. The optimal strategy is proved in this case to be the strategy which reflects the process X at a certain barrier level; this strategy is known as the barrier strategy.

Traditionally, the reserves of the insurance company are modeled by a compound Poisson process with negative jumps plus a drift. This representation, known as the Cramér-Lundberg model, is a more natural representation than the Brownian motion one, since the drift can be seen as the premium the company collects over time and the jumps of the compound Poisson process can be seen as claims made by the insured. In the Cramér-Lundberg setting, the generator is an integro-differential operator which makes the control problem a lot more difficult than in the case of linear Brownian motion. This is for instance illustrated by the article of Azcue and Muler [6] in which heavy analytical machinery concerning this integro-differential operator is used in order to tackle the optimal dividend problem. In the literature no examples of Cramér-Lundberg processes have been given for which the optimal strategy can be described explicitly; the only exception being the case when the jumps of the compound Poisson process have an exponentially distribution. In the latter case Gerber [22] proved that, as in the Brownian case, a barrier strategy is optimal.

In this thesis, the problem of de Finetti is considered, but now X takes the form of a spectrally negative Lévy process, which is a generalization of both the Cramér-Lundberg process and the Brownian motion with drift. In this case one faces the same difficulty of the generator being a nonlocal operator as in the Cramér-Lundberg model, but an additionally complexity arises since due to the possibility of a Lévy process having an infinite amount of jumps in a (small) time interval, the technique of 'conditioning on arrival of the first jump', often applied in the case of Cramér-Lundberg processes, is no longer feasible. Despite these difficulties though, one can as it turns out, still get quality results on the problem by using fluctuation theory and the theory of scale functions for spectrally negative Lévy processes.

In particular, we show in this thesis, by using the results from Avram, Palmowski and Pistorius [5] who were the first to consider this problem for a general spectrally negative Lévy processes, that whether the barrier strategy is optimal or not depends on the shape of the scale function. Then combining the relation between scale functions of spectrally negative Lévy processes and renewal (or potential) functions of subordinators (due to the Wiener-Hopf factorization) with known results on analytical properties of the latter class of functions and on complete Bernstein functions, we show that the barrier strategy is optimal for the control problem if the Lévy measure has a density which is completely monotone. This vastly extends the number of explicit examples of processes for which the barrier strategy is optimal in the dividends problem of de Finetti.

The thesis itself consists of four chapters which are self-contained. This results in there being some overlap between the different chapters. The first chapter has been accepted for publication in the Annals of Applied Probability as [47], whereas the other chapters have all been submitted.

In Chapter 1 we deal with the classical de Finetti problem mentioned above. The results derived and the methods used in this chapter form the backbone of the later Chapters 2 and 3.

In Chapter 2 we consider a generalization of the de Finetti problem in which the value function contains an additional term. This optimal dividends problem has been studied and solved by Boguslavskaya [11] in case X is a Brownian motion plus drift and by Shreve, Lehoczky and Gaver [57] in case X is a diffusion. Additionally, Thonhauser and Albrecher [65] solved the problem when X is a Cramér-Lundberg risk process with exponentially distributed claims. We generalize the theorems obtained in Chapter 1 and show in particular that, with the exception of a small class for which we show that the so-called take-the-money-and-run strategy is optimal, an optimal strategy is formed by a barrier strategy in case the Lévy measure has a completely monotone density.

Chapter 3 deals with the impulse control version of the de Finetti problem and was first studied by Jeanblanc and Shiryaev [34] for X being a Brownian motion plus drift. The difference with the de Finetti problem is that only pure jump dividend processes are now allowed and that at each time a dividend is paid out, a transaction cost is incurred. We show that the strategy we call the (c_1, c_2) policy is optimal when the Lévy measure has a density which is log-convex. This is done by again employing heavily the results in Chapter 1 as well as utilizing the additional results of the de Finetti problem obtained by Kyprianou, Rivero and Song [42] who by going deeper into the theory of potential functions of subordinators and Bernstein functions, showed that the barrier strategy is optimal when the Lévy measure has a log-convex density.

The fourth chapter is joint work with Andreas Kyprianou and differs from the first three. In this chapter we will deal with processes we call refracted Lévy processes, which are (spectrally negative) Lévy processes where one subtracts off a linear drift whenever the process is above a certain threshold. The motivation comes again from risk theory; when the Lévy measure is a finite measure, the refracted Lévy process can be seen as a Cramér-Lundberg risk process with a two-step premium rate or alternatively as a Cramér-Lundberg risk process where dividends are paid out at a certain rate each time the process is above the threshold, the so-called threshold (dividend) strategy. Using fluctuation theory, we show the existence of refracted Lévy processes with respect to a general spectrally negative Lévy process (a matter which turns out to be nontrivial) and further establish a number of identities concerning one and two sided exit problems. Associated to refracted Lévy processes is another offshoot of the de Finetti optimal control problem, whereby now the control/dividend process has to be absolutely continuous with respect to the Lebesgue measure with a density bounded by a certain constant. This control problem has been studied and solved by Jeanblanc and Shiryaev [34] and by Asmussen and Taksar [3] in case X is a Brownian motion plus drift and by Gerber and Shiu [26] for the case that X is a Cramér-Lundberg risk process with exponentially distributed jumps; in both cases the optimal strategy being of threshold type. We remark that we do not treat this control problem here and that it is still an open question whether the analogue results of the first three chapters hold in this case.

This thesis would never have been written without my supervisor Andreas Kyprianou and I would like to thank him for his excellent guidance and help during all three years of my study. Further I would like to thank all the people from the Department of Mathematical Sciences at the University of Bath and from the Department of Actuarial Mathematics and Statistics at Heriot-Watt University in Edinburgh, the latter being the place at which the first year of the research has been carried out. I also thank EPSRC and the University of Bath for financial support.

Chapter 1

On optimality of the barrier strategy in de Finetti's dividend problem for spectrally negative Lévy processes

We consider the classical optimal dividend control problem which was proposed by de Finetti [14]. Recently Avram et al. [5] studied the case when the risk process is modeled by a general spectrally negative Lévy process. We draw upon their results and give sufficient conditions under which the optimal strategy is of barrier type, thereby helping to explain the fact that this particular strategy is not optimal in general. As a consequence, we are able to extend considerably the class of processes for which the barrier strategy proves to be optimal.

1.1 Introduction

De Finetti [14] introduced the dividend model in risk theory. In this model the insurance company has the option to pay out dividends of its surplus to its beneficiaries up to the moment of ruin. De Finetti [14] argued that this should be done in an optimal way, namely such that the expected sum of the discounted paid out dividends from time zero until ruin is maximized. He proved that if the risk/surplus process evolves as a random walk with step sizes ± 1, then an optimal way of paying out dividends is according to a barrier strategy, i.e. there exists a constant $a^* \geq 0$, such that at each time epoch the excess of the net risk process over the level a^* is paid out. In the case of continuous-time models, the problem of finding the optimal dividend strategy has been studied extensively in the Brownian motion setting [3, 34, 52, 64] and in the Cramér-Lundberg setting [6, 12, 22, 56], where by the former is meant that the risk process $X = \{X_t : t \geq 0\}$ is modeled by a Brownian motion plus drift and by the latter that

$$X_t - X_0 = ct - \sum_{i=1}^{N_t} C_i,$$

where $C_1, C_2, \ldots$ are i.i.d. positive random variables representing the claims, $c > 0$ represents the premium rate and $N = \{N_t : t \geq 0\}$ is an independent Poisson process with arrival rate λ. Note that traditionally in the Cramér-Lundberg model it is assumed that X drifts to infinity, but this condition is not necessary to formulate the problem. Very recently, Avram et al. [5] considered the case where the risk process is given by a general spectrally negative Lévy process. Explanations for why this particular process serves as an appropriate generalization of the classical compound Poisson risk process can be found in e.g. [21, 29, 39]. It has been proved that in the Brownian motion setting and in the Cramér-Lundberg setting with exponentially distributed claims, an optimal dividend strategy is formed by a barrier strategy. No other explicit examples of spectrally negative Lévy processes have been given for which the same can be said. On the other hand Azcue and Muler [6, Section 10.1] have found an example for which the optimal strategy is not a barrier strategy. Further, Avram et al. [5] have given a sufficient condition involving the generator of the Lévy process for optimality of the barrier strategy. Besides finding the optimal strategy, a large body of literature exists [15, 20, 23, 24, 31, 39, 44, 46, 54, 70, 73] in which expressions are derived for e.g. the expected time of ruin, the moments of the expected paid out dividends and the Gerber-Shiu discounted penalty function, under the assumption that the insurance company pays out dividends according to a barrier strategy; the main motivation being the fact that the barrier strategy is optimal in (at least) the aforementioned two examples.

In this chapter motivated by the long history and broad interest of this control problem, we will shed new light on optimality of the barrier strategy when the risk process is modeled by a spectrally negative Lévy process. Using the setup and results from Avram et al. [5], we show that the shape of the so-called scale functions of spectrally negative Lévy processes plays a central role. Further we will prove optimality of the barrier strategy if an easily checked analytical condition is imposed on the jump measure of the underlying Lévy process. This enables us to extend considerably the class of processes for which this strategy is optimal.

The outline of this chapter is as follows. In Sections 2 and 3 we state the problem and briefly introduce scale functions. We present our main results in Section 4 and prove them in Section 5 using some earlier results from Avram et al. [5]. We then conclude by giving some explicit examples to illustrate our results.

1.2 Problem setting

Let $X = \{X_t : t \geq 0\}$ be a spectrally negative Lévy process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$ satisfying the usual conditions. We denote by $\{\mathbb{P}_x, x \in \mathbb{R}\}$ the family of probability measures corresponding to a translation of X such that $X_0 = x$, where we write $\mathbb{P} = \mathbb{P}_0$. Further $\mathbb{E}_x$ denotes the expectation with respect to $\mathbb{P}_x$ with $\mathbb{E}$ being used in the obvious way. Let the Lévy triplet of X be given by (γ, σ, ν), where $\gamma \in \mathbb{R}$, $\sigma \geq 0$ and ν is a measure on $(0, \infty)$ satisfying

$$\int_{(0,\infty)} \left(1 \wedge x^2\right) \nu(\mathrm{d}x) < \infty.$$

Note that even though X only has negative jumps, for convenience we choose the Lévy measure to have only mass on the positive instead of the negative half line. The Laplace exponent of X is given by

$$\psi(\theta) = \log\left(\mathbb{E}\left(\mathrm{e}^{\theta X_1}\right)\right) = \gamma\theta + \frac{1}{2}\sigma^2\theta^2 - \int_{(0,\infty)} \left(1 - \mathrm{e}^{-\theta x} - \theta x \mathbf{1}_{\{0<x<1\}}\right) \nu(\mathrm{d}x)$$

and is well defined for $\theta \geq 0$. Note that in the Cramér-Lundberg setting $\sigma = 0$, $\nu(\mathrm{d}x) = \lambda F(\mathrm{d}x)$ where F is the law of C_1 and $\gamma = c - \int_{(0,1)} x\nu(\mathrm{d}x)$. We exclude the case that X has monotone paths. The process X will represent the risk/surplus process of an insurance company before dividends are deducted.

We denote a dividend or control strategy by π, where $\pi = \{L_t^\pi : t \geq 0\}$ is a nondecreasing, left-continuous $\mathbb{F}$-adapted process which starts at zero. The random variable L_t^π will represent the cumulative dividends the company has paid out until time t under the control π. We define the controlled (net) risk process $U^\pi = \{U_t^\pi : t \geq 0\}$ by $U_t^\pi = X_t - L_t^\pi$. Let $\sigma^\pi = \inf\{t > 0 : U_t^\pi < 0\}$ be the ruin time and define the value function of a dividend strategy π by

$$v_\pi(x) = \mathbb{E}_x\left[\int_0^{\sigma^\pi} \mathrm{e}^{-qt}\mathrm{d}L_t^\pi\right],$$

where $q > 0$ is the discount rate. By definition it follows that $v_\pi(x) = 0$ for $x < 0$. A strategy π is called admissible if ruin does not occur by a dividend payout, i.e. $L_{t+}^\pi - L_t^\pi \leq U_t^\pi \vee 0$ for $t \leq \sigma^\pi$. Let Π be the set of all admissible dividend policies. The control problem consists of finding the optimal value function v_* given by

$$v_*(x) = \sup_{\pi\in\Pi} v_\pi(x)$$

and an optimal strategy $\pi_* \in \Pi$ such that

$$v_{\pi_*}(x) = v_*(x) \quad \text{for all } x \geq 0.$$

We denote by $\pi_a = \{L_t^a : t \geq 0\}$ the barrier strategy at level a which is defined by $L_0^a = 0$ and

$$L_t^a = \left(\sup_{0\leq s<t} X_s - a\right) \vee 0 \quad \text{for } t > 0.$$

Note that $\pi_a \in \Pi$. Let v_a denote the value function when using the dividend strategy π_a. In this chapter we find sufficient conditions such that $v_*(x) = v_a(x)$ for all $x \geq 0$ for a certain specified a.

1.3 Scale functions

For each $q \geq 0$ there exists a function $W^{(q)} : \mathbb{R} \to [0, \infty)$, called the ($q$-)scale function of X, which satisfies $W^{(q)}(x) = 0$ for $x < 0$ and is characterized on $[0, \infty)$ as a strictly increasing and continuous function whose Laplace transform is given by

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x) \mathrm{d}x = \frac{1}{\psi(\theta) - q} \quad \text{for } \theta > \Phi(q),$$

where $\Phi(q) = \sup\{\theta \geq 0 : \psi(\theta) = q\}$ is the right-inverse of ψ. We write $W = W^{(0)}$. We will later on use the following relation:

$$W^{(q)}(x) = \mathrm{e}^{\Phi(q)x} W_{\Phi(q)}(x). \tag{1.1}$$

Here $W_{\Phi(q)}$ is the (0-)scale function of X under the measure $\mathbb{P}^{\Phi(q)}$, where this measure is defined by the change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}^{\Phi(q)}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \mathrm{e}^{\Phi(q)X_t - qt}.$$

The process X under the measure $\mathbb{P}^{\Phi(q)}$ is still a spectrally negative Lévy process, but with a different Lévy triplet. In particular its Lévy measure is now given by $\mathrm{e}^{-\Phi(q)x}\nu(\mathrm{d}x)$. We refer to [38, Chapter 8] for more information on scale functions.

Throughout this chapter we will use the term sufficiently smooth, whereby we mean the following. A function $f : \mathbb{R} \to \mathbb{R}$ which vanishes on $(-\infty, 0)$ is called sufficiently smooth at a point $x > 0$ if f is continuously differentiable at x when X is of bounded variation and is twice continuously differentiable at x when X is of unbounded variation. A function is then called sufficiently smooth if it is sufficiently smooth at all $x > 0$; see [13] for conditions under which the scale function $W^{(q)}$ is sufficiently smooth. The derivative of $x \mapsto W^{(q)}(x)$ is denoted by $W^{(q)\prime}$.

Avram et al. [5] showed that the value of the barrier strategy can be expressed in terms of scale functions in the following way.

Proposition 1. *Assume $W^{(q)}$ is continuously differentiable on $(0, \infty)$. The value function of the barrier strategy at level $a \geq 0$ is given by*

$$v_a(x) = \begin{cases} \frac{W^{(q)}(x)}{W^{(q)\prime}(a)} & \text{if } x \leq a, \\ x - a + \frac{W^{(q)}(a)}{W^{(q)\prime}(a)} & \text{if } x > a. \end{cases}$$

The proof of Proposition 1 given in [5] is based on excursion theory. An alternative proof where only basic fluctuation identities are used in conjunction with the strong Markov property, is given in [54, 74]. Define now the (candidate) optimal barrier level by

$$a^* = \sup\left\{a \geq 0 : W^{(q)\prime}(a) \leq W^{(q)\prime}(x) \text{ for all } x \geq 0\right\},$$

where $W^{(q)\prime}(0)$ is understood to be equal to $\lim_{x\downarrow 0} W^{(q)\prime}(x)$. It follows that $a^* < \infty$

since $\lim_{x\to\infty} W^{(q)\prime}(x) = \infty$. Note that our definition of the optimal barrier level is slightly different than the one given by Avram et al. [5]. It is easily seen that if an optimal strategy is formed by a barrier strategy, then the barrier strategy at a^* has to be an optimal strategy.

1.4 Main results

We will now present the main results of this chapter which give sufficient conditions for optimality of the barrier strategy π_{a^*}.

Theorem 2. *Suppose $W^{(q)}$ is sufficiently smooth and*

$$W^{(q)\prime}(a) \leq W^{(q)\prime}(b) \quad \textit{for all } a^* \leq a \leq b. \tag{1.2}$$

Then the barrier strategy at a^* is an optimal strategy.

A drawback of condition (1.2) is that it involves the scale function for which closed form expressions are only known in a few cases. It would be better to have a condition which is directly given in terms of the Lévy triplet (γ, σ, ν) and the discount rate q. The second theorem entails exactly such a condition.

Theorem 3. *Suppose that the Lévy measure ν of X has a completely monotone density, i.e. $\nu(\mathrm{d}x) = \mu(x)\mathrm{d}x$, where $\mu : (0,\infty) \to [0,\infty)$ has derivatives $\mu^{(n)}$ of all orders which satisfy*

$$(-1)^n \mu^{(n)}(x) \geq 0 \quad \textit{for } n = 0, 1, 2, \ldots .$$

Then $W^{(q)\prime}$ is strictly convex on $(0,\infty)$ for all $q > 0$. Consequently, (1.2) holds and the barrier strategy at a^* is an optimal strategy for the control problem.

1.5 Proof of main results

Before proving the main results, we give two lemmas. Both lemmas are lifted from Avram et al. [5]. We therefore do not give a proof of the first lemma which is a verification lemma involving a Hamilton-Jacobi-Bellman inequality. However, we do include a short proof of the second one as various arguments will be instructive to refer back to in the proof of Theorem 2.

Let Γ be the operator acting on sufficiently smooth functions f, defined by

$$\Gamma f(x) = \gamma f'(x) + \frac{\sigma^2}{2} f''(x) + \int_{(0,\infty)} [f(x-y) - f(x) + f'(x) y \mathbf{1}_{\{0<y<1\}}] \nu(\mathrm{d}y).$$

Lemma 4 (Verification lemma)**.** *Suppose π is an admissible dividend strategy such that v_π is sufficiently smooth and for all $x > 0$*

$$\max\{\Gamma v_\pi(x) - q v_\pi(x), 1 - v_\pi'(x)\} \leq 0. \quad \textit{(HJB-inequality)}$$

Then $v_\pi(x) = v_*(x)$ for all $x \in \mathbb{R}$.

Lemma 5. *Suppose $W^{(q)}$ is sufficiently smooth and suppose that*

$$(\Gamma - q)v_{a^*}(x) \le 0 \quad \textit{for } x > a^*. \tag{1.3}$$

Then $v_{a^*}(x) = v_*(x)$ for all $x \in \mathbb{R}$.

Proof of Lemma 5. It suffices to show that under the conditions of Lemma 5, v_{a^*} satisfies the conditions of the verification lemma. When $a^* = 0$ this is trivial because of (1.3), so we assume without loss of generality that $a^* > 0$. Because $W^{(q)}$ is sufficiently smooth and by Proposition 1, it follows that for any $a \ge 0$, $v_a(x)$ is sufficiently smooth at all $x \in (0,\infty)\backslash\{a\}$. By definition of a^* and the assumed smoothness, we have $W^{(q)\prime\prime}(a^*) = 0$ when X is of unbounded variation and hence $v_{a^*}(x)$ is also sufficiently smooth at $x = a^*$. Further $v'_{a^*}(x) \ge 1$ by definition of a^*. Since $\left(\mathrm{e}^{-q(t\wedge\tau_0^-\wedge\tau_a^+)}W^{(q)}(X_{t\wedge\tau_0^-\wedge\tau_a^+})\right)_{t\ge 0}$ is a $\mathbb{P}_x$-martingale, one can deduce that

$$(\Gamma - q)v_a(x) = 0 \quad \text{for } 0 < x < a \text{ and } a > 0. \tag{1.4}$$

(Note that for $a \neq a^*$, $v_a(x)$ is not necessarily twice continuously differentiable in $x = a$ even if $W^{(q)\prime\prime}$ is continuous in a. Therefore $(\Gamma - q)v_a(x)$ is not necessarily continuous in a and so (1.4) does not hold for $x = a$ in general.) In particular (1.4) holds for $a = a^*$. Hence together with (1.3), v_{a^*} satisfies the HJB-inequality. ■

Proof of Theorem 2. First, we claim that

$$\lim_{y\uparrow x}(\Gamma - q)(v_{a^*} - v_x)(y) \le 0 \quad \text{for } x > a^*. \tag{1.5}$$

We prove the claim for X being of unbounded variation (the case of bounded variation is slightly easier). Let $x > a^*$. By assumption on the smoothness of the scale function, v_x and v_{a^*} are twice continuously differentiable on $(0,\infty)$, except for the possibility that $\lim_{y\uparrow x} v''_x(y) \neq \lim_{y\downarrow x} v''_x(y)$. We can use the dominated convergence theorem to deduce

$$\begin{aligned}
&\lim_{y\uparrow x}(\Gamma - q)(v_{a^*} - v_x)(y) \\
&\qquad = \gamma(v'_{a^*} - v'_x)(x) + \frac{\sigma^2}{2}(v''_{a^*}(x) - \lim_{y\uparrow x} v''_x(y)) - q(v_{a^*} - v_x)(x) + \\
&\qquad\int_{(0,\infty)} \left\{[(v_{a^*} - v_x)(x-z) - (v_{a^*} - v_x)(x)] + (v'_{a^*} - v'_x)(x)z\mathbf{1}_{\{0<z<1\}}\right\}\nu(\mathrm{d}z).
\end{aligned}$$

Since we have by using Proposition 1

(i) $\lim_{y\uparrow x} v''_x(y) \ge 0 = v''_{a^*}(x)$ where the inequality is by (1.2),

(ii) $(v'_{a^*} - v'_x)(u) \ge 0$ for $u \in [0,x]$, since for $u \in [0,a^*]$ $(v'_{a^*} - v'_x)(u) \ge 0$ by definition of a^* and for $u \in (a^*,x]$ $(v'_{a^*} - v'_x)(u) \ge 0$ by (1.2); this implies that $(v_{a^*} - v_x)(x-z) \le (v_{a^*} - v_x)(x)$ for all $z \ge 0$,

(iii) $(v_{a^*} - v_x)(x) \ge 0$ which follows from $v_{a^*}(a^*) \ge v_x(a^*)$ and (ii),

(iv) $v'_{a^*}(x) = v'_x(x) = 1$,

the claim follows.

We now prove by contradiction that (1.3) holds; the theorem is then proved by applying Lemma 5. Suppose there exist $x > a^* \geq 0$ such that $(\Gamma - q)v_{a^*}(x) > 0$. Then by (1.5) and the continuity of $(\Gamma - q)v_{a^*}$ we have $\lim_{y\uparrow x}(\Gamma - q)v_x(y) > 0$ which contradicts (1.4). ■

Proof of Theorem 3. Since $\nu_{\Phi(q)}(\mathrm{d}x) = \mathrm{e}^{-\Phi(q)x}\mu(x)\mathrm{d}x$ is the Lévy measure of the process X under the measure $\mathbb{P}^{\Phi(q)}$, we have that $\nu_{\Phi(q)}(\mathrm{d}x)$ has a completely monotone density, since the product of two completely monotone functions is completely monotone. It follows that $x \longmapsto \nu_{\Phi(q)}(x,\infty)$ is completely monotone, since $\frac{\mathrm{d}}{\mathrm{d}x}\nu_{\Phi(q)}(x,\infty) = -\mathrm{e}^{-\Phi(q)x}\mu(x)$.

Let $\{\widehat{H}_t : t \geq 0\}$ be the descending ladder height process of X. As $q > 0$, the process X under $\mathbb{P}^{\Phi(q)}$ drifts to infinity and it follows that the process $\widehat{H}$ under $\mathbb{P}^{\Phi(q)}$ (under a suitably chosen constant appearing in the local time at the minimum) is a killed subordinator with Lévy measure given by $\nu_{\Phi(q)}(x,\infty)\mathrm{d}x$ (see e.g. [38, Exercise 6.5]). Hence the Lévy measure of $\widehat{H}$ under $\mathbb{P}^{\Phi(q)}$ has a completely monotone density and consequently the Laplace exponent of $\widehat{H}$ under $\mathbb{P}^{\Phi(q)}$ is a complete Bernstein function (see [32, Theorem 3.9.29]). We may now use a result from Rao et al. [53, Theorem 2.3] combined with [59, Remark 2.2] to conclude that the renewal function of $\widehat{H}$ under $\mathbb{P}^{\Phi(q)}$ defined by $\widehat{U}_{\Phi(q)}(x) = \mathbb{E}^{\Phi(q)}\left(\int_0^\infty \mathbf{1}_{\{\widehat{H}_t\in[0,x]\}}\mathrm{d}t\right)$ has a completely monotone derivative.

It is well known that the scale function of a spectrally negative Lévy process which does not drift to minus infinity is equal (up to a multiplicative constant appearing in the local time) to the renewal function of the descending ladder height process (see e.g. [10, Chapter VII.2]). So we can say that $W_{\Phi(q)}(x) = \widehat{U}_{\Phi(q)}(x)$ and therefore $W'_{\Phi(q)}$ is completely monotone. A nonnegative function on $(0,\infty)$ with a completely monotone derivative is also known as a Bernstein function.

Because $W_{\Phi(q)}|_{(0,\infty)}$ is a Bernstein function, it admits the following representation, which is closely related to Bernstein's theorem, (see e.g. [32, Chapter 3.9]):

$$W_{\Phi(q)}(x) = a + bx + \int_{(0,\infty)} (1 - \mathrm{e}^{-xt})\xi(\mathrm{d}t) \quad x > 0, \tag{1.6}$$

where $a, b \geq 0$ and ξ is a measure on $(0,\infty)$ satisfying $\int_{(0,\infty)}(t \wedge 1)\xi(\mathrm{d}t) < \infty$; in other words $W_{\Phi(q)}$ is the Laplace exponent of some (possibly killed) subordinator. From (1.6) and (1.1) it follows that

$$W^{(q)}(x) = \mathrm{e}^{\Phi(q)x}(a + bx) + \int_{(0,\infty)} (\mathrm{e}^{\Phi(q)x} - \mathrm{e}^{-x(t-\Phi(q))})\xi(\mathrm{d}t).$$

By repeatedly using the dominated convergence theorem, we can now deduce

$$\begin{aligned}W^{(q)\prime\prime\prime}(x) =& f'''(x) + \int_{(0,\infty)} \left(\Phi(q)^3 \mathrm{e}^{\Phi(q)x} + (t-\Phi(q))^3 \mathrm{e}^{-x(t-\Phi(q))}\right) \xi(\mathrm{d}t) \\ =& f'''(x) + \int_{(0,\Phi(q)]} \left(\Phi(q)^3 \mathrm{e}^{\Phi(q)x} - (\Phi(q)-t)^3 \mathrm{e}^{(\Phi(q)-t)x}\right) \xi(\mathrm{d}t) \\ &+ \int_{(\Phi(q),\infty)} \left(\Phi(q)^3 \mathrm{e}^{\Phi(q)x} + (t-\Phi(q))^3 \mathrm{e}^{-x(t-\Phi(q))}\right) \xi(\mathrm{d}t),\end{aligned}$$

where $f(x) = \mathrm{e}^{\Phi(q)x}(a+bx)$. Hence $W^{(q)\prime\prime\prime}(x) > 0$ for all $x > 0$ and so $W^{(q)\prime}$ is strictly convex on $(0,\infty)$. Since $W^{(q)}$ is infinitely differentiable, we can now apply Theorem 2 to deduce that the barrier strategy at a^* is optimal. ■

1.6 Examples

Example from Theorem 2 We now give an example to illustrate Theorem 2. Let X be given by the Cramér-Lundberg model perturbed by Brownian motion, i.e.

$$X_t = x + ct - \sum_{i=1}^{N_t} C_i + \sigma B_t,$$

where we let $C_1 \sim \text{Erlang}(2,\alpha)$ (i.e. sum of two independent exponentially random variables with parameter α). Note that the Lévy measure $\nu(\mathrm{d}x) = \lambda\alpha^2 x \mathrm{e}^{-\alpha x}\mathrm{d}x$ (where λ is the arrival rate of the Poisson process $\{N_t : t \geq 0\}$) does not have a completely monotone density. For this example a closed form expression for the q-scale function in terms of the roots of $\psi(u) = q$ can easily be found by inverting its Laplace transform by the method of partial fraction expansion. Indeed, we can write (for $q > 0$ and $\sigma > 0$)

$$\begin{aligned}\frac{1}{\psi(u)-q} =& \frac{1}{cu - \lambda + \frac{\lambda\alpha^2}{(\alpha+u)^2} + \frac{1}{2}\sigma^2 u^2 - q} \times \frac{(\alpha+u)^2}{(\alpha+u)^2} \\ =& \frac{(\alpha+u)^2}{\frac{1}{2}\sigma^2 \prod_{j=1}^4 (u-\theta_j)} = \sum_{j=1}^4 \frac{D_j}{u-\theta_j},\end{aligned}$$

where $(\theta_j)_{j=1}^4$ are the (possibly complex) zeros (which are assumed to be distinct) of the polynomial $(\psi(u)-q)(\alpha+u)^2$ and $(D_j)_{j=1}^4$ are given by

$$D_j = \left.\frac{1}{\psi(u)-q}(u-\theta_j)\right|_{u=\theta_j} = \frac{(\alpha+\theta_j)^2}{\frac{1}{2}\sigma^2 \prod_{k=1,k\neq j}^4 (\theta_j - \theta_k)}.$$

The scale function is then given by

$$W^{(q)}(x) = \sum_{j=1}^4 D_j \mathrm{e}^{\theta_j x} \quad \text{for } x \geq 0.$$

We now choose the values of the parameters as follows: $c = 21.4$, $\lambda = 10$, $\alpha = 1$, $q = 0.1$ and for σ we consider two cases, the case when $\sigma = 1.4$ and $\sigma = 2$. (For these choices of the parameter values, the zeros $(\theta_j)_{j=1}^4$ are indeed distinct.) Note that when $\sigma = 0$, this is exactly the example given by Azcue and Muler [6] for which the optimal strategy is not of barrier type. In the two figures the graphs of $W^{(q)\prime}$ and $(\Gamma - q)v_{a^*}(x)$ for the chosen parameters are plotted with the help of Matlab. When $\sigma = 1.4$, $a^* \approx 0.4$ and we see from Figure 1-1 that (1.2) and also (1.3) do not hold. When $\sigma = 2$ the minimum of the derivative has shifted; now $a^* \approx 10.5$ and we see from Figure 1-2 that (1.2) does hold. Consequently by (the proof of) theorem 2, (1.3) must hold, which is confirmed by the figure.

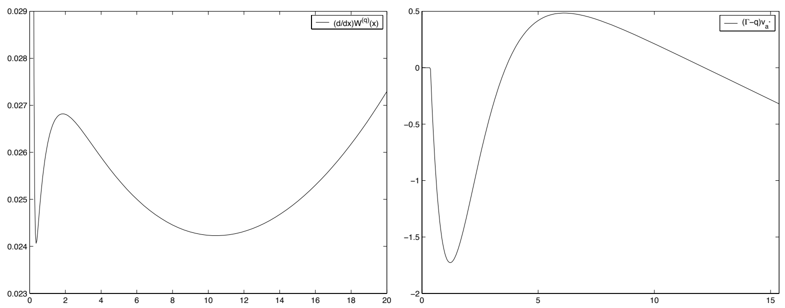

Figure 1-1: $\sigma = 1.4$; left: $x \mapsto W^{(q)\prime}(x)$, right: $x \mapsto (\Gamma - q)v_{a^*}(x)$.

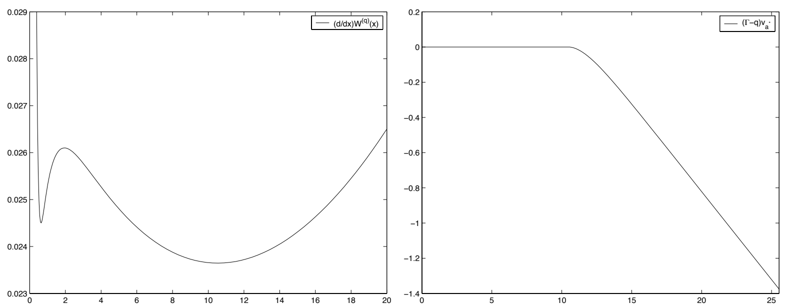

Figure 1-2: $\sigma = 2$; left: $x \mapsto W^{(q)\prime}(x)$, right: $x \mapsto (\Gamma - q)v_{a^*}(x)$.

Examples from Theorem 3 By Theorem 3, we have that when the Lévy measure is completely monotone, then the barrier strategy at a^* is always an optimal strategy. There are many examples of spectrally negative Lévy processes which have such a feature and which have been used in the literature to model the risk process. We name as examples the α-stable process which has Lévy density

$$\mu(x) = \lambda x^{-1-\alpha} \quad \text{with } \lambda > 0 \text{ and } \alpha \in (0, 2)$$

and is used in [21] and the (one-sided) tempered stable process which has Lévy density given by

$$\mu(x) = \lambda x^{-1-\alpha} \mathrm{e}^{-\beta x} \quad \text{with } \lambda, \beta > 0 \text{ and } -1 \leq \alpha < 2.$$

The latter process includes other familiar Lévy processes, like the gamma process ($\alpha = 0$) which is considered in [17] and the inverse Gaussian process ($\alpha = 1/2$) which is used in [16] to model the risk process.

We can also conclude that the barrier strategy at a^* is optimal, when we are in the Cramér-Lundberg setting where the claims have a distribution with a completely monotone probability density function. Some examples of these types of claim distributions which have been used in risk theory (see [2, Chapter I.2]) are the heavy-tailed Weibull distribution

$$\mu(x) = crx^{r-1}\mathrm{e}^{-cx^r} \quad \text{with } c > 0 \text{ and } 0 < r < 1,$$

the Pareto distribution

$$\mu(x) = \alpha(1+x)^{-\alpha-1} \quad \text{with } \alpha > 0$$

and the hyperexponential distribution

$$\mu(x) = \sum_{j=1}^{n} A_j \beta_j \mathrm{e}^{-\beta_j x} \quad \text{with } \beta_j, A_j > 0,\ j = 1, \dots, n \text{ and } \sum_{j=1}^{n} A_j = 1.$$

Note that since in Theorem 3 there is no condition on the value of the Gaussian component σ, a barrier strategy will still form an optimal strategy if any one of the above examples is perturbed by Brownian motion.

For most spectrally negative Lévy processes an explicit expression for the q-scale function (and hence a^*) cannot be obtained. However, very recently Hubalek and Kyprianou [28] have found some new examples (including where the Lévy measure has a completely monotone density) for which the q-scale function is completely explicit.

Chapter 2

An optimal dividends problem with a terminal value for spectrally negative Lévy processes with a completely monotone jump density

We consider a modified version of the classical optimal dividends problem of de Finetti in which the objective function is altered by adding in an extra term which takes account of the ruin time of the risk process, the latter being modeled by a spectrally negative Lévy process. We show that, with the exception of a small class, a barrier strategy forms an optimal strategy under the condition that the Lévy measure has a completely monotone density. As a prerequisite for the proof we show that under the aforementioned condition on the Lévy measure, the q-scale function of the spectrally negative Lévy process has a derivative which is strictly log-convex.

2.1 Introduction

In this chapter we consider the classical de Finetti's optimal dividends problem but with an extra component regarding the ruin time added to the objective function. Within this problem we assume that the underlying dynamics of the risk process is described by a spectrally negative Lévy process which is now widely accepted and used as a replacement for the classical Cramér-Lundberg process (cf. [1,5,16,17,21,29,39,42,54]). Recall that a Cramér-Lundberg risk process $\{X_t : t \geq 0\}$ corresponds to

$$X_t = x + ct - \sum_{i=1}^{N_t} C_i,$$

where $x > 0$ denotes the initial surplus, the claims $C_1, C_2, \dots$ are i.i.d. positive random variables with expected value μ, $c > 0$ represents the premium rate and $N = \{N_t : t \geq 0\}$ is an independent Poisson process with arrival rate λ. Traditionally it is assumed in the Cramér-Lundberg model that the net profit condition $c > \lambda\mu$ holds, or equivalently that X drifts to infinity. In this chapter X will be a general spectrally negative Lévy process and the condition that X drifts to infinity will not be assumed.

We will now state the control problem considered in this chapter. As mentioned before, $X = \{X_t : t \geq 0\}$ is a spectrally negative Lévy process which is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$ satisfying the usual conditions. Within the definition of a spectrally negative Lévy process it is implicitly assumed that X does not have monotone paths. We denote by $\{\mathbb{P}_x, x \in \mathbb{R}\}$ the family of probability measures corresponding to a translation of X such that $X_0 = x$, where we write $\mathbb{P} = \mathbb{P}_0$. Further $\mathbb{E}_x$ denotes the expectation with respect to $\mathbb{P}_x$ with $\mathbb{E}$ being used in the obvious way. The Lévy triplet of X is given by (γ, σ, ν), where $\gamma \in \mathbb{R}$, $\sigma \geq 0$ and ν is a measure on $(0, \infty)$ satisfying

$$\int_{(0,\infty)} \left(1 \wedge x^2\right) \nu(\mathrm{d}x) < \infty.$$

Note that even though X only has negative jumps, for convenience we choose the Lévy measure to have only mass on the positive instead of the negative half line. The Laplace exponent of X is given by

$$\psi(\theta) = \log\left(\mathbb{E}\left(\mathrm{e}^{\theta X_1}\right)\right) = \gamma\theta + \frac{1}{2}\sigma^2\theta^2 - \int_{(0,\infty)} \left(1 - \mathrm{e}^{-\theta x} - \theta x \mathbf{1}_{\{0<x<1\}}\right) \nu(\mathrm{d}x)$$

and is well defined for $\theta \geq 0$. Note that the Cramér-Lundberg process corresponds to the case that $\sigma = 0$, $\nu(\mathrm{d}x) = \lambda F(\mathrm{d}x)$ where F is the law of C_1 and $\gamma = c - \int_{(0,1)} x\nu(\mathrm{d}x)$. The process X will represent the risk/surplus process of an insurance company before dividends are deducted.

We denote a dividend or control strategy by π, where $\pi = \{L_t^\pi : t \geq 0\}$ is a non-decreasing, left-continuous $\mathbb{F}$-adapted process which starts at zero. The random variable L_t^π will represent the cumulative dividends the company has paid out until time t under the control π. We define the controlled (net) risk process $U^\pi = \{U_t^\pi : t \geq 0\}$ by $U_t^\pi = X_t - L_t^\pi$. Let $\sigma^\pi = \inf\{t > 0 : U_t^\pi < 0\}$ be the ruin time and define the value function of a dividend strategy π by

$$v_\pi(x) = \mathbb{E}_x\left[\int_0^{\sigma^\pi} \mathrm{e}^{-qt}\mathrm{d}L_t^\pi + S\mathrm{e}^{-q\sigma^\pi}\right],$$

where $q > 0$ is the discount rate and $S \in \mathbb{R}$ is the terminal value. By definition it follows that $v_\pi(x) = S$ for $x < 0$. A strategy π is called admissible if ruin does not occur due to a lump sum dividend payment, i.e. $L_{t+}^\pi - L_t^\pi \leq U_t^\pi \vee 0$ for $t \leq \sigma^\pi$. Let Π be the set of all admissible dividend policies. The control problem consists of finding the optimal value function v_* given by

$$v_*(x) = \sup_{\pi \in \Pi} v_\pi(x)$$

and an optimal strategy $\pi_* \in \Pi$ such that

$$v_{\pi_*}(x) = v_*(x) \quad \text{for all } x \geq 0.$$

When $S = 0$ the above optimal control problem transforms, albeit within the more general framework of a spectrally negative Lévy risk process, to the original optimal dividends problem introduced firstly in a discrete time setting by de Finetti [14] and later studied in, amongst others, [5, 6, 22, 42]. The general case when $S \in \mathbb{R}$ we consider here is not new. Thonhauser and Albrecher [65] have studied in the Cramér-Lundberg setting the case $S < 0$. In that case the extra term added to the value function penalizes early ruin and so this model can be used if, besides the value of the dividend payments, one also wants to take into consideration the lifetime of the risk process. The parameter S can then be used to find the desired 'balance' between optimizing the value of the dividends and maximizing the ruin time. When $S > 0$, the model can be used if the company, when it becomes bankrupt, has a salvage value equaling S which is distributed to the same beneficiaries as the dividends are, see also the discussion in Radner and Shepp [52, Section 3]. In a Brownian motion/diffusion setting this control problem has been studied in [11, 57].

We will now introduce two types of dividend strategies and state our main theorem. We denote by $\pi_a = \{L_t^a : t \geq 0\}$ the barrier strategy at level $a \geq 0$ with corresponding value function v_a and ruin time σ^a. This strategy is defined by $L_0^a = 0$ and

$$L_t^a = \left(\sup_{0 \leq s < t} X_s - a \right) \vee 0 \quad \text{for } t > 0.$$

Note that $\pi_a \in \Pi$. So if dividends are paid out according to a barrier strategy with the barrier placed at a, then the corresponding controlled risk process will be a spectrally negative Lévy process reflected in a.

We further introduce the take-the-money-and-run strategy $\pi_{\text{run}} = \{L_t^{\text{run}} : t \geq 0\}$ which is the strategy where directly all of the surplus of the company is paid out and immediately thereafter ruin is forced (note that ruin is defined as the state when the controlled risk process is *strictly* below zero). The value of this strategy is $v_{\text{run}}(x) = x + S$ for $x \geq 0$. In case X is not a Cramér-Lundberg risk process, this strategy is the same as the barrier strategy with the barrier placed at zero (i.e. almost surely, $L_t^0 = L_t^{\text{run}}$ for all $t \geq 0$). But if X is a Cramér-Lundberg risk process, then the barrier strategy at zero does not imply immediate ruin; ruin occurs only after the first jump/claim which takes an exponentially distributed with parameter $\nu(0, \infty)$ amount of time. Therefore the value of the latter strategy might be different than the value of the take-the-money-and-run strategy. In particular for large terminal values, v_{run} might be bigger than v_0 since it can be beneficial to become ruined as soon as possible. Note that in the Cramér-Lundberg case, ruin can be forced in an admissible way by paying out dividends at a rate which is larger than the premium rate immediately after taking out all the surplus.

Recall that an infinitely differentiable function $f : (0, \infty) \to [0, \infty)$ is completely monotone if its derivatives alternate in sign, i.e. $(-1)^n f^{(n)}(x) \geq 0$ for all $n = 0, 1, 2, \ldots$ for all $x > 0$. The main theorem of this chapter reads now as follows.

Theorem 1. *Suppose the Lévy measure of the spectrally negative Lévy process X with Lévy triplet (γ, σ, ν), has a completely monotone density. Let $c = \gamma + \int_0^1 x\nu(\mathrm{d}x)$. Then the following holds.*

(i) If $\sigma > 0$, or $\nu(0,\infty) = \infty$, or $\nu(0,\infty) < \infty$ and $S \le c/q$, then an optimal strategy for the control problem is formed by a barrier strategy.

(ii) If $\sigma = 0$ and $\nu(0,\infty) < \infty$ and $S > c/q$, then the take-the-money-and-run strategy is an optimal strategy for the control problem.

For X being equal to a Brownian motion with drift, this control problem has been solved in [11,57]. In the case when X is a Cramér-Lundberg process with exponentially distributed claims, the control problem was solved by Gerber [22] for $S = 0$ and by Thonhauser and Albrecher [65] for $S < 0$. Note that both cases are examples for which the Lévy measure has a completely monotone density. Some other examples of spectrally negative Lévy processes which have a Lévy measure with a completely monotone density can be found in [47].

Building on the work of Avram et al. [5], Loeffen [47] proved Theorem 1 for $S = 0$. In particular, it was shown that optimality of the barrier strategy depends on the shape of the so-called scale function of a spectrally negative Lévy process. To be more specific, the q-scale function of X, $W^{(q)} : \mathbb{R} \to [0,\infty)$ where $q \ge 0$, is the unique function such that $W^{(q)}(x) = 0$ for $x < 0$ and on $[0,\infty)$ is a strictly increasing and continuous function characterized by its Laplace transform which is given by

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\theta) - q} \quad \text{for } \theta > \Phi(q),$$

where $\Phi(q) = \sup\{\theta \ge 0 : \psi(\theta) = q\}$ is the right-inverse of ψ. Loeffen [47] showed that when $W^{(q)}$ is sufficiently smooth and $W^{(q)\prime}$ is increasing on (a^*,∞) where a^* is the largest point where $W^{(q)\prime}$ attains its global minimum, then the barrier strategy at a^* is optimal for the control problem (in the $S = 0$ case). Here $W^{(q)}$ being sufficiently smooth means that $W^{(q)}$ is once/twice continuously differentiable when X is of bounded/unbounded variation. It was then shown in [47] that when X has a Lévy measure which has a completely monotone density, these conditions on the scale function are satisfied and in particular that $W^{(q)\prime}$ is strictly convex on $(0,\infty)$. Shortly thereafter, Kyprianou et al. [42] showed that $W^{(q)\prime}$ is strictly convex on (a^*,∞) (but not necessarily on $(0,\infty)$, see [42, Section 3]) under the weaker condition that the Lévy measure has a density which is log-convex. Though the scale function is in that case not necessarily sufficiently smooth, Kyprianou et al. [42] were able to circumvent this problem and proved that the barrier strategy at a^* is still optimal when the Lévy measure has a log-convex density. Note that without a condition on the Lévy measure the barrier strategy is not optimal in general. Indeed Azcue and Muler [6] have given an example for which no barrier strategy is optimal.

The proof of Theorem 1 in the case when $S \neq 0$, relies on the assumption that $W^{(q)\prime}$ is strictly log-convex on $(0,\infty)$. Though in [47] it was only shown under the complete monotonicity assumption on the Lévy measure, that $W^{(q)\prime}$ is strictly convex on $(0,\infty)$, we will show in Section 2 that the stronger property of strict log-convexity actually holds in that case. Then in Section 3 the proof of Theorem 1 will be given.

2.2 Scale functions

Associated to the functions $\{W^{(q)} : q \geq 0\}$ mentioned in the previous section are the functions $Z^{(q)} : \mathbb{R} \to [1, \infty)$ defined by

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y)\mathrm{d}y$$

for $q \geq 0$. Together, the functions $W^{(q)}$ and $Z^{(q)}$ are collectively known as scale functions and predominantly appear in almost all fluctuation identities for spectrally negative Lévy processes. As an example we mention the one sided exit below problem for which

$$\mathbb{E}_x \left(\mathrm{e}^{-q\tau_0^-} \mathbf{1}_{(\tau_0^- < \infty)} \right) = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x), \tag{2.1}$$

where $\tau_0^- = \inf\{t > 0 : X_t < 0\}$.

We will now recall some properties of scale functions which we will need later on. When the Lévy process drifts to infinity or equivalently $\psi'(0+) > 0$, the 0-scale function $W^{(0)}$ (which will be denoted from now on by W) is bounded and has a limit $\lim_{x \to 0} W(x) = 1/\psi'(0+)$. Further for $q \geq 0$ there is the following relation between scale functions

$$W^{(q)}(x) = \mathrm{e}^{\Phi(q)x} W_{\Phi(q)}(x), \tag{2.2}$$

where $W_{\Phi(q)}$ is the (0-)scale function of X under the measure $\mathbb{P}^{\Phi(q)}$ defined by

$$\left. \frac{\mathrm{d}\mathbb{P}^{\Phi(q)}}{\mathrm{d}\mathbb{P}} \right|_{\mathcal{F}_t} = \mathrm{e}^{\Phi(q) X_t - qt}.$$

The process X under the measure $\mathbb{P}^{\Phi(q)}$ is still a spectrally negative Lévy process and its Laplace exponent is given by $\psi_{\Phi(q)}(\theta) = \psi(\Phi(q) + \theta) - \psi(\Phi(q))$. When $q > 0$ it is known that $\psi'_{\Phi(q)}(0+) = \psi'(\Phi(q)) > 0$.

When X does not drift to minus infinity then from [38, p.220] it follows that for $x, a > 0$

$$\log(W(x)) = \log(W(a)) + \int_a^x g(t)\mathrm{d}t,$$

where g is a decreasing function and hence $\log(W(x))$ is concave on $(0, \infty)$ (see e.g. [66, Theorem 1.13]). From (2.2) it now follows that for $q \geq 0$, $\log(W^{(q)}(x))$ is concave on $(0, \infty)$ and thus $W^{(q)}$ is log-concave on $(0, \infty)$ for all $q \geq 0$.

The initial value of the scale function $W^{(q)}(0)$ is equal to $1/c$, where c is as in Theorem 1. Note that if X is of unbounded variation, then $c = \infty$ and thus $W^{(q)}(0) = 0$.

The initial value of the derivative of the scale function is given by (see e.g. [40])

$$W^{(q)\prime}(0) := \lim_{x\downarrow 0} W^{(q)\prime}(x) = \begin{cases} 2/\sigma^2 & \text{when } \sigma > 0 \\ (\nu(0,\infty)+q)/c^2 & \text{when } \sigma = 0 \text{ and } \nu(0,\infty) < \infty \\ \infty & \text{otherwise.} \end{cases}$$

Despite the fact that the scale function is in general only implicitly known through its Laplace transform, there are plenty examples of spectrally negative Lévy processes for which there exists closed-form expressions for their scale functions, although most of these examples only deal with the $q = 0$ scale function. In case no explicit formula for the scale function exists, one can use numerical methods as described in [63] to invert the Laplace transform of the scale function. We refer to the papers [28, 41, 42] for an updated account on explicit examples of scale functions and their properties.

In the sequel for $a \in \mathbb{R}$, a function f and a Borel measure μ, we will use the notation $\int_a^\infty f(x)\mu(\mathrm{d}x)$ and $\int_{a+}^\infty f(x)\mu(\mathrm{d}x)$ to mean integration over the interval $[a,\infty)$ in the first case and integration over the interval (a,∞) in the second case. In particular, $\int_a^\infty f(x)\mu(\mathrm{d}x) = f(a)\mu\{a\} + \int_{a+}^\infty f(x)\mu(\mathrm{d}x)$. We recall Bernstein's theorem which says that a real-valued function f is completely monotone if and only if there exists a Borel measure μ such that $f(x) = \int_0^\infty \mathrm{e}^{-xt}\mu(\mathrm{d}t)$, $x > 0$. We now strengthen the conclusion of Theorem 3 in [47]. First we need the following proposition.

Proposition 2. *Suppose $q > 0$. Then*

$$\liminf_{x\to\infty} \mathrm{e}^{\Phi(q)x} W'_{\Phi(q)}(x) = 0.$$

Proof. Taking derivatives on both sides in (2.1) and using (2.2), we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}_x\left(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{(\tau_0^-<\infty)}\right) = -\frac{q}{\Phi(q)}\mathrm{e}^{\Phi(q)x}W'_{\Phi(q)}(x).$$

Suppose now that the conclusion of the proposition does not hold. Then $\mathrm{e}^{\Phi(q)x}W'_{\Phi(q)}(x)$ will eventually be bounded from below by a strictly positive constant. It follows then that

$$\lim_{x\to\infty}\mathbb{E}_x\left(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{(\tau_0^-<\infty)}\right) = -\infty,$$

which contradicts the positivity of the expectation. ■

Theorem 3. *Suppose the Lévy measure ν has a completely monotone density and $q > 0$. Then the q-scale function can be written as*

$$W^{(q)}(x) = \frac{\mathrm{e}^{\Phi(q)x}}{\psi'(\Phi(q))} - f(x), \quad x > 0,$$

where f is a completely monotone function.

Proof. It was shown in [47] that if the Lévy measure ν has a completely monotone density, then $W_{\Phi(q)}$ is a Bernstein function and therefore admits the representation

$$W_{\Phi(q)}(x) = a + bx + \int_{0+}^{\infty} (1 - \mathrm{e}^{-xt})\xi(\mathrm{d}t) \quad x > 0, \tag{2.3}$$

where $a, b \geq 0$ and ξ is a measure on $(0, \infty)$ satisfying $\int_{0+}^{\infty} (t \wedge 1)\xi(\mathrm{d}t) < \infty$. Since $q > 0$, $W_{\Phi(q)}$ will be bounded and therefore $b = 0$ and by using Fatou's lemma

$$\begin{aligned}\xi(0, \infty) &= \int_{0+}^{\infty} \lim_{x\to\infty} (1 - \mathrm{e}^{-xt})\xi(\mathrm{d}t) \leq \lim_{x\to\infty} \int_{0+}^{\infty} (1 - \mathrm{e}^{-xt})\xi(\mathrm{d}t) \\ &= \lim_{x\to\infty} W_{\Phi(q)}(x) - a < \infty.\end{aligned}$$

We now deduce from Proposition 2, (2.3) and Fatou's lemma

$$\begin{aligned}0 &= \liminf_{x\to\infty} \mathrm{e}^{\Phi(q)x} W'_{\Phi(q)}(x) = \liminf_{x\to\infty} \int_{0+}^{\infty} \mathrm{e}^{-x(t-\Phi(q))} t\xi(\mathrm{d}t) \\ &\geq \int_{0+}^{\infty} \liminf_{x\to\infty} \mathrm{e}^{-x(t-\Phi(q))} t\xi(\mathrm{d}t) \geq \Phi(q)\xi(0, \Phi(q)].\end{aligned}$$

It follows that $\xi(0, \Phi(q)] = 0$ and using (2.2) and (2.3), we can write

$$\begin{aligned}W^{(q)}(x) &= \mathrm{e}^{\Phi(q)x} \left(a + \xi(\Phi(q), \infty)\right) - \int_{\Phi(q)+}^{\infty} \mathrm{e}^{-x(t-\Phi(q))}\xi(\mathrm{d}t) \\ &= \mathrm{e}^{\Phi(q)x} \left(a + \xi(\Phi(q), \infty)\right) - \int_{0+}^{\infty} \mathrm{e}^{-xt}\xi(\mathrm{d}t + \Phi(q)).\end{aligned} \tag{2.4}$$

Now the conclusion of the theorem follows by Bernstein's theorem and the fact that $a + \xi(\Phi(q), \infty) = \lim_{x\to\infty} W_{\Phi(q)}(x) = 1/\psi'(\Phi(q))$. ■

Denote by $W^{(q,n)}(x)$ the n-th derivative of $W^{(q)}(x)$ for $x > 0$ and $n = 0, 1, 2, \ldots$.

Corollary 4. *Suppose the Lévy measure ν has a completely monotone density, $q > 0$ and n is an odd integer. Then $\log\left(W^{(q,n)}(x)\right)$ has a strictly positive second derivative for all $x > 0$. Consequently, the function $W^{(q,n)}$ is strictly log-convex on $(0, \infty)$.*

Proof. Suppose that the Lévy measure has a completely monotone density, $q > 0$ and n is an odd integer. Let $f(x) = \frac{\mathrm{e}^{\Phi(q)x}}{\psi'(\Phi(q))} - W^{(q)}(x)$ and $g(x) = -f'(x)$. By Theorem 3, f and g are completely monotone functions and

$$W^{(q,n)}(x) = \frac{\Phi(q)^n}{\psi'(\Phi(q))}\mathrm{e}^{\Phi(q)x} + g^{(n-1)}(x), \tag{2.5}$$

where $g^{(n-1)}$ is the $(n-1)$-th derivative of g. Define

$$h_n(x) = \left(W^{(q,n)}(x)\right)^2 \left[\log\left(W^{(q,n)}(x)\right)\right]'' = W^{(q,n)}W^{(q,n+2)}(x) - \left(W^{(q,n+1)}\right)^2.$$

We need to prove that $h_n(x) > 0$ for all $x > 0$. Using (2.5) we get

$$h_n(x) = \left[g^{(n-1)}(x)g^{(n+1)}(x) - \left(g^{(n)}(x)\right)^2\right] + \frac{\Phi(q)^n}{\psi'(\Phi(q))}\mathrm{e}^{\Phi(q)x}\left\{\Phi(q)^2 g^{(n-1)}(x) + g^{(n+1)}(x) - 2\Phi(q)g^{(n)}(x)\right\}.$$

Since n is odd, $g^{(n-1)}$ is completely monotone and because a completely monotone function is log-convex, the expression between the square brackets is positive. Further, the complete monotonicity of $g^{(n-1)}$ implies that each of the terms between the curly brackets is positive and hence $h_n(x) \geq 0$. As $q > 0$, $\Phi(q) > 0$ and it suffices to prove that one of the terms between the curly brackets, say $g^{(n+1)}(x)$, is strictly positive. We do this by contradiction. Suppose $g^{(n+1)}(x) = 0$. Then it is easily seen from Bernstein's theorem that the function f has to be equal to a constant. In that case (2.4) implies that $f \equiv 0$. But this means that for $\lambda > \Phi(q)$

$$\frac{1}{\psi(\lambda) - q} = \int_0^\infty \mathrm{e}^{-\lambda x} W^{(q)}(x)\mathrm{d}x = \int_0^\infty \frac{\mathrm{e}^{-(\lambda-\Phi(q))x}}{\psi'(\Phi(q))}\mathrm{d}x = \frac{1}{(\lambda - \Phi(q))\psi'(\Phi(q))}.$$

Thus $\psi(\lambda)$ is the Laplace exponent of a subordinator (consisting of just a single drift term). But subordinators were excluded from the definition of a spectrally negative Lévy process, which gives us the desired contradiction. ■

2.3 Proof of main theorem

In this section the proof of Theorem 1 will be given with the aid of a series of lemmas. The approach is similar to [5] and [47], namely calculating the value of a barrier strategy where the barrier is arbitrary, then choosing the 'optimal' barrier and finally putting this particular barrier strategy (or the take-the-money-and-run strategy) through a verification lemma.

First we recall what we mean by the term sufficiently smooth. A function $f : \mathbb{R} \to \mathbb{R}$ which vanishes on $(-\infty, 0)$ and which is right-continuous at zero, is called sufficiently smooth at a point $x > 0$ if f is continuously differentiable at x when X is of bounded variation and is twice continuously differentiable at x when X is of unbounded variation. A function is then called sufficiently smooth if it is sufficiently smooth at all $x > 0$. Note that we implicitly assume that a sufficiently smooth function is right-continuous at zero. We let Γ be the operator acting on sufficiently smooth functions f, defined by

$$\Gamma f(x) = \gamma f'(x) + \frac{\sigma^2}{2}f''(x) + \int_{0+}^\infty [f(x-y) - f(x) + f'(x)y\mathbf{1}_{\{0<y<1\}}]\nu(\mathrm{d}y).$$

Lemma 5 (Verification lemma)**.** *Suppose $\hat{\pi}$ is an admissible dividend strategy such that $(v_{\hat{\pi}} - S)$ is sufficiently smooth, $v_{\hat{\pi}} \geq S$ and for all $x > 0$*

$$\max\{\Gamma v_{\hat{\pi}}(x) - q v_{\hat{\pi}}(x), 1 - v'_{\hat{\pi}}(x)\} \leq 0. \tag{2.6}$$

Then $v_{\hat{\pi}}(x) = v_*(x)$ for all $x \geq 0$ and hence $\hat{\pi}$ is an optimal strategy.

Proof. By definition of v_*, it follows that $v_{\hat{\pi}}(x) \leq v_*(x)$ for all $x \geq 0$. Let now $w := v_{\hat{\pi}}$ and denote by Π_0 the following set of admissible dividend strategies

$$\Pi_0 = \{\pi \in \Pi : \inf\{t > 0 : U_t^\pi \leq 0\} = \sigma^\pi \ \mathbb{P}_x\text{-a.s. for all } x > 0\}.$$

Note that when X is of unbounded variation, $\Pi_0 = \Pi$, but that Π_0 is a strictly smaller set than Π when X is of bounded variation. We will show that $w(x) \geq v_\pi(x)$ for all $\pi \in \Pi$ for all $x > 0$. Since any $\pi \in \Pi$ can be approximated by dividend strategies from Π_0 (i.e. for all $\epsilon > 0$ there exists $\pi_\epsilon \in \Pi_0$ such that $v_\pi(x) \leq v_{\pi_\epsilon}(x) + \epsilon$; take e.g. π_ϵ to be the strategy where you do not pay out any dividends until L^π is at least ϵ, then at that time point pay out a dividend equal to the size of the overshoot of L^π over ϵ and afterwards follow the same strategy as π until ruin occurs for the latter strategy at which point you force ruin immediately), we assume without loss of generality that $\pi \in \Pi_0$.

Suppose $x > 0$ and let $\widetilde{L}^\pi, \widetilde{U}^\pi$ be the right-continuous modifications of L^π, U^π. Note that since the filtration $\mathbb{F}$ was assumed to be right-continuous, $\widetilde{L}^\pi$ and $\widetilde{U}^\pi$ are adapted processes. Let $(T_n)_{n\in\mathbb{N}}$ be the sequence of stopping times defined by $T_n = \inf\{t > 0 : \widetilde{U}_t^\pi > n \text{ or } \widetilde{U}_t^\pi < \frac{1}{n}\}$. Since $\widetilde{U}^\pi$ is a cadlag semi-martingale and w is sufficiently smooth - in particular w and its derivatives are bounded on $[1/n, n]$ for each n - we can use the change of variables/Itô's formula (cf. [51, Theorem II.31 & II.32]) on $\mathrm{e}^{-q(t\wedge T_n)} w(\widetilde{U}^\pi_{t\wedge T_n})$ and after some similar calculations as in [5, Proposition 4] using (2.6), we get

$$w(\widetilde{U}_0^\pi) \geq \int_{0+}^{t\wedge T_n} \mathrm{e}^{-qs} w'(\widetilde{U}^\pi_{s-}) \mathrm{d}\widetilde{L}_s^\pi + \mathrm{e}^{-q(t\wedge T_n)} w(\widetilde{U}^\pi_{t\wedge T_n}) + M_t,$$

where $\{M_t : t \geq 0\}$ is a zero-mean $\mathbb{P}_x$-martingale. Using the assumption that $w \geq S$, taking expectations, letting t and n go to infinity and using the monotone convergence theorem we get

$$w(\widetilde{U}_0^\pi) \geq \mathbb{E}_x \left(\int_{0+}^{\sigma^\pi} \mathrm{e}^{-qs} \mathrm{d}\widetilde{L}_s^\pi \right) + S\mathbb{E}_x \left(\mathrm{e}^{-q\sigma^\pi} \right).$$

Note that we used here that $T_n \nearrow \sigma^\pi$ $\mathbb{P}_x$-a.s. which follows because $\pi \in \Pi_0$. Now using the mean value theorem together with the assumption that $w'(\cdot) \geq 1$ on $(0, \infty)$, we get

$$w(\widetilde{U}_0^\pi) = w(x - L^\pi_{0+}) \leq w(x) - L^\pi_{0+}$$

and combining with

$$\mathbb{E}_x \left(\int_{0+}^{\sigma^\pi} \mathrm{e}^{-qs} \mathrm{d}\widetilde{L}_s^\pi \right) = \mathbb{E}_x \left(\int_{0}^{\sigma^\pi} \mathrm{e}^{-qs} \mathrm{d}L_s^\pi \right) - L^\pi_{0+} = v_\pi(x) - S\mathbb{E}_x \left(\mathrm{e}^{-q\sigma^\pi} \right) - L^\pi_{0+},$$

we deduce $w(x) \geq v_\pi(x)$ and hence we proved $w(x) \geq v_*(x)$ for all $x > 0$.

To finish the proof, note that v_* is an increasing function and hence because w is right-continuous at zero, $v_*(0) \leq \lim_{x\downarrow 0} v_*(x) \leq \lim_{x\downarrow 0} w(x) = w(0)$. ■

Proposition 6. *Assume $W^{(q)}$ is continuously differentiable on $(0,\infty)$. The value function of the barrier strategy at level $a \geq 0$ is given by*

$$v_a(x) = \begin{cases} SZ^{(q)}(x) + W^{(q)}(x)\left(\frac{1-qSW^{(q)}(a)}{W^{(q)\prime}(a)}\right) & \text{if } x \leq a \\ x - a + SZ^{(q)}(a) + W^{(q)}(a)\left(\frac{1-qSW^{(q)}(a)}{W^{(q)\prime}(a)}\right) & \text{if } x > a. \end{cases}$$

Proof. Clearly the proposition only needs to be proved for $0 \leq x \leq a$. Let $x \in [0,a]$. By Avram et al. [5, Proposition 1], it follows that

$$\mathbb{E}_x\left[\int_0^{\sigma^a} \mathrm{e}^{-qt}\mathrm{d}L_t^a\right] = \frac{W^{(q)}(x)}{W^{(q)\prime}(a)}.$$

Since

$$\sigma^a = \inf\{t > 0 : X_t - L_t^a < 0\} = \inf\{t > 0 : \left(\sup_{0\leq s<t} X_s\right) \vee a - X_t > a\},$$

it follows by Avram et al. [4, Theorem 1] that

$$\mathbb{E}_x\left[\mathrm{e}^{-q\sigma^a}\right] = Z^{(q)}(x) - W^{(q)}(x)\frac{qW^{(q)}(a)}{W^{(q)\prime}(a)}.$$

■

Define the function $\zeta : [0,\infty) \to \mathbb{R}$ by

$$\zeta(x) = \frac{1 - qSW^{(q)}(x)}{W^{(q)\prime}(x)} \quad \text{for } x > 0$$

and $\zeta(0) = \lim_{x\downarrow 0}\zeta(x)$. We now define the (candidate) optimal barrier level by

$$a^*(S) = \sup\left\{a \geq 0 : \zeta(a) \geq \zeta(x) \text{ for all } x \geq 0\right\}.$$

Hence $a^*(S)$ is the last point where ζ attains its global maximum. Note that $a^*(0)$ is the point a^* mentioned in Section 2.1. In the sequel we will write a^* instead of $a^*(0)$.

Proposition 7. *Suppose $W^{(q)}$ is continuously differentiable on $(0,\infty)$. Then $a^*(S) < \infty$.*

Proof. Define

$$f(x) = \zeta(x) + \frac{qS}{\Phi(q)} = \frac{1 + qS\left(\Phi(q)^{-1}W^{(q)\prime}(x) - W^{(q)}(x)\right)}{W^{(q)\prime}(x)}.$$

Since $\lim_{x\to\infty}\frac{W^{(q)}(x)}{W^{(q)\prime}(x)} = \frac{1}{\Phi(q)}$ (see e.g. [5, Section 3.3]) and $W^{(q)}$ is continuously differentiable, it follows that $\lim_{x\to\infty} f(x) = 0$ and f is continuous. Hence $a^*(S) < \infty$ if there exists $x \geq 0$ such that $f(x) > 0$. But by (2.2)

$$f(x) = \frac{1 + \frac{qS}{\Phi(q)} \mathrm{e}^{\Phi(q)x} W'_{\Phi(q)}(x)}{W^{(q)\prime}(x)}$$

and thus by Proposition 2, there exists $x \geq 0$ such that $f(x) > 0$. ■

Note that when $a^*(S) > 0$ and $W^{(q)}$ is twice continuously differentiable, then $\zeta'(a^*(S)) = 0$. Further, it is easily seen that if an optimal strategy is formed by a barrier strategy, then the barrier strategy at $a^*(S)$ has to be an optimal strategy.

Lemma 8. *Suppose $W^{(q)}$ is sufficiently smooth and that*

$$\zeta(a) \geq \zeta(b) \quad \textit{for all } a, b \textit{ such that } a^*(S) \leq a \leq b. \tag{2.7}$$

Then the following holds.

(i) If $\zeta(a^*(S)) \geq 0$, then the barrier strategy at $a^*(S)$ is an optimal strategy.

(ii) If $a^*(S) = 0$ and $\zeta(0) \leq 0$, then the take-the-money-and-run strategy is optimal.

Note that Lemma 8 is a generalization of Theorem 2 in [47]. Indeed when $S = 0$, $\zeta(a^*) = 1/W^{(q)\prime}(a^*) > 0$ and condition (2.7) transforms into the condition that $W^{(q)\prime}$ is increasing on (a^*, ∞).

Proof. We first prove (i) by showing that $v_{a^*(S)}$ satisfies the conditions of the verification lemma. Using (2.7), all the conditions of the verification lemma can be proved following the same arguments as in the proofs of Lemma 5 and Theorem 2 in [47], with the exception being the condition that $v_{a^*(S)}(x) \geq S$ for all $x \geq 0$. (Note that in deducing the analogue of equation (4) in [47], one also uses the fact that $\left(\mathrm{e}^{-q(t\wedge\tau_0^-\wedge\tau_a^+)} Z^{(q)}(X_{t\wedge\tau_0^-\wedge\tau_a^+})\right)_{t\geq 0}$ is a $\mathbb{P}_x$-martingale, cf. [38, p.229].) The missing condition now follows from $v'_{a^*(S)}(x) \geq 1$ for $x > 0$ and

$$v_{a^*(S)}(0) = SZ^{(q)}(0) + W^{(q)}(0)\zeta(a^*(S)) \geq S,$$

where the inequality follows from the assumption that $\zeta(a^*(S)) \geq 0$.

For case (ii) we prove that v_{run} satisfies the conditions of the verification lemma. Note that since $v_{\mathrm{run}}(x) = x + S$ for $x \geq 0$, the only non-trivial thing to show is that $(\Gamma - q)v_{\mathrm{run}}(x) \leq 0$ for all $x > 0$. This can be achieved by mimicking the proof of Theorem 2 in [47], which involves proving that

$$\lim_{y\uparrow x} (\Gamma - q)(v_{\mathrm{run}} - v_x)(y) \leq 0 \quad \text{for } x > 0.$$

Note that in order to prove the above inequality, one uses that $v_{\mathrm{run}}(0) \geq v_x(0)$ which follows from $\zeta(x) \leq 0$ and the latter is due to the assumption that $\zeta(0) \leq 0$ and $a^*(S) = 0$ (combined with (2.7)). ■

Proof of Theorem 1. Since the case $S = 0$ was proved in Loeffen [47], we assume without loss of generality that $S \neq 0$. Note that by Theorem 3, $W^{(q)}$ is infinitely differentiable (this was proved for the first time in [13]) and therefore certainly smooth enough. Further note that $W^{(q)\prime\prime}$ is strictly negative on $(0, a^*)$, strictly positive on (a^*, ∞) and if $a^* > 0$, then $W^{(q)\prime\prime}(a^*) = 0$. We will show that

$$\zeta \text{ is strictly increasing on } (0, a^*(S)) \text{ and strictly decreasing on } (a^*(S), \infty), \tag{2.8}$$

from which it follows that $a^*(S)$ is the only point where ζ has a local/global maximum and that (2.7) holds.

First note that with $g(x) = -qSW^{(q)\prime}(x)/W^{(q)\prime\prime}(x)$ for $x \in (0, \infty)\backslash\{a^*\}$, the following differential equation holds for ζ

$$\zeta'(x) = -\frac{W^{(q)\prime\prime}(x)}{W^{(q)\prime}(x)} \left(\zeta(x) - g(x)\right), \quad x \in (0, \infty)\backslash\{a^*\}.$$

From this it follows that

$$\begin{aligned} &\text{for } x \in (0, a^*) \quad \zeta'(x) > 0 (< 0, = 0) \text{ iff } \zeta(x) > g(x) (< g(x), = g(x)), \\ &\text{for } x \in (a^*, \infty) \quad \zeta'(x) > 0 (< 0, = 0) \text{ iff } \zeta(x) < g(x) (> g(x), = g(x)). \end{aligned} \tag{2.9}$$

Suppose that $S > 0$. Since

$$\zeta'(x) = \frac{qS\left[W^{(q)}(x)W^{(q)\prime\prime}(x) - \left(W^{(q)\prime}(x)\right)^2\right] - W^{(q)\prime\prime}(x)}{\left(W^{(q)\prime}(x)\right)^2} \tag{2.10}$$

and the expression between square brackets is negative due to the log-concavity of $W^{(q)}$, it follows that $\zeta'(x) < 0$ on (a^*, ∞) and therefore $a^*(S) \leq a^*$. If $a^* = 0$, (2.8) now holds, so we can assume without loss of generality that $a^* > 0$. Then $\lim_{x \uparrow a^*} g(x) = \infty$ and (2.9) imply $a^*(S) \neq a^*$ and thus $a^*(S) < a^*$. By the strict log-convexity of $W^{(q)\prime}$ (Corollary 4), g is strictly increasing on $(0, a^*)$. The foregoing and (2.9) imply then that either ζ intersects g exactly once on $(0, \infty)$ (at $a^*(S)$) and (2.8) holds or that $\zeta'(x) < 0$ for all $x > 0$ and in that case $a^*(S) = 0$. Hence (2.8) holds when $S > 0$.

Suppose now that $S < 0$ and $a^* > 0$. Then ζ is strictly positive on $(0, \infty)$ by definition and g is strictly negative on $(0, a^*)$. Hence $a^*(S) \geq a^*$. Due to the strict log-convexity of $W^{(q)\prime}$, g is in this case strictly decreasing on (a^*, ∞) and combined with (2.9) and the fact that $\lim_{x \downarrow a^*} g(x) = \infty$, this implies that ζ and g intersect each other exactly once, $a^*(S) > a^*$ and that (2.8) holds.

This leaves the final case when $S < 0$ and $a^* = 0$. If $\zeta(0) \geq g(0)$, then (2.9) and g being strictly decreasing on $(0, \infty)$ implies ζ is strictly decreasing on $(0, \infty)$ and hence $a^*(S) = 0$. If $\zeta(0) < g(0)$, then $a^*(S) > 0$ and further (2.8) holds by the same arguments as before.

Suppose now that $\sigma > 0$, or $\nu(0, \infty) = \infty$, or $\nu(0, \infty) < \infty$ and $S \leq c/q$. Then from the values of $W^{(q)}(0)$ and $W^{(q)\prime}(0)$ given in Section 2.2, it follows that $\zeta(0) \geq 0$ and hence $\zeta(a^*(S)) \geq 0$ by definition of $a^*(S)$. Thus part (i) of the theorem follows from Lemma 8(i).

To prove part (ii), suppose that $\sigma = 0$ and $\nu(0,\infty) < \infty$ and $S > c/q$. This implies $S > 0$ and $\zeta(0) < 0$. If $a^* = 0$ then $a^*(S) = 0$ since $S > 0$. If $a^* > 0$ then $g(0) > 0$ and hence by (2.9) and (2.10), $\zeta'(x) < 0$ for all $x > 0$ and therefore $a^*(S) = 0$. Part (ii) follows now from Lemma 8(ii). ■

Chapter 3

An optimal dividends problem with transaction costs for spectrally negative Lévy processes

We consider an optimal dividends problem with transaction costs where the reserves are modeled by a spectrally negative Lévy process. We make the connection with the classical de Finetti problem and show in particular that when the Lévy measure has a log-convex density, then an optimal strategy is given by paying out a dividend in such a way that the reserves are reduced to a certain level c_1 whenever they are above another level c_2. Further we describe a method to numerically find the optimal values of c_1 and c_2.

3.1 Introduction

In this chapter we consider an offshoot of the classical de Finetti's optimal dividends problem in continuous time for which a transaction cost is incurred each time a dividend payment is made. Because of this fixed cost, it is no longer feasible to pay out dividends at a certain rate and therefore only lump sum dividend payments are possible.

Within this problem we assume that the underlying dynamics of the risk process is described by a spectrally negative Lévy process which is now widely accepted and used as a replacement for the classical Cramér-Lundberg process (cf. [1, 5, 16, 17, 21, 29, 39, 42, 54]). Recall that a Cramér-Lundberg risk process $\{X_t : t \geq 0\}$ corresponds to

$$X_t = x + ct - \sum_{i=1}^{N_t} C_i, \tag{3.1}$$

where $x > 0$ denotes the initial surplus, the claims $C_1, C_2, \dots$ are i.i.d. positive random variables with expected value μ, $c > 0$ represents the premium rate and $N = \{N_t : t \geq 0\}$ is an independent Poisson process with arrival rate λ. Traditionally it is assumed in the Cramér-Lundberg model that the net profit condition $c > \lambda\mu$ holds, or equivalently that X drifts to infinity. In this chapter X will be a general spectrally negative Lévy process and the condition that X drifts to infinity will not be assumed.

We will now state the control problem considered in this chapter. As mentioned before, $X = \{X_t : t \geq 0\}$ is a spectrally negative Lévy process which is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$ satisfying the usual conditions. Within the definition of a spectrally negative Lévy process it is implicitly assumed that X does not have monotone paths. We denote by $\{\mathbb{P}_x, x \in \mathbb{R}\}$ the family of probability measures corresponding to a translation of X such that $X_0 = x$, where we write $\mathbb{P} = \mathbb{P}_0$. Further $\mathbb{E}_x$ denotes the expectation with respect to $\mathbb{P}_x$ with $\mathbb{E}$ being used in the obvious way. The Lévy triplet of X is given by (γ, σ, ν), where $\gamma \in \mathbb{R}$, $\sigma \geq 0$ and ν is a measure on $(0, \infty)$ satisfying

$$\int_{(0,\infty)} \left(1 \wedge x^2\right) \nu(\mathrm{d}x) < \infty.$$

Note that even though X only has negative jumps, for convenience we choose the Lévy measure to have only mass on the positive instead of the negative half line. The Laplace exponent of X is given by

$$\psi(\theta) = \log\left(\mathbb{E}\left(\mathrm{e}^{\theta X_1}\right)\right) = \gamma\theta + \frac{1}{2}\sigma^2\theta^2 - \int_{(0,\infty)} \left(1 - \mathrm{e}^{-\theta x} - \theta x \mathbf{1}_{\{0<x<1\}}\right) \nu(\mathrm{d}x)$$

and is well defined for $\theta \geq 0$. Note that the Cramér-Lundberg process corresponds to the case that $\sigma = 0$, $\nu(\mathrm{d}x) = \lambda F(\mathrm{d}x)$ where F is the law of C_1 and $\gamma = c - \int_{(0,1)} x\nu(\mathrm{d}x)$. The process X will represent the risk process/reserves of the company before dividends are deducted.

We denote a dividend or control strategy by π, where $\pi = \{L_t^\pi : t \geq 0\}$ is a non-decreasing, left-continuous $\mathbb{F}$-adapted process which starts at zero. Further we assume that the process L^π is a pure jump process, i.e.

$$L_t^\pi = \sum_{0 \leq s < t} \Delta L_s^\pi \quad \text{for all } t \geq 0. \tag{3.2}$$

Here we mean by $\Delta L_s^\pi = L_{s+}^\pi - L_s^\pi$ the jump of the process L^π at time s.

The random variable L_t^π will represent the cumulative dividends the company has paid out until time t under the control π. We define the controlled (net) risk process $U^\pi = \{U_t^\pi : t \geq 0\}$ by $U_t^\pi = X_t - L_t^\pi$. Let $\sigma^\pi = \inf\{t > 0 : U_t^\pi < 0\}$ be the ruin time and define the value function of a dividend strategy π by

$$v_\pi(x) = \mathbb{E}_x\left[\int_0^{\sigma^\pi} \mathrm{e}^{-qt}\mathrm{d}\left(L_t^\pi - \sum_{0 \leq s < t} \beta \mathbf{1}_{\{\Delta L_s^\pi > 0\}}\right)\right],$$

where $q > 0$ is the discount rate and $\beta > 0$ is the transaction cost incurred for each dividend payment. By definition it follows that $v_\pi(x) = 0$ for $x < 0$. A strategy π is called admissible if ruin does not occur due to a lump sum dividend payment, i.e. $\Delta L_t^\pi \leq U_t^\pi \vee 0$ for $t \leq \sigma^\pi$. Let Π be the set of all admissible dividend policies. The control problem consists of finding the optimal value function v_* given by

$$v_*(x) = \sup_{\pi\in\Pi} v_\pi(x)$$

and an optimal strategy $\pi_* \in \Pi$ such that

$$v_{\pi_*}(x) = v_*(x) \quad \text{for all } x \geq 0.$$

Since control strategies of the form (3.2) are known as impulse controls, we refer to this problem as the impulse control problem.

An important type of strategy for the impulse control problem is the one we call in this chapter the $(c_1; c_2)$ policy and which is similar to the well known (s, S) policy appearing in inventory control models, see e.g. [7,62]. The $(c_1; c_2)$ policy is the strategy where each time the reserves are above a certain level c_2, a dividend payment is made which brings the reserves down to another level c_1 and where no dividends are paid out when the reserves are below c_2. In case X is a Brownian motion plus drift, Jeanblanc and Shiryaev [34] showed that an optimal strategy for the impulse control problem is formed by a $(c_1; c_2)$ policy. Paulsen [49] considered the case when X is modeled by a diffusion process and showed that under certain conditions a $(c_1; c_2)$ policy is optimal. Note that in Paulsen [49] this type of strategy is referred to as a lump sum dividend barrier strategy. In this chapter we will investigate when an optimal strategy for our impulse control problem is formed by a $(c_1; c_2)$ policy.

When the assumption (3.2) is dropped and the transaction cost β is taken to be equal to zero, then the impulse control problem transforms into the classical de Finetti optimal dividends problem. The latter optimal dividends problem will be referred to as the de Finetti problem in the remainder of the chapter. This particular problem was introduced by de Finetti [14] in a discrete time setting for the case that the risk process evolves as a simple random walk. Thereafter the de Finetti problem has been studied in a continuous time setting for the case that X is a Cramér-Lundberg risk process [6, 22] and for the case that the risk process is a general spectrally negative Lévy process [5,42,47]. For this problem an important strategy is the so called barrier strategy. The barrier strategy at level a is the strategy where initially (in case the starting value of the reserves are above a) a lump sum dividend payment is made to bring the reserves back to level a and thereafter each time the reserves reach the level a, non-lump sum dividend payments are made in such a way that the reserves do not exceed the level a, but where no dividends are paid out when the reserves are strictly below a. Mathematically this corresponds to reflecting the risk process X at a. The barrier strategy at level a may be seen (at least intuitively) as a limit of $(c_1; c_2)$ policies where c_1 and c_2 converge to the barrier a.

Gerber [22] proved that an optimal strategy for the de Finetti problem is formed by a barrier strategy in the case where X is a Cramér-Lundberg risk process with exponentially distributed claims. Building on the work of Avram et al. [5], Loeffen [47] showed that optimality of the barrier strategy for the de Finetti problem depends on the shape of the so-called scale function of a spectrally negative Lévy process. To be more specific, the q-scale function of X, $W^{(q)} : \mathbb{R} \to [0, \infty)$ where $q \geq 0$, is the unique function such that $W^{(q)}(x) = 0$ for $x < 0$ and on $[0, \infty)$ is a strictly increasing and continuous function characterized by its Laplace transform which is given by

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\theta) - q} \quad \text{for } \theta > \Phi(q), \tag{3.3}$$

where $\Phi(q) = \sup\{\theta \geq 0 : \psi(\theta) = q\}$ is the right-inverse of ψ. Theorem 2 of Loeffen [47] then says that if $W^{(q)}$ is sufficiently smooth and if $W^{(q)\prime}$ is increasing on (a^*, ∞) where a^* is the largest point where $W^{(q)\prime}$ attains its global minimum, then the barrier strategy at a^* is optimal for the de Finetti problem. Here $W^{(q)}$ being sufficiently smooth means that $W^{(q)}$ is once/twice continuously differentiable when X is of bounded/unbounded variation. It was then shown in [47] that when X has a Lévy measure which has a completely monotone density, these conditions on the scale function are satisfied and in particular that $W^{(q)\prime}$ is strictly convex on $(0, \infty)$. (Note that it was shown in Chapter 2 that $W^{(q)\prime}$ is actually strictly log-convex.) Shortly thereafter, Kyprianou et al. [42] proved that $W^{(q)\prime}$ is strictly convex on (a^*, ∞) under the weaker condition that the Lévy measure has a density which is log-convex and then used Theorem 2 from [47] mentioned above, to conclude that the barrier strategy at a^* is optimal (though they needed to relax the sufficiently smoothness assumption). It is important to note that without a condition on the Lévy measure the barrier strategy is not optimal in general. Indeed Azcue and Muler [6] have given an example for which no barrier strategy is optimal.

In this chapter we will show that the results for the de Finetti problem mentioned in the previous paragraph have their counterparts for the impulse control problem, whereby the role of the barrier strategy is now played by the $(c_1; c_2)$ policy. In particular we will give a theorem similar to Theorem 2 in [47] and then use this theorem to show that a certain $(c_1; c_2)$ policy is optimal if the Lévy measure has a log-convex density. Moreover we give an example for which no $(c_1; c_2)$ policy is optimal.

The outline of this chapter is as follows. In the next section we review some properties concerning scale functions and in Section 3 we give sufficient conditions under which the $(c_1; c_2)$ policy is optimal. We treat the case when the Lévy measure has a log-convex density in Section 4 and show that the optimal strategy is formed by a unique $(c_1; c_2)$ policy. Further we show how to numerically find the optimal values of c_1 and c_2. In the last section we treat two explicit examples including one for which we show that no $(c_1; c_2)$ policy is optimal.

3.2 Scale functions

The scale function, defined via its Laplace transform given by (3.3), appears in almost all fluctuation identities for spectrally negative Lévy processes. As an example we mention the two sided exit above problem for which

$$\mathbb{E}_x \left(\mathrm{e}^{-q\tau_a^+} \mathbf{1}_{(\tau_a^+ < \tau_0^-)} \right) = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{3.4}$$

where $x \leq a$, $\tau_0^- = \inf\{t > 0 : X_t < 0\}$ and $\tau_a^+ = \inf\{t > 0 : X_t > a\}$. For background on scale functions we refer to Chapter 8 of Kyprianou [38].

We will now recall some properties of scale functions which we will need later on. The initial value of the scale function $W^{(q)}(0)$ is equal to $1/c$ when X is of bounded variation and is equal to 0 when X is of unbounded variation. Here $c = \gamma + \int_0^1 x\nu(\mathrm{d}x)$ stands for the drift of X when it is of bounded variation. The initial value of the derivative of the scale function is given by (see e.g. [40])

$$W^{(q)\prime}(0) := \lim_{x \downarrow 0} W^{(q)\prime}(x) = \begin{cases} 2/\sigma^2 & \text{when } \sigma > 0 \\ (\nu(0,\infty) + q)/c^2 & \text{when } \sigma = 0 \text{ and } \nu(0,\infty) < \infty \\ \infty & \text{otherwise.} \end{cases}$$

The scale function is log-concave for all $q \geq 0$ (see Chapter 2) and thus $\frac{W^{(q)\prime}(x)}{W^{(q)}(x)}$ is a decreasing function (in the weak sense). For $q \geq 0$ there is the following relation between scale functions

$$W^{(q)}(x) = \mathrm{e}^{\Phi(q)x} W_{\Phi(q)}(x), \tag{3.5}$$

where $W_{\Phi(q)}$ is the (0-)scale function of X under the measure $\mathbb{P}^{\Phi(q)}$ defined by

$$\left.\frac{\mathrm{d}\mathbb{P}^{\Phi(q)}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \mathrm{e}^{\Phi(q)X_t - qt}.$$

When the Lévy measure has a density which is log-convex, Kyprianou et al. [42] proved that $W^{(q)\prime}$ is strictly increasing and strictly convex on (a^*, ∞), where a^* is defined (as in Section 3.1) by

$$a^* = \sup\left\{a \geq 0 : W^{(q)\prime}(a) \leq W^{(q)\prime}(x) \text{ for all } x \geq 0\right\}$$

which is necessarily finite since $\lim_{x\to\infty} W^{(q)\prime}(x) = \infty$. Note also that by the log-concavity of $W^{(q)}$, it follows that $W^{(q)\prime}(a^*) > 0$. In the next proposition we show that slightly more can be said in this case.

Proposition 1. *If the Lévy measure has a log-convex density, then there exists* $0 \leq a' \leq a^*$ *such that* $W^{(q)\prime}$ *is strictly decreasing on* $(0, a')$*, constant on* (a', a^*) *and strictly increasing and strictly convex on* (a^*, ∞)*.*

Proof. Under the condition that the Lévy measure has a log-convex density, Kyprianou, Rivero & Song [42, Proof of Theorem 2.6] proved that $u_q(x) = \mathrm{e}^{\Phi(q)x} W'_{\Phi(q)}(x)$ is convex on $(0, \infty)$. Therefore the function $k : (0, \infty) \to \mathbb{R}$ defined by $k(y) = \Phi(q)W^{(q)\prime}(y) + u_q'^{+}(y)$ is well defined and $u_q'^{+}$ is an increasing function. Here $u_q'^{+}$ stands for the right-derivative of u_q. Using (3.5) we can write for arbitrary $a > 0$

$$\begin{aligned} k(y) =& \mathrm{e}^{\Phi(q)y} \left\{ \Phi^2(q) W_{\Phi(q)}(y) + 2\Phi(q) W'_{\Phi(q)}(y) + W''^{+}_{\Phi(q)}(y) \right\} \\ =& \mathrm{e}^{\Phi(q)y} \left\{ u'^{+}_q(y) \mathrm{e}^{-\Phi(q)y} + \int_a^y u'^{+}_q(z) \Phi(q) \mathrm{e}^{-\Phi(q)z} \mathrm{d}z \right. \\ & \left. + \Phi^2(q) W_{\Phi(q)}(a) + \Phi(q) W'_{\Phi(q)}(a) \right\}. \end{aligned}$$

Suppose now that $x, y > 0$ with $x \leq y$ and let

$$M = \frac{\int_x^y u'^{+}_q(z) \Phi(q) \mathrm{e}^{-\Phi(q)z} \mathrm{d}z}{\mathrm{e}^{-\Phi(q)x} - \mathrm{e}^{-\Phi(q)y}}.$$

Since u'^{+}_q is an increasing function, it follows that $M \in [u'^{+}_q(x), u'^{+}_q(y)]$ and hence we deduce

$$\begin{aligned} & k(y)\mathrm{e}^{-\Phi(q)y} - k(x)\mathrm{e}^{-\Phi(q)x} \\ & \qquad = u'^{+}_q(y)\mathrm{e}^{-\Phi(q)y} - u'^{+}_q(x)\mathrm{e}^{-\Phi(q)x} + M(\mathrm{e}^{-\Phi(q)x} - \mathrm{e}^{-\Phi(q)y}) \\ & \qquad \qquad = \left(u'^{+}_q(y) - M\right)\mathrm{e}^{-\Phi(q)y} + \left(M - u'^{+}_q(x)\right)\mathrm{e}^{-\Phi(q)x} \geq 0. \end{aligned}$$

Hence $y \mapsto k(y)\mathrm{e}^{-\Phi(q)y}$ is an increasing function and it follows that there exists $0 \leq a_1 \leq a_2 \leq \infty$ such that k is strictly negative on $(0, a_1)$, zero on (a_1, a_2) and strictly positive and strictly increasing on (a_2, ∞). Since we can use (3.5) to write for arbitrary $a > 0$

$$W^{(q)\prime}(x) = \Phi(q) W^{(q)}(x) + u_q(x) = W^{(q)\prime}(a) + \int_a^x k(y)\mathrm{d}y \quad \text{for } x > 0,$$

the statement of the proposition follows with $a' = a_1$ and $a^* = a_2$. Note also that the fact that $\lim_{x\to\infty} W^{(q)\prime}(x) = \infty$ forces a_2 to be finite. ■

Despite the fact that the scale function is in general only implicitly known through its Laplace transform, there are plenty examples of spectrally negative Lévy processes for which there exists closed-form expressions for their scale functions, although most of these examples only deal with the $q = 0$ scale function. In case no explicit formula for the scale function exists, one can use numerical methods as described in [63] to invert the Laplace transform of the scale function. We refer to the papers [28, 41, 42] for an updated account on explicit examples of scale functions and their properties.

3.3 Conditions for optimality of a $(c_1; c_2)$ policy

A description of a $(c_1; c_2)$ policy was given in Section 3.1. We now define this strategy mathematically. For $c_2 > c_1 \geq 0$, let $\{T_i^{c_1,c_2}, i = 1, 2, \ldots\}$ be the set of stopping times defined by

$$T_i^{c_1,c_2} = \inf\{t > 0 : X_t > X_0 \vee c_2 + (c_2 - c_1)(i-1)\}, \quad i = 1, 2, \ldots.$$

Then $\pi_{c_1,c_2} = \{L_t^{c_1,c_2} : t \geq 0\}$ is defined by

$$L_t^{c_1,c_2} = \mathbf{1}_{\{T_1^{c_1,c_2} < t\}} (X_0 \vee c_2 - c_1) + \sum_{i=2}^{\infty} \mathbf{1}_{\{T_i^{c_1,c_2} < t\}} (c_2 - c_1), \quad t \geq 0.$$

Note that with $U_t^{c_2,c_2} = X_t - L_t^{c_1,c_2}$ the above defined stopping times can then be identified as $T_1^{c_1,c_2} = \inf\{t > 0 : U_t^{c_1,c_2} > c_2\}$ and $T_{i+1}^{c_1,c_2} = \inf\{t > T_i^{c_1,c_2} : U_t^{c_1,c_2} > c_2\}$ for $i \geq 1$. Let v_{c_1,c_2} denote the value function of the strategy π_{c_1,c_2}.

Proposition 2. *The value function of the strategy π_{c_1,c_2} is given by*

$$v_{c_1,c_2}(x) = \begin{cases} \frac{c_2 - c_1 - \beta}{W^{(q)}(c_2) - W^{(q)}(c_1)} W^{(q)}(x) & \text{if } x \leq c_2, \\ x - c_1 - \beta + \frac{c_2 - c_1 - \beta}{W^{(q)}(c_2) - W^{(q)}(c_1)} W^{(q)}(c_1) & \text{if } x > c_2. \end{cases}$$

Proof. Since U^{c_1,c_2} is a Markov process, the proposition only needs to be proved for $0 \leq x \leq c_2$. Let $x \in [0, c_2]$. Since no dividends are paid out until X reaches the level c_2, we get by applying the strong Markov property at $\tau_{c_2}^+$ and (3.4)

$$v_{c_1,c_2}(x) = \mathbb{E}_x \left(\mathrm{e}^{-q\tau_{c_2}^+} \mathbf{1}_{\{\tau_{c_2}^+ < \tau_0^-\}} \right) v_{c_1,c_2}(c_2) = \frac{W^{(q)}(x)}{W^{(q)}(c_2)} v_{c_1,c_2}(c_2). \tag{3.6}$$

When $X_0 = c_2$, a dividend payment of size $c_2 - c_1$ is made immediately plus a transaction cost of size β is incurred and so by using the above equation we get

$$v_{c_1,c_2}(c_2) = c_2 - c_1 - \beta + v_{c_1,c_2}(c_1) = c_2 - c_1 - \beta + \frac{W^{(q)}(c_1)}{W^{(q)}(c_2)} v_{c_1,c_2}(c_2).$$

Now solving for $v_{c_1,c_2}(c_2)$ and plugging the result in (3.6) will give us the desired expression for $v_{c_1,c_2}(x)$. ■

We now want to find the values of $(c_1; c_2)$ which is likely to give us the best $(c_1; c_2)$ policy. A good guess would be the values of $(c_1; c_2)$ that minimizes

$$g(c_1, c_2) = \frac{W^{(q)}(c_2) - W^{(q)}(c_1)}{c_2 - c_1 - \beta},$$

where the domain of g is given by $\mathrm{dom}(g) = \{(c_1; c_2) : c_1 \geq 0, c_2 > c_1 + \beta\}$. Let C^* be the set of minimizers of g, i.e.

$$C^* = \{(c_1^*; c_2^*) \in \mathrm{dom}(g) : g(c_1^*, c_2^*) = \inf_{(c_1; c_2) \in \mathrm{dom}(g)} g(c_1, c_2)\}.$$

Proposition 3. *Assume $W^{(q)} \in C^1(0, \infty)$. Then the set C^* is non-empty and for each*

$(c_1^*; c_2^*) \in C^*$ we have

$$W^{(q)\prime}(c_2^*) = \frac{W^{(q)}(c_2^*) - W^{(q)}(c_1^*)}{c_2^* - c_1^* - \beta} \tag{3.7}$$

and further one of the following holds: (i) $W^{(q)\prime}(c_1^*) = W^{(q)\prime}(c_2^*)$ or (ii) $c_1^* = 0$.

Proof. First, by the mean value theorem

$$g(c_1, c_2) \geq \min_{x \in [c_1, c_2]} W^{(q)\prime}(x) \frac{c_2 - c_1}{c_2 - c_1 - \beta} > \min_{x \in [c_1, c_2]} W^{(q)\prime}(x) \tag{3.8}$$

and since $\lim_{x\to\infty} W^{(q)\prime}(x) = \infty$, this implies that an infimum of g is not reached when $c_1 \to \infty$. Hence there exists $C_1 > 0$ such that

$$\inf_{\mathrm{dom}(g)} g = \inf_{\mathrm{dom}(g), c_1 \leq C_1} g(c_1, c_2).$$

Second,

$$\begin{aligned}
\lim_{c_2\to\infty} \inf_{c_1 \in [0, C_1]} g(c_1, c_2) &= \lim_{c_2\to\infty} \inf_{c_1 \in [0, C_1]} \left(\frac{W^{(q)}(c_2)}{c_2 - c_1 - \beta} - \frac{W^{(q)}(c_1)}{c_2 - c_1 - \beta} \right) \\
&\geq \lim_{c_2\to\infty} \left(\frac{W^{(q)}(c_2)}{c_2 - \beta} - \frac{W^{(q)}(C_1)}{c_2 - C_1 - \beta} \right) = \infty
\end{aligned}$$

and hence an infimum of g is also not reached when $c_2 \to \infty$. Finally, by the mean value theorem

$$g(c_1, c_2) \geq \frac{W^{(q)\prime}(a^*)(c_2 - c_1)}{c_2 - c_1 - \beta} \geq W^{(q)\prime}(a^*) \frac{\beta}{c_2 - c_1 - \beta}$$

and thus since $W^{(q)\prime}(a^*) > 0$, an infimum of g is not reached when $(c_1; c_2)$ converges to the line $c_2 = c_1 + \beta$.

By the previous conclusions and the continuity of g it follows that C^* is non-empty and that for each $(c_1^*; c_2^*) \in C^*$ either $c_1^* = 0$ or $(c_1^*; c_2^*)$ is an interior point of $\mathrm{dom}(g)$. In the latter case it follows since g is partial differentiable in c_1 and c_2 (which follows from the hypothesis $W^{(q)} \in C^1(0, \infty)$) that

$$\frac{\partial g(c_1, c_2)}{\partial c_1}(c_1^*, c_2^*) = 0 \quad \text{and} \quad \frac{\partial g(c_1, c_2)}{\partial c_2}(c_1^*, c_2^*) = 0,$$

which in turn implies (3.7) and (i). In the case that $c_1^* = 0$, we have that c_2^* minimizes the function $g_0 : (\beta, \infty) \to (0, \infty)$ defined by $g_0(c_2) = g(0, c_2) = \frac{W^{(q)}(c_2) - W^{(q)}(0)}{c_2 - \beta}$ and hence $g_0'(c_2^*) = 0$ which implies (3.7). ■

Corollary 4. *Assume* $W^{(q)} \in C^1(0,\infty)$. *Then for each* $(c_1^*; c_2^*)$ *in* C^*

$$v_{c_1^*,c_2^*}(x) = \begin{cases} \frac{W^{(q)}(x)}{W^{(q)\prime}(c_2^*)} & \text{for } x \leq c_2^*, \\ x - c_2^* + \frac{W^{(q)}(c_2^*)}{W^{(q)\prime}(c_2^*)} & \text{for } x > c_2^* \end{cases}$$

and so $v_{c_1^*,c_2^*}(x) = v_{c_2^*}(x)$, *where* $v_{c_2^*}$ *is the value of the barrier strategy at level* c_2^* *in the de Finetti problem. Moreover,* $W^{(q)\prime}(c_2^*) > W^{(q)\prime}(a^*)$.

Proof. The corollary follows directly from the two previous propositions and (3.8). Note that the formula for the value of a barrier strategy was given by Avram et al. [5]. ■

We now give some definitions in order to state the verification lemma which we will need to show that under certain conditions a particular $(c_1; c_2)$ policy is optimal. Note that the definition of sufficiently smooth given below is slightly weaker than the one given in [47].

Definition 5. Given a spectrally negative Lévy process X, we call a function $f : [0,\infty) \to \mathbb{R}$ sufficiently smooth if f is right-continuous at zero, $f \in C^1(0,\infty)$ and additionally when X is of unbounded variation then f' can be written for any $x, a > 0$ as $f'(x) = f'(a) + \int_a^x h(y)\mathrm{d}y$ where $h : (0,\infty) \to \mathbb{R}$ is a measurable function which is bounded on sets of the form $[1/n, n]$, $n \geq 1$.

Definition 6. Given a spectrally negative Lévy process X with triplet (γ, σ, ν), let Γ be the operator acting on smooth functions f, defined by

$$\Gamma f(x) = \gamma f'(x) + \frac{\sigma^2}{2} h(x) + \int_{(0,\infty)} [f(x-y) - f(x) + f'(x) y \mathbf{1}_{\{0<y<1\}}] \nu(\mathrm{d}y),$$

where h is as in the previous definition in the case where X is of unbounded variation.

Remark 7. It can be verified that when f is sufficiently smooth then $\Gamma f(x)$ is absolutely convergent for all $x > 0$ and $\Gamma f(x) - \frac{1}{2}\sigma^2 h(x)$ is continuous for all $x \in (0,\infty)$, see e.g. [42, Lemma 4.1]. Further Kyprianou et al. [42, Theorem 2.6 and 2.9] show that the scale function $W^{(q)}$ is sufficiently smooth in any of the following three cases: (i) $\sigma > 0$, (ii) X is of bounded variation and the Lévy measure has no atoms and (iii) the Lévy measure has a log-convex density. Note that actually in case (i) the stronger statement that $W^{(q)} \in C^2(0,\infty)$ holds.

Lemma 8 (Verification lemma)**.** *Let* $\hat{\pi}$ *be an admissible strategy such that* $v_{\hat{\pi}}$ *is sufficiently smooth and the following three conditions hold*

(i) $(\Gamma - q) v_{\hat{\pi}}(x) \leq 0$ *for all* $x > 0$,

(ii) $v_{\hat{\pi}}(x) - v_{\hat{\pi}}(y) \geq x - y - \beta$ *for all* $x \geq y \geq 0$,

(iii) $v_{\hat{\pi}}(x) \geq 0$ *for all* $x > 0$.

Then $v_*(x) = v_{\hat\pi}(x)$ for all $x \geq 0$ and hence $\hat\pi$ is an optimal strategy for the control problem.

Proof. By definition of v_*, it follows that $v_{\hat\pi}(x) \leq v_*(x)$ for all $x \geq 0$. We write $w := v_{\hat\pi}$ and show that $w(x) \geq v_\pi(x)$ for all $\pi \in \Pi$ for all $x \geq 0$. First we suppose $x > 0$. We define for $\pi \in \Pi$ the stopping time σ_0^π by $\sigma_0^\pi = \inf\{t > 0 : U_t^\pi \leq 0\}$ and denote by Π_0 the following set of admissible dividend strategies

$$\Pi_0 = \Big\{\pi \in \Pi : \int_0^{\sigma_0^\pi} \mathrm{e}^{-qt}\mathrm{d}\left(L_t^\pi - \beta \mathbf{1}_{\{\Delta L_t^\pi > 0\}}\right) = \int_0^{\sigma^\pi} \mathrm{e}^{-qt}\mathrm{d}\left(L_t^\pi - \beta \mathbf{1}_{\{\Delta L_t^\pi > 0\}}\right) \ \mathbb{P}_x\text{-a.s. for all } x > 0\Big\}.$$

Note that when X is of unbounded variation, then $\sigma_0^\pi = \sigma^\pi$ a.s. and hence $\Pi_0 = \Pi$, but that Π_0 is a strictly smaller set than Π when X is of bounded variation. We claim that any $\pi \in \Pi$ can be approximated by dividend strategies from Π_0 in the sense that for all $\epsilon > 0$ there exists $\pi_\epsilon \in \Pi_0$ such that $v_\pi(x) \leq v_{\pi_\epsilon}(x) + \epsilon$ and therefore it is enough to show that $w(x) \geq v_\pi(x)$ for all $\pi \in \Pi_0$. Indeed, we can take π_ϵ to be the strategy where you do not pay out any dividends until L^π is at least ϵ, then at that time point pay out a lump-sum dividend equal to the size of the overshoot of L^π over ϵ and afterwards follow the same strategy as π until ruin occurs for the latter strategy at which point you stop paying out any dividends. It is hereby important to note that σ_0^π and σ^π are first entry times for the controlled risk process U^π; for the spectrally negative Lévy process X, the first entry time in $(-\infty, 0]$ is equal almost surely to the first entry time in $(-\infty, 0)$, provided $X_0 > 0$.

We now assume without loss of generality that $\pi \in \Pi_0$ and we let $\widetilde{L}^\pi, \widetilde{U}^\pi$ be the right-continuous modifications of L^π, U^π. Note that since the filtration $\mathbb{F}$ was assumed to be right-continuous, $\widetilde{L}^\pi$ and $\widetilde{U}^\pi$ are adapted processes. Let $(T_n)_{n\in\mathbb{N}}$ be the sequence of stopping times defined by $T_n = \inf\{t > 0 : \widetilde{U}_t^\pi > n \text{ or } \widetilde{U}_t^\pi < \frac{1}{n}\}$. Since $\widetilde{U}^\pi$ is a cadlag semi-martingale and w is sufficiently smooth - in particular w and its derivatives are bounded on $[1/n, n]$ for each n - we can when X is of bounded variation use the change of variables formula (cf. [51, Theorem II.31]) and when X is of unbounded variation use the extant second derivative Meyer-Itô formula (cf. [51, Theorem IV.71]) on $\mathrm{e}^{-q(t\wedge T_n)}w(\widetilde{U}_{t\wedge T_n}^\pi)$, to deduce

$$\begin{aligned}\mathrm{e}^{-q(t\wedge T_n)}w(\widetilde{U}_{t\wedge T_n}^\pi) - w(\widetilde{U}_0^\pi) &= \int_{0+}^{t\wedge T_n} \mathrm{e}^{-qs}\left(\frac{\sigma^2}{2}h(\widetilde{U}_{s-}^\pi) - qw(\widetilde{U}_{s-}^\pi)\right)\mathrm{d}s \\ &\quad + \int_{0+}^{t\wedge T_n} \mathrm{e}^{-qs}w'(\widetilde{U}_{s-}^\pi)\mathrm{d}X_s \\ &\quad + \sum_{0<s\leq t\wedge T_n} \mathrm{e}^{-qs}[\Delta w(\widetilde{U}_s^\pi) - w'(\widetilde{U}_{s-}^\pi)\Delta X_s],\end{aligned} \tag{3.9}$$

where we use the following notation: $\Delta\widetilde{U}_s^\pi = \widetilde{U}_s^\pi - \widetilde{U}_{s-}^\pi$, $\Delta w(\widetilde{U}_s^\pi) = w(\widetilde{U}_s^\pi) - w(\widetilde{U}_{s-}^\pi)$. Note that to derive (3.9), we used that $\widetilde{L}^\pi$ is a pure jump process. One can easily verify that

$$\sum_{0<s\le t\wedge T_n} \mathrm{e}^{-qs}[\Delta w(\widetilde{U}^\pi_s) - w'(\widetilde{U}^\pi_{s-})\Delta X_s] =$$

$$\sum_{0<s\le t\wedge T_n} \mathrm{e}^{-qs}[\Delta w(\widetilde{U}^\pi_{s-}+\Delta X_s) - w'(\widetilde{U}^\pi_{s-})\Delta X_s]$$

$$- \sum_{0<s\le t\wedge T_n} \mathrm{e}^{-qs}[w(X_s-\widetilde{L}^\pi_{s-}) - w(X_s-\widetilde{L}^\pi_{s-}-\Delta\widetilde{L}^\pi_s)]. \quad (3.10)$$

Since by admissibility of L^π, we have $\Delta\widetilde{L}^\pi_s \le X_s - \widetilde{L}^\pi_{s-}$ and so by assumption (ii)

$$w(X_s-\widetilde{L}^\pi_{s-})-w(X_s-\widetilde{L}^\pi_{s-}-\Delta\widetilde{L}^\pi_s) \ge \left(\Delta\widetilde{L}^\pi_s - \beta\right)\mathbf{1}_{\{\Delta\widetilde{L}^\pi_s\neq 0\}} \quad \text{for } 0<s<t\wedge T_n. \quad (3.11)$$

Combining (3.9), (3.10) and (3.11) leads to

$$\begin{aligned}
\mathrm{e}^{-q(t\wedge T_n)}w(\widetilde{U}^\pi_{t\wedge T_n}) - w(\widetilde{U}^\pi_0) &\le \int_{0+}^{t\wedge T_n} \mathrm{e}^{-qs}\left(\frac{\sigma^2}{2}h(\widetilde{U}^\pi_{s-}) - qw(\widetilde{U}^\pi_{s-})\right)\mathrm{d}s \\
&+ \int_{0+}^{t\wedge T_n} \mathrm{e}^{-qs}w'(\widetilde{U}^\pi_{s-})\mathrm{d}X_s + \sum_{0<s\le t\wedge T_n} \mathrm{e}^{-qs}[\Delta w(\widetilde{U}^\pi_{s-}+\Delta X_s) - w'(\widetilde{U}^\pi_{s-})\Delta X_s] \\
&- \sum_{0<s\le t\wedge T_n} \mathrm{e}^{-qs}\left(\Delta\widetilde{L}^\pi_s - \beta\right)\mathbf{1}_{\{\Delta\widetilde{L}^\pi_s\neq 0\}} \\
&= \int_{0+}^{t\wedge T_n} \mathrm{e}^{-qs}(\Gamma-q)w(\widetilde{U}^\pi_{s-})\mathrm{d}s - \sum_{0<s\le t\wedge T_n} \mathrm{e}^{-qs}\left(\Delta\widetilde{L}^\pi_s - \beta\right)\mathbf{1}_{\{\Delta\widetilde{L}^\pi_s\neq 0\}} \\
&+ \left\{\int_{0+}^{t\wedge T_n} \mathrm{e}^{-qs}w'(\widetilde{U}^\pi_{s-})\mathrm{d}[X_s - \gamma s - \sum_{0<u\le s}\Delta X_u \mathbf{1}_{\{|\Delta X_u|\ge 1\}}]\right\} \\
&+ \left\{\sum_{0<s\le t\wedge T_n} \mathrm{e}^{-qs}[\Delta w(\widetilde{U}^\pi_{s-}+\Delta X_s) - w'(\widetilde{U}^\pi_{s-})\Delta X_s\mathbf{1}_{\{|\Delta X_s|<1\}}]\right. \\
&\left. - \int_{0+}^{t\wedge T_n}\int_{0+}^{\infty} \mathrm{e}^{-qs}\left[w(\widetilde{U}^\pi_{s-}-y) - w(\widetilde{U}^\pi_{s-}) + w'(\widetilde{U}^\pi_{s-})y\mathbf{1}_{\{0<y<1\}}\right]\nu(\mathrm{d}y)\mathrm{d}s\right\}.
\end{aligned}$$

By the Lévy-Itô decomposition the expression between the first pair of curly brackets is a zero-mean martingale and by the compensation formula (cf. [38, Corollary 4.6]) the expression between the second pair of curly brackets is also a zero-mean martingale.

Using the assumptions (i) and (iii) and taking expectations we get

$$w(\widetilde{U}^\pi_0) \ge \mathbb{E}_x\left(\sum_{0<s\le t\wedge T_n} \mathrm{e}^{-qs}\left(\Delta\widetilde{L}^\pi_s - \beta\right)\mathbf{1}_{\{\Delta\widetilde{L}^\pi_s\neq 0\}}\right).$$

Letting t and n go to infinity and using the monotone convergence theorem we get

$$w(\widetilde{U}_0^\pi) \geq \mathbb{E}_x\left(\sum_{0<s\leq\sigma^\pi} \mathrm{e}^{-qs}\left(\Delta\widetilde{L}_s^\pi - \beta\right)\mathbf{1}_{\{\Delta\widetilde{L}_s^\pi\neq 0\}}\right).$$

Note that we used here that $T_n \nearrow \sigma_0^\pi$ $\mathbb{P}_x$-a.s. and that $\pi \in \Pi_0$. Further we have

$$w(\widetilde{U}_0^\pi) = w(x - L_{0+}^\pi) \leq w(x) - \left(L_{0+}^\pi - \beta\right)\mathbf{1}_{\{L_{0+}^\pi>0\}},$$

where the inequality is due to assumption (ii). Combining this with

$$\begin{aligned}&\mathbb{E}_x\left(\sum_{0<s\leq\sigma^\pi} \mathrm{e}^{-qs}\left(\Delta\widetilde{L}_s^\pi - \beta\right)\mathbf{1}_{\{\Delta\widetilde{L}_s^\pi\neq 0\}}\right)\\ &\qquad= \mathbb{E}_x\left(\sum_{0\leq s\leq\sigma^\pi} \mathrm{e}^{-qs}\left(\Delta L_s^\pi - \beta\right)\mathbf{1}_{\{\Delta L_s^\pi\neq 0\}}\right) - \left(L_{0+}^\pi - \beta\right)\mathbf{1}_{\{L_{0+}^\pi>0\}}\\ &\qquad= v_\pi(x) - \left(L_{0+}^\pi - \beta\right)\mathbf{1}_{\{L_{0+}^\pi>0\}},\end{aligned}$$

we deduce $w(x) \geq v_\pi(x)$ and hence it follows that $w(x) \geq v_*(x)$ for all $x > 0$.

To finish the proof, note that v_* is an increasing function and hence because w is right-continuous at zero $v_*(0) \leq \lim_{x\downarrow 0} v_*(x) \leq \lim_{x\downarrow 0} w(x) = w(0)$. ∎

Remark 9. When $\sigma > 0$, condition (i) in Lemma 8 can be relaxed in the sense that the inequality only needs to hold for a.e. $x > 0$ instead of for all $x > 0$. Indeed with $w = v_{\hat{\pi}}$, let $A = \{x \in (0,\infty) : (\Gamma - q)w(x) > 0\}$ and $B = \{s \in [0,t] : \widetilde{U}_s^\pi \in A\}$. If we assume that $\sigma > 0$ and $\mathrm{Leb}(A) = 0$ ($\mathrm{Leb}(\cdot)$ being the Lebesgue measure), then by using the occupation formula for the semi-martingale local time (see e.g. [51, Corollary 1, p.219]), we get a.s.

$$\int_0^t \mathbf{1}_{\{s\in B\}}\sigma^2 \mathrm{d}s = \int_0^t \mathbf{1}_{\{s\in B\}}\mathrm{d}[\widetilde{U}^\pi, \widetilde{U}^\pi]_s^c = \int_{-\infty}^{\infty} \mathcal{L}_t^a \mathbf{1}_{\{a\in A\}}\mathrm{d}a = 0$$

with $\mathcal{L}^a$ being the semi-martingale local time at a of $\widetilde{U}^\pi$. It follows that $\mathrm{Leb}(B) = 0$ and hence $\int_{0+}^{t\wedge T_n} \mathrm{e}^{-qs}(\Gamma - q)w(\widetilde{U}_{s-}^\pi)\mathrm{d}s \leq 0$ almost surely. Therefore the proof of Lemma 8 still works and moreover this shows that the above verification lemma does not depend on the choice of h.

Lemma 10. *Let $(c_1^*; c_2^*) \in C^*$. Then for $x \geq y \geq 0$,*

$$v_{c_1^*,c_2^*}(x) - v_{c_1^*,c_2^*}(y) \geq x - y - \beta.$$

Proof. Note that since $v_{c_1^*,c_2^*}$ is an increasing function, we can assume without loss of generality that $x - y > \beta$. First suppose $x \geq y \geq c_2^*$, then $v_{c_1^*,c_2^*}(x) - v_{c_1^*,c_2^*}(y) = x - y \geq$

$x - y - \beta$. Second, if $c_2^* \geq x \geq y$, then

$$v_{c_1^*,c_2^*}(x) - v_{c_1^*,c_2^*}(y) = \frac{W^{(q)}(x) - W^{(q)}(y)}{W^{(q)\prime}(c_2^*)} \geq x - y - \beta, \tag{3.12}$$

where the inequality follows since $(c_1^*; c_2^*) \in C^*$ and therefore with the help of (3.7)

$$W^{(q)\prime}(c_2^*) = \frac{W^{(q)}(c_2^*) - W^{(q)}(c_1^*)}{c_2^* - c_1^* - \beta} \leq \frac{W^{(q)}(x) - W^{(q)}(y)}{x - y - \beta}.$$

Finally, suppose $x \geq c_2^* \geq y$, then using Corollary 4

$$\begin{aligned} v_{c_1^*,c_2^*}(x) - v_{c_1^*,c_2^*}(y) =& x - c_2^* + \frac{W^{(q)}(c_2^*) - W^{(q)}(y)}{W^{(q)\prime}(c_2^*)} \\ \geq& x - c_2^* + c_2^* - y - \beta, \end{aligned}$$

where the inequality follows from (3.12). ■

The theorem below gives sufficient conditions for which a particular $(c_1; c_2)$ policy is optimal for the impulse control problem and is similar in nature to Theorem 2 in [47] which concerns optimality of the barrier strategy at a^* for the de Finetti problem.

Theorem 11. *Suppose that $W^{(q)}$ is sufficiently smooth and that there exists $(c_1^*; c_2^*) \in C^*$ such that*

$$W^{(q)\prime}(a) \leq W^{(q)\prime}(b) \quad \textit{for all } c_2^* \leq a \leq b. \tag{3.13}$$

Then the strategy $\pi_{c_1^*,c_2^*}$ is an optimal strategy for the impulse control problem.

Proof. We prove the theorem by showing that $v_{c_1^*,c_2^*} = v_{c_2^*}$ (see Corollary 4) satisfies the conditions of Lemma 8. First note that $W^{(q)}$ being sufficiently smooth implies that $v_{c_2^*}$ is sufficiently smooth. Condition (iii) from Lemma 8 is trivial and condition (ii) follows from Lemma 10. We now prove condition (i). Let $\tau_{0,a} = \inf\{t > 0 : X_t > a \text{ or } X_t < 0\}$ where $a > 0$. Since for all $0 < x < a$, $\left\{e^{-q(t\wedge\tau_{0,a})}W^{(q)}(X_{t\wedge\tau_{0,a}})\right\}_{t\geq 0}$ is a $\mathbb{P}_x$-martingale (see e.g. [38, p.229]), one deduces by an application of Itô's formula that for all $0 < x < a$

$$\int_0^{t\wedge\tau_{0,a}} e^{-qs}(\Gamma - q)W^{(q)}(X_s)ds = 0 \quad \mathbb{P}_x\text{-almost surely for all } t \geq 0. \tag{3.14}$$

Since $W^{(q)}$ is sufficiently smooth and when $\sigma > 0$ even twice continuously differentiable, it follows that $(\Gamma - q)W^{(q)}$ is continuous on $(0, \infty)$ (see Remark 7). This together with (3.14) and the right-continuity of the paths of X gives us that $(\Gamma - q)W^{(q)}(x) = 0$ for all $x > 0$. Since on $(0, c_2^*)$, $v_{c_2^*}$ and its derivatives are equal up to a multiplicative constant to $W^{(q)}$ and its derivatives, it follows that $(\Gamma - q)v_{c_2^*}(x) = 0$ for $0 < x < c_2^*$. For $x > c_2^*$ the property that $(\Gamma - q)v_{c_2^*}(x) \leq 0$ follows by mimicking the proof of Theorem 2 in Loeffen [47]. Note that it is here that one uses condition (3.13). Finally, when $\sigma = 0$ we have that $(\Gamma - q)v_{c_2^*}$ is continuous (see Remark 7) and therefore $(\Gamma - q)v_{c_2^*}(c_2^*) = 0$, which finishes the proof in this case. When $\sigma > 0$, one can either pick a smart choice for h w.r.t. $v_{c_2^*}$ in Definition 5 such that $(\Gamma - q)v_{c_2^*}(c_2^*) \leq 0$ or one can note that by Remark 9, condition (i) only needs to hold almost everywhere in order to conclude that the proof is done also in this case. ■

3.4 Log-convex density

Throughout this section it is assumed that the Lévy measure has a log-convex density. Let a' be as in Proposition 1. We know then that $W^{(q)\prime}$ is strictly decreasing on $(0, a')$, constant on (a', a^*) and strictly increasing on (a^*, ∞). Moreover $W^{(q)\prime\prime+}(x)$ and $W^{(q)\prime\prime-}(x)$ exist for all $x > 0$ and so in particular the scale function is sufficiently smooth. Here $W^{(q)\prime\prime+}$ and $W^{(q)\prime\prime-}$ stand for respectively the right- and left-derivative of $W^{(q)\prime}$.

It is then easy to see from Proposition 3 and (3.8) that for each $(c_1^*, c_2^*) \in C^*$ we have $c_1^* \leq a'$ and $c_2^* > a^*$ and hence by Theorem 11 the strategy $\pi_{c_1^*, c_2^*}$ is optimal. Indeed, when $c_1^* > 0$ then $W^{(q)\prime}(c_1^*) = W^{(q)\prime}(c_2^*)$ and thus since $W^{(q)\prime}(c_2^*) > W^{(q)\prime}(a^*)$ we must have $c_1^* < a'$ and $c_2^* > a^*$. When $c_1^* = 0$, then by (3.7) and (3.8) it follows that c_2^* cannot be smaller or equal to a^*.

Further it is straightforward to show that C^* consists of only one element and hence there is a unique (c_1, c_2) policy which is optimal for the control problem. Indeed, suppose that (c_1, c_2) and (c_1', c_2') are both in C^*. By (3.7) we then have $W^{(q)\prime}(c_2) = W^{(q)\prime}(c_2')$ and since $c_2, c_2' > a^*$ and $W^{(q)\prime}$ is increasing on (a^*, ∞), this implies that $c_2 = c_2'$. Similar arguments show that c_1 and c_1' can only be different if one of them is zero and the other strictly positive. Suppose without loss of generality that $c_1' = 0$ and $c_1 > 0$. Then by Proposition 3, $W^{(q)\prime}(c_1) = W^{(q)\prime}(c_2) = (W^{(q)}(c_2) - W^{(q)}(0))/(c_2 - \beta)$ and hence by using Proposition 3 again, the mean value theorem and $W^{(q)\prime}$ being strictly decreasing on $(0, c_1)$, we get the following contradiction

$$\begin{aligned} W^{(q)\prime}(c_1) =& \frac{\left\{W^{(q)\prime}(c_1)(c_2 - \beta) + W^{(q)}(0)\right\} - W^{(q)}(c_1)}{c_2 - c_1 - \beta} \\ =& \frac{W^{(q)\prime}(c_1)(c_2 - \beta) - W^{(q)\prime}(\xi)c_1}{c_2 - c_1 - \beta} \\ <& W^{(q)\prime}(c_1). \end{aligned}$$

Here $\xi \in (0, c_1)$ is the number such that $W^{(q)\prime}(\xi)c_1 = W^{(q)}(c_1) - W^{(q)}(0)$. It follows that c_1 has to be equal to c_1'.

We now denote by (c_1^*, c_2^*) the unique element of C^* and give some conditions which specify whether $c_1^* = 0$ or $c_1^* > 0$. We first introduce some new functions and parameters. Let $\varsigma_2 : (0, a') \to (a^*, \infty)$ be the function implicitly defined by $W^{(q)\prime}(x) = W^{(q)\prime}(\varsigma_2(x))$. Then ς_2 is a strictly decreasing function. This together with the fact that

$W^{(q)\prime}$ is left- and right-differentiable and strictly increasing on (a^*, ∞) implies that

$$\begin{aligned}\lim_{x\downarrow a}\frac{\varsigma_2(x)-\varsigma_2(a)}{x-a} &= \lim_{x\downarrow a}\frac{\varsigma_2(x)-\varsigma_2(a)}{W^{(q)\prime}(\varsigma_2(x))-W^{(q)\prime}(\varsigma_2(a))}\frac{W^{(q)\prime}(x)-W^{(q)\prime}(a)}{x-a}\\ &= \lim_{y\uparrow\varsigma_2(a)}\frac{y-\varsigma_2(a)}{W^{(q)\prime}(y)-W^{(q)\prime}(\varsigma_2(a))}\lim_{x\downarrow a}\frac{W^{(q)\prime}(x)-W^{(q)\prime}(a)}{x-a}\\ &= \frac{W^{(q)\prime\prime+}(a)}{W^{(q)\prime\prime-}(\varsigma_2(a))}\end{aligned}$$

for $a \in (0, a')$ and thus ς_2 is right-differentiable. (A similar calculation shows that ς_2 is left-differentiable.) Note that from the proof of Proposition 1 it follows that $W^{(q)\prime\prime+}(x) < 0$ for all $x < a^*$ and similarly we can deduce that $W^{(q)\prime\prime-}(x) > 0$ for all $x > a^*$. Hence $\varsigma_2'^{+}(x) < 0$ for all $x \in (0, a')$.

Let

$$c_{1max} = \inf\{c_1 \in (0, a') : \varsigma_2(c_1) - c_1 \leq \beta\},$$

where we put $c_{1max} = 0$ when $\lim_{x\downarrow 0}\varsigma_2(x) \leq \beta$ and then define the function $g_1 : (0, c_{1max}) \to (0, \infty)$ by

$$g_1(c_1) = g(c_1, \varsigma_2(c_1)) = \frac{W^{(q)}(\varsigma_2(c_1)) - W^{(q)}(c_1)}{\varsigma_2(c_1) - c_1 - \beta}.$$

Further define the function $g_0 : (\beta, \infty) \to (0, \infty)$ by

$$g_0(c_2) = g(0, c_2) = \frac{W^{(q)}(c_2) - W^{(q)}(0)}{c_2 - \beta}.$$

From the construction of the functions g_1 and g_0 and the existence of a unique minimizer for g, it is easy to see that if $c_1^* > 0$, then c_1^* is the unique minimizer of g_1 and that if $c_1^* = 0$, then c_2^* is the unique minimizer of g_0. For g_1 and g_0 we have the following differential equations

$$\begin{aligned}g_1'^{+}(c_1) &= \frac{\varsigma_2'^{+}(c_1) - 1}{\varsigma_2(c_1) - c_1 - \beta}\left(W^{(q)\prime}(c_1) - g_1(c_1)\right),\\ g_0'(c_2) &= \frac{1}{c_2 - \beta}\left(W^{(q)\prime}(c_2) - g_0(c_2)\right)\end{aligned}$$

and hence we get since $\varsigma_2'^{+}(c_1) < 0$,

$$\begin{aligned}g_1'^{+}(c_1) < 0(> 0, = 0) \ \text{ iff } \ g_1(c_1) < W^{(q)\prime}(c_1)(> W^{(q)\prime}(c_1), = W^{(q)\prime}(c_1)),\\ g_0'(c_2) < 0(> 0, = 0) \ \text{ iff } \ g_0(c_2) > W^{(q)\prime}(c_2)(< W^{(q)\prime}(c_2), = W^{(q)\prime}(c_2)).\end{aligned} \tag{3.15}$$

We now show that g_0 has a unique minimizer. Note that $\lim_{x\downarrow\beta} g_0(x) = \infty > W^{(q)\prime}(\beta)$ and that further for x large enough, $g_0(x) \leq \frac{W^{(q)}(x)}{(1-\epsilon)x}$, for any $\epsilon \in (0, 1)$ and by (3.5) $W^{(q)\prime}(x) \geq \Phi(q)W^{(q)}(x)$, which implies that $g_0(x) < W^{(q)\prime}(x)$ for x large enough. This combined with (3.15), the behaviour of $W^{(q)\prime}$ and (3.8) implies that that there exists a unique point $\hat{c}_2 \in (\beta \vee a^*, \infty)$ such that g_0 is strictly decreasing on $(\beta, \hat{c}_2)$ and strictly increasing on $(\hat{c}_2, \infty)$. Further, we have $g_0(\hat{c}_2) = W^{(q)\prime}(\hat{c}_2)$. Hence if $c_1^* = 0$, then $c_2^* = \hat{c}_2$.

Note that when $W^{(q)\prime}(0) < \infty$ and $a' > 0$, then $\varsigma_2(0) := \lim_{x\downarrow 0} \varsigma_2(x) < \infty$ and therefore the following parameter β_{max} is well defined,

$$\beta_{max} = \begin{cases} \infty & \text{if } W^{(q)\prime}(0) = \infty, \\ \varsigma_2(0) - \frac{W^{(q)}(\varsigma_2(0)) - W^{(q)}(0)}{W^{(q)\prime}(0)} & \text{if } W^{(q)\prime}(0) < \infty \text{ and } a' > 0. \end{cases}$$

Consider now the following three cases: (i) $a' > 0$ and $\beta < \beta_{max}$, (ii) $a' > 0$ and $\beta \geq \beta_{max} < \infty$ and (iii) $a' = 0$.

Suppose we are in case (i). We show that then g_1 also has a unique minimizer. When $\beta_{max} = \infty$, we have $W^{(q)\prime}(0) = \infty$ and when $\beta_{max} < \infty$, then

$$g_1(0) = \frac{W^{(q)}(\varsigma_2(0)) - W^{(q)}(0)}{\varsigma_2(0) - \beta} < \frac{W^{(q)}(\varsigma_2(0)) - W^{(q)}(0)}{\varsigma_2(0) - \beta_{max}} = W^{(q)\prime}(0).$$

This together with $\lim_{x\uparrow c_{1max}} g_1(x) = \infty > W^{(q)\prime}(c_{1max})$, (3.15) and the fact that $W^{(q)\prime}$ is strictly decreasing on $(0, a')$ implies that there exists a unique point $\hat{c}_1 \in (0, c_{1max})$ such that g_1 is strictly decreasing on $(0, \hat{c}_1)$ and strictly increasing on $(\hat{c}_1, c_{1max})$. Also $g_1(\hat{c}_1) = W^{(q)\prime}(\hat{c}_1)$.

From earlier considerations we now see that in case (i), $(c_1^*; c_2^*)$ is either equal to $(0; \hat{c}_2)$ or $(\hat{c}_1; \varsigma_2(\hat{c}_1))$. We will show that $(c_1^*; c_2^*)$ is equal to the latter. First note that by (3.15) and $\hat{c}_1$ being strictly positive, we have $g_1(\hat{c}_1) = W^{(q)\prime}(\hat{c}_1) < W^{(q)\prime}(0)$. This implies that if $g_0(\hat{c}_2) \geq W^{(q)\prime}(0)$, then $g_1(\hat{c}_1) < g_0(\hat{c}_2)$ and so g is minimized in $(\hat{c}_1; \varsigma_2(\hat{c}_1))$. Assume now that $g_0(\hat{c}_2) < W^{(q)\prime}(0)$. Then we have

$$\lim_{x\downarrow 0} \frac{\partial}{\partial x} g(x, \hat{c}_2) = \frac{1}{\hat{c}_2 - \beta} \left(g_0(\hat{c}_2) - W^{(q)\prime}(0) \right) < 0$$

and hence g is not minimized in $(0; \hat{c}_2)$. It follows that $(c_1^*; c_2^*) = (\hat{c}_1; \varsigma_2(\hat{c}_1))$.

Now assume that we are in case (ii). Then $\lim_{x\downarrow 0} g_1(x) \geq W^{(q)\prime}(0)$ and hence by (3.15) and the fact that $W^{(q)\prime}$ is strictly decreasing on $(0, a')$, we have that g_1 is strictly increasing on $(0, c_{1max})$. Hence $(c_1^*; c_2^*) = (0; \hat{c}_2)$.

Finally, suppose that we are in case (iii). Then $W^{(q)\prime}$ is an increasing function on $(0, \infty)$ and hence we conclude $(c_1^*; c_2^*) = (0; \hat{c}_2)$.

We put the conclusions of this section in the following theorem.

Theorem 12. *If the Lévy measure has a log-convex density, then there is a unique $(c_1; c_2)$ policy which is optimal for the impulse control problem. Further, $c_1^* = 0$ if and only if $\beta \geq \beta_{max}$ or $a' = 0$, where c_1^* is the unique optimal value of c_1.*

3.5 Examples

In order to obtain the (candidate) optimal $(c_1; c_2)$ policy one has to find the element(s) in C^*. In order to do this one first needs to evaluate the scale function which often has to be done by inverting the Laplace transform via numerical methods. But, even when there is a (simple) explicit expression for the scale function, it is not possible to give an explicit formula for the optimal parameters c_1^* and c_2^* (if they exist); see e.g. Theorem B in Jeanblanc and Shiryaev [34] for the case when X is a Brownian motion plus drift. Hence one has to resort to numerical methods to find the minimizer(s) of g. One possibility is to minimize the function g over c_1 and c_2 via a numerical program, but it might be that one ends up with a local instead of a global minimum. However, from the previous section we know that when the Lévy measure has a log-convex density, we can find the optimal parameters by minimizing either g_0 or g_1, both being functions of just one variable and with only one local minimum. In case the Lévy measure does not have a log-convex density, the element(s) of C^* might still be found by applying some of the methods described in Section 3.4 locally. We give an example of both cases. The figures and calculations in these examples are all made with the help of Matlab.

Example 1 The first example concerns the case when X is a spectrally negative stable process with index $\alpha \in (1,2)$. Its Laplace exponent is given by $\psi(\theta) = \theta^\alpha$. An explicit expression for its scale function was found by Bertoin [9] and is given by $W^{(q)}(x) = \alpha x^{\alpha-1} E'_\alpha(qx^\alpha)$ for $q, x \geq 0$, where E'_α is the derivative of the Mittag-Leffler function of index α given by $E_\alpha(x) = \sum_{n=0}^\infty \frac{x^n}{\Gamma(1+\alpha n)}$. Since the Lévy measure of a stable process, given by $\nu(\mathrm{d}x) = x^{-1-\alpha}\mathrm{d}x$, has a completely monotone density and $W^{(q)\prime}(0) = \infty = \beta_{max}$, we know by Theorem 12 that the set C^* consists of exactly one point, denoted by $(c_1^*; c_2^*)$ and that $c_1^* > 0$. Further we know from Section 3.4 that c_1^* is the unique minimum of the function g_1 on $(0, c_{1max})$ and the only intersection point of $W^{(q)\prime}$ and g_1; moreover, c_2^* is given by the unique point in (a^*, ∞) such that $W^{(q)\prime}(c_2^*) = W^{(q)\prime}(c_1^*)$. In our example the parameters are chosen as follows: $\alpha = 1.5$, $q = 0.1$ and $\beta = 1$. In Figure 3-1 the graphs of $W^{(q)\prime}$ and g_1 are plotted and the optimal levels are found to be equal to $(c_1^*; c_2^*) = (0.41; 4.85)$, whereas the parameter $c_{1max} = 1.33$.

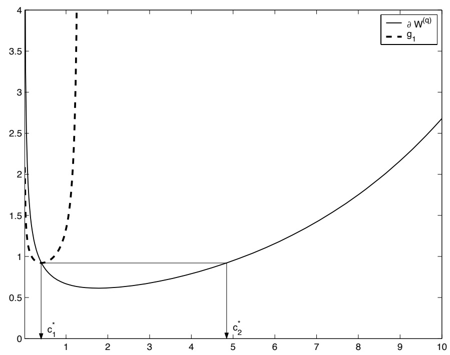

Figure 3-1: Stable with index 1.5

Example 2 For the second example we consider, we let X be a Cramér-Lundberg risk process as in (3.1) with Lévy measure given by $\nu(\mathrm{d}x) = \lambda\alpha^2 x \mathrm{e}^{-\alpha x}$. This means that the claims are Erlang$(2,\alpha)$ distributed. The scale function for X, which can be derived by the method of partial fraction, is given by

$$W^{(q)}(x) = \sum_{i=1}^{3} D_i \mathrm{e}^{\theta_i x}, \quad x \geq 0,$$

where $\{\theta_i : i = 1,2,3\}$ are the (distinct) roots of

$$\psi(\theta) - q = c\theta - \lambda + \frac{\lambda\alpha^2}{(\alpha+\theta)^2} - q,$$

with $\theta_1 > 0$ and $\theta_2, \theta_3 < 0$ and where $\{D_i : i = 1,2,3\}$ are given by $D_i = 1/\psi'(\theta_i)$. We now choose the parameters as follows: $c = 21.4$, $\lambda = 10$, $\alpha = 1$, $q = 0.1$ and for β we consider two cases, the case when $\beta = 0.015$ and $\beta = 0.2$. This example corresponds to the one in Azcue and Muler [6] for which they showed that the optimal strategy for the De Finetti problem is not a barrier strategy. Note that the Lévy measure does not have a log-convex density and therefore Theorem 12 does not apply.

The derivative of this scale function is plotted in Figure 3-2. We see from Figure 3-2 that the absolute minimum of $W^{(q)\prime}$ is attained at $x = 0$, but that this function further also has a local maximum and a second local minimum. Denote by a_1 resp. a_2 the point on the x-axis at which $W^{(q)\prime}$ has this local maximum resp. local minimum. Further denote by $\varsigma(a_1) \in (a_2, \infty)$ the point such that $W^{(q)\prime}(\varsigma(a_1)) = W^{(q)\prime}(a_1)$ and by $\varsigma(a_2) \in (0, a_1)$ the point such that $W^{(q)\prime}(\varsigma(a_2)) = W^{(q)\prime}(a_2)$. Note that one can see from the figure that these points exist.

We now want to find for a given β the elements of C^*; we can then use Theorem 11 to find out if a certain $(c_1; c_2)$ policy is optimal. Let $(c_1^*; c_2^*) \in C^*$. Since by Proposition 3 we must have $c_1^* = 0$ or $W^{(q)\prime}(c_1^*) = W^{(q)\prime}(c_2^*)$, it follows that there are three possible cases: (i) $c_1^* = 0$, (ii) $c_2^* \in (a_2, \varsigma(a_1))$, $c_1^* \in (\varsigma(a_2), a_2)$ and (iii) $c_2^* \in (a_1, a_2)$, $c_1^* \in (\varsigma(a_2), a_1)$. Though we will now show that (iii) cannot happen. Let $\tilde{\varsigma}_2 : (\varsigma(a_2), a_1) \to (a_1, a_2)$ be the function implicitly defined by $W^{(q)\prime}(\tilde{\varsigma}_2(x)) = W^{(q)\prime}(x)$ and let $\tilde{g}_1(x) = g(x, \tilde{\varsigma}_2(x)) = \frac{W^{(q)}(\tilde{\varsigma}_2(x)) - W^{(q)}(x)}{\tilde{\varsigma}_2(x) - x - \beta}$, where we take the domain of $\tilde{g}_1$ to be all $x \in (\varsigma(a_2), a_1)$ big enough such that the denominator of $\tilde{g}_1(x)$ is strictly positive. By the mean value theorem and the fact that $W^{(q)\prime}$ is strictly increasing on $(\varsigma(a_2), a_1)$ and strictly decreasing on (a_1, a_2), we get for all x in the domain of $\tilde{g}_1$

$$\tilde{g}_1(x) \geq \min_{\xi \in [x, \tilde{\varsigma}_2(x)]} W^{(q)\prime}(\xi) \frac{\tilde{\varsigma}_2(x) - x}{\tilde{\varsigma}_2(x) - x - \beta} > W^{(q)\prime}(x)$$

and thus by Proposition 3, case (iii) is not possible.

This leaves the remaining two cases (i) and (ii). To find out which value(s) $(c_1^*; c_2^*)$ takes, we introduce the function $\varsigma_2 : (\varsigma(a_2), a_2) \to (a_2, \varsigma(a_1))$ implicitly defined by $W^{(q)\prime}(\varsigma_2(x)) = W^{(q)\prime}(x)$ and let

$$D = \{c_1 \in (\varsigma(a_2), a_2) : \varsigma_2(c_1) - c_1 > \beta\}.$$

As in Section 3.4, we define the functions $g_1 : D \to (0, \infty)$ and $g_0 : (\beta, \infty) \to (0, \infty)$

given by $g_1(x) = g(x, \varsigma_2(x))$ and $g_0(x) = g(0, x)$. Note that the minimum of g will be equal to either the minimum of g_0 or to the minimum of g_1, whichever one lies lower; in case (i) the minimum of g_0 will lie lower, in case (ii) it will be the other way around.

In Figure 3-2 the graphs of g_0, g_1 and $W^{(q)\prime}$ are plotted for both values of β. For $\beta = 0.015$, we see that the minimum of g_0 lies lower than the minimum of g_1 and hence $c_1^* = 0$; the other level is then found to be equal to $c_2^* = 0.316$. Further we see that condition (3.13) of Theorem 11 is not satisfied. Hence we cannot conclude at this stage that an optimal strategy for the impulse control problem is formed by a $(c_1; c_2)$ policy.

But this does not mean that the strategy $\pi_{c_1^*,c_2^*}$ is not optimal, since Theorem 11 only gives sufficient, and not necessary, conditions for a particular $(c_1; c_2)$ policy to be optimal. To see that actually no $(c_1; c_2)$ policy is optimal for the impulse control problem, we first note that by the representation for the value function of a $(c_1; c_2)$ policy given in Proposition 2 and the fact that $(c_1^*; c_2^*)$ is the only minimizer of g, that for all $(c_1; c_2) \neq (c_1^*; c_2^*)$,

$$v_{c_1^*,c_2^*}(x) > v_{c_1,c_2}(x) \quad \text{for all } 0 < x \leq c_2^* \wedge c_2$$

and hence the only $(c_1; c_2)$ policy which can be optimal is the one with the levels equal to c_1^* and c_2^*. But if the parameters $c_1' = 8$ and $c_2' = 12$ are taken, one can calculate that for the starting value $x = 6$,

$$v_{c_1',c_2'}(6) = 8.235 > 7.883 = v_{c_1^*,c_2^*}(6)$$

and thus $\pi_{c_1^*,c_2^*}$ is not optimal and therefore no $(c_1; c_2)$ policy is optimal for the impulse control problem in this case.

In the other case when $\beta = 0.2$, we see that $(c_1^*; c_2^*)$ satisfies (3.13) and hence we can conclude that $\pi_{c_1^*,c_2^*}$ is an optimal strategy for the impulse control problem.

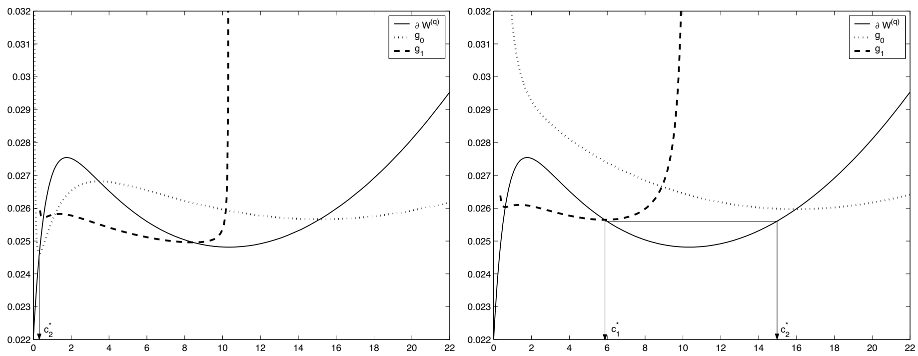

Figure 3-2: Cramér-Lundberg with Erlang$(2, 1)$ claims; left: $\beta = 0.015$, right: $\beta = 0.2$

Chapter 4

Refracted Lévy processes [1]

Motivated by classical considerations from risk theory, we investigate boundary crossing problems for refracted Lévy processes. The latter is a Lévy process whose dynamics change by subtracting off a fixed linear drift (of suitable size) whenever the aggregate process is above a pre-specified level. More formally, whenever it exists, a refracted Lévy process is described by the unique strong solution to the stochastic differential equation

$$\mathrm{d}U_t = -\delta \mathbf{1}_{\{U_t > b\}}\mathrm{d}t + \mathrm{d}X_t$$

where $X = \{X_t : t \geq 0\}$ is a Lévy process with law $\mathbb{P}$ and $b, \delta \in \mathbb{R}$ such that the resulting process U may visit the half line (b, ∞) with positive probability. We consider in particular the case that X is spectrally negative and establish a suite of identities for the case of one and two sided exit problems. All identities can be written in terms of the q-scale function of the driving Lévy process and its perturbed version describing motion above the level b. We remark on a number of applications of the obtained identities to (controlled) insurance risk processes.

4.1 Introduction

In this chapter we are interested in understanding the dynamics of a one-dimensional Lévy process when its path is perturbed in a simple way. Informally speaking, a linear drift at rate $\delta > 0$ is subtracted from the increments of a Lévy process whenever it exceeds a pre-specified positive level. More formally, suppose that $X = \{X_t : t \geq 0\}$ is Lévy process. If we denote the level by $b > 0$, a natural way to model such processes is to consider them as solutions to the stochastic differential equation

$$U_t = X_t - \delta \int_0^t \mathbf{1}_{\{U_s > b\}}\mathrm{d}s, \; t \geq 0 \tag{4.1}$$

[1] Based on joint work with A.E. Kyprianou.

assuming that at least a unique weak solution exists and such that $U = \{U_t : t \geq 0\}$ visits (b, ∞) with positive probability.

As a first treatment of (4.1) we shall restrict ourselves to the case that X is a process with no positive jumps and such that $-X$ is not a subordinator (also henceforth referred to as spectrally negative Lévy processes). As a special case of the latter, suppose that X may be written in the form

$$X_t = ct - S_t,\ t \geq 0 \tag{4.2}$$

where $c > 0$ is a constant and $S = \{S_t : t \geq 0\}$ is a pure jump compound Poisson subordinator. In that case it is easy to see that, under the hypothesis $c > \delta$, a solution to (4.1) may be constructed pathwise utilizing the fact that b is always crossed by X from below on the path of a linear part of the trajectory at a discrete set of times and is always crossed by $X_t - \delta t$ from above by a jump. Note that the trajectory of the process U is piecewise linear and 'bent' as it crosses the level b in the fashion that a light ray refracts from one medium to another. Inspired by this mental picture, we refer to solutions of (4.1) when the driving process X is a general one dimensional Lévy process as a *refracted Lévy process.* [2]

The special case (4.2) with compound Poisson jumps described above may also be seen as an example of a Cramér-Lundberg process as soon as $\mathbb{E}(X_1) > 0$. This provides a specific motivation for the study of the dynamics of (4.1). Indeed very recent studies of problems related to ruin in insurance risk has seen some preference to working with general spectrally negative Lévy processes in place of the classical Cramér-Lundberg process (which is itself an example of the former class). See for example [5, 21, 29, 30, 36, 37, 39, 54, 60]. This preference is largely thanks to the robust mathematical theory which has been developed around certain path decompositions of such processes as well as the meaningful interpretation of the general spectrally negative Lévy process as an insurance risk process (see for example the discussion in Section 4.10 or [36, 60]).

Under such a general model, the solution to the stochastic differential equation (4.1) may now be thought of as the aggregate of the insurance risk process when dividends are paid out at a rate δ whenever it exceeds the level b. Quantities which have been of persistent interest in the literature invariably pertain to the behaviour of (4.1) up to the ruin time $\kappa_0^- = \inf\{t > 0 : U_t < 0\}$. For example, the probability of ruin, $\mathbb{P}_x(\kappa_0^- < \infty)$, the net present value of the dividends paid out until ruin, $\mathbb{E}_x\left(\int_0^{\kappa_0^-} e^{-qt}\delta \mathbf{1}_{\{U_t > b\}} \mathrm{d}s\right)$, where $q > 0$, and the overshoot and undershoot at ruin, $\mathbb{P}_x(U_{\kappa_0^-} \in A, U_{\kappa_0^- -} \in B)$ where $A \subset (-\infty, 0)$, $B \subset [0, \infty)$ and $U_{\kappa_0^- -} = \lim_{t \uparrow \kappa_0^-} U_t$. Whilst expressions for the expected discounted value of the dividends, the Laplace transform of the ruin probability and the joint law of the undershoot and overshoot have been established before for refracted Lévy processes (cf. [45], [68], [71], [72], [74]) none of them go beyond the case of a compound Poisson jump structure. Moreover, existing identities in these cases are not often written in the modern language of scale functions (defined in Section 4.2 below).

[2]See for example the diagram on page 80 of [26] and the text above it which also makes reference to 'refraction' in the case of compound Poisson jumps. The article [25] also uses the terminology 'refraction' for the case that X is a linear Brownian motion.

The latter has some advantage given the analytical properties and families of examples that are now known for such functions (cf. [28, 41]).

Our objectives in this chapter are three fold. Firstly to show that refracted Lévy processes exist as a unique solution to (4.1) in the strong sense whenever X is a spectrally negative Lévy process (establishing the existence and uniqueness turns out to be not as simple as (4.1) looks for some cases of driving process X). Secondly to study their dynamics by establishing a suite of identities, written in terms of scale functions, related to one and two sided exit problems and thirdly to cite the relevance of such identities in context of a number of recent and classical applications of spectrally negative Lévy processes within the context of ruin problems.

The remainder of the chapter is structured as follows. In the next section we compile all of our main results together. Principally these consist of showing the existence and uniqueness of solutions to (4.1) which turns out to be in the strong sense. The principal difficulties that arise in handling (4.1) lie with the case that X has unbounded variation paths with no Gaussian part which seemingly falls outside of many standard results on existence and uniqueness of solutions to stochastic differential equations driven by Lévy processes. Then in Sections 4.3-9 we give the proofs of our main results. Finally, in Section 4.10 we return to the discussion on applications in (controlled) risk processes where explicit examples are given.

4.2 Main results

Henceforth the process $(X, \mathbb{P})$ will always denote a spectrally negative Lévy process. It is well known that spectral negativity allows us to talk about the Laplace exponent $\psi(\theta) = \log \mathbb{E}(e^{\theta X_1})$ for $\theta \geq 0$. Further the Laplace exponent is known to necessarily take the form

$$\psi(\theta) = \left\{\frac{1}{2}\sigma^2\theta^2\right\} + \left\{\gamma\theta - \int_{(1,\infty)} (1 - e^{-\theta x})\Pi(\mathrm{d}x)\right\} - \left\{\int_{(0,1)} (1 - e^{-\theta x} - \theta x)\Pi(\mathrm{d}x)\right\} \tag{4.3}$$

for $\gamma \in \mathbb{R}$, $\sigma \geq 0$ and Lévy measure Π satisfying

$$\Pi(-\infty, 0) = 0 \text{ and } \int_{(0,\infty)} (1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$$

(even though X only has negative jumps, for convenience we choose the Lévy measure to have only mass on the positive instead of the negative half line). Note that when $\Pi(0, \infty) = \infty$ the process X enjoys a countably infinite number of jumps over each finite time horizon. We shall also denote by $\{\mathbb{P}_x : x \in \mathbb{R}\}$ probabilities of X such that under $\mathbb{P}_x$, the process X is issued from x. Moreover, $\mathbb{E}_x$ will be the expectation operator associated to $\mathbb{P}_x$. For convenience in the case that $x = 0$ we shall always write $\mathbb{P}$ and $\mathbb{E}$ instead of $\mathbb{P}_0$ and $\mathbb{E}_0$.

We need the following hypothesis which will be in force throughout the remainder of the chapter:

(H) *the constant* $0 < \delta < \gamma + \int_{(0,1)} x\Pi(\mathrm{d}x)$ *if* X *has paths of bounded variation.*

Note that when X is a spectrally negative Lévy process with bounded variation paths, it can always be written in the form (4.2) where $c > 0$ and S is a pure jump subordinator. In that case, one sees that the hypothesis (H) simply says that $c > \delta > 0$. Write $\mathcal{S}$ for the space of spectrally negative Lévy processes satisfying (H). As well as writing $X \in \mathcal{S}$, we shall also abuse our notation and write $(\gamma, \sigma, \Pi) \in \mathcal{S}$ if (γ, σ, Π) is the triplet associated to X. Below, our first result concerns existence and uniqueness of solutions to (4.1).

Theorem 1. *For a fixed* $X_0 = x \in \mathbb{R}$, *there exists a unique strong solution to (4.1) when* X *is in the class* $\mathcal{S}$.

Remark 2. The existence of a unique strong solution to (4.1) is, to some extent, no surprise within the class of solutions driven by a general Lévy processes (not necessarily spectrally negative) with non-zero Gaussian component. Indeed for the latter class, existence of a strong unique solution is known, for example, from the work of Veretennikov [67] and Theorem 305 of the monograph of Situ [58]. The strength of Theorem 1 thus lies in dealing with the case that $X \in \mathcal{S}$ with no Gaussian component. In fact it will turn out that the real difficulties lie with the case that X has paths of unbounded variation with no Gaussian part. Such stochastic differential equations, in particular with drift coefficients which are neither Lipschitzian nor continuous but just bounded and measurable, are called degenerate and less seems to be known about them in the literature for the case of a driving Lévy process. See for example the remark proceeding Theorem III.2.34 on p159 of [33] as well as the presentation in [58].

Remark 3. Standard arguments show that the existence of a unique strong solution to (4.1) for each point of issue $x \in \mathbb{R}$, implies that U is a Strong Markov Process. Indeed suppose that T is a stopping time with respect to the natural filtration generated by X. Then define a process $\widehat{U}$ whose dynamics are those of $\{U_t : t \leq T\}$ issued from x and, on the event that $\{T < \infty\}$, it evolves on the time horizon $[T, \infty)$ as the unique solution, say $\widetilde{U}$, to (4.1) driven by the Lévy process $\widetilde{X} = \{X_{T+s} - X_T : s \geq 0\}$ when issued from the random starting point U_T. Note that by construction, on $\{T < \infty\}$, the dependency of $\{\widehat{U}_t : t \geq T\}$ on $\{\widehat{U}_t : t \leq T\}$ occurs only through the value $\widehat{U}_T = U_T$. Note also that for $t > 0$

$$\begin{aligned}
\widehat{U}_{T+t} &= \widetilde{U}_t \\
&= \widehat{U}_T + \widetilde{X}_t - \delta \int_0^t \mathbf{1}_{\{\widetilde{U}_s > b\}} \mathrm{d}s \\
&= x + X_T - \delta \int_0^T \mathbf{1}_{\{U_s > b\}} \mathrm{d}s + (X_{T+t} - X_T) - \delta \int_0^t \mathbf{1}_{\{\widehat{U}_{T+s} > b\}} \mathrm{d}s \\
&= x + X_{T+t} - \delta \int_0^{T+t} \mathbf{1}_{\{\widehat{U}_s > b\}} \mathrm{d}s
\end{aligned}$$

showing that $\widehat{U}$ solves (4.1) issued from x. Given there is strong uniqueness of solutions to (4.1), we may identify this solution to be $\widehat{U}$ and thus in possession of the Strong Markov Property.

Before proceeding to the promised fluctuation identities we must first recall a few facts concerning *scale functions* for spectrally negative Lévy processes, in terms of which all identities will be written. For each $q \geq 0$ define $W^{(q)} : \mathbb{R} \to [0, \infty)$ such that $W^{(q)}(x) = 0$ for all $x < 0$ and on $(0, \infty)$ $W^{(q)}$ is the unique continuous function with Laplace transform

$$\int_0^\infty e^{-\beta x} W^{(q)}(x)\,\mathrm{d}x = \frac{1}{\psi(\beta) - q} \tag{4.4}$$

for all $\beta > \Phi(q)$, where $\Phi(q) = \sup\{\theta \geq 0 : \psi(\theta) = q\}$. For convenience, we write W instead of $W^{(0)}$. Associated to the functions $W^{(q)}$ are the functions $Z^{(q)} : \mathbb{R} \to [1, \infty)$ defined by

$$Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)\,\mathrm{d}y$$

for $q \geq 0$. Together, the functions $W^{(q)}$ and $Z^{(q)}$ are collectively known as scale functions and predominantly appear in almost all fluctuation identities for spectrally negative Lévy processes. Indeed, several such identities which are well known (cf. Chapter 8 of [38]) are given in Theorem 23 in the Appendix and will be of repeated use throughout the remainder of the text.

Note also that by considering the Laplace transform of $W^{(q)}$, it is straightforward to deduce that $W^{(q)}(0+) = 1/c$ when X has bounded variation and therefore is (necessarily) written in the form $ct - S_t$ where $S = \{S_t : t \geq 0\}$ is a driftless subordinator and $c > 0$. Otherwise $W^{(q)}(0+) = 0$ for the case of unbounded variation. In all cases, if X drifts to ∞ then $W(\infty) = 1/\mathbb{E}(X_1)$. In general the derivative of the scale function is well defined except for at most countably many points. However, when X has unbounded variation or Π has no atoms, then for any $q \geq 0$, the restriction of $W^{(q)}$ to the positive half line belongs to $C^1(0, \infty)$. See for example [43] and [42]. In [13] it was also shown that when X has a Gaussian component ($\sigma > 0$), then $W^{(q)} \in C^2(0, \infty)$. Finally it is worth mentioning that as the Laplace exponent ψ is continuous in its Lévy triplet (continuity for the Lévy measure is understood in the sense of weak convergence), it follows by the Continuity Theorem for Laplace transforms that $W^{(q)}$ is also continuous in its underlying Lévy triplet. Moreover, performing an integration by parts, one obtains

$$\int_{[0,\infty)} e^{-\beta x} W^{(q)}(\mathrm{d}x) = \frac{\beta}{\psi(\beta) - q}$$

for all $\beta > \Phi(q)$ which, by the same reasoning as before, shows that $W^{(q)\prime}(x)$ is also continuous in its underlying Lévy triplet for all $x > 0$.

We are now ready to state our main conclusions with regard to certain fluctuation identities. In all theorems, the process $U = \{U_t : t \geq 0\}$ is the solution to (4.1) when driven by $X \in \mathcal{S}$ and the level $b > 0$. We shall frequently refer to the stopping times

$$\kappa_a^+ := \inf\{t > 0 : U_t > a\} \text{ and } \kappa_0^- := \inf\{t > 0 : U_t < 0\}.$$

where $a > 0$. Further, let $Y = \{Y_t := X_t - \delta t : t \geq 0\}$. For each $q \geq 0$, $W^{(q)}$ and $Z^{(q)}$ are the q-scale functions associated with X and $\mathbb{W}^{(q)}$ and $\mathbb{Z}^{(q)}$ is the q-scale function associated with Y. Moreover φ is defined as the right inverse of the Laplace exponent of Y so that

$$\varphi(q) = \sup\{\theta \geq 0 : \psi(\theta) - \delta\theta = q\}.$$

Theorem 4 (Two sided exit problem)**.**

(i) *For $q \geq 0$ and $0 \leq x, b \leq a$ we have*

$$\mathbb{E}_x(e^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\kappa_0^-\}}) = \frac{W^{(q)}(x) + \delta\mathbf{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y)\mathrm{d}y}{W^{(q)}(a) + \delta\int_b^a \mathbb{W}^{(q)}(a-y)W^{(q)\prime}(y)\mathrm{d}y}. \tag{4.5}$$

(ii) *For $q \geq 0$ and $0 \leq x, b \leq a$ we have*

$$\begin{aligned}\mathbb{E}_x\left(\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^-<\kappa_a^+\}}\right) &= Z^{(q)}(x) + \delta\mathbf{1}_{\{x\geq b\}}q\int_b^x \mathbb{W}^{(q)}(x-y)W^{(q)}(y)\mathrm{d}y \\ &- \frac{Z^{(q)}(a) + \delta q\int_b^a \mathbb{W}^{(q)}(a-y)W^{(q)}(y)\mathrm{d}y}{W^{(q)}(a) + \delta\int_b^a \mathbb{W}^{(q)}(a-y)W^{(q)\prime}(y)\mathrm{d}y} \\ &\cdot\left(W^{(q)}(x) + \delta\mathbf{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y)\mathrm{d}y\right).\end{aligned}$$

Theorem 5 (One sided exit problem)**.**

(i) *For $q \geq 0$ and $x, b \leq a$ we have*

$$\mathbb{E}_x(e^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+<\infty\}}) = \frac{\mathrm{e}^{\Phi(q)x} + \delta\Phi(q)\mathbf{1}_{\{x\geq b\}}\int_b^x \mathrm{e}^{\Phi(q)z}\mathbb{W}^{(q)}(x-z)\mathrm{d}z}{\mathrm{e}^{\Phi(q)a} + \delta\Phi(q)\int_b^a \mathrm{e}^{\Phi(q)z}\mathbb{W}^{(q)}(a-z)\mathrm{d}z}$$

(ii) *For $x, b \geq 0$ and $q > 0$*

$$\begin{aligned}\mathbb{E}_x(e^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^-<\infty\}}) &= Z^{(q)}(x) + \delta\mathbf{1}_{\{x\geq b\}}q\int_b^x \mathbb{W}^{(q)}(x-y)W^{(q)}(y)\mathrm{d}y \\ - \frac{q\int_b^\infty \mathrm{e}^{-\varphi(q)y}W^{(q)}(y)\mathrm{d}y}{\int_b^\infty \mathrm{e}^{-\varphi(q)y}W^{(q)\prime}(y)\mathrm{d}y} &\left(W^{(q)}(x) + \delta\mathbf{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y)\mathrm{d}y\right).\end{aligned}$$

If in addition $0 < \delta < \mathbb{E}(X_1)$, then letting $q \downarrow 0$ one has the ruin probability

$$\mathbb{P}_x(\kappa_0^- < \infty) = 1 - \frac{\mathbb{E}(X_1) - \delta}{1 - \delta W(b)}\left(W(x) + \delta\mathbf{1}_{\{x\geq b\}}\int_b^x \mathbb{W}(x-y)W'(y)\mathrm{d}y\right).$$

Theorem 6 (Resolvents)**.** *Fix the Borel set $B \subseteq \mathbb{R}$.*

(i) *For* $q \geq 0$ *and* $0 \leq x, b \leq a$,

$$\begin{aligned}
&\mathbb{E}_x\left(\int_0^\infty e^{-qt}\mathbf{1}_{\{U_t\in B,\, t<\kappa_0^-\wedge\kappa_a^+\}}\mathrm{d}s\right)\\
&= \int_{B\cap[b,a]}\left\{\frac{W^{(q)}(x)+\delta\mathbf{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z}{W^{(q)}(a)+\delta\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z}\mathbb{W}^{(q)}(a-y)\right.\\
&\left.-\mathbb{W}^{(q)}(x-y)\right\}\mathrm{d}y+\int_{B\cap[0,b)}\left\{\frac{W^{(q)}(x)+\delta\mathbf{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z}{W^{(q)}(a)+\delta\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z)\mathrm{d}z}\right.\\
&\cdot\left(W^{(q)}(a-y)+\delta\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z-y)\mathrm{d}z\right)\\
&\left.-\left(W^{(q)}(x-y)+\delta\mathbf{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z-y)\mathrm{d}z\right)\right\}\mathrm{d}y. \qquad (4.6)
\end{aligned}$$

(ii) *For* $x, b \geq 0$ *and* $q > 0$,

$$\begin{aligned}
&\mathbb{E}_x\left(\int_0^\infty e^{-qt}\mathbf{1}_{\{U_t\in B,\, t<\kappa_0^-\}}\mathrm{d}s\right)\\
&= \int_{B\cap[b,\infty)}\left\{\frac{W^{(q)}(x)+\delta\mathbf{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z}{\delta\int_b^\infty \mathrm{e}^{-\varphi(q)z}W^{(q)\prime}(z)\mathrm{d}z}\mathrm{e}^{-\varphi(q)y}\right.\\
&\left.-\mathbb{W}^{(q)}(x-y)\right\}\mathrm{d}y+\int_{B\cap[0,b)}\left\{\frac{\int_b^\infty \mathrm{e}^{-\varphi(q)z}W^{(q)\prime}(z-y)\mathrm{d}z}{\int_b^\infty \mathrm{e}^{-\varphi(q)z}W^{(q)\prime}(z)\mathrm{d}z}\right.\\
&\cdot\left(W^{(q)}(x)+\delta\mathbf{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z\right)\\
&\left.-\left(W^{(q)}(x-y)+\delta\mathbf{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z-y)\mathrm{d}z\right)\right\}\mathrm{d}y.
\end{aligned}$$

(iii) *For* $x, b \leq a$ *and* $q \geq 0$,

$$\begin{aligned}
&\mathbb{E}_x\left(\int_0^\infty e^{-qt}\mathbf{1}_{\{U_t\in B,\, t<\kappa_a^+\}}\mathrm{d}s\right)\\
&= \int_{B\cap[b,a]}\left\{\frac{\mathrm{e}^{\Phi(q)x}+\delta\Phi(q)\mathbf{1}_{\{x\geq b\}}\int_b^\infty \mathrm{e}^{\Phi(q)z}\mathbb{W}^{(q)}(x-z)\mathrm{d}z}{\mathrm{e}^{\Phi(q)a}+\delta\Phi(q)\int_b^\infty \mathrm{e}^{\Phi(q)z}\mathbb{W}^{(q)}(a-z)\mathrm{d}z}\mathbb{W}^{(q)}(a-y)\right.\\
&\left.-\mathbb{W}^{(q)}(x-y)\right\}\mathrm{d}y+\int_{B\cap(-\infty,b)}\left\{\frac{\mathrm{e}^{\Phi(q)x}+\delta\Phi(q)\mathbf{1}_{\{x\geq b\}}\int_b^\infty \mathrm{e}^{\Phi(q)z}\mathbb{W}^{(q)}(x-z)\mathrm{d}z}{\mathrm{e}^{\Phi(q)a}+\delta\Phi(q)\int_b^\infty \mathrm{e}^{\Phi(q)z}\mathbb{W}^{(q)}(a-z)\mathrm{d}z}\right.\\
&\cdot\left(W^{(q)}(a-y)+\delta\int_b^a \mathbb{W}^{(q)}(a-z)W^{(q)\prime}(z-y)\mathrm{d}z\right)\\
&\left.-\left(W^{(q)}(x-y)+\delta\mathbf{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z-y)\mathrm{d}z\right)\right\}\mathrm{d}y.
\end{aligned}$$

(iv) *For $x, b \in \mathbb{R}$ and $q > 0$,*

$$
\begin{aligned}
&\mathbb{E}_x\left(\int_0^\infty e^{-qt}\mathbf{1}_{\{U_t \in B\}}\mathrm{d}s\right) \\
&= \int_{B\cap[b,\infty)} \Bigg\{ \left(\mathrm{e}^{\Phi(q)(x-b)} + \delta\Phi(q)\mathrm{e}^{-\Phi(q)b}\mathbf{1}_{\{x\geq b\}}\int_b^x \mathrm{e}^{\Phi(q)z}\mathbb{W}^{(q)}(x-z)\mathrm{d}z\right) \\
&\qquad\qquad \cdot \frac{\varphi(q)-\Phi(q)}{\delta\Phi(q)}\mathrm{e}^{-\varphi(q)(y-b)} - \mathbb{W}^{(q)}(x-y)\Bigg\}\mathrm{d}y \\
&+ \int_{B\cap(-\infty,b)} \Bigg\{ \left(\mathrm{e}^{\Phi(q)(x-b)} + \delta\Phi(q)\mathrm{e}^{-\Phi(q)b}\mathbf{1}_{\{x\geq b\}}\int_b^x \mathrm{e}^{\Phi(q)z}\mathbb{W}^{(q)}(x-z)\mathrm{d}z\right) \\
&\qquad\qquad \cdot \frac{\varphi(q)-\Phi(q)}{\Phi(q)}\mathrm{e}^{\varphi(q)b}\int_b^\infty \mathrm{e}^{-\varphi(q)z}W^{(q)\prime}(z-y)\mathrm{d}z \\
&\qquad - \left(W^{(q)}(x-y) + \delta\mathbf{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z-y)\mathrm{d}z\right)\Bigg\}\mathrm{d}y.
\end{aligned}
$$

Theorem 7 (Creeping)**.** *For all $x, b \geq 0$ and $q > 0$,*

$$
\begin{aligned}
\mathbb{E}_x\left(\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{U_{\kappa_0^-}=0\}}\right) &= \frac{\sigma^2}{2}\Bigg\{W^{(q)\prime}(x) + \delta\mathbf{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime\prime}(z)\mathrm{d}z \\
&- \frac{\int_b^\infty \mathrm{e}^{-\varphi(q)z}W^{(q)\prime\prime}(z)\mathrm{d}z}{\int_b^\infty \mathrm{e}^{-\varphi(q)z}W^{(q)\prime}(z)\mathrm{d}z}\left(W^{(q)}(x) + \delta\mathbf{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-z)W^{(q)\prime}(z)\mathrm{d}z\right)\Bigg\},
\end{aligned}
$$

where the right hand side should be understood to be equal to zero when $\sigma = 0$.

Remark 8 (Identities in Theorems 6 and 7 when $q = 0$)*.* In the previous two theorems the parameter q was taken to be strictly positive for some of the identities. The case that $q = 0$ can be handled by taking limits as $q = 0$ on both left and right hand sides of these identities.

Remark 9*.* Note that in the above (and subsequent) expressions the derivative of the scale function appears, despite the fact that in general $W^{(q)\prime}$ may not be well defined for a countable number of points. However, since $W^{(q)\prime}$ only appears in the integrand of an ordinary Lebesgue integral, this does not present a problem.

Remark 10*.* As is the case with any presentation which expresses identities in terms of scale functions of spectrally negative Lévy processes, one may argue that one has only transferred the issue of 'solving the problem' into finding explicit examples of scale functions. Although in general scale functions are only semi-explicitly known through their Laplace transform, there are now quite a number of cases for which they can be calculated explicitly. See for example [28] and [41] for an updated account including a variety of new, explicit examples.

For the cases where no explicit formula is known for the scale function, [55] and [63] advocate simple methods of numerical Laplace inversion. Numerical computation of scale functions has already proved to be of practical value in, for example, the work of [18] and [27].

4.3 Proof of Theorem 1 in a subclass $\mathcal{S}^{(\infty)} \subseteq \mathcal{S}$

In this section, our objective is to define a subclass $\mathcal{S}^{(\infty)}$ of $\mathcal{S}$ and to show that Theorem 1 holds with $\mathcal{S}$ being replaced by this subclass $\mathcal{S}^{(\infty)}$. To this end, by taking advantage of the fact that when X has bounded variation, 0 is irregular for $(-\infty, 0)$, let us construct a pathwise solution to (4.1) for X having bounded variation and satisfying (H) (which will shortly turn out to be the unique solution within that class). Define the times T_n and S_n recursively as follows. We set $S_0 = 0$ and for $n = 1, 2, \ldots$

$$T_n = \inf\{t > S_{n-1} : X_t - \delta \sum_{i=1}^{n-1} (S_i - T_i) \geq b\},$$
$$S_n = \inf\{t > T_n : X_t - \delta \sum_{i=1}^{n-1} (S_i - T_i) - \delta(t - T_n) < b\}.$$

Since 0 is irregular for $(-\infty, 0)$, the difference between two consecutive times is strictly positive (except possibly for S_0 and T_1). Now we construct a solution to (4.1), $U = \{U_t : t \geq 0\}$, as follows. The process is issued from $X_0 = x$ and

$$U_t = \begin{cases} X_t - \delta \sum_{i=1}^{n} (S_i - T_i) & \text{for } t \in [S_n, T_{n+1}) \text{ and } n = 0, 1, 2, \ldots \\ X_t - \delta \sum_{i=1}^{n-1} (S_i - T_i) - \delta(t - T_n) & \text{for } t \in [T_n, S_n) \text{ and } n = 1, 2, \ldots \end{cases}$$

Note that in particular the times T_n and S_n for $n = 1, 2, \ldots$ can then be identified as

$$T_n = \inf\{t > S_{n-1} : U_t \geq b\}, \quad S_n = \inf\{t > T_n : U_t < b\}$$

and moreover

$$U_t = X_t - \delta \int_0^t \mathbf{1}_{\{U_s > b\}} \mathrm{d}s.$$

The next Lemma is the first step in showing that any solution to (4.1) which is driven by a spectrally negative Lévy process of unbounded variation, can be shown to exist uniquely as the result of strong approximation by solutions to (4.1) driven by a sequence of bounded variation processes respecting the condition (H). In order to state it we shall introduce some notation.

Definition 11. It is known (cf. p.210 of [10] for example) that for any spectrally negative Lévy process with unbounded variation paths, X, there exists a sequence of bounded variation spectrally negative Lévy processes, $X^{(n)}$, such that for each $t > 0$,

$$\lim_{n \uparrow \infty} \sup_{s \in [0,t]} |X_s^{(n)} - X_s| = 0$$

almost surely and moreover, when $X^{(n)}$ is written in the form (4.2) the drift coefficient tends to infinity as $n \uparrow \infty$. The latter fact implies that for all n sufficiently large, the sequence $X^{(n)}$ will automatically fulfil the condition (H). Such a sequence, $X^{(n)}$ will be referred to as *strongly approximating* for X. Rather trivially we may also talk of a strongly approximating sequence for processes of bounded variation respecting (H).

Lemma 12. *Suppose that X is a spectrally negative Lévy process satisfying (H) and that $X^{(n)}$ is a strongly approximating sequence. Denote by $U^{(n)}$ the sequence of pathwise solutions associated with each $X^{(n)}$ which are constructed pathwise in the manner described above. Then there exists a stochastic process $U^{(\infty)} = \{U_t^{(\infty)} : t \geq 0\}$ such that for each fixed $t > 0$,*

$$\lim_{n\uparrow\infty} \sup_{s\in[0,t]} |U_s^{(n)} - U_s^{(\infty)}| = 0$$

almost surely.

***Proof*.** It suffices to give a proof for the case that X has paths of unbounded variation. Fix the constant $\eta > 0$. Let $N \in \mathbb{N}$ be such that for all $n, m \geq N$, $\sup_{s\in[0,t]} |X^{(n)}(s) - X^{(m)}(s)| < \eta$. We will prove that for each fixed $t > 0$

$$\sup_{s\in[0,t]} |U_s^{(n)} - U_s^{(m)}| < 2\eta \tag{4.7}$$

from which we deduce that $\{U_s^{(n)} : s \in [0,t]\}$ is a Cauchy sequence in the Banach space consisting of $D[0,t]$ equipped with the supremum norm where $D[0,t]$ is the space of cadlag mappings from $[0,t]$. Note that the limit $U^{(\infty)}$ does not depend on t. Indeed, if $U^{(\infty,t_i)}$ for $i = 1,2$ are the limits obtained over two different time horizons $0 < t_1 < t_2 < \infty$ then a simple application of the triangle inequality shows that

$$\sup_{s\in[0,t_1]} |U_s^{(\infty,t_1)} - U_s^{(\infty,t_2)}| \leq \lim_{n\uparrow\infty} \sup_{s\in[0,t_1]} |U_s^{(\infty,t_1)} - U_s^{(n)}| + \lim_{n\uparrow\infty} \sup_{s\in[0,t_1]} |U_s^{(\infty,t_2)} - U_s^{(n)}| = 0$$

almost surely.

Returning to the proof of (4.7), define $\Delta^{(n,m)}U_s = U_s^{(n)} - U_s^{(m)}$ and $\Delta^{(n,m)}X_s = X_s^{(n)} - X_s^{(m)}$. Moreover, set

$$A_s^{(n,m)} := \Delta^{(n,m)}U_s - \Delta^{(n,m)}X_s = \delta \int_0^s \left(\mathbf{1}_{\{U_v^{(m)} > b, U_v^{(n)} \leq b\}} - \mathbf{1}_{\{U_v^{(m)} \leq b, U_v^{(n)} > b\}} \right) \mathrm{d}v. \tag{4.8}$$

We shall proceed now to show that, almost surely $\sup_{s\in[0,t]} |A_s^{(n,m)}| \leq \eta$ from which (4.7) follows.

Suppose the latter claim is not true. Then since $A^{(n,m)}$ is continuous and $A_0^{(n,m)} = 0$ there exists $0 < s < t$ such that either (i) $A_s^{(n,m)} = \eta$ and for all $\epsilon > 0$ sufficiently small there exists $r \in (s, s+\epsilon)$ such that $A_r^{(n,m)} > \eta$ or (ii) $A_s^{(n,m)} = -\eta$ and and for all $\epsilon > 0$ sufficiently small there exists $r \in (s, s+\epsilon)$ such that $A_r^{(n,m)} < -\eta$.

In case (i) it follows that $\Delta^{(n,m)}U_s > 0$ since $\Delta^{(n,m)}X_s \in (-\eta, \eta)$ and thus by right-continuity there exists $\epsilon > 0$ such that $\Delta^{(n,m)}U_r > 0$ for all $r \in [s, s+\epsilon)$. Hence considering the integrand in (4.8), the first indicator is necessarily zero when $v \in$

$[s, s+\epsilon)$. It follows that $A_r \leq \eta$ for all $r \in [s, s+\epsilon)$ which forms a contradiction. A similar argument by contradiction excludes case (ii). ■

We may now introduce the class $\mathcal{S}^{(\infty)} \subseteq \mathcal{S}$ for which we will be able to prove that the statement of Theorem 1 holds.

Definition 13. The class $\mathcal{S}^{(\infty)} = \mathcal{S}^{(\infty)}(x)$ consists of all processes $X \in \mathcal{S}$ (issued from x) such that for the associated process $U^{(\infty)}$ it holds that $\mathbb{P}_x(U_t^{(\infty)} = b) = 0$ for Lebesgue almost every $t \geq 0$.

Remark 14. Note in particular that $\mathcal{S}^{(\infty)}$ contains all solutions to (4.1) for which X is of bounded variation satisfying (H).

Proposition 15. *When $X \in \mathcal{S}^{(\infty)}$, the process $U^{(\infty)}$ is the unique strong solution of* (4.1) *and consequently Theorem 1 holds when the class $\mathcal{S}$ is replaced by $\mathcal{S}^{(\infty)}$.*

Proof. The fact that $U^{(\infty)}$ is a strong solution to (4.1) is immediate as soon as it is clear that for each fixed $t > 0$

$$\lim_{n\uparrow\infty} \int_0^t \mathbf{1}_{\{U_s^{(n)} > b\}} \mathrm{d}s = \int_0^t \mathbf{1}_{\{U_s^{(\infty)} > b\}} \mathrm{d}s$$

almost surely. However this is an immediate consequence of Lemma 12 and the assumption that $X \in \mathcal{S}^{(\infty)}$.

For pathwise uniqueness of this solution we use an argument which is based on ideas found in Example 2.4 on p286 of [35]. Suppose that $U^{(1)}$ and $U^{(2)}$ are two strong solutions to (4.1) then writing

$$\Delta_t = U_t^{(1)} - U_t^{(2)} = -\delta \int_0^t (\mathbf{1}_{\{U_s^{(1)} > b\}} - \mathbf{1}_{\{U_s^{(2)} > b\}}) \mathrm{d}s$$

it follows from classical calculus that

$$\Delta_t^2 = -2\delta \int_0^t \Delta_s (\mathbf{1}_{\{U_s^{(1)} > b\}} - \mathbf{1}_{\{U_s^{(2)} > b\}}) \mathrm{d}s.$$

Now note that thanks to the fact that $\mathbf{1}_{\{x > b\}}$ is an increasing function, it follows from the above representation that, for all $t \geq 0$, $\Delta_t^2 \leq 0$ and hence $\Delta_t = 0$ almost surely. This concludes the proof of existence and uniqueness amongst the class of strong solutions. ■

4.4 A key analytical identity

The main goal of this section is to establish a key analytical identity which will play an important role throughout the remainder of the chapter.

Theorem 16. *Suppose X is a spectrally negative Lévy process that has paths of bounded variation and let $0 < \delta < c$, where $c = \gamma + \int_{(0,1)} x\Pi(\mathrm{d}x)$. Then for $v \geq u > m \geq 0$*

$$\int_0^\infty \int_{(z,\infty)} W^{(q)}(z-\theta+m)\Pi(\mathrm{d}\theta)\left[\frac{\mathbb{W}^{(q)}(v-m-z)}{\mathbb{W}^{(q)}(v-m)}\mathbb{W}^{(q)}(u-m) - \mathbb{W}^{(q)}(u-m-z)\right]\mathrm{d}z$$

$$= -\frac{\mathbb{W}^{(q)}(u-m)}{\mathbb{W}^{(q)}(v-m)}\left(W^{(q)}(v) + \delta\int_m^v \mathbb{W}^{(q)}(v-z)W^{(q)\prime}(z)\mathrm{d}z\right)$$

$$+ W^{(q)}(u) + \delta\int_m^u \mathbb{W}^{(q)}(u-z)W^{(q)\prime}(z)\mathrm{d}z. \quad (4.9)$$

Proof. We denote $p(x,\delta) = \mathbb{E}_x(e^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+ < \kappa_0^-\}})$. Suppose that $x \leq b$. Then by conditioning on U until it passes above b, we have

$$p(x,\delta) = \mathbb{E}_x\left(\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_0^- > \tau_b^+\}}\right)p(b,\delta) = \frac{W^{(q)}(x)}{W^{(q)}(b)}p(b,\delta). \quad (4.10)$$

where in the last equality we have used (4.25) from the Appendix. Let now $x \geq b$ and $x \leq a$. Recall the process $Y = \{Y_t : t \geq 0\}$ where $Y_t = X_t - \delta t$ and denote by P_x the law of the process Y when issued from x (with E_x as the associated expectation operator). Using respectively that 0 is irregular for $(-\infty, 0)$ for Y, (4.25), the Strong Markov Property, (4.10) and (4.27), we have

$$\begin{aligned}
p(x,\delta) &= \mathrm{E}_x\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_0^- > \tau_a^+\}}\mathbf{1}_{\{\tau_b^- > \tau_a^+\}}\right) + \mathrm{E}_x\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_0^- > \tau_a^+\}}\mathbf{1}_{\{\tau_b^- < \tau_a^+\}}\right)\\
&= \frac{\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)} + \mathrm{E}_x\left(\mathrm{e}^{-q\tau_b^-}\mathbf{1}_{\{\tau_b^- < \tau_a^+\}}\mathbb{E}_{U_{\tau_b^-}}\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_0^- > \tau_a^+\}}\right)\right)\\
&= \frac{\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)} + \frac{p(b,\delta)}{W^{(q)}(b)}\mathrm{E}_x\left(\mathrm{e}^{-q\tau_b^-}\mathbf{1}_{\{\tau_b^- < \tau_a^+\}}W^{(q)}(Y_{\tau_b^-})\right)\\
&= \frac{\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)} + \frac{p(b,\delta)}{W^{(q)}(b)}\int_0^{a-b}\int_{(y,\infty)} W^{(q)}(b+y-\theta)\\
&\quad\cdot\left[\frac{\mathbb{W}^{(q)}(x-b)\mathbb{W}^{(q)}(a-b-y)}{\mathbb{W}^{(q)}(a-b)} - \mathbb{W}^{(q)}(x-b-y)\right]\Pi(\mathrm{d}\theta)\mathrm{d}y. \quad (4.11)
\end{aligned}$$

By setting $x = b$ in (4.11) we can now get an explicit expression for $p(b,\delta)$ using that $\mathbb{W}^{(q)}(0) = 1/(c-\delta)$

$$p(b,\delta) = W^{(q)}(b)\Bigg\{(c-\delta)\mathbb{W}^{(q)}(a-b)W^{(q)}(b)$$

$$-\int_0^{a-b}\int_{(y,\infty)} W^{(q)}(b+y-\theta)\mathbb{W}^{(q)}(a-b-y)\Pi(\mathrm{d}\theta)\mathrm{d}y\Bigg\}^{-1}. \quad (4.12)$$

We now start with the second step which concerns simplifying the term involving the double integral in above expression. Noting that for $\delta = 0$ (the case that there is no refraction) we have by (4.25) for all $x \geq 0$

$$p(b,0) = \mathbb{E}_b\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_0^- > \tau_a^+\}}\right) = \frac{W^{(q)}(b)}{W^{(q)}(a)}, \tag{4.13}$$

it follows from (4.12) and (4.13) that

$$\int_0^{a-b}\int_{(y,\infty)} W^{(q)}(y-\theta+b)W^{(q)}(a-b-y)\Pi(\mathrm{d}\theta)\mathrm{d}y$$
$$= cW^{(q)}(b)W^{(q)}(a-b) - W^{(q)}(a). \tag{4.14}$$

As $a \geq b$ is taken arbitrarily, we set $a = x$ in the above identity and take Laplace transforms from b to ∞ of both sides of the above expression. Denote by $\mathcal{L}_b$ the operator which satisfies $\mathcal{L}_b f[\lambda] := \int_b^\infty \mathrm{e}^{-\lambda x} f(x)\mathrm{d}x$. Let $\lambda > \Phi(q)$. For the left hand side of (4.14) we get by using Fubini's Theorem

$$\int_b^\infty \mathrm{e}^{-\lambda x}\int_0^\infty\int_{(y,\infty)} W^{(q)}(y-\theta+b)W^{(q)}(x-b-y)\mathrm{d}y\Pi(\mathrm{d}\theta)\mathrm{d}x$$
$$= \frac{\mathrm{e}^{-\lambda b}}{\psi(\lambda)-q}\int_0^\infty\int_{(y,\infty)} \mathrm{e}^{-\lambda y}W^{(q)}(y-\theta+b)\Pi(\mathrm{d}\theta)\mathrm{d}y.$$

For the right hand side of (4.14) we get

$$\int_b^\infty \mathrm{e}^{-\lambda x}\left(W^{(q)}(x-b)cW^{(q)}(b) - W^{(q)}(x)\right)\mathrm{d}x$$
$$= \frac{\mathrm{e}^{-\lambda b}}{\psi(\lambda)-q}cW^{(q)}(b) - \int_b^\infty \mathrm{e}^{-\lambda x}W^{(q)}(x)\mathrm{d}x$$

and so

$$\int_0^\infty\int_{(y,\infty)} \mathrm{e}^{-\lambda y}W^{(q)}(y-\theta+b)\Pi(\mathrm{d}\theta)\mathrm{d}y = cW^{(q)}(b) - (\psi(\lambda)-q)\mathrm{e}^{\lambda b}\mathcal{L}_b W^{(q)}[\lambda] \tag{4.15}$$

for $\lambda > \Phi(q)$. Our objective is now to use (4.15) to show that for $q \geq 0$, for $x \geq b$, we have

$$\int_0^\infty\int_{(y,\infty)} W^{(q)}(b+y-\theta)\Pi(\mathrm{d}\theta)\mathbb{W}^{(q)}(x-b-y)\mathrm{d}y$$
$$= -W^{(q)}(x) + (c-\delta)W^{(q)}(b)\mathbb{W}^{(q)}(x-b) - \delta\int_b^x \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y)\mathrm{d}y. \tag{4.16}$$

The latter identity then implies the statement of the theorem.

The equality in (4.16) follows by taking Laplace transforms on both sides in x. To this end note that by (4.15) it follows that the Laplace transform of the left hand side equals (for $\lambda > \varphi(q)$)

$$\int_b^\infty \mathrm{e}^{-\lambda x} \int_0^\infty \int_{(y,\infty)} W^{(q)}(b+y-\theta)\mathbb{W}^{(q)}(x-b-y)\Pi(\mathrm{d}\theta)\mathrm{d}y\mathrm{d}x$$
$$= \frac{\mathrm{e}^{-\lambda b}}{\psi(\lambda)-\delta\lambda-q}\left(cW^{(q)}(b)-(\psi(\lambda)-q)\mathrm{e}^{\lambda b}\mathcal{L}_b W^{(q)}\right). \quad (4.17)$$

Since $\mathcal{L}_b\left(\int_b^x f(x-y)g(y)\mathrm{d}y\right)[\lambda] = (\mathcal{L}_0 f)[\lambda](\mathcal{L}_b g)[\lambda]$ and $\mathcal{L}_b W^{(q)\prime}[\lambda] = \lambda\mathcal{L}_b W^{(q)}[\lambda] - \mathrm{e}^{-\lambda b}W^{(q)}(b)$ (which follows by integration by parts) it follows that the Laplace transform of the right hand side of (4.16) is equal to the right hand side of (4.17). Hence (4.16) holds for almost every $x \geq b$. Because both sides of (4.16) are continuous in x, we have that (4.16) holds for all $x \geq b$. ■

Remark 17. Close examination of the proof of the last theorem shows that one may easily obtain the identity in Theorem 4 (i) for the case that X has paths of bounded variation. Indeed plugging (4.9) into (4.11) and (4.12) and then using (4.12) in (4.10) and (4.11) gives the required identity.

Note however that although this method may be used to obtain other identities in the case of bounded variation paths, it is not sufficient to reach the entire family of identities presented in this chapter which explains why the forthcoming line of reasoning does not necessarily appeal directly to the observation above.

4.5 Some calculations for resolvents

In this section, we shall always take X to be of bounded variation satisfying (H). Recall for this class of driving Lévy processes, we know that (4.1) has a unique strong solution by Proposition 15 which has been described piecewise at the beginning of Section 4.3.

Let $\overline{U}_t = \sup_{0\leq s\leq t} U_t$, $\underline{U}_t = \inf_{0\leq s\leq t} U_t$ and define for $q > 0$ and Borel $B \in [0,\infty)$,

$$V^{(q)}(x,B) = \int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(U_t \in B,\, \overline{U}_t \leq a,\, \underline{U}_t \geq 0)\mathrm{d}t = \int_0^\infty \mathbb{P}_x(U_t \in B,\, t < \kappa_0^- \wedge \kappa_a^+)\mathrm{d}t.$$

The identity in Theorem 16 will be instrumental in establishing the following result.

Proposition 18. *When X is of bounded variation satisfying (H) the conclusion of Theorem 6 (i) holds.*

Proof. Recall that the process $Y = \{Y_t : t \geq 0\}$ is given by $Y_t = X_t - \delta t$ and its law is denote by P_x when issued from x. We have for $x \leq b$ by the Strong Markov Property, (4.25) and (4.26),

$$\begin{aligned}V^{(q)}(x,B) =&\mathbb{E}_x\left(\int_0^{\tau_b^+}\mathrm{e}^{-qt}\mathbf{1}_{\{U_t\in B, t<\tau_a^+\wedge\tau_0^-\}}\mathrm{d}t\right)\\&+\mathbb{E}_x\left(\int_{\tau_b^+}^{\infty}\mathrm{e}^{-qt}\mathbf{1}_{\{U_t\in B, t<\tau_a^+\wedge\tau_0^-, \tau_b^+<\tau_0^-\}}\mathrm{d}t\right)\\=&\mathbb{E}_x\left(\int_0^{\tau_b^+\wedge\tau_0^-}\mathrm{e}^{-qt}\mathbf{1}_{\{X_t\in B\}}\mathrm{d}t\right)+\mathbb{E}_x\left(\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tau_0^-\}}\right)V^{(q)}(b,B)\\=&\int_B\left(\frac{W^{(q)}(b-y)}{W^{(q)}(b)}W^{(q)}(x)-W^{(q)}(x-y)\right)\mathrm{d}y+\frac{W^{(q)}(x)}{W^{(q)}(b)}V^{(q)}(b,B).\end{aligned}\tag{4.18}$$

Moreover, for $b\le x\le a$ we have

$$\begin{aligned}&V^{(q)}(x,B)\\&=\mathrm{E}_x\left(\int_0^{\tau_b^-\wedge\tau_a^+}\mathrm{e}^{-qt}\mathbf{1}_{\{Y_t\in B\cap[b,a]\}}\mathrm{d}t\right)+\mathrm{E}_x\left(\mathbf{1}_{\{\tau_b^-<\tau_a^+\}}\int_{\tau_b^-}^{\tau_a^+\wedge\tau_0^-}\mathrm{e}^{-qt}\mathbf{1}_{\{U_t\in B\}}\mathrm{d}t\right)\\&=\int_0^\infty\mathrm{e}^{-qt}\mathbb{P}_x\left(Y_t\in B\cap[b,a], t<\tau_b^-\wedge\tau_a^+\right)\mathrm{d}t+\mathrm{E}_x\left(\mathbf{1}_{\{\tau_b^-<\tau_a^+\}}\mathrm{e}^{-q\tau_b^-}V^{(q)}(Y_{\tau_b^-},B)\right)\\&=\int_{B\cap[b,a]}\left(\frac{\mathbb{W}^{(q)}(a-z)}{\mathbb{W}^{(q)}(a-b)}\mathbb{W}^{(q)}(x-b)-\mathbb{W}^{(q)}(x-z)\right)\mathrm{d}z\\&\quad+\int_0^\infty\int_{(z,\infty)}\left\{\int_B\left[\frac{W^{(q)}(b-y)}{W^{(q)}(b)}W^{(q)}(z-\theta+b)-W^{(q)}(z-\theta+b-y)\right]\mathrm{d}y\right.\\&\quad\left.+\frac{V^{(q)}(b,B)}{W^{(q)}(b)}W^{(q)}(z-\theta+b)\right\}\\&\quad\cdot\left[\frac{\mathbb{W}^{(q)}(a-b-z)}{\mathbb{W}^{(q)}(a-b)}\mathbb{W}^{(q)}(x-b)-\mathbb{W}^{(q)}(x-b-z)\right]\Pi(\mathrm{d}\theta)\mathrm{d}z,\end{aligned}$$

where in the second equality we have used the Strong Markov Property and in the third equality (4.18), (4.26) and (4.27). Next we shall apply the identity proved in Theorem 16 twice in order to simplify the expression for $V^{(q)}(x,B)$, $a\ge x\ge b$. We use it once by setting $m=b$, $u=x$, $v=a$ and once by setting $m=b-y$ and $u=x-y$, $v=a-y$

for $y \in [0,b]$. One obtains

$$
\begin{aligned}
V^{(q)}(x,B) = &\int_{B\cap[b,a]} \left(\frac{\mathbb{W}^{(q)}(a-z)}{\mathbb{W}^{(q)}(a-b)} \mathbb{W}^{(q)}(x-b) - \mathbb{W}^{(q)}(x-z) \right) \mathrm{d}z \\
&+ \int_{B\cap[0,b)} \Bigg\{ \frac{W^{(q)}(b-y)}{W^{(q)}(b)} \Bigg(- \frac{\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)} \\
&\cdot \left(W^{(q)}(a) + \delta \int_b^a \mathbb{W}^{(q)}(a-z) W^{(q)\prime}(z) \mathrm{d}z \right) \\
&+ W^{(q)}(x) + \delta \int_b^x \mathbb{W}^{(q)}(x-z) W^{(q)\prime}(z) \mathrm{d}z \Bigg) \\
&- \Bigg(- \frac{\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)} \left(W^{(q)}(a-y) + \delta \int_{b-y}^{a-y} \mathbb{W}^{(q)}(a-y-z) W^{(q)\prime}(z) \mathrm{d}z \right) \\
&+ W^{(q)}(x-y) + \delta \int_{b-y}^{x-y} \mathbb{W}^{(q)}(x-y-z) W^{(q)\prime}(z) \mathrm{d}z \Bigg) \Bigg\} \mathrm{d}y \\
&+ \frac{V^{(q)}(b,B)}{W^{(q)}(b)} \Bigg(- \frac{\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)} \left(W^{(q)}(a) + \delta \int_b^a \mathbb{W}^{(q)}(a-z) W^{(q)\prime}(z) \mathrm{d}z \right) \\
&+ W^{(q)}(x) + \delta \int_b^x \mathbb{W}^{(q)}(x-z) W^{(q)\prime}(z) \mathrm{d}z \Bigg).
\end{aligned}
\tag{4.19}
$$

Setting $x = b$ in (4.19), one then gets an expression for $V^{(q)}(b,B)$ in terms of itself. Solving this and then putting the resulting expression for $V^{(q)}(b,B)$ in (4.18) and (4.19) leads to (4.6) which proves the proposition. ■

Keeping with the setting that X is a Lévy process of bounded variation fulfilling (H), we may proceed to use the conclusion of the above proposition to establish an identity for the resolvent of U without killing which we denote by

$$
R^{(q)}(x,B) = \int_0^\infty e^{-qt} \mathbb{P}_x(U_t \in B) \mathrm{d}t \tag{4.20}
$$

for $q > 0$ and Borel $B \in \mathbb{R}$.

Corollary 19. *The conclusion of Theorem 6 (iv) is valid when X has paths of bounded variation satisfying (H).*

Proof. By taking the expression given in Proposition 18 and letting $a \uparrow \infty$ one gets by the Monotone Convergence Theorem the expression for the one sided exit below resolvent given in Theorem 6 (ii) in case X is of bounded variation. It should be noted that here one uses the relation (cf. Chapter 8 of [38]) $W^{(q)}(x) = \mathrm{e}^{\Phi(q)x} W_{\Phi(q)}(x)$ (and similarly $\mathbb{W}^{(q)}(x) = \mathrm{e}^{\varphi(q)} \mathbb{W}_{\varphi(q)}(x)$), where $\Phi(q)$ is the right inverse of the Laplace exponent of X and $W_{\Phi(q)}$ plays the role of the $(q=0)$ scale function for the spectrally negative Lévy process with Laplace exponent $\psi(\theta + \Phi(q)) - q$. Moreover one should use the known fact that $W_{\Phi(q)}(\infty) < \infty$ when $q > 0$.

In the same spirit, replacing b by $b+\theta$, x by $x+\theta$, B by $B+\theta$ and then letting $\theta \uparrow \infty$ in the expression for the one sided exit below resolvent obtained above, one may recover the expression for $R^{(q)}(x,B)$ given in Theorem 6 (iv). Here one uses L'Hôpital's rule and the known fact that the (left-) derivative of $W_{\Phi(q)}$ is bounded on intervals of the form (x_0,∞) where $x_0>0$ and tends to zero at infinity. ■

We close this section with a result which says that if $X^{(n)}$ strongly approximates X and the latter has unbounded variation, then by taking $n \uparrow \infty$, even though we are not yet able to necessarily identify the limiting process $U^{(\infty)}$ as the solution to (4.1), the limit of the associated resolvents to $U^{(n)}$ say $R_n^{(q)}$ still exists as $n \uparrow \infty$ and it is absolutely continuous with density which is equal to the limiting density of $R_n^{(q)}$.

Lemma 20. *Suppose that X has paths of unbounded variation with strongly approximating sequence $X^{(n)}$. For $x \in \mathbb{R}$ and bounded interval B we have*

$$\lim_{n\uparrow\infty} R_n^{(q)}(x,B) = \int_B \lim_{n\uparrow\infty} r_n^{(q)}(x,y)\mathrm{d}y,$$

where $r_n^{(q)}(x,y)$ is the density of $R_n^{(q)}(x,\mathrm{d}y)$. In particular $\lim_{n\uparrow\infty} r_n^{(q)}(x,y)$ is equal to the density in the right hand side of (4.6).

Proof. The proof is a direct consequence of the Dominated Convergence Theorem and the fact that, by the Continuity Theorem for Laplace transforms, both $W^{(q)}$, $\mathbb{W}^{(q)}$ and $W^{(q)\prime}$ are continuous with respect to the Lévy triplet of the underlying Lévy process. ■

4.6 Proof of Theorem 1

Our objective in this section is to use the resolvents of the previous section to prove the following result.

Lemma 21. *It holds that $\mathcal{S}^{(\infty)}$ contains all spectrally negative Lévy processes of unbounded variation and hence by Remarks 2 and 14 and Proposition 15 it follows that Theorem 1 holds.*

Proof. It suffices to show that for all driving Lévy processes X with paths of unbounded variation we have that, when x is fixed, $\mathbb{P}_x(U_t^{(\infty)}=b)=0$ for Lebesgue almost every $t \geq 0$. In fact we shall prove something slightly more general (for future convenience).

Let X be strongly approximated by the sequence $X^{(n)}$. Note that for each $t,\eta>0$ and $a \in \mathbb{R}$, thanks to Lemma 12,

$$\begin{aligned}\{U_t^{(\infty)}=a\} &\subseteq \liminf_{n\uparrow\infty}\{U_t^{(n)} \in (a-\eta,a+\eta)\}\\ &:= \{U_t^{(n)} \in (a-\eta,a+\eta) \text{ eventually as } n\uparrow\infty\}.\end{aligned}$$

Standard measure theory (cf. Exercise 3.1.12 of [61]) now gives us for each $\eta>0$

$$\mathbb{P}_x(U_t^{(\infty)}=a) \leq \liminf_{n\uparrow\infty} \mathbb{P}_x(U_t^{(n)} \in (a-\eta,a+\eta)).$$

Now applying Fatou's Lemma followed by the conclusion of Lemma 20 we have for $q > 0$,

$$\begin{aligned}\int_0^\infty e^{-qt}\mathbb{P}_x(U_t^{(\infty)} = a)\mathrm{d}t &\leq \liminf_{n\uparrow\infty}\int_0^\infty e^{-qt}\mathbb{P}_x(U_t^{(n)} \in (a-\eta, a+\eta))\mathrm{d}t \\ &= \liminf_{n\uparrow\infty} R_n^{(q)}(x, (a-\eta, a+\eta)) \\ &= \int_{a-\eta}^{a+\eta} r^{(q)}(x,y)\mathrm{d}y\end{aligned}$$

where $r^{(q)}(x,y)$ is the density on the right hand side of (4.6). Note that, uniformly in η, the integral on the right hand side above is bounded by $1/q$ thanks to Lemma 20 and the fact that for all n, $R_n^{(q)}(x,\mathbb{R}) \leq 1/q$ on account of (4.20). Since the quantity η is arbitrary the required statement that $\mathbb{P}_x(U_t^{(\infty)} = a) = 0$ for Lebesgue almost every $t > 0$ follows. ■

Before concluding this section, it is worth registering the following corollary for the next section which follows directly from the conclusion and proof above.

Corollary 22. *For all $X \in \mathcal{S}$, we have for each given $x, a \in \mathbb{R}$ that the unique strong solution U to (4.1) satisfies $\mathbb{P}_x(U_t = a) = 0$ for Lebesgue almost every $t \geq 0$.*

4.7 Proof of Theorem 6

Firstly let us note that parts (ii), (iii) and (iv) follow from part (i) by taking limits much in the spirit of the proof of Corollary 19. As before such calculations are straightforward and hence, for the sake of brevity, are left to the reader.

To establish part (i) we have already seen that (4.6) is true for case that X has bounded variation and satisfies (H). To deal with the case that X has paths of unbounded variation we consider as usual a strongly approximating sequence $X^{(n)}$. As before, we will write the left hand side of the identity in (4.6) when the driving process is $X^{(n)}$ in the form

$$V_n^{(q)}(x,B) = \int_0^\infty e^{-qt}\mathbb{P}_x(U_t^{(n)} \in B,\, \overline{U}_t^{(n)} \leq a,\, \underline{U}_t^{(n)} \geq 0)\mathrm{d}t$$

for $q > 0$ and Borel $B \in \mathbb{R}$.

Recall from Lemma 21 that $U^{(\infty)}$ defined in Lemma 12 is the unique solution to (4.1). We shall henceforth refer to it as just U. In the spirit of Lemma 20 we may prove that for open intervals B,

$$\lim_{n\uparrow\infty} V_n^{(q)}(x,B) = \int_B v^{(q)}(x,y)\mathrm{d}y \tag{4.21}$$

where $v^{(q)}(x,y)$ is the density which appears on the right hand side of the identity

(4.6). It is known (see for example Lemma 13.4.1 of [69]) that

$$|\overline{U}_t^{(n)} - \overline{U}_t| \vee |\underline{U}_t^{(n)} - \underline{U}_t| \leq \sup_{s\in[0,t]} |U_s^{(n)} - U_s|$$

Thanks to Lemma 12, it follows that for each $t > 0$, in the almost sure sense,

$$\lim_{n\uparrow\infty}(U_t^{(n)}, \overline{U}_t^{(n)}, \underline{U}_t^{(n)}) = (U_t, \overline{U}_t, \underline{U}_t).$$

This tells us that by the Dominated Convergence Theorem

$$\lim_{n\uparrow\infty} V_n^{(q)}(x, B) = \int_0^\infty e^{-qt}\mathbb{P}_x(U_t \in B, \overline{U}_t \leq a, \underline{U}_t \geq 0)\mathrm{d}t$$

provided the boundary of $\{U_t \in B,\, \overline{U}_t \leq a,\, \underline{U}_t \geq 0\}$ is not charged by $\mathbb{P}_x$. To rule the latter out it suffices to show that

$$\mathbb{P}_x(U_t \in \partial B) = \mathbb{P}_x(\underline{U}_t = 0) = \mathbb{P}_x(\overline{U}_t = a) = 0. \tag{4.22}$$

for Lebesgue almost every $t \geq 0$.

To this end, note that if $\kappa^{[0,\epsilon)} = \inf\{t > 0 : U_t \in [0, \epsilon)\}$ where $\epsilon > 0$ then it is easy to see from (4.1) that on $\{\kappa^{[0,\epsilon)} < \infty\}$

$$U_{\kappa^{[0,\epsilon)}+s} \leq U_{\kappa^{[0,\epsilon)}} + \widetilde{X}_s$$

where $\widetilde{X}$ is a copy of X which is independent of $\{U_s : s \leq \kappa^{[0,\epsilon)}\}$. Let $g(x,t) = \mathbb{P}_x(\inf_{s\leq t} X_s \geq 0)$ and note that it is increasing in x and decreasing in t. Moreover, by regularity of $(-\infty, 0)$ for X we have that for each fixed $t > 0$, $g(0,t) = 0$. It follows that for all $\epsilon > 0$

$$\begin{aligned}
&\mathbb{P}_x(\underline{U}_t = 0)\\
&\quad\leq \mathbb{E}_x[\mathbf{1}_{\{\kappa^{[0,\epsilon)}\leq t\}}\mathbb{P}_x(U_{\kappa^{[0,\epsilon)}} + \inf_{s\leq t-\kappa^{[0,\epsilon)}} \widetilde{X}_s \geq 0|\mathcal{F}_{\kappa^{[0,\epsilon)}})]\\
&\quad\leq \mathbb{E}_x[\mathbf{1}_{\{\kappa^{[0,\epsilon)}\leq t\}}g(\epsilon, t - \kappa^{[0,\epsilon)})].
\end{aligned}\tag{4.23}$$

By monotonicity there exist $\kappa^{\{0\}} := \lim_{\epsilon\downarrow 0}\kappa^{[0,\epsilon)}$ and the event $\{\kappa^{\{0\}} = t\}$ implies almost surely that $U_{t-} = 0 = U_t$ where the last equality follows on account of the fact that t is a jump time with probability zero. Hence by dominated convergence

$$\mathbb{P}_x(\underline{U}_t = 0) \leq \mathbb{E}_x[\mathbf{1}_{\{\kappa^{\{0\}}<t\}}\lim_{\epsilon\downarrow 0} g(\epsilon, t - \kappa^{\{0\}})] + \mathbb{P}_x(U_t = 0).$$

The preceding remarks concerning $g(x,t)$ and the conclusion of Corollary 22 now imply that $\mathbb{P}_x(\underline{U}_t = 0) = 0$ for Lebesgue almost every $t \geq 0$. A similar argument can be employed to show that $\mathbb{P}_x(\overline{U}_t = a) = 0$ for Lebesgue almost every $t \geq 0$. It is also a simple consequence of Corollary 22 that $\mathbb{P}_x(U_t \in \partial B) = 0$ for Lebesgue almost every $t \geq 0$. Thus (4.22) is satisfied and referring back to (4.21) we see that the proof is complete. ■

4.8 Proof of Theorems 4 and 5

The idea of the proofs is to make use of the identities in parts (i)–(iv) of Theorem 6. We give only the important ideas of the proof as the details of the computations are straightforward and so left to the reader, again, for the sake of brevity. In doing so, one will need to make use of the following identity for $q, a \geq 0$

$$\delta \int_0^a \mathbb{W}^{(q)}(a-y) W^{(q)}(y)\mathrm{d}y = \int_0^a \mathbb{W}^{(q)}(y)\mathrm{d}y - \int_0^a W^{(q)}(y)\mathrm{d}y,$$

which can be proved by showing that the Laplace transforms on both sides are equal.

One obtains the result in Theorem 5 (ii) by noting that

$$\mathbb{E}_x\left(\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^- < \infty\}}\right) = 1 - \mathbb{P}_x(\underline{U}_{\mathbf{e}_q} \geq 0) = 1 - q\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(U_t \in \mathbb{R}, t < \kappa_0^-)\mathrm{d}t.$$

For the proof of Theorem 4 (i), it suffices to note that for $q > 0$, by applying the Strong Markov Property, one has that

$$\mathbb{P}_x(\underline{U}_{\mathbf{e}_q} \geq 0, \overline{U}_{\mathbf{e}_q} > a) = \mathbb{E}_x(\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+ < \kappa_0^-\}})\mathbb{P}_a(\underline{U}_{\mathbf{e}_q} \geq 0).$$

The first and the last probabilities above can be obtained directly from the potential measures given in Theorem 6 since

$$\begin{aligned}\mathbb{P}_x(\underline{U}_{\mathbf{e}_q} \geq 0, \overline{U}_{\mathbf{e}_q} > a) &= \mathbb{P}_x(\underline{U}_{\mathbf{e}_q} \geq 0) - \mathbb{P}_x(\underline{U}_{\mathbf{e}_q} \geq 0, \overline{U}_{\mathbf{e}_q} \leq a) \\ &= q\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(U_t \in \mathbb{R}, t < \tau_0^-) - q\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(U_t \in [0,a], t < \tau_a^+ \wedge \tau_0^-)\mathrm{d}t\end{aligned}$$

and

$$\mathbb{P}_a(\underline{U}_{\mathbf{e}_q} \geq 0) = q\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_a(U_t \in [0,\infty), t < \tau_0^-)\mathrm{d}t.$$

By using the Strong Markov Property for (4.1) at the specific stopping time κ_a^+ and the fact that $U_{\kappa_a^+} = a$ on $\{\kappa_a^+ < \infty\}$ we now have that

$$\begin{aligned}\mathbb{E}_x\left(\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^- < \kappa_a^+\}}\right) &= \mathbb{E}_x\left(\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^- < \infty\}}\right) - \mathbb{E}_x\left(\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_a^+ < \kappa_0^-\}}\right) \\ &= \mathbb{E}_x\left(\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^- < \infty\}}\right) - \mathbb{E}_x\left(\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+ < \kappa_0^-\}}\right)\mathbb{E}_a\left(\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{\kappa_0^- < \infty\}}\right),\end{aligned}$$

for $0 \leq x, b \leq a$. This gives the required identity in Theorem 4 (ii).

For part (i) of Theorem 5 one notes that

$$\mathbb{E}_x\left(\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+ < \infty\}}\right) = 1 - \mathbb{P}_x(\overline{U}_{\mathbf{e}_q} \leq a) = 1 - q\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(U_t \in (-\infty, a], t < \kappa_a^+)\mathrm{d}t.$$

However, it seems difficult to derive the required expression in this way. In place of this method one may obtain the result by using the expression for $\mathbb{E}_x(\mathrm{e}^{-q\kappa_a^+}\mathbf{1}_{\{\kappa_a^+ < \kappa_0^-\}})$, namely first replace x by $x + \theta$, a by $a + \theta$ and b by $b + \theta$ and then let $\theta \uparrow \infty$. ■

4.9 Proof of Theorem 7

It is a well established fact (cf. Chapter VI of [10]) that a spectrally negative Lévy process creeps downward if and only if it has a Gaussian component. For this reason it is obvious that the probability that U creeps downward is zero as soon as X has no Gaussian component. We therefore restrict ourselves to the case that X has a Gaussian component.

Suppose that for $x, b \geq a$, $w^{(q)}(x, y, a, b)$ is the resolvent density for U with killing on exiting the interval $[a, \infty)$. Note that by spatial homogeneity $w^{(q)}(x, y, a, b) = w^{(q)}(x - a, y - a, 0, b - a)$ and therefore an expression for this density is already given in Theorem 6. Since U is the sum of a continuous process and a Lévy process, it is quasi-left continuous and hence we can use Proposition 1(i) in [50] to deduce

$$\mathbb{E}_x\left(\mathrm{e}^{-q\kappa_0^-}\mathbf{1}_{\{U_{\kappa_0^-}=0\}}\right) = \lim_{\epsilon\downarrow 0}\mathbb{E}_x\left(\mathrm{e}^{-q\kappa^{\{0\}}}\mathbf{1}_{\{\kappa^{\{0\}}<\kappa_{-\epsilon}^-\}}\right) = \lim_{\epsilon\downarrow 0}\frac{w^{(q)}(x,0,-\epsilon,b)}{w^{(q)}(0,0,-\epsilon,b)}$$
$$= \lim_{\epsilon\downarrow 0}\frac{w^{(q)}(x+\epsilon,\epsilon,0,b+\epsilon)}{w^{(q)}(\epsilon,\epsilon,0,b+\epsilon)},$$

where $\kappa^{\{0\}} = \inf\{t > 0 : U_t = 0\}$. The limit can then be computed by using l'Hôptital's rule, the Dominated Convergence Theorem and the fact that $W^{(q)\prime}(0) = 2/\sigma^2$ when $\sigma > 0$. ■

4.10 Applications in ruin theory

As alluded to in the introduction, modern perspectives on the theory of ruin has seen preference for working with spectrally negative Lévy processes. Indeed one may understand the third bracket in (4.3) as the part of a risk process corresponding to countably infinite number of arbitrarily small claims compensated by a deterministic positive drift (which may be infinite in the case that $\int_{(0,1)} x\Pi(\mathrm{d}x) = \infty$) corresponding to the accumulation of premiums over an infinite number of contracts. Roughly speaking, the way in which claims occur is such that in any arbitrarily small period of time $\mathrm{d}t$, a claim of size x is made independently with probability $\Pi(\mathrm{d}x)\mathrm{d}t + o(\mathrm{d}t)$. The insurance company thus counterbalances such claims by ensuring that it collects premiums in such a way that in any $\mathrm{d}t$, $x\Pi(\mathrm{d}x)\mathrm{d}t$ of its income is devoted to the compensation of claims of size x. The second bracket in (4.3) we may understand as coming from large claims which occur occasionally and are compensated against by a steady income at rate $\gamma > 0$ as in the Cramér-Lundberg model. Here 'large' is taken to mean claims of size one or more. Finally the first bracket in (4.3) may be seen as a stochastic perturbation of the system of claims and premium income.

As mentioned earlier, a quantity which is of particular value is the probability of ruin. This is given precisely in the second half of Theorem 5 (i). Another quantity of interest mentioned in the introduction is the net present value of the dividends paid out until ruin. Such a quantity is easily obtained from Theorem 6 and it is equal to

$$\mathbb{E}_x\left(\int_0^{\kappa_0^-} e^{-qt}\delta \mathbf{1}_{\{U_t>b\}}\mathrm{d}s\right) = \frac{\delta}{q}\left(1-\mathbb{Z}^{(q)}(x-b)\right) + \frac{W^{(q)}(x)+\delta\mathbf{1}_{\{x\geq b\}}\int_b^x \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y)\mathrm{d}y}{\varphi(q)\int_0^\infty \mathrm{e}^{-\varphi(q)y}W^{(q)\prime}(y+b)\mathrm{d}y}. \quad (4.24)$$

As U is a semi-martingale whose jumps are described by the same Poisson point process of jumps which describes the jumps of the driving Lévy process, one may apply the compensation formula in a straightforward way together with the resolvent in part (ii) of Theorem 6 to deduce the following expression for the joint law of the overshoot and undershoot at ruin (see for example the spirit of the discussion at the beginning of Section 8.4 of [38]) in the case that $0<\delta<\mathbb{E}(X_1)$.

Let $A\subset(-\infty,0)$ and $B\subset[0,\infty)$ be Borel-sets and let $U_{\kappa_0^- -}=\lim_{t\uparrow\kappa_0^-}U_t$. For $x\in\mathbb{R}$

$$\begin{aligned}
&\mathbb{P}_x(U_{\kappa_0^-}\in A, U_{\kappa_0^- -}\in B)\\
&\quad=\int_B \Pi(y-A)\frac{1-\delta W(b-y)}{1-\delta W(b)}\mathrm{d}y\left(W(x)+\delta\mathbf{1}_{\{x\geq b\}}\int_b^x \mathbb{W}(x-z)W'(z)\mathrm{d}z\right)\\
&\quad-\int_{B\cap[0,b)}\Pi(y-A)\left(W(x-y)+\delta\mathbf{1}_{\{x\geq b\}}\int_b^x\mathbb{W}(x-z)W'(z-y)\mathrm{d}z\right)\mathrm{d}y\\
&\quad-\int_{B\cap[b,\infty)}\Pi(y-A)\mathbb{W}(x-y)\mathrm{d}y.
\end{aligned}$$

As mentioned in the introduction, expressions for the expected discounted value of the dividends, the Laplace transform of the ruin probability and the joint law of the undershoot and overshoot have been established before for refracted Lévy processes, but only for the case $\Pi(0,\infty)<\infty$. Moreover, the identities we have obtained here, aside from being more generally applicable, arguably appear in a simpler form, being expressed in terms of scale functions. For example, considering the expression for the value of the dividends, denoted by $V(x)$, given in (4.24), we see that we can easily differentiate that expression with respect to x (providing $W^{(q)},\mathbb{W}^{(q)}\in C^1(0,\infty)$). In that case, it follows that there is smooth pasting, i.e. $\lim_{x\uparrow b}V'(x)=\lim_{x\downarrow b}V'(x)$, if and only if X has paths of unbounded variation or b is chosen such that $\varphi(q)\int_0^\infty \mathrm{e}^{-\varphi(q)y}W^{(q)\prime}(y+b)\mathrm{d}y = W^{(q)\prime}(b)$. Having an expression for the derivative of V is very important regarding a certain optimal control problem involving refracted Lévy processes, see Gerber and Shiu [26] who solve this control problem for an extremely particular example of a refracted Lévy process (the compound Poisson case with exponentially distributed jumps). Besides in the Cramér-Lundberg model, the threshold strategy (and/or corresponding control problem) has also been considered in a Brownian motion setting, see e.g. [3,25,34]. Refracted Lévy processes have also been recently studied in the context of queuing theory, see e.g. Bekker et al. [8] and references therein. By comparison, the setting here operates at a greater degree of generality however.

We conclude this section with two concrete examples.

Example 1

Suppose that we take X to be a spectrally negative α-stable process for $\alpha \in (1,2)$ with positive linear drift $c > \delta$. It is known that for such processes (cf. [21]),

$$W(x) = \frac{1}{c}\left(1 - E_{\alpha-1}(-cx^{\alpha-1})\right)$$

where $E_{\alpha-1}(x) = \sum_{n\geq 0} x^n/\Gamma((\alpha-1)n+1)$ is the one-parameter Mittag-Leffer function with index $\alpha - 1$. It follows that when X is refracted with rate δ, then the ruin probability is given by

$$\begin{aligned}\mathbb{P}_x(\kappa_0^- < \infty) = 1 - &\frac{c-\delta}{c-\delta+\delta E_{\alpha-1}(-cb^{\alpha-1})}\Big\{1 - E_{\alpha-1}(-cx^{\alpha-1}) \\ &- \mathbf{1}_{\{x\geq b\}}\delta(\alpha-1)\int_b^x [1 - E_{\alpha-1}(-(c-\delta)(x-y)^{\alpha-1})]E'_{\alpha-1}(-cy^{\alpha-1})y^{\alpha-2}\mathrm{d}y\Big\}.\end{aligned}$$

Example 2

Let X be a spectrally negative Lévy process of bounded variation with compound Poisson jumps such that the Lévy measure is given by

$$\Pi(\mathrm{d}x) = \lambda\sum_{k=1}^{n} A_k \mathrm{e}^{-\alpha_k x}\mathrm{d}x, \quad \lambda, A_k, \alpha_k > 0, \sum_{k=1}^{n} A_k = 1$$

and when written in the form (4.2) the drift coefficient is taken to be c such that $\mathbb{E}(X_1) > 0$. This corresponds to the case of a Cramér-Lundberg process with premium rate c and claims which are hyper-exponentially distributed. Moreover we assume that $q > 0$ and that $0 < \delta < c$. Then the Laplace exponent of X is well defined and given by

$$\log\mathbb{E}\left(\mathrm{e}^{\theta X_1}\right) = c\theta - \lambda + \lambda\sum_{k=1}^{n} A_k\frac{\alpha_k}{\alpha_k+\theta} \quad \text{for } \theta > \min\{\alpha_1,\dots,\alpha_n\}.$$

Denote (with slight abuse of notation) by $\psi(\theta)$ the right hand side of above equation and note that this expression is well defined for all $\theta \in \mathbb{R}\backslash\{-\alpha_1,\dots,-\alpha_n\}$. By using the partial fraction method, we can then write for all $\theta \in \mathbb{R}\backslash\{-\alpha_1,\dots,-\alpha_n\}$,

$$\begin{aligned}\frac{1}{\psi(\theta)-q} =& \frac{1}{c\theta - \lambda + \lambda\sum_{k=1}^{n} A_k\frac{\alpha_k}{\alpha_k+\theta} - q}\cdot\frac{\prod_{k=1}^{n}(\alpha_k+\theta)}{\prod_{k=1}^{n}(\alpha_k+\theta)} \\ =& \frac{\prod_{k=1}^{n}(\alpha_k+\theta)}{c\prod_{i=0}^{n}(\theta-\theta_i)} = \sum_{i=0}^{n}\frac{D_i}{\theta-\theta_i}.\end{aligned}$$

Here $\{\theta_i : i = 0, 1, ..., n\}$ are the roots of $\psi(\theta) - q$, with $\theta_0 = \Phi(q) > 0$ and the other roots being strictly negative. Further $\{D_i : i = 0, 1, ..., n\}$ are given by $D_i = 1/\psi'(\theta_i)$.

It follows that the scale function of X is given by

$$W^{(q)}(x) = \sum_{i=0}^{n} D_i \mathrm{e}^{\theta_i x}, \quad x \geq 0.$$

Similarly, the scale function of the process $\{X_t - \delta t : t \geq 0\}$ is given by

$$\mathbb{W}^{(q)}(x) = \sum_{j=0}^{n} \tilde{D}_j \mathrm{e}^{\tilde{\theta}_j x}, \quad x \geq 0,$$

where $\{\tilde{\theta}_j : j = 0, 1, ..., n\}$ are the roots of $\psi(\theta) - \delta\theta - q$ with $\tilde{\theta}_0 = \varphi(q) > 0$ and $\tilde{D}_j = 1/\psi'(\tilde{\theta}_j)$. We now want to give an explicit expression for the value of the dividends, denoted by V, for which the generic formula was given in (4.24) above.

We can write

$$\begin{aligned}\int_b^x \mathbb{W}^{(q)}(x-z) W^{(q)\prime}(z) \mathrm{d}z =& \sum_{j=0}^{n} \sum_{i=0}^{n} \frac{\tilde{D}_j}{\theta_i - \tilde{\theta}_j} D_i \theta_i \left(\mathrm{e}^{\theta_i x} - \mathrm{e}^{\theta_i b} \mathrm{e}^{\tilde{\theta}_j (x-b)} \right) \\ =& -\frac{1}{\delta} W^{(q)}(x) - \sum_{j=0}^{n} \sum_{i=0}^{n} \frac{\tilde{D}_j}{\theta_i - \tilde{\theta}_j} D_i \theta_i \mathrm{e}^{\theta_i b} \mathrm{e}^{\tilde{\theta}_j (x-b)},\end{aligned}$$

where the second equality follows since

$$\sum_{j=0}^{n} \frac{\tilde{D}_j}{\theta_i - \tilde{\theta}_j} = \frac{1}{\psi(\theta_i) - \delta\theta_i - q} = -\frac{1}{\delta\theta_i}.$$

Further we have

$$\varphi(q) \int_0^\infty \mathrm{e}^{-\varphi(q) y} W^{(q)\prime}(y+b) \mathrm{d}y = \tilde{\theta}_0 \sum_{i=0}^{n} \frac{D_i \theta_i}{\tilde{\theta}_0 - \theta_i} \mathrm{e}^{\theta_i b}$$

and since $\sum_{j=0}^{n} \tilde{D}_j / \tilde{\theta}_j = -1/(\psi(0) - \delta \cdot 0 - q)$, we get

$$\frac{\delta}{q} \left(1 - \mathbb{Z}^{(q)}(x-b) \right) = -\delta \sum_{j=0}^{n} \frac{\tilde{D}_j}{\tilde{\theta}_j} \left(\mathrm{e}^{\tilde{\theta}_j (x-b)} - 1 \right) = \frac{\delta}{q} - \delta \sum_{j=0}^{n} \frac{\tilde{D}_j}{\tilde{\theta}_j} \mathrm{e}^{\tilde{\theta}_j (x-b)}.$$

Hence the value of the dividends V is given for $x \leq b$ by

$$V(x) = \left(\tilde{\theta}_0 \sum_{i=0}^{n} \frac{D_i \theta_i}{\tilde{\theta}_0 - \theta_i} \mathrm{e}^{\theta_i b} \right)^{-1} \cdot \sum_{i=0}^{n} D_i \mathrm{e}^{\theta_i x}$$

and for $x \geq b$ by

$$V(x) = \frac{\delta}{q} + \sum_{j=0}^{n} \left\{ \left(\tilde{\theta}_0 \sum_{i=0}^{n} \frac{D_i \theta_i}{\tilde{\theta}_0 - \theta_i} \mathrm{e}^{\theta_i b} \right)^{-1} \sum_{i=0}^{n} \frac{\tilde{D}_j}{\tilde{\theta}_j - \theta_i} D_i \theta_i \mathrm{e}^{\theta_i b} - \frac{\tilde{D}_j}{\tilde{\theta}_j} \right\} \delta \mathrm{e}^{\tilde{\theta}_j (x-b)}.$$

Note that the $j = 0$ term between the curly brackets is zero. The above formulas for V are an improvement upon the calculations made in Appendix A of Gerber and Shiu [26].

Appendix

The theorem below is a collection of known fluctuation identities which have been used in the preceding text. See for example Chapter 8 of [38] for proofs and the origin of these identities.

Theorem 23. *Recall that X is a spectrally negative Lévy process and let*

$$\tau_a^+ = \inf\{t > 0 : X_t > a\} \text{ and } \tau_0^- = \inf\{t > 0 : X_t < 0\}.$$

(i) For $q \geq 0$ and $x \leq a$

$$\mathbb{E}_x \left(\mathrm{e}^{-q\tau_a^+} \mathbf{1}_{\{\tau_0^- > \tau_a^+\}} \right) = \frac{W^{(q)}(x)}{W^{(q)}(a)}. \tag{4.25}$$

(ii) For any $a > 0$, $x, y \in [0, a]$, $q \geq 0$

$$\int_0^\infty \mathbb{P}_x(X_t \in \mathrm{d}y,\, t < \tau_a^+ \wedge \tau_0^-)\mathrm{d}t = \left\{ \frac{W^{(q)}(x) W^{(q)}(a-y)}{W^{(q)}(a)} - W^{(q)}(x-y) \right\} \mathrm{d}y. \tag{4.26}$$

(iii) Let $a > 0$, $x \in [0, a]$, $q \geq 0$ and f, g be positive, bounded measurable functions. Further suppose that X is of bounded variation or $f(0)g(0) = 0$. Then

$$\begin{aligned} &\mathbb{E}_x(e^{-q\tau_0^-} f(X_{\tau_0^-}) g(X_{\tau_0^- -}) \mathbf{1}_{\{\tau_0^- < \tau_a^+\}}) \\ &= \int_0^a \int_{(y,\infty)} f(y-\theta) g(y) \left\{ \frac{W^{(q)}(x) W^{(q)}(a-y)}{W^{(q)}(a)} - W^{(q)}(x-y) \right\} \Pi(\mathrm{d}\theta)\mathrm{d}y. \end{aligned} \tag{4.27}$$

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