Conditioned Stable Lévy Processes

submitted by

Tsogzolmaa Saizmaa

for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

December 2021

Declaration of any previous Submission of the Work

The material presented here for examination for the award of a higher degree by research has not been incorporated into a submission for another degree.

. . . .

Tsogzolmaa Saizmaa

Declaration of Authorship

I am the author of this thesis, and the work described therein was carried out by myself personally, in collaboration with my supervisors Sandra Palau and Andreas E. Kyprianou, and external collaborator Mateusz Kwaśnicki.

. . . .

Tsogzolmaa Saizmaa

Summary

The aim of the thesis is to characterise new ways of path-conditioning of a d-dimensional isotropic stable Lévy process and consider their time-reversed paths. In our analysis, we use both of classical potential theory approach and recently-developed methods around the theory of self-similar Markov processes. In doing so, we have the opportunity to consider the role of certain harmonic/excessive functions that have not been previously studied.

In the first part of the thesis, Chapter 3, we consider an oscillatory conditioned attraction of the stable Lévy process to a subset of the unit sphere or a hyperplane. We characterise the hitting distribution as well as the time-reversed process of this conditioned processes in the sense of Hunt-Nagasawa duality for Markov processes. The resulting time-reversed processes have the same distribution as the unconditioned stable Lévy process itself when issued from the corresponding subset of the unit sphere or an hyperplane.

In the second part, Chapter 4, we condition the stable Lévy process to remain either outside or inside of the sphere and we characterise the same conditioned attraction to a subset of the unit sphere. The methods of the previous chapter are not applicable in this case. We use instead recent developments in the representation of d-dimensional isotropic stable Lévy processes as a self-similar Markov processes. In particular, we use a characterisation of the point of closest/furthest reach from the unit sphere for the stable Lévy process. As in the first part, we characterise the hitting distribution as well as the time-reversed process of the newly conditioned processes via Hunt-Nagasawa duality. The resulting time-reversed processes have the law of stable Lévy processes conditioned to stay away from the unit sphere and the process conditioned to stay inside the unit sphere and continuously absorbed at the origin correspondingly. We also extend the conditioned processes as well as its time-reversed processes to be issued from the boundary of the domain in which they live.

Finally, we would like point out that the methodology that we use in the Chapter 4 was only possible for a subset of the unit sphere and we could not extend to the setting of the hyperplane. The reason for this is that we use the law of the point of closest reach from the unit sphere to define the conditioning. There is no such result for the case of an hyperplane so far. Thus, for further work, we aim to characterise one-sided attraction to a subset of an hyperplane.

Acknowledgements

First and foremost, I would like to express my sincere gratitude to my supervisor Andreas Kyprianou, for all his encouragement and support since we have met during his visit at National University of Mongolia, for his guidance, patience, and enthusiasm during my study at Bath, and for initiating and providing collaborative activities with Mongolia over these years regardless of his increasingly busy schedule. I would also deeply thank my supervisor Sandra Palau for her guidance, encouragement, and friendly support from whom I have learnt a lot. Besides my supervisors, I would like to thank my collaborators, Mateusz Kwaśnicki who worked together to produce the paper supporting this thesis, and Minmin Wang who had many fruitful discussions and worked together on the projects during my study at Bath.

I am very grateful to my sponsors University of Bath and Schlumberger Inc. who supports women in mathematics through its Faculty for the Future Scholarship. Receiving such prestigious scholarships was not possible without the help and advices from Andreas and Susie whom I thank a lot. I would also like to thank Department of Mathematics, particularly Prob-L@B, for providing me full of resources and range of seminars which was incredibly interesting and helpful.

Also, I want to thank Susie and Jess at SAMBa office for always creating truly supportive and friendly environment for all SAMBa students. During all these years, I was so lucky to have such supportive friends at SAMBa cohorts, especially Emma, Dorka, Isaac, Malena, Emico, and Daniel.

Studying and living at Bath was full of the happy memories. I am very thankful to Jargalmaa, Lhama, Joel, and Jessica for always being there to share those happy moments. My special thanks goes to Jargalmaa who made all this journey possible and made my dream come true.

Finally, I would like to thank my parents, my husband Gerelt-Ireedui and son Khantushig, for their unconditional love and support. I could not have done this without you.

Contents

Chapter 1

Introduction

In this thesis, we want to condition a process to a zero probability event. Since we cannot use the classical conditional probability, the way to do it is to perform a change of measure using a super-martingale and a limiting procedure.

More precisely, let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. A probability measure $\mathbf{Q}$, defined on the measurable space $(\Omega, \mathcal{F})$, is said to be absolutely continuous with respect to $\mathbf{P}$ if, for every $A \in \mathcal{F}$, we have

$$\mathbf{P}(A) = 0 \Rightarrow \mathbf{Q}(A) = 0. \tag{1.1}$$

Then, the Radon-Nikodym theorem implies that, if $\mathbf{Q}$ is absolutely continuous with respect to $\mathbf{P}$, then there exists a σ-measurable function $M \colon \Omega \to [0, \infty)$ such that for any $A \in \mathcal{F}$,

$$\mathbf{Q}(A) = \int_A M(\omega) d\mathbf{P}(\omega) \tag{1.2}$$

and we denote it as

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = M.$$

Conversely, whenever we find such measurable function M for a given probability measure $\mathbf{P}$, so that the function $\mathbf{Q}$ computed by Equation (1.2) forms a probability measure, then we can change the measure space $(\Omega, \mathcal{F}, \mathbf{P})$ to $(\Omega, \mathcal{F}, \mathbf{Q})$.

Let $X = (X_t, t \geq 0)$ be a stochastic process in this space with values in $\mathbb{R}^d$, and $(\mathcal{F}_t, t \geq 0)$ be a filtration. A stochastic process X is said to be adapted to the filtration $(\mathcal{F}_t, t \geq 0)$, if X_t is measurable by $\mathcal{F}_t$ for all $t \geq 0$. The notion of adaptability means that, at time t, information regarding a stochastic process X up to time t is known but it is not known for the further times. An example of filtration is $\mathcal{F}_t := \sigma(X_s, s \leq t)$, that is the σ-algebra generated by $\{X_s, s \leq t\}$. This filtration is called the natural filtration of the stochastic process X. It is clear that a stochastic process X is adapted with respect to its natural filtration.

For the stochastic process $(X_t, t \geq 0)$ issued from x with the initial distribution $\mathbf{P}_x$, we have the stronger result due to Girsanov's theorem which states that, under the above change of measure, if

$$\frac{d\mathbf{Q}_x}{d\mathbf{P}_x}\bigg|_{\mathcal{F}_t} = M_t \tag{1.3}$$

is a $\mathbf{P}$ super-martingale then it is a $\mathbf{Q}$ super-martingale, in particular M_t is measurable in $\mathcal{F}_t$ and $\mathbf{E}_x^{\mathbf{P}} M_t$ as well as $\mathbf{E}_x^{\mathbf{Q}} M_t$ are non-decreasing in t. Conversely, whenever we find a $\mathbf{P}$ super-martingale M_t for a given probability measure $\mathbf{P}_x$, so that the function $\mathbf{Q}_x$ computed by the equation (1.3), then we get the distribution of the new stochastic process when the initial distribution is changed from $\mathbf{P}_x$ to $\mathbf{Q}_x$.

The way to construct a super-martingale is to find an "excessive" function h with the main property that

$$\mathbf{E}_x h(X_t) \leq h(x), \quad t \geq 0, \quad x \in \mathbb{R}^d. \tag{1.4}$$

Then, we can define a super-martingale $M_t := \frac{h(X_t)}{h(x)}$ and

$$\frac{d\mathbf{P}^*{}_x}{d\mathbf{P}_x}\Big|_{\mathcal{F}_t} = \frac{h(X_t)}{h(x)} \quad t \geq 0, \tag{1.5}$$

is a change of measure. Methods for finding such excessive functions or super-martingales to conditioning Markov processes include a potential theory approach and a probabilistic approach.

In this thesis, we consider path conditioning of a stable Lévy process. Such processes lie at the intersection of Lévy processes and self-similar Markov processes. In other words, stable Lévy processes have both of the characteristics of Lévy processes and self-similar Markov processes.

A Lévy process $Y = (Y_t, t \geq 0)$ is a stochastic process with the following characterizing properties:

(i) it has paths that are almost surely right-continuous with left-limits (càdlàg);

(ii) it has stationary increments, which means that for fixed two times $s < t$, the distribution of the increment $Y_t - Y_s$ depends only on $t - s$;

(iii) it has independent increments, which means that for fixed times $0 \leq t_1 < ... < t_n$, the increments $Y_{t_n} - Y_{t_{n-1}}, ..., Y_{t_2} - Y_{t_1}$ are independent of each other.

A self-similar process $Z = (Z_t, t \geq 0)$ is a Hunt process on $\mathbb{R}^d$ if there exists a constant $\alpha > 0$ such that, for any $x \in \mathbb{R}^d \setminus \{0\}$ and $c > 0$,

$$\text{the law of} \quad (cZ_{c^{-\alpha}t}, t \geq 0) \quad \text{under} \quad P_x \quad \text{is the same as} \quad (Z_t, t \geq 0) \quad \text{under} \quad P_{cx}, \tag{1.6}$$

where P_x is the law of Z starting from x. The property (1.6) is also called a scaling property and α is called the scaling index or self-similarity index.

Throughout this thesis, we will consider isotropic stable Lévy processes. This means that X is a stable Lévy process such that for every orthogonal transformation $U\colon \mathbb{R}^d \to \mathbb{R}^d$, the law of UX under $\mathbb{P}_x$ is equal to the law of X under $\mathbb{P}_{Ux}$.

The conditioning of a Markov process as a limiting procedure has seen a series of results for the setting of random walks (RWs) and Lévy processes in one dimension. First, Bertoin and Doney [2] showed the connection between the limiting conditioning procedures for one dimensional RWs conditioned to stay positive and harmonic/martingale functions. Then, Chaumont and Doney [5] extended the definition of conditioning to stay positive to the setting of Lévy processes, including the boundary point 0 for the point of issue. Bertoin and Savov [3] identified the role of duality in describing the time-reversed conditioned processes (see also [1]). As RWs can be thought of as discrete analogues of Brownian motion, the similar results relating to limiting conditioning procedures for RWs/Lévy processes can be sought for the setting of Brownian motion or for the more general Markov processes.

To investigate the phenomenon of conditioning in more complex settings, we choose the candidate Markov process to be an isotropic stable Lévy process in dimension $d \geq 2$ as it inherits the same properties as that of a Brownian motion except it has discontinuous paths. More precisely, we want to characterise the conditionings of a stable Lévy process X to hit

(i) a subset $\mathsf{S} \in \mathbb{S}^{d-1}$ from both sides of the sphere $\mathbb{S}^{d-1}$;

(ii) a subset $\mathsf{D} \in \mathbb{H}^{d-1}$ from both side of the hyperplane $\mathbb{H}^{d-1}$; and

(iii) a subset $\mathsf{S} \in \mathbb{S}^{d-1}$ from one side of the sphere

and investigate their time-reversed process from the hitting time.

1.1 Outline of thesis

The aim of the thesis is to characterise some newly derived path-conditionings of d-dimensional isotropic stable Lévy processes and their time-reversed dual processes. The remainder of the thesis consists of 3 chapters and a conclusion. Chapters 3 and 4 contain research papers, which were written in collaboration with my supervisors Professor Andreas E. Kyprianou and Dr. Sandra Palau, and collaborator Professor Mateusz Kwaśnicki.

1. **Preliminary (Chapter 2).** To characterise path conditionings of the stable processes in later chapters, we use recent developments in the representation of a d-dimensional isotropic stable Lévy process as a self-similar Markov process. In Chapter 2 we review these results from the literature. We are also interested in extending the definition of conditioned stable process to be outside (or inside) a region to include the case when the process starts at the boundary of the region. In addition, understanding time-reversal of conditioned stable processes also features in our analysis.

   In-line with these objectives, the preliminary chapter begins by summarising some standard properties of Markov process in Section 2.1. Among the exposition, we discuss the notion of a Doob-h transformation and conditioning of Markov processes and a well-known result from potential analysis concerning the equilibrium measure in [6].

   We introduce Lévy processes in Section 2.2, stressing their characteristic properties, as well as introducing essential notion and results relating to its path structure. Then, we introduce the special case of isotropic Stable Lévy process in Section 2.3, focusing on fluctuation results. This is followed by an introduction of self-similar Markov processes in Section 2.4 and their recent developments, [10], which serves as the base of some of the analysis in Chapter 4.

Since we want to consider issuing our conditioned processes from the boundary as well as time-reversed dual processes, we briefly introduce the theory of excursions for Lévy processes, radial excursion theory for stable Lévy processes in Section 2.5 and time-reversed duality concepts in Section 2.6.

2. **Oscillatory attraction and repulsion from a subset of the unit sphere or hyperplane for isotropic stable Lévy processes (Chapter 3).** In this chapter, we consider a stable process $X = (X_t, t \geq 0)$ with lifetime ζ valued in $\mathbb{R}^d (d \geq 2)$ and want to condition it to continuously approach a subset of the unit sphere. Such conditioning has been explored when $d = 1$ [12]. Moreover, we want to explore repulsion from a subset of the unit sphere using the constructed attraction process.

   We begin our analysis defining what we mean by conditioning stable process to continuously approach a subset of the unit sphere. As we will see in the preliminaries, when $\alpha \in (1, 2)$, stable processes will hit the unit sphere with positive probability and otherwise, when $\alpha \in (0, 1]$, it hits the unit sphere with probability zero; see e.g. [13] or [9]. Thus, the aforesaid conditioning is only of interest when $\alpha \in (0, 1]$. Moreover, as we will see in the preliminaries, such conditioning can be extended to any sphere in $\mathbb{R}^d$ thanks to the self similarity as well as stationary and independent increments of the stable process.

   Suppose that S is a closed set of the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ in dimension $d \geq 2$, which has positive surface measure. We want to construct the law of the stable process conditioned to converge continuously to $\mathsf{S} \subset \mathbb{S}^{d-1}$ whilst visiting both $\mathbb{B}_d := \{x \in \mathbb{R}^d : |x| < 1\}$ and $\bar{\mathbb{B}}_d^c := \mathbb{R}^d \setminus \bar{\mathbb{B}}_d$ infinitely often at arbitrarily small times prior to hitting S. We shall denote the associated probabilities by $\mathbb{P}^{\mathsf{S}} = (\mathbb{P}^{\mathsf{S}}_x, x \in \mathbb{R}^d)$. For a more precise definition, we introduce the stopping times,

$$\tau_\beta = \inf\{t > 0 : \beta^{-1} < |X_t| < \beta\}, \qquad \text{for } \beta > 1. \tag{1.7}$$

   Whenever it is well defined, we write, for $t \geq 0$, $\Lambda \in \mathcal{F}_t$ and $x \notin \mathsf{S}$,

$$\mathbb{P}^{\mathsf{S}}_x(\Lambda,\, t < \zeta) = \lim_{\beta \to 1} \lim_{\varepsilon \to 0} \mathbb{P}_x\left(\Lambda, t < \tau_\beta \middle| \tau_{\mathsf{S}_\varepsilon} < \infty\right), \tag{1.8}$$

   where ζ is the first time that X_t hits $\mathbb{S}^{d-1}$ and

$$\tau_{\mathsf{S}_\varepsilon} = \inf\{t > 0 : X_t \in \mathsf{S}_\varepsilon\} \text{ and } \mathsf{S}_\varepsilon := \{x \in \mathbb{R}^d : 1 - \varepsilon \leq |x| \leq 1 + \varepsilon \text{ and } \arg(x) \in \mathsf{S}\}.$$

   Our first main result shows that $(X, \mathbb{P}^{\mathsf{S}})$ is indeed well defined. The proof of the result relies on establishing the asymptotic leading order behaviour of $\mathbb{P}_x(\tau_{\mathsf{S}_\varepsilon} < \infty)$, the probability of hitting the set S_ε. For this, we use a potential-theoretic method and develop a 'guess and verify' approach. In particular, since we are not chasing an exact formula for $\mathbb{P}_x(\tau_{\mathsf{S}_\varepsilon} < \infty)$, we can 'guess' a measure, say μ_ε, supported on S_ε, such that

$$U\mu_\varepsilon^{\mathsf{S}}(x) = \int_{\mathsf{S}_\varepsilon} |x-y|^{\alpha-d}\mu_\varepsilon(dy) = 1 + o(1), \qquad x \in \mathsf{S}_\varepsilon \text{ as } \varepsilon \to 0, \tag{1.9}$$

so that

$$(1+o(1))\mathbb{P}_x(\tau_{\mathsf{S}_\varepsilon} < \infty) = U\mu_\varepsilon^{\mathsf{S}}(x), \qquad x \notin \mathsf{S}_\varepsilon, \tag{1.10}$$

from which, we can draw out the leading order decay in ε.

In addition to constructing the conditioned process, we develop an expression for the law of the limiting point of contact on S. Moreover, we show that, when time reversed from the strike point on S, the resulting process can be described as the underlying stable Lévy process itself. The extreme case $\mathsf{S} = \mathbb{S}^{d-1}$ (the whole unit sphere) is included in our analysis. For $d = 1$, [12] explored that conditioned process. In order to make our results pertinent, we restrict ourselves to the case that $d \geq 2$.

The second part of the chapter is devoted to proving analogous results in the case when S is replaced by D, a closed bounded subset of the hyperplane $\{x \in \mathbb{R}^d : (x, v) = 0\}$ with positive surface measure, where v is a unit vector and $(\cdot, \cdot)$ is the usual Euclidean inner product. The methodology we used in the first part is robust enough to prove the results in the second part with the appropriate choice of the measure μ_ε.

Preprint: arXiv: 2011.07402. *to appear in Special Issue in Honor of Ron Doney's 80th Birthday, Birkhäuser.*

3. **Attraction to and repulsion from a subset of the unit sphere for isotropic stable Lévy processes (Chapter 4).** In this chapter, we restrict the stable process be attracted to, or repelled from, a subset of the unit sphere, S, from either the exterior or the interior of the unit sphere. The method we use in this chapter is different from the method of Chapter 3. In this chapter we use a probabilistic approach that uses recent developments in the representation of d-dimensional isotropic stable Lévy processes as a self-similar Markov process.

   We characterise the law of the stable Lévy process conditioned to approach S continuously from either inside or outside of the sphere. More precisely, if S is not a point, we define $A_\varepsilon = \{r\theta : r \in (1, 1+\varepsilon), \theta \in \mathsf{S}\}$ and $B_\varepsilon = \{r\theta : r \in (1-\varepsilon, 1), \theta \in \mathsf{S}\}$, for $0 < \varepsilon < 1$ and define the corresponding events $C_\varepsilon^\vee := \{X_{\underline{G}(\infty)} \in A_\varepsilon\}$ and $C_\varepsilon^\wedge := \{X_{\overline{G}(\tau_1^\ominus -)} \in B_\varepsilon\}$. Here, $X_{\underline{G}(\infty)}$ is the point of closest reach to the origin in the range of X, and $X_{\overline{G}(\tau_1^\ominus -)}$ is the point of furthest reach from the origin prior to exiting $\mathbb{B}_d$. More precisely,

$$\underline{G}(t) = \sup\{s \leq t \colon |X_s| = \inf_{u \leq s} |X_u|\}, \quad \overline{G}(t) = \sup\{s \leq t \colon |X_s| = \sup_{u \leq s} |X_u|\},$$

   where $\tau_1^\ominus = \inf\{t > 0 : |X_t| > 1\}$. We are interested in the asymptotic conditioning

$$\mathbb{P}_x^\vee(A, \, t < \mathtt{k}) = \lim_{\varepsilon \downarrow 0} \mathbb{P}_x(A, \, t < \tau_1^\oplus | C_\varepsilon^\vee), \tag{1.11}$$

when $x \in \bar{\mathbb{B}}_d^c$ where $\tau_1^{\oplus} = \inf\{t > 0 : |X_t| < 1\}$ and

$$\mathbb{P}_x^{\wedge}(A,\, t < \mathtt{k}) = \lim_{\varepsilon \downarrow 0} \mathbb{P}_x(A,\, t < \tau_1^{\ominus} | C_\varepsilon^{\wedge}), \tag{1.12}$$

when $x \in \mathbb{B}_d$, for all $A \in \mathcal{F}_t$.

When $\mathsf{S} = \{\vartheta\} \in \mathbb{S}^{d-1}$, we need to adapt slightly the sets A_ε and B_ε so that $A_\varepsilon = \{r\phi\colon r \in (1, 1+\varepsilon), \phi \in \mathbb{S}^{d-1}, |\phi - \vartheta| < \varepsilon\}$ and $B_\varepsilon = \{r\phi\colon r \in (1-\varepsilon, 1), \phi \in \mathbb{S}^{d-1}, |\phi - \vartheta| < \varepsilon\}$.

Our first main result shows that $(X, \mathbb{P}^{\wedge})$ and $(X, \mathbb{P}^{\vee})$ are well defined. The proof of the result relies on recent fluctuation identities related to the deep factorisation of stable processes, cf. [8, 10, 11]. Moreover, we are able to characterise the law of the limiting point of contact on S.

We also extend the characterisation of these two conditioned processes by including the case when X is issued from the unit sphere itself but not within S, i.e. $\mathbb{S}^{d-1} \setminus \overline{\mathsf{S}}$. The methodology we use for the proof relies on radial excursion theory, introduced in Chapter 2.

Additionally, in the sense of time reversal, we show that $(X, \mathbb{P}^{\wedge})$ and $(X, \mathbb{P}^{\vee})$ are in duality with, respectively, the stable process conditioned to remain inside the sphere and absorb continuously at the origin and the stable process conditioned to remain outside of the sphere, respectively. The extensions of these processes when issued from the boundary $\mathbb{S}^{d-1}$ are also characterised using the same methodology, i.e. radial excursion theory.

The extreme cases $\mathsf{S} = \mathbb{S}^{d-1}$ (the whole unit sphere) and $\mathsf{S} = \{\vartheta\} \in \mathbb{S}^{d-1}$ (a single point on the unit sphere) are included in our analysis, however, we will otherwise insist that the Lebesgue surface measure of S is strictly positive.

Our results extend the recent contributions of [12], where similar conditioning is considered, albeit in one dimension, as well as providing analogues of the same and very classical results for Brownian motion [7]. We also note that, the choice of limiting conditioning procedure that we have used reflects a similar approach taken in [12] in one dimension.

Published: Stochastic Processes and their Applications, 137 (2021), 272-293.

4. **Conclusions.** The final chapter is a short summary of the relevance of the work in this thesis, with an outlook to future work.

This thesis is presented in the alternative format which includes publications. This means the research chapters are developed independent of the introduction and supposed to be self-contained. Hence, it is inevitable that there will be some inconsistencies in notation and redundant content in the introduction chapter.

Bibliography

[1] Bertoin, J. (1996) *Lévy processes.* Cambridge University Press.

[2] Bertoin, J. and Doney, R.A. (1994) On conditioning a random walk to stay nonnegative *Ann. Probab.* **22**, 2152–2167.

[3] Bertoin, J. and Savov, R.A. (2011) Some applications of duality for Lévy processes in a half-line. *Bulletin of the London Mathematical Society* **43 (1)**, 97–110.

[4] Chaumont, L. (1996) Conditionings and path decompositions for Lévy processes. *Stochastic processes and their applications.* **64 (1)**, 39–54.

[5] Chaumont, L. and Doney, R. A. (2005) On Lévy processes conditioned to stay positive. *Electron. J. Probab.* **10**, no. 28, 948–961.

[6] Chung, K. L. and Walsh, J. B. (2005) *Markov processes, Brownian motion, and time symmetry.* Second edition. Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), 249. Springer, New York.

[7] Doob, J.L. (1957) Conditional brownian motion and the boundary limits of harmonic functions *Bulletin de la S. M. F.*, **85**, 431–458.

[8] Kyprianou, A.E. (2016) Deep factorisation of the stable process. *Electron. J. Probab.* **21**, no. 23, 1–28.

[9] Kyprianou, A.E. (2018) Stable processes, self-similarity and the unit ball *ALEA, Lat. Am. J. Probab. Math. Stat.* 15, 617–690.

[10] Kyprianou, A.E., Rivero V., and Satitkanitkul, W. (2018) Deep factorization of the stable process III: Radial excursion theory and the point of closest reach. *arXiv:1706.09924v3*

[11] Kyprianou, A.E., Rivero V., and Sengul, B. (2017) Conditioning subordinators embedded in Markov processes. *Stoch. Proc. Appl.* **127**, 1234–1254.

[12] Philip, H.W. (2018) *Lévy processes conditioned to avoid and hit intervals.* PhD Thesis, Mannheim.

[13] Port, S. C. (1969) First hitting distribution of a sphere for symmetric stable processes. *Trans. Amer. Math. Soc.* **135**, 115–125.

Chapter 2

Preliminaries

In this preliminary chapter, we begin by introducing Markov processes in Section 2.1. Next, we introduce Lévy processes in Section 2.2 focusing on their characterising properties as well as results relating to its path structure. The introduction of an isotropic Stable Lévy process in Section 2.3 focusing on the fluctuation results regarding a sphere [15, 14]. Section 2.4 introduces self-similar Markov processes and their recent developments in [16], which serves as a base of our analysis in Chapter 4.

Finally, as it will be of continued relevance throughout the thesis, we briefly introduce excursion theory in Section 2.5 and time-reversed duality concepts in Section 2.6, respectively.

2.1 Markov processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space, E be a locally compact metric space and $\mathcal{B}$ be the Borel field of E. In this thesis, we restrict ourselves to $E = \mathbb{R}^d$ for $d \geq 2$. For each $t \in [0, \infty)$, let $X_t(w) = X(t, w) : \Omega \to E$ be a random variable. Let us associate to $X = (X_t, t \geq 0)$ the filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ where, for each $t \geq 0$, $\mathcal{F}_t$ is the natural enlargement of $\sigma(X_s : s \leq t)$, i.e $\mathcal{F}_t = \sigma(X_s : s \leq t) \cup \{\mathbb{P} - \text{null sets}\}$.

Definition 2.1.1. *The process $X = (X_t, t \geq 0)$ possesses the Markov property if, for each Borel set $B \in \mathbb{R}^d$ and $s, t \geq 0$,*

$$P(X_{t+s} \in B | \mathcal{F}_t) = p_s(X_t, B) \tag{2.1}$$

where, for all $x \in \mathbb{R}^d$ and $s \geq 0$, $p_s(x, B) := P(X_s \in B | X_0 = x)$.

We now define stopping times and the strong Markov property.

Definition 2.1.2. *A non-negative random variable τ is called a stopping time if*

$$\{\tau \leq t\} \in \mathcal{F}_t, \quad \textit{for all} \quad t \geq 0.$$

An example of a stopping time can be the random time when a Markov process X enters

$A \subset \mathbb{R}^d$ where A is either an open or closed set, that is

$$\tau_A = \inf\{t > 0 \colon X_t \in A\}.$$

For any measurable function $f \colon \mathbb{R}^d \to (0, \infty)$ and $u \geq 0$, another example of a stopping time can be defined as

$$\tau_u^f = \inf\left\{\int_0^t f(X_s)ds > u\right\}.$$

Each stopping time has an associated $\sigma-$algebra defined as

$$\mathcal{F}_\tau := \{A \in \mathcal{F} \colon A \cap \{\tau \leq t\} \in \mathcal{F}_t \quad \text{for all} \quad t \geq 0\}.$$

We define the family of probability measure $\mathbf{P}_x, x \in \mathbb{R}^d$, where $\mathbf{P}_x(\cdot) = \mathbf{P}(\cdot | X_0 = x)$.

Definition 2.1.3. *The process X is said to satisfy the strong Markov property if, for each stopping time τ,*

$$P(Y_{\tau+s} \in B | \mathcal{F}_\tau) = P_{Y_\tau}(Y_s \in B)$$

on $\{\tau < \infty\}$ for all Borel set $B \in \mathbb{R}^d$.

Definition 2.1.4. *We define the semigroup of X, $\mathcal{P} = (\mathcal{P}_t, t \geq 0)$, as follows*

$$\mathcal{P}_t[f](x) := \mathbf{E}_x[f(X_t)], \quad x \in \mathbb{R}^d,$$

for any bounded measurable function $f \colon \mathbb{R}^d \to \mathbb{R}$.

When $\mathcal{F}_0$ is the trivial σ-algebra, $\mathcal{P}_0[f](x) = f(x)$. Moreover, $\mathcal{P}$ has the semigroup property i.e. $\mathcal{P}_t[\mathcal{P}_s[f](\cdot)](x) = \mathcal{P}_{t+s}[f](x)$ for all $x \in \mathbb{R}^d$ and $s, t \geq 0$ due to the Markov property.

Feller processes: We now define Feller processes which are a subspace of Markov processes. Let $C_0(\mathbb{R}^d)$ be the Banach space of bounded measurable functions which decay to 0 as $|x| \to \infty$, equipped with the supremum norm.

Definition 2.1.5. *The semigroup $\mathcal{P}$ is said to be Feller if it has the Feller property. That is,*

(i) for all $t \geq 0, \mathcal{P}_t \colon C_0(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$,

(ii) for all $x \in \mathbb{R}^d$ and $f \in C_0(\mathbb{R}^d)$, $\lim_{t \downarrow 0} \mathcal{P}_t[f](x) = f(x)$.

Hunt processes: From the Proposition 5 in Chapter 2.2 of [9], a Feller process restricted to any countable dense subset $S \in [0, \infty)$ has paths with right limits in $[0, \infty)$ and left limits in $(0, \infty)$ for almost every $w \in \Omega$. Moreover, for any countable dense subset $S \in [0, \infty)$, define

$$X_t^{\leftarrow}(w) = \lim_{s \in S, s \downarrow t} X_s(w), \quad X_t^{\rightarrow}(w) = \lim_{s \in S, s \uparrow t} X_s(w), \quad t \geq 0 \tag{2.2}$$

for each $w \in \Omega_* \subset \Omega$ for some Ω_* such that $\mathbb{P}(\Omega_*) = 1$. Then, it is clear that $X_t^{\leftarrow}(w)$ is right continuous in t with left limits and $X_t^{\rightarrow}(w)$ is left continuous in t with right limits at each $t > 0$.

Finally, due to Theorem 6 in Chapter 2.2 of [9], $X_t^{\rightarrow}$ and $X_t^{\leftarrow}$ are versions of X_t, i.e they are modifications of X_t; hence each of these is a Feller process with the same transition semigroup $\mathcal{P}_t$ as X_t since $\mathcal{F}_t$ is the natural enlargement of $\sigma(X_s\colon s \le t)$. These facts will be the base of the connection between Feller process and Hunt process defined below.

Definition 2.1.6. *Let $X = (X_t, \mathcal{F}_t, t \ge 0)$ be a temporally homogeneous Markov process with state space $(E, \mathcal{B})$ and semigroup $\mathcal{P}$. Then, X is called a Hunt process if and only if*

(i) it is right continuous,

(ii) it has the strong Markov property,

(iii) it is quasi left continuous, in a sense that, for any sequence of stopping times T_n which increases to a stopping time T, $\lim_{n\to\infty} X_{T_n} = X_T$ on $\{T < \infty\}$ almost surely.

Note that quasi left continuity implies left continuity as each $t \ge 0$ is a stopping time. In Chapter 9.5 of [7], it is shown that any Feller process can be constructed as a Hunt process. Such a version of the Feller process is called a regular Feller process.

2.1.1 Quitting time and equilibrium measure

Let $X = (X_t, t \ge 0)$ be a Hunt process on $(E, \mathcal{B})$ and μ be a σ-finite measure on $\mathcal{B}$.

Definition 2.1.7. *For some $q > 0$, the q-resolvent kernel $U^{(q)}$ is defined as*

$$U^{(q)}(x, A) = \int_0^\infty e^{-qt} \mathbf{P}_x(X_t \in A) dt, \tag{2.3}$$

and the potential kernel U is defined as

$$U(x, A) = \int_0^\infty \mathbf{P}_x(X_t \in A) dt. \tag{2.4}$$

Suppose that the potential kernel U has a density u with respect to μ, i.e.

$$U(x, A) = \int_A u(x, y)\mu(dy), \quad x \in E, A \in \mathcal{B}, \tag{2.5}$$

where u is non-negative and measurable on $\mathcal{E} \times \mathcal{E}$. Let $A \in \mathcal{E}$ and

$$\Delta_A = \{w | \exists t > 0\colon X_t(w) \in A\} \tag{2.6}$$

which is a measurable set. Now, define the quitting time or last exit time $\gamma_A(w)$ for a set A as follows:

$$\gamma_A(w) = \sup\{t > 0\colon X_t(w) \in A\}, \quad \text{if} \quad w \in \Delta_A, \tag{2.7}$$

otherwise $\gamma_A(w) = 0$. Define the hitting time $\tau_A(w)$ for $w \in \Omega$

$$\tau_A(w) = \inf\{t > 0\colon X_t(w) \in A\}, \quad \text{if} \quad w \in \Delta_A, \tag{2.8}$$

otherwise $\tau_A(w) = \infty$, e.g. for $w \in \Omega \setminus \Delta_A$. From these definitions, it is clear that

$$\{\gamma_A > 0\} = \{\tau_A < \infty\} \tag{2.9}$$

almost surely.

Definition 2.1.8. *The set A is called transient (recurrent) if and only if* $\quad \gamma_A < \infty \quad (\gamma_A = \infty)$ *almost surely.*

The following theorem gives a last passage time characterisation in terms of a potential density.

Theorem 2.1.1. *([9]) Let X be a Hunt process with potential kernel U. Suppose that its potential density u exists and has the following properties:*

(i) For each $x \in E$, $y \to u(x, y)^{-1}$ is finite continuous,

(ii) $u(x, y) = \infty$ if and only if $x = y$, and we set $u(x, x)^{-1} = 0$.

Then, for each transient set A, there exists a Radon measure μ_A such that for any $x \in E$ and $B \in \mathcal{B}$,

$$\mathbf{E}_x(\gamma_A > 0; X(\gamma_A -) \in B) = \int_B u(x, y)\mu_A(dy). \tag{2.10}$$

If almost all paths of the process are continuous, then μ_A has support in ∂A. In general, if A is open then μ_A has support in $\overline{A}$.

The measure μ_A is called the *equilibrium measure* of A. Moreover, the corollary below is a key feature of potential analysis.

Corollary 2.1.1. *([9]) Under the assumptions of the Theorem 2.1.1, we have*

$$\mathbf{P}_x(\tau_A < \infty) = \int_E u(x, y)\mu_A(dy). \tag{2.11}$$

Corollary 2.1.1 will be a key element of the proofs in Chapter 3.

2.1.2 Conditioning of a Markov process: Doob-h transformation

Let $E_\delta = E \cup \{\delta\}$ be a locally compact metric space E with an isolated cemetery point δ and $\mathcal{P} = (\mathcal{P}_t, t \geq 0)$ be a Borel-measurable semigroup on E_δ. Assume that, for a probability measure μ on E_δ, there exists a strong Markov process $X = (X_t, t \geq 0)$ with state space E_δ, probabilities $\mathbf{P} = (\mathbf{P}_x, x \in E)$ and transition semigroup $\mathcal{P}$, satisfying:

(i) $\mathbf{P}(X_0 \in A) = \mu\mathcal{P}_0(A) := \int \mu(dx)\mathcal{P}_0(x, A)$,

(ii) there exists a random variable ζ, $0 \leq \zeta \leq \infty$, such that

$$X_t(w) \in E \quad \text{if} \quad t < \zeta(w) \quad \text{and} \quad X_t(w) = \delta \quad \text{if} \quad \zeta(w) \leq t,$$

(iv) $t \to X_t(w)$ is right continuous on $[0, \infty)$ and has left limits in E on $(0, \zeta)$.

An example of a random variable ζ could be the first entrance time to a Borel set $B \in E$. In this case, X_t will be a killed Markov process upon entering B and the corresponding semigroup $\mathcal{P}^B = (\mathcal{P}_t^B, t \geq 0)$ will be the killed semigroup.

Definition 2.1.9. *A measurable function $h\colon E \to \mathbb{R}^+$ is called an excessive function relative to $\mathcal{P}_t$ if*

(i) for all $x \in E$ and $t \geq 0$, $\mathcal{P}_t[h](x) \leq h(x)$,

(ii) for all $x \in E$, $\lim_{t \downarrow 0} \mathcal{P}_t[f](x) = f(x)$.

An excessive function is called harmonic if $\mathcal{P}_t[h](x) = h(x)$, for all $x \in E$. If h is an excessive (harmonic) function relative to the killed semigroup $\mathcal{P}^B$, then h is called an excessive (harmonic) function on the set B.

Let h be an excessive function and $E_h := \{x\colon 0 < h(x) < \infty\}$. Define

$$ {}_h\mathcal{P}_t[f](x) := \begin{cases} \frac{1}{h(x)}\mathcal{P}_t[hf](x), & x \in E_h \\ 0, & x \in E \setminus E_h. \end{cases} \tag{2.12}$$

In Chapter 11.2 in [9], it is shown that, for an excessive function h, ${}_h\mathcal{P}_t$ is a sub-Markov semigroup on E. In other words, it is a Markov semigroup such that ${}_h\mathbf{P}_t(x, E) \leq 1$ and it satisfies

$$ {}_h\mathbf{P}_t(x, E \setminus E_h) = 0, \quad \text{for all} \quad x \in E_h \quad \text{and} \quad t \geq 0. \tag{2.13}$$

Moreover, by setting ${}_h\mathbf{P}_t(\delta, \{\delta\}) = 1$ and ${}_h\mathbf{P}_t(x, \{\delta\}) = 1 - {}_h\mathbf{P}_t(x, E)$, a sub-Markov semigroup ${}_h\mathcal{P}_t$ can be extended to a Markov semigroup on E. This also suggests that, if h is an harmonic function, ${}_h\mathcal{P}_t$ is directly a Markov semigroup.

Indeed, in [9], a stochastic process with semigroup ${}_h\mathcal{P}$ is constructed. Such a process is called a Doob-h transformation of the process of X and denoted X^h. Moreover, it is shown that X^h is a right-continuous strong Markov process which has left limits except possibly at the time of death ζ. Also, it is shown that, for any T stopping time, $\Lambda \in \mathcal{F}_{T+}$, and $x \in E_h$, we have

$$ {}_h\mathbf{P}_x(\Lambda; T < \zeta) = \frac{1}{h(x)}\mathbf{E}_x[h(X_T); \Lambda], \tag{2.14}$$

where $\mathbf{P}_x(\Lambda; T < \zeta) = \mathbf{P}_x(\Lambda \cap \{T < \zeta\})$ and $\mathbf{E}_x[h(X_T); \Lambda] = \int_\Lambda h(X_T(w))\mathbf{P}_x(dw)$.

2.2 Lévy Processes

Definition 2.2.1. *A stochastic process $X := \{X_t : t \geq 0\}$ valued in $\mathbb{R}^d$ and defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is called a* Lévy process *if*

(i) it is issued from the origin (i.e. $\mathbb{P}\{X_0 = 0\} = 1$),

(ii) it has stationary increments (i.e. $X_t - X_s$ is equal in distribution to X_{t-s}, for $0 \leq s \leq t$),

(iii) it has independent increments (i.e. $X_t - X_s$ is independent of $\{X_u : u \leq s\}$),

(iv) it is $\mathbb{P}$-almost surely right-continuous (i.e. $\mathbb{P}\{X_{t+} := \exists \lim_{s \downarrow t} X_s = X_t\} = 1$) with left-limits (i.e. $\mathbb{P}\{X_{t-} := \exists \lim_{s \uparrow t} X_s\} = 1$) (in short, having cádlág *paths).*

Examples of a Lévy process include Brownian motion, Poison processes, and compound-Poisson processes. While a Brownian motion has continuous paths and normally distributed increments, a Poisson process has discontinuous paths and non-negative increments. This shows the variety of path types that occur within the class of Lévy processes.

From the definition of a Lévy process, one can easily check that it has the infinite divisibility property, which justifies the definition below.

Definition 2.2.2. *For a d-dimensional Lévy process $X = (X_t)_{t \geq 0}$, the function*

$$\Psi(\theta) := -\frac{1}{t} \log \mathbb{E}[e^{i\theta \cdot X_t}] = -\log \mathbb{E}[e^{i\theta \cdot X_1}], \quad \theta \in \mathbb{R}^d,$$

is called the characteristic exponent *of X.*

The characteristic exponent of a Lévy process uniquely describes the path structure of the process via the following theorem.

Theorem 2.2.1. *(Lévy-Khintchine formula for Lévy processes) Suppose that $a \in \mathbb{R}^d$, $\mathbb{Q}$ is a $d \times d$ Gaussian covariance matrix, and Π is a measure concentrated on $\mathbb{R}^d \backslash \{0\}$ such that $\int_{\mathbb{R}^d} (1 \wedge x^2) \Pi(dx) < \infty$. From this triple, we define for each $\theta \in \mathbb{R}^d$*

$$\Psi(\theta) = ia \cdot \theta + \frac{1}{2} \theta \cdot \mathbb{Q}\theta + \int_{\mathbb{R}^d} (1 - e^{i\theta \cdot x} + i\theta \cdot x \mathbf{1}_{(|x|<1)}) \Pi(dx). \tag{2.15}$$

If Ψ is the characteristic exponent of a Lévy process, then it necessarily satisfies (2.15). Conversely, given (2.15), there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which a d-dimensional Lévy process is defined and have Ψ as its characteristic exponent.

Regarding the path structure of a Lévy process, the presence of $\mathbb{Q}$ implies the inclusion of an independent d-dimensional linear Brownian motion with covariance matrix $\mathbb{Q}$, while the measure Π describes the jump sizes and rates. The condition on Π implies that $\Pi(A) < \infty$ for all Borel set $A \in \mathbb{R}^d$ such that 0 is in the interior of A^c. The proof of the theorem is discussed in [12].

2.2.1 Main Properties of a Lévy process

In this section, we will briefly state properties of a Lévy process as connections to other classes of stochastic processes. Proofs can be found from [12, 9, 2].

Markov property: For the process X, let us denote by $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ the natural filtration, that is $\mathcal{F}_t = \sigma(X_u, u \leq t), \quad \forall t \geq 0$. Since, in a Lévy process X, the law of $X_{t+s} - X_t$ is independent of $\mathcal{F}_t$, for all $s, t \geq 0$, then Lévy processes satisfy the Markov property.

Strong Markov property: Lévy process possesses the strong Markov property which also could be stated via a stronger statement as in the following theorem. The proof can be found in [12].

Theorem 2.2.2. *Let τ be a stopping time. On $\{\tau < \infty\}$, define the process $\tilde{X} = (\tilde{X}_t, t \geq 0)$*

$$\tilde{X}_t = X_{\tau+t} - X_\tau, \quad t \geq 0.$$

Then, on the event $\{\tau < \infty\}$, the process $\tilde{X}$ is independent of the σ-algebra associated with τ, that is $\mathcal{F}_\tau := \{A \in \mathcal{F} \colon A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}$, and has the same law as X. Therefore it is a Lévy process.

Feller and Hunt properties: Thanks to the shifting property of a Lévy process e.g. $\mathcal{P}_t[f](x) = \mathbb{E}_x[f(X_t)] = \mathbb{E}[f(x + X_t)]$, we can show that a Lévy process has the Feller property. It is shown in general that any Feller process is a Hunt process in the Chapter 2 in [9]. Thus, Lévy process is a Hunt process with the state space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

Time-reversal duality: Since a Lévy process has stationary and independent increments, its time-reversed process has stationary and independent increments which agree with those of $-Y$, which is called a dual process of Y and denoted by $\hat{Y}$. More specifically, we have the following Duality lemma for a Lévy process, see [2].

Lemma 2.2.1. *For each fixed $t > 0$, the time-reversed process $(Y_{(t-s)-} - Y_t, 0 \leq s \leq t)$ and the dual process $\hat{Y} = (-Y_s, 0 \leq s \leq t)$ have the same law under $\mathbb{P}$.*

Duality relationships can also be expressed via the corresponding semigroups $(\mathcal{P}_t, t \geq 0)$ and $(\hat{\mathcal{P}}_t, t \geq 0)$ or q-resolvents $U^{(q)}$ and $\hat{U}^{(q)}$ as stated in [2] via the following lemma.

Lemma 2.2.2. *Suppose that f and g are non-negative, bounded and measurable functions. Then,*

$$\mathbb{E}_x[g(x)f(X_t)] = \mathbb{E}_x[f(x)g(\hat{X}_t)], \quad \int_{\mathbb{R}^d} g(x)\mathcal{P}_t[f](x)dx = \int_{\mathbb{R}^d} f(x)\hat{\mathcal{P}}_t[g](x)dx \tag{2.16}$$

and for $q > 0$,

$$\int_{\mathbb{R}^d} g(x)U_t^{(q)}[f](x)dx = \int_{\mathbb{R}^d} f(x)\hat{U}_t^{(q)}[g](x)dx. \tag{2.17}$$

These duality results are also true when the Lévy process is killed when entering an open or closed subset $D \in \mathbb{R}^d$. Indeed, let $(\mathcal{P}_t^D, t \geq 0)$ be a semigroup of a killed Lévy process Y_t upon entering D. That means,

$$\mathcal{P}_t^D[f](x) = \mathbb{E}_x[f(Y_t); t < \tau_D] \tag{2.18}$$

where f is a bounded and measurable function and $\tau_D := \inf\{t > 0 : Y_t \in D\}$. Then, the following result called Hunt's switching identity is proven in [2].

Lemma 2.2.3. *(Hunt's switching identity) Suppose that f and g are non-negative measurable functions and D is an open or closed domain. Then,*

$$\int_{\mathbb{R}^d} g(x)\mathcal{P}_t^D[f](x)dx = \int_{\mathbb{R}^d} f(x)\hat{\mathcal{P}}_t^D[g](x)dx \tag{2.19}$$

and for $q > 0$,

$$\int_{\mathbb{R}^d} g(x) U_D^{(q)}[f](x) dx = \int_{\mathbb{R}^d} f(x) \hat{U}_D^{(q)}[g](x) dx \tag{2.20}$$

where $(U_D^{(q)}, q \geq 0)$ and $(\hat{U}_D^{(q)}, q \geq 0)$ are corresponding q-resolvents.

The above duality results are for the time reversal from the deterministic fixed time t and valid for a Lévy process only.

2.2.2 Notions of reflecting path behaviour

In this section, we will briefly discuss notions that reflect how a Lévy process explores $\mathbb{R}^d$.

Transience and recurrence: A Lévy process X is said to be transient if, for all $a > 0$,

$$\mathbb{P}\Big(\int_0^\infty \mathbf{1}_{(|X_t| < a)} dt < \infty \Big) = 1$$

and recurrent if, for all $a > 0$,

$$\mathbb{P}\Big(\int_0^\infty \mathbf{1}_{(|X_t| < a)} dt = \infty \Big) = 1$$

It turns out that a Lévy process is either transient or recurrent as the theorem states below.

Theorem 2.2.3. *Let Ψ be a characteristic exponent of a Lévy process X. Then X is transient if and only if, for some sufficiently small $\varepsilon > 0$,*

$$\int_{|z| < \varepsilon} Re\Big(\frac{1}{\Psi(z)} \Big) dz < \infty, \tag{2.21}$$

and otherwise X is recurrent.

The proof of the theorem can be found in [12]. Transience and recurrence properties have also an interpretation on how a Lévy process behaves pathwise.

Theorem 2.2.4. *Let X be a Lévy process. Then,*

(i) X is transient if and only if

$$\lim_{t \to \infty} |X_t| = \infty$$

almost surely. In other words, the transience property for a Lévy process implies that it drifts to infinity.

(ii) If X is not a compound Poisson process, then it is recurrent if and only if, for all $x \in \mathbb{R}^d$,

$$\lim \inf_{t \to \infty} |X_t - x| = 0$$

almost surely. This means that, if a Lévy process is not a compound Poisson process and is recurrent, then we can always find a point on the path of X sufficiently close to any fixed point $x \in \mathbb{R}^d$ and vice-versa.

The proof of the theorem is discussed in [12].

Polarity of points: A point $y \in \mathbb{R}^d$ is said to be

(i) polar if for every x,

$$\mathbb{P}_x(X_t = y \quad \text{for some} \quad t > 0) = 0, \tag{2.22}$$

(i) essentially polar if for Lebesgue almost every x,

$$\mathbb{P}_x(X_t = y \quad \text{for some} \quad t > 0) = 0. \tag{2.23}$$

Theorem 2.2.5. *For dimension $d \geq 2$, all points are essentially polar.*

It is clear that polar points are essentially polar and the theorem below gives a condition when we have polarity for essentially polar points.

Definition 2.2.3. *For a Lévy process, the q-resolvent kernel is defined as*

$$U^{(q)}(x, dy) = \int_0^\infty e^{-qt}\mathbb{P}_x(X_t \in dy)dt, \quad x \in \mathbb{R}^d. \tag{2.24}$$

When $q = 0$, $U := U^{(0)}$ is called the resolvent.

Theorem 2.2.6. *If the resolvent kernel is absolutely continuous, then every essentially polar point is polar.*

Although main concern of the thesis is on the d-dimensional stable Lévy process for $d \geq 2$, which will be defined in Section 2.3, we will also use the Lamperti transformation (Section 2.4.1) where such processes can be expressed via one dimensional Lévy process through a particular space and time change. Hence, we will briefly state notions reflecting path behaviour of one dimensional Lévy process.

Notions for a real-valued Lévy process: In order to understand the fluctuations of a d-dimensional stable Lévy process around the sphere or hyperplane, we need to understand hitting points, path variation and regularity of the half line for a real-valued Lévy process.

Definition 2.2.4. *A real-valued Lévy process X is said to hit a point $x \in \mathbb{R}$ if*

$$\mathbb{P}(X_t = x \quad \textit{for some} \quad t > 0) > 0.$$

The following theorem provides a criteria if a Lévy process can hit points. The proof of the theorem can be found in [2].

Theorem 2.2.7. *Suppose that a real-valued Lévy process X is not a compound Poisson process. Then, X can hit points if and only if*

$$\int_{\mathbb{R}} Re\Big(\frac{1}{1 + \Psi(z)}\Big)dz < \infty. \tag{2.25}$$

Path variation: For a real-valued Lévy process X, its Lévy-Khintchine exponent takes the form

$$\Psi(z) = iaz + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (1 - e^{izx} + izx\mathbf{1}_{(|x|<1)})\Pi(dx). \tag{2.26}$$

To discuss boundary issuance problems for Lévy processes in Chapter 4, we need notions of bounded variation and regularity of points.

Definition 2.2.5. *A stochastic process X is said to have a bounded variation if, for all $n \in \mathbb{N}$ and division $G : 0 = t_0 < t_1 < ...t_n$*

$$\sup_{n,G} \Sigma_{i=1}^n |X_{t_{i+1}} - X_{t_i}| < \infty. \tag{2.27}$$

We will now state a theorem that reveals a path of a Lévy process has bounded variation based on the Lévy-Khintchine exponent given in (2.26). The proof of the theorem can be found in [12].

Theorem 2.2.8. *A real-valued Lévy process with Lévy-Khintchine exponent (2.26) corresponding to the triple (a, σ, Π) has paths of bounded variation if and only if*

$$\sigma = 0 \quad \textit{and} \quad \int_{\mathbb{R}} (1 \wedge |x|)\Pi(dx) < \infty. \tag{2.28}$$

Finiteness of the integral in (2.28) also allows for the Lévy-Khintchine exponent of any such bounded variation process to be rewritten in the form

$$\Psi(z) = -ibz + \int_{\mathbb{R}} (1 - e^{izx})\Pi(dx), \tag{2.29}$$

where $b \in \mathbb{R}$ is called the *drift coefficient* and relates to a and Π via

$$b = -\Big(a + \int_{(|x|<1)} x\Pi(dx)\Big). \tag{2.30}$$

Thanks to the representation (2.29), it is easy to detect the presence of a drift term for a bounded variation Lévy process as

$$\lim_{|z|\to\infty} \frac{\Psi(z)}{z} = -ib. \tag{2.31}$$

Regularity of the half line: Finally, we would like to discuss when a real-valued Lévy process immediately enters to the upper or the lower half line. Let $\tau_0^+ = \inf\{t > 0 \colon X_t > 0\}$ for a Lévy process X. Due to the Blumenthal $0-1$ law, the probability $\mathbb{P}(\tau_0^+ = 0)$ is zero or one. Based on this fact, we can give the following definition.

Definition 2.2.6. *For a real-valued Lévy process X, we say that 0 is*

(i) regular for $(0, \infty)$, if $\mathbb{P}(\tau_0^+ = 0) = 1$,

(ii) irregular for $(0, \infty)$, if $\mathbb{P}(\tau_0^+ = 0) = 0$, and

(iii) regular for $(-\infty, 0)$*, if* $-X$ *is regular for the upper half line.*

The regularity property helps to analyse issuance from the boundary problems [16].

Theorem 2.2.9. *For a real-valued Lévy process* X*, the point* 0 *is regular for* $(0, \infty)$ *if and only if one of the following three situations occurs:*

(i) X *is a process of unbounded variation,*

(ii) X *is a process of bounded variation and* $b > 0$ *where* b *is the drift in the representation (2.29),*

(iii) X *is a process of bounded variation,* $b = 0$*, and*

$$\int_0^1 \frac{x\Pi(dx)}{\int_0^x \Pi(-\infty, -y)dy} = \infty. \tag{2.32}$$

We now shift to the main section of the introduction where we define d-dimensional stable Lévy processes and we study their path behaviour around the unit sphere or an hyperplane.

2.3 Isotropic Stable Lévy Processes

In this section, we will present isotropic stable Lévy processes. We will also state their main properties as well as fluctuation results regarding the unit sphere or an hyperplane.

Definition 2.3.1. *A process* $X = (X_t, t \geq 0)$ *with probabilities* $\{\mathbb{P}_x, x \in \mathbb{R}^d\}$ *is called a d-dimensional stable process if it is a Lévy process and if there exists a stability index* α *such that, for* $c > 0$*, and* $x \in \mathbb{R}^d \setminus \{0\}$*,*

$$\text{under} \quad \mathbb{P}_x, \quad \text{the law of} \quad (cX_{c^{-\alpha}t}, t \geq 0) \quad \text{is equal to the law of} \quad (X_t, t \geq 0) \quad \text{under} \quad \mathbb{P}_{cx}.$$

As a Lévy process, a stable process must have the infinite divisibility property. It turns out that α needs to be in $(0, 2]$ (see Section 1.2.6 in [15]). The case $\alpha = 2$ corresponds to the d-dimensional Brownian motion, which has a continuous path. The processes we construct in this thesis are more interesting in the jump setting and thus we restrict ourselves to the pure jump setting of $\alpha \in (0, 2)$.

Although the distribution of any 2-stable Lévy process is invariant under any orthogonal transformation, this is not necessarily the case when $\alpha \in (0, 2)$. Hence, we give the following definition for an isotropic stable Lévy process.

Definition 2.3.2. X *is called an isotropic stable process if, for all orthogonal transformations* $B : \mathbb{R}^d \mapsto \mathbb{R}^d$ *and* $x \in \mathbb{R}^d$*,*

$$\text{under} \quad \mathbb{P}_x, \quad \text{the law of} \quad (BX_t, t \geq 0) \quad \text{is equal to the law of} \quad (X_t, t \geq 0) \quad \text{under} \quad \mathbb{P}_{Bx}.$$

As a Lévy process, an isotropic stable process of index $(0, 2)$ has a characteristic triplet $(0, 0, \Pi)$ [15], where the jump measure Π satisfies

$$\Pi(B) = C \int_B \frac{1}{|y|^{\alpha+d}} dy, \quad B \subset \mathcal{B}(\mathbb{R}^d),$$

where C could be any constant. For normalisation purposes, the coefficient C is usually chosen as

$$C := \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2} |\Gamma(-\alpha/2)|}$$

to give normalised characteristic exponent Ψ. Equivalently, this means that $X = \{X_t : t \geq 0\}$ is an isotropic α-stable process in $\mathbb{R}^d$ with characteristic exponent

$$\Psi(\theta) = -\log \mathbb{E}(e^{i\theta X_t}) = |\theta|^\alpha, \quad \theta \in \mathbb{R}.$$

From now on, we restrict our work to the isotropic setting and we use the term stable process to mean isotropic stable Lévy process for short.

2.3.1 Path behaviour

To understand how a stable process explores $\mathbb{R}^d$, let us first focus on properties of transience, recurrence, and polarity.

Theorem 2.3.1. *For dimension $d \geq 2$, a stable process is transient if and only if $\alpha < d$.*

Indeed, for small ε using polar coordinates, we have

$$\int_{-\varepsilon}^{\varepsilon} \frac{1}{|z|^\alpha} dz = C_d \int_0^{\varepsilon} \frac{1}{r^{\alpha-d+1}} dr$$

where C_d is a positive constant that only depends on d. Then, due to the integral test (2.21), we have that it is transient if and only if $\alpha < d$.

Since throughout the thesis, we will work only on the cases $\alpha \in (0, 2)$ and $d \geq 2$, we always have the transience property, which, in turn, implies the existence of a unique resolvent measure. Recall that, for a Lévy process, the q-resolvent measure is defined as

$$U^{(q)}(dx) = \int_0^\infty e^{-qt} \mathbb{P}(X_t \in dx) dt, \quad x \in \mathbb{R}^d. \tag{2.33}$$

When $q = 0$, $U := U^{(0)}$ is called the resolvent.

Theorem 2.3.2. *When X is a transient stable process, its resolvent exists and is absolutely continuous with respect to the Lebesque measure with resolvent density given by*

$$u(x) = 2^{-\alpha} \pi^{-d/2} \frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)} |x|^{\alpha-d}. \tag{2.34}$$

The proof of the theorem can be found from [15]. Thanks to Theorem 2.3.2 together with Theorems 2.2.5 and 2.2.6, we have polarity for all points. This means any stable process in higher dimensions (e.g. when $d \geq 2$) cannot hit any points with positive probabilities.

2.3.2 Spatial fluctuations and the unit sphere

In this subsection, we are interested in the existing fluctuation identities, which explain how a stable process behaves in relation to a sphere centered around the origin. Results relating to other spheres can be obtained by a shifted stable process due to the shifting property. Let us denote $\mathbb{S}^{d-1}(0,a) = \{x \in \mathbb{R}^d \colon |x| = a\}$ for some $a > 0$.

Hitting of a sphere: Let $\tau_a^{\odot} = \inf\{t > 0 \colon |X_t| = a\}$ and $\sigma_a(dz)$ be the surface measure on $\mathbb{S}^{d-1}(0,a)$ normalized to have unit total mass. In [14], the hitting probability and the hitting distribution of the stable process is characterised and the results are given in the theorem below.

Theorem 2.3.3. *For any $a > 0$ and any stable process X issued from $x \in \mathbb{R}^d \setminus \{0\}$, we have the following results:*

(i) If $\alpha \in (0,1)$, X cannot hit the sphere $\mathbb{S}^{d-1}(0,a)$ for all $|x| \neq a$, e.g. $\mathbb{P}_x(\tau_a^{\odot} = \infty) = 1$.

(i) If $\alpha \in (1,2)$, X can hit the sphere $\mathbb{S}^{d-1}(0,a)$ with the probability of

$$\mathbb{P}_x(\tau_a^{\odot} < \infty) = \frac{\Gamma(\frac{\alpha+d}{2}-1)\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d}{2})\Gamma(\alpha-1)} \begin{cases} {}_2F_1\big(\frac{d-\alpha}{2}, 1-\frac{\alpha}{2}, \frac{d}{2}; \frac{|x|^2}{a^2}\big), & a > |x|, \\ \big(\frac{|x|}{a}\big)^{\alpha-d} {}_2F_1\big(\frac{d-\alpha}{2}, 1-\frac{\alpha}{2}, \frac{d}{2}; \frac{a^2}{|x|^2}\big), & a \leq |x|, \end{cases} \tag{2.35}$$

and its hitting distribution is given by

$$\mathbb{P}_x(X_{\tau_a^{\odot}} \in dy) = \frac{\Gamma(\frac{\alpha+d}{2}-1)\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d}{2})\Gamma(\alpha-1)} \frac{||x|^2-a^2|^{\alpha-1}a^{d-\alpha}}{|x-y|^{\alpha+d-2}} \sigma_a(dy)\mathbf{1}_{(|x|\neq a)} + \sigma_x(dy)\mathbf{1}_{(|x|=a)}. \tag{2.36}$$

The proof of the theorem can be found in [14].

First entrance and exit of a ball: Now, let us denote $\tau_a^{\oplus} = \inf\{t > 0 \colon |X_t| < a\}$ and $\tau_a^{\ominus} = \inf\{t > 0 \colon |X_t| > a\}$ for some $a > 0$. Since a stable process is transient when $d \geq 2$, we have $\mathbb{P}_x(\tau_a^{\ominus} < \infty) = 1$ for any $|x| < a$, while

$$\mathbb{P}_x(\tau_a^{\oplus} = \infty) = \frac{\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)} \int_0^{(|x|^2/a^2)-1} (u+1)^{-d/2} u^{\alpha/2-1} du \tag{2.37}$$

for any $|x| > a$. To provide the hitting distribution, let us first denote the function

$$g_a(x,y) = \pi^{-(d/2+1)}\Gamma(d/2)\sin(\pi\alpha/2)\frac{|a^2-|x|^2|^{\alpha/2}}{|a^2-|y|^2|^{\alpha/2}}|x-y|^{-d}$$

for any $x, y \in \mathbb{R}^d \setminus \mathbb{S}^{d-1}(0,a)$ and state the following theorem which is proven in [14].

Theorem 2.3.4. *When $|x| < a$, we have $\mathbb{P}_x(X_{\tau_a^{\ominus}} \in dy) = g_a(x,y)dy$, for any $|y| > a$, and when $|x| > a$, we have $\mathbb{P}_x(X_{\tau_a^{\oplus}} \in dy, \tau_a^{\oplus} < \infty) = g_a(x,y)dy$, for any $|y| \leq a$.*

Killed stable process upon entering/exiting a ball: Let us describe some additional identities which will be useful for our dicsussion. These identities pertains to how stable process explores $\mathbb{R}^d$ regarding a sphere. With the notions of $\mathbb{B}_d := \{x \in \mathbb{R}^d : |x| < 1\}$ and $\tau_1^{\oplus} := \inf\{t > 0 : X_t \in \mathbb{B}_d\}$, let $X^{\mathbb{B}_d} := (X_t, t \le \tau_1^{\oplus})$ be the process X_t killed upon entering $\mathbb{B}_d$. The law of $X^{\mathbb{B}_d}$ is characterized in [5]. In particular, for any $|x| > 1$ and $|y| > 1$, let $R_x(dy)$ be the resolvent of the killed process defined as

$$R_x(dy) = \int_0^\infty dt \mathbb{P}_x(X_t \in dy, t < \tau_1^{\oplus}) = \mathbb{E}_x \left[\int_0^\infty dt \mathbf{1}_{\{t < \tau_1^{\oplus}, X_t \in dy\}} \right]. \tag{2.38}$$

According to Theorem III.3.9 in [14], its potential density $r_x(y)$ defined by $R_x(dy) = r_x(y)dy$ exists and is given by

$$r_x(y) = 2^{-\alpha} \pi^{-d/2} \frac{\Gamma(d/2)}{\Gamma(\alpha/2)^2} |x-y|^{(\alpha-d)} \int_0^{\zeta^+(x,y)} u^{\frac{\alpha}{2}-1}(u+1)^{-\frac{d}{2}} du \tag{2.39}$$

with $\zeta^+(x,y) = ||x|^2 - 1||{|y|^2 - 1}|/|x-y|^2$ for any $x, y \in \mathbb{B}_d^c$. This identity is also true when $|x| < 1$ and X_t is killed upon entering $\mathbb{B}_d^c$ for any $x, y \in \mathbb{B}_d$.

Point of closest/furthest reach from the origin: For the d-dimensional isotropic stable process X with $d \ge 2$, the facts that X is transient, any sphere of radius $r > 0$ is regular for both its interior and exterior for X, and X has càdlàg paths ensure that the process $X_{\underline{G}(t)}$, where

$$\underline{G}(t) := \sup\{s \le t : |X_s| = \inf_{u \le s} |X_u|\}, \quad t \ge 0,$$

is well defined as the point of closest reach to the origin up to time t in the sense that $X_{\underline{G}(t)-} = X_{\underline{G}(t)}$ and $|X_{\underline{G}(t)}| = \inf_{0 \le s \le t} |X(s)|$. Moreover, since the process $(\underline{G}(t), t \ge 0)$ is monotone increasing, we can define $\underline{m} := \lim_{t \to \infty} \underline{G}(t)$ almost surely. Then, due to the transient property of X, almost surely, $\underline{m} = \underline{G}(t)$ for all t sufficiently large and

$$|X_{\underline{m}}| = \inf_{s \ge 0} |X(s)|.$$

In Theorem 1.1 in [16], the law of $X_{\underline{m}}$ is given by

$$\mathbb{P}_x(X_{\underline{m}} \in dz) \quad = \quad c_{\alpha,d} \frac{(|x|^2 - |z|^2)^{\alpha/2}}{|z|^\alpha} |x-z|^{-d} dz, \quad |z| < |x|$$

where $c_{\alpha,d}$ is a suitable constant that only depends on α and d.

On the other hand, define $\overline{G}(t) = \sup\{s \le t : |X_s| = \sup_{u \le s} |X_u|\}, t \ge 0$, and write

$$\overline{G}(\tau_1^{\ominus}-) = \sup\{s < \tau_1^{\ominus} : |X_s| = \sup_{u \le s} |X_u|\}, \quad t \ge 0$$

for the instant of furthest reach from the origin immediately before first exit from $\mathbb{B}_d$. From the definition, it is clear that $X_{\overline{m}} = X_{\overline{G}(\tau_1^{\ominus}-)}$. Then, due to Corollary 1.3(ii) in [16],

$$\mathbb{P}_x(X_{\overline{G}(\tau_1^{\ominus}-)} \in dz, X_{\tau_1^{\ominus}} \in dv) = c_{\alpha,d}\frac{(|z|^2 - |x|^2)^{\alpha/2}}{(|v|^2 - |z|^2)^{\alpha/2}|z-v|^d|z-x|^d}dzdv,$$

for $|x| < |z| < 1$ and $|v| > 1$.

Riesz-Bogdan-Żak transform: Finally, let $Kx = \frac{x}{|x|^2}, x \in \mathbb{R}^d$ be the transformation in $\mathbb{R}^d$ that converts the interior of $\mathbb{B}_d$ to its exterior and vice versa while $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ is mapped to itself through this transformation. Let us define a change of measure for an isotropic d-dimensional stable process $(X, \mathbb{P}_x), x \in \mathbb{R}^d \backslash \{0\}$ given by

$$\left.\frac{d\mathbb{P}_x^{\circ}}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \quad t \geq 0. \tag{2.40}$$

Then, the following theorem in [6] is used to convert the results derived for the stable process outside of the unit ball to the one inside of the ball and vice versa.

Theorem 2.3.5. *(d-dimensional Riesz-Bogdan-Żak transform, $d \geq 2$) Suppose that X is a d-dimensional isotropic stable procees with $d \geq 2$. Define*

$$\eta(t) = \inf\left\{s > 0 \colon \int_0^s |X_u|^{-2\alpha} du > t\right\}, \quad t \geq 0.$$

Then, for all $x \in \mathbb{R}^d \setminus \{0\}, (KX_{\eta(t)}, t \geq 0)$ under $\mathbb{P}_x$ is equal in law to $(X_t, t \geq 0)$ under $\mathbb{P}_{Kx}^{\circ}$.

As stable process is transient when $d \geq 2$, $|X|$ tends to infinity as $t \to \infty$. Then, due to the above theorem, the converted process KX tends to 0. That is natural since the change of measure 2.40 is proven to be the measure to condition the stable process to hit the origin [15].

2.4 Self-similar Markov processes

Recall that every Feller process has a modification with paths which are almost surely right-continuous with left limits and thus it has quasi-left continuous paths, that is paths which are left-continuous at increasing sequence of stopping times. Such Feller process is said to be a regular Feller process.

Definition 2.4.1. *A $[0, \infty)$-valued regular Feller process $Z = (Z_t, t \geq 0)$ is called a positive self-similar Markov process (pssMp) with the index α, if there exists a constant $\alpha > 0$ such that, for any $x > 0$ and $c > 0$,*

$$\text{the law of} \quad (cZ_{c^{-\alpha}t}, t \geq 0) \quad \text{under} \quad P_x \quad \text{is equal to the law of} \quad (Z_t, t \geq 0) \quad \text{under} \quad P_{cx}, \tag{2.41}$$

where P_x is the law of Z when it is issued from x.

2.4.1 The Lamperti transform

For a Lévy process ξ, define the integrated exponential process $I := (I_t, t \geq 0)$ via

$$I_t = \int_0^t e^{\alpha \xi_t} ds, \quad t \geq 0. \tag{2.42}$$

Since the process I is increasing, we can define a time change to ξ by using I as

$$\varphi(t) = \inf\{s > 0 \colon I_s > t\} \quad t \geq 0. \tag{2.43}$$

Then, the following connection between the class of exponentially killed Lévy process and positive self-similar Markov process up to an absorption time ζ to the origin, that is with the lifetime $\zeta = \inf\{t > 0 \colon Z_t = 0\}$, is proved by Lamperti in [17].

Theorem 2.4.1. *(The Lamperti transform) For a fixed $\alpha > 0$, we have*

(i) If $(Z, P_x), x > 0$, is a positive self-similar Markov process with index of self-similarity α, then up to absorption at the origin, it can be represented as

$$Z_t \mathbf{1}_{(t<\zeta)} = e^{\xi_{\varphi(t)}}, \quad t \geq 0, \tag{2.44}$$

such that $\xi_o = \log x$ and either

(1) $P_x(\zeta = \infty) = 1$ for all $x > 0$, in which case, ξ is a Lévy process satisfying $\limsup_{t\uparrow\infty} \xi_t = \infty$,

(2) $P_x(\zeta < \infty$ and $Z_{\zeta-} = 0) = 1$ for all $x > 0$, in which case, ξ is a Lévy process satisfying $\limsup_{t\uparrow\infty} \xi_t = -\infty$, or

(3) $P_x(\zeta < \infty$ and $Z_{\zeta-} > 0) = 1$ for all $x > 0$, in which case, ξ is a Lévy process killed at an indpendent and exponentially distributed random time.

In all cases, we may have $\zeta = I_\infty$.

(ii) Conversely, for each $x > 0$, suppose that ξ is a given (killed) Lévy process issued from $\log x$. Define

$$Z_t = e^{\xi_{\varphi(t)}} \mathbf{1}_{(t<I_\infty)}, \quad t \geq 0. \tag{2.45}$$

Then, Z defines a positive self-similar Markov process, with the self-similarity index α, up to its absorption time $\zeta = I_\infty$ which satisfies $Z_0 = x$.

2.4.2 MAPs and the Lamperti-Kiu transform

In the previous section, a representation of the positive self-similar process is given through the Lamperti transform. Now, we summarise the analogous result regarding the $\mathbb{R}^d$-valued self-similar Markov processes.

Definition 2.4.2. *A regular Feller process $Z = (Z_t, t \geq 0)$ on $\mathbb{R}^d \setminus \{0\}$, with the cemetery state at the origin, is called a self-similar Markov process (ssMp) with the index α, if there exists a constant $\alpha > 0$ such that, for any $x \in \mathbb{R}^d \setminus \{0\}$ and $c > 0$,*

$$\text{the law of} \quad (cZ_{c^{-\alpha}t}, t \geq 0) \quad \text{under} \quad P_x \quad \text{is} \quad P_{cx}, \tag{2.46}$$

where P_x is the law of Z when it is issued from x.

Now, we would like to decompose any $\mathbb{R}^d$-valued self-similar Markov process into its radial part and angular part in the polar coordination and represent the radial part in a way similar to the Lamperti transform. To do so, we need the concept of a Markov additive process defined below. Let E be a locally compact, complete, and separable metric space with cemetery state δ.

Definition 2.4.3. *A regular Feller process $(\xi, \Theta) = ((\xi_t, \Theta_t), t \geq 0)$ defined on $\mathbb{R} \times E$ with initial probabilities $\mathbf{P}_{x,\theta}$, for each $x \in \mathbb{R}, \theta \in E$, is called a Markov additive process (MAP), if for every bounded measurable function $f \colon \mathbb{R} \times E \to \mathbb{R}, t, s \geq 0$, and $(x, \theta) \in \mathbb{R} \times E$, on $\{\Theta_t = \phi, t < \zeta\}$,*

$$\mathbf{E}_{x,\theta}[f(\xi_{t+s} - \xi_t, \Theta_{t+s})\mathbf{1}_{(t+s<\zeta)}|\mathcal{G}_t] = \mathbf{E}_{0,\phi}[f(\xi_s, \Theta_s)\mathbf{1}_{(s<\zeta)}], \tag{2.47}$$

where $\zeta = \inf\{t > 0 \colon \Theta_t = \delta\}, \xi_\zeta = -\infty$, and $(\mathcal{G}_t, t \geq 0)$ is the filtration generated by the natural enlargement of the MAP.

In the above definition, ξ is called the ordinate and Θ is called the modulator. Moreover, it is possible to show that Θ alone is a regular Feller process. In the scope of the thesis, we will choose $E = \mathbb{S}^{d-1} := \{x \in \mathbb{R}^d \colon |x| = 1\}$ to represent Θ as an angular part and ξ as a radial part of the polar decomposition of the $\mathbb{R}^d$-valued self-similar Markov process to make use of the Lamperti-Kiu transform due to below theorem.

Theorem 2.4.2. *(Generalised Lamperti-Kiu transform) The process Z is a ssMp with index $\alpha > 0$ if and only if there exists a killed MAP, (ξ, Θ) on $\mathbb{R} \times \mathbb{S}^{d-1}$ such that*

$$Z_t = e^{\xi_{\varphi(t)}}\Theta_{\varphi(t)}, \quad t \leq I_\zeta$$

where

$$\varphi(t) = \inf\{s > 0 \colon \int_0^s e^{\alpha\xi_u} du > t\}, \quad t \leq I_\zeta,$$

and $I_\zeta = \int_o^\zeta e^{\alpha\xi_s} ds$ is the lifetime of Z until absorption at the origin.

In the same way as we define isotropic property for a stable process, a self-similar Markov process $Z = (Z_t, t \geq 0)$ is called isotropic if the law of $(U^{-1}Z, P_x)$ is equal to that of $(Z, P_{U^{-1}x})$ for every orthogonal d-dimensional matrix U and $x \in \mathbb{R}^d$. For the isotropic self-similar Markov process, we have more specific results given below.

Theorem 2.4.3. *Let $Z = (Z_t, t \geq 0)$ be a self-similar Markov process with underlying MAP (ξ, Θ) through Lamperti-Kiu representation. Then,* *(i) Z is an isotropic ssMp if and only if $((\xi, U^{-1}\Theta), \mathbf{P}_{x,\theta})$ is equal in law to $((\xi, \Theta), \mathbf{P}_{x,U^{-1}\theta})$, that means Θ is also an isotropic Markov process;*

(ii) If Z is an isotropic ssMp, then $(|Z|, t \geq 0)$, where $|\cdot|$ denotes the Euclidian norm, is equal in law to a positive self-similar Markov process, thus ξ is a Lévy process.

The proof of the above results are given in Chapter 11.5 of [15].

Example 2.4.1. ***Radial part of an isotropic stable process:*** *For an isotropic stable process X with an index $\alpha \in (0, 2)$, denote by R_t its radial part, i. e. $R_t = |X_t|, t \geq 0$. We know that a d-dimensional stable process X never hits a point, thus never hits the origin, when $d > \alpha$. Thus R_t never hits 0. Then, the above theorem states that R is a positive self-similar Markov process and the underlying ξ through the Lamperti transformation is a Lévy process. Moreover, it is shown in [15] that the underlying Lévy process ξ belongs to the class of hypergeometric Lévy processes with the characteristic exponent given as*

$$\Psi_\xi(z) = 2^\alpha \frac{\Gamma(\frac{1}{2})(-iz + \alpha)}{\Gamma(-\frac{1}{2}iz)} \frac{\Gamma(\frac{1}{2})(iz + d)}{\Gamma(\frac{1}{2}(iz + d - \alpha))}, \quad z \in \mathbb{R}. \tag{2.48}$$

Since X is transient when $d > \alpha$, R is also transient making ξ drifts towards ∞. Moreover, ξ is regular for $(-\infty, 0)$ and $(0, \infty)$.

2.5 Excursions for Markov processes

2.5.1 Excursions away from a point

For any Markov process $X = (X_t, t \geq 0)$ and any fixed point $b \in \mathbb{R}^d$, we would like to decompose the path of X in terms of the random times $C := \{t | X_t = b\}$ and the resulting pieces of path will be defined over the complement of the closure $\overline{C}$. These pieces of path are called "excursions away from b". To give a precise definition, we need to understand on additive functionals and local times of a Markov processes.

Additive functionals: A family of functions $\{A_t, t \geq 0\} : \Omega \to [0, \infty]$ is called an *additive functional of X* if

(a) $t \to A_t(w)$ is non-decreasing, right continuous and $A_0(w) = 0$ except for $w \in \Lambda$ where Λ is a negligible set, that is $\mathbb{P}_x(\Lambda) = 0$ for all measure $\mathbb{P}_x$,

(b) for each t, A_t is measurable in $\mathcal{F}_t$, and

(c) for each t and s, $A_{t+s} = A_t + A_s \cdot \theta_t$, where θ_t is a shift operator, almost surely for all $\mathbb{P}_x$.

An example of an additive functional can be constructed by setting

$$A_t(w) = \int_0^t f(X_s(w)) ds \tag{2.49}$$

for a positive, bounded, measurable function f. It is also known that continuous additive functional has a strong additive property, that is, if T is a stopping time then almost surely

$$L_{t+T} = L_T + L_t \cdot \theta_T. \tag{2.50}$$

A useful feature of additive functionals could be explained by the following theorem. Recall that, for a fixed $\lambda > 0$, a positive, measurable function φ is said to be λ-excessive for the process X, if

$$e^{-\lambda t}\mathcal{P}_t\varphi \leq \varphi \tag{2.51}$$

and

$$e^{-\lambda t}\mathcal{P}_t\varphi \to \varphi \quad \text{as} \quad t \to 0. \tag{2.52}$$

Moreover, a λ-excessive function is said to be uniformly λ-excessive if the convergence in (2.52) is uniform. The theorem below gives connection between excessive functions and additive functionals. Its proof can be found in [4].

Theorem 2.5.1. *[4]. For a fixed $\lambda > 0$, if a bounded uniformly λ-excessive function f satisfies $e^{-\lambda t}\mathcal{P}_t f \to 0$ as $t \to \infty$, then there exists an additive functional L, such that f is the $\lambda-$potential of L, that is*

$$f(x) = \mathbb{P}_x \int_0^\infty e^{-\lambda t} dL_t. \tag{2.53}$$

Moreover, for all w except that is in negligible sets for all $\mathbb{P}_x$, $L_t(w)$ is continuous and satisfies $L_{t+s}(w) = L_t(w) + L_s(\theta_t w)$ for all s and t, as well as such additive functional L is unique for all t.

Local time: Let σ be the time of hitting the point $\{b\}$, that is $\sigma = \inf\{t > 0\colon X_t = b\}$. To define excursions away from the point, we consider only a point regular itself, hence we assume that $\{b\}$ is regular itself, that is $\mathbb{P}_b(\sigma = 0) = 1$. Thanks to the previous theorem, by choosing the λ-excessive function $f(x) = \mathbb{E}_x e^{-\lambda\sigma}$ and $\lambda = 1$, we can find a unique continuous additive functional $\{L_t; t \geq 0\}$ such that

$$\mathbb{E}_x e^{-\sigma} = \mathbb{E}_x \int_0^\infty e^{-t} dL_t \tag{2.54}$$

and such $\{L_t; t \geq 0\}$ is called the *local time at $\{b\}$ for the process X.*

In [4], the justification why this particular continuous additive functional L is called "local time at $\{b\}$" is given based on the fact that

$$L^R \subset \{t | X_t = b\} \subset L^I, \tag{2.55}$$

where L^I and L^R are respectively the set of points of increase and of right increase defined as

$$L^I = L^I(w) = \{t | L_{t-\varepsilon}(w) < L_{t+\varepsilon}(w) \quad \text{for all} \quad \varepsilon > 0\},$$

and

$$L^R = L^R(w) = \{t | L_t(w) < L_{t+\varepsilon}(w) \quad \text{for all} \quad \varepsilon > 0\}.$$

It is also shown that the difference of the above two sets is countable and $dL_t(w)$ puts no mass at points. Thus, the measure $dL_t(w)$ puts all its mass on $L^I(w)$ hence on $L^R(w)$. Therefore, for any positive measurable function f, the identity

$$\int_0^t f(X_s)dL_s = f(b)L_t \tag{2.56}$$

holds for all t almost surely. These facts show that $L = (L_t; t \geq 0)$ grows exactly on $\{t | X_t = b\}$. Note that if L is a local time, then kL is also a local time for any constant $k > 0$. Hence, local times can be defined uniquely up to a multiplicative constant.

Inverse local time: For a local time $L = (L_t; t \geq 0)$, its *right continuous inverse* $L^{-1} = (L^{-1}(t), t \geq 0)$ is defined as

$$L^{-1}(t) := \inf\{s \colon L(s) > t\} \quad t \geq 0. \tag{2.57}$$

Note that, in line with the above definition, we have

$$L^{-1}(t-) := \inf\{s \colon L(s) \geq t\} = \lim_{s \to t-} L^{-1}(s) \quad t > 0. \tag{2.58}$$

In Proposition 7, Chapter 4 of [2], the below statement regarding inverse local time is proven.

(i) For every $t \geq 0$, both $L^{-1}(t)$ and $L^{-1}(t-)$ are stopping times,

(ii) The process L^{-1} is increasing, right-continuous and adapted to the filtration $\mathcal{F}_{L_t^{-1}}$,

(iii) For all $t > 0$, we have

$$L^{-1}(L(t)) = \inf\{L^{-1}(u) : L^{-1}(u) > t\} = \inf\{s > t : X_s = b\}$$

and

$$L^{-1}(L(t)-) = \sup\{L^{-1}(u) : L^{-1}(u) < t\} = \sup\{s > t : X_s = b\},$$

Note that L^{-1} is a Lévy process and it has non-decreasing path. Such a Lévy process is called a *subordinator*.

Excursion measure: Since the point b is regular, the complement of $\overline{C}$, that is $[0, \infty) \setminus \overline{C}$, will be a countable union of disjoint open intervals which is called an *excursion intervals*. Let $G = G(w)$ denote the strictly positive left ends of each of such excursion interval. Then, it is shown in [4] that there exist an excursion measure $\hat{P}$ associated with the process X_t killed at time σ and with the local time L, such that the excursion formula

$$\mathbb{E}_x \sum_{s \in G} Z_s f \cdot \theta_s = \mathbb{E}_x \int_0^\infty Z_t \hat{P}(f) dL_t \tag{2.59}$$

holds for all x, all positive previsible Z and all positive measurable f.

2.5.2 Excursions away from a set

In this section, we briefly state the notion of an excursion away from a set in an analogue way to the notion of an excursion away from a set. The concept of an excursion away from a set relies on the particular properties of additive functionals which will be stated below.

Regular λ-potential: For a continuous additive functional $\{A_t, t \geq 0\}$ and a positive λ, let its λ-potential

$$\mathbb{E}_x \int_0^\infty e^{-\lambda t} dA_t = f(x)$$

be finite for all $x \in E$. Then, it is clear that f is λ-excessive and for any increasing sequence of stopping times T_n with the limit T, we have

$$P_{T_n}^\lambda f(x) := \mathbb{E}_x \int_{T_n}^\infty e^{-\lambda t} dA_t \to P_T^\lambda f(x), \qquad n \to \infty.$$

Such λ-potential is called a *regular λ-potential* and the theorem below provides possibility to define an excursion away from a set for any Markov process.

Theorem 2.5.2. *For a regular λ-potential function f, there exists a unique continuous additive functional A such that*

$$f(x) = \mathbb{E}_x \int_0^\infty e^{-\lambda t} dA_t, \quad x \in E. \tag{2.60}$$

The proof of the theorem is given in [4].

Local time on a set: Let $B \subset E$ be a closed subset in E and $\sigma_B := \inf\{t > 0 | X_t \in B\}$. Also, let every point of B is regular for B, that is

$$\mathbb{P}_x\{\sigma_B = 0\} = 1$$

for all $x \in B$. Then, the function

$$f(x) = \mathbb{E}_x e^{-\sigma_B} \tag{2.61}$$

is 1-excessive and finite, thus a regular 1-potential. Hence, due to Theorem 2.5.2, there exists a continuous additive functional $\{L_B(t), t \geq 0\}$ such that

$$\mathbb{E}_x e^{-\sigma_B} = \mathbb{E}_x \int_0^\infty e^{-t} dL_B(t). \tag{2.62}$$

In a similar way for the case of a local time on a point, we can see that L_B grows exactly at $\{t | X_t \in B\}$. Hence, L_B is called a *local time on B*.

Excursions measure: Moreover, let $C_B := \{t | X_t(w) \in B\}$ and $G_B := G_B(w)$ is the strictly positive left ends of the open intervals making up the complement of $\overline{C}_B$. Then, as was in the previous section, there exists a continuous additive functional $\{L_t, t \geq 0\}$ which grows on C_B

and a family of excursion measures $\{\hat{P}_x, x \in B\}$ associated with the process X_t killed at time σ_B and with the local time L, such that the excursion formula

$$\mathbb{E}_x \sum_{s \in G_B} Z_s f \cdot \theta_s = \mathbb{E}_x \int_0^\infty Z_t \hat{P}_{X_t}(f) dL_t \tag{2.63}$$

holds for all x, all positive previsible Z and all positive measurable f. The proof and the detailed discussion is given in [4]. The pair $(\hat{P}, L)$ is also called as the *an exit system.*

2.5.3 Excursions for a Lévy process

Assume a Lévy process Y nor $-Y$ is not a subordinator and not killed. Define the *running maximum* and *running minimum* processes as

$$\overline{Y}_t = \sup_{s \le t} Y_s \quad \text{and} \quad \underline{Y}_t = \inf_{s \le t} Y_s$$

respectively. Then, both of the processes $\overline{Y} - Y$ and $Y - \underline{Y}$ are strong Markov processes [12].
Local times: It is shown in [2] that there exists a random measure, called as a *local time at maximum*, L on $[0, \infty)$ with the following properties:

(i) a continuous, non-decreasing, $[0, \infty)$-valued, and adapted process,

(ii) its support agrees with the closure of the set $\{t \ge 0\colon Y_t = \overline{Y}_t\}$,

(iii) if T is any stopping time such that $Y_T = \overline{Y}_T$ on $\{T < \infty\}$, then $((Y_{T+t} - Y_T, \overline{Y}_{T+t} - Y_T, L_{T+t} - L_T), t \ge 0)$ is independent of $\mathcal{F}_T$ on $\{T < \infty\}$ and has the same law as $((Y_t, \overline{Y}_t - Y_t, L_t), t \ge 0)$ under $\mathbb{P}$.

Similarly, there exists a *local time at minimum*, if we use the above findings to the dual process $-Y$. We will denote it as $\hat{L}$.
Excursions: As the processes L and $\hat{L}$ are also a local times at 0 for the reflected processes $\overline{Y} - Y$ and $Y - \underline{Y}$ correspondingly, their inverse $(L_t^{-1}, t \ge 0)$ and $(\hat{L}_t^{-1}, t \ge 0)$ are subordinators and killed when $L_\infty < \infty$ or $\hat{L}_\infty < \infty$. Moreover, for the countable set of times $\{t > 0 : \Delta_t^{-1} := L_t^{-1} - L_{t-}^{-1} > 0\}$, the excursion of Y from its maximum can be identified as

$$\epsilon_t(s) = Y_{L_{t-}^{-1}+s} - Y_{L_{t-}^{-1}}, \quad 0 \le s \le \Delta L_t^{-1} \tag{2.64}$$

which has right-continuous paths with left limits and strictly negative on $(0, \zeta)$, where ζ is its path lifetime. From the definition, it is clear that $\epsilon(\zeta) \ge 0$ when $\zeta < \infty$. Hence, the lifetime ζ can also be defined as $\zeta := \inf\{t > 0\colon \epsilon(t) > 0\}$.

The excursions $(\epsilon_t, t < L_\infty)$ form a stopped Poisson point process on $[0, \infty) \times \overline{\mathcal{U}}(\mathbb{R})$, where $\overline{\mathcal{U}}(\mathbb{R})$ is the space of paths of right-continuous with left limits and strictly negative-valued on $(0, \zeta)$, and with the intensity measure $dt \times \mathrm{d}\overline{n}$. The excursion of Y from its minimum can be defined in a similar way using $\hat{L}$, the local time at minimum.

The Ascending and Descending Ladder Processes: Moreover, we can define

$$H_t = \begin{cases} Y_{L_t^{-1}}, & t < L_\infty \\ \infty, & t = L_\infty \end{cases} \tag{2.65}$$

and the range of the process H agrees with the range of $\overline{Y}$. Similarly,

$$\hat{H}_t = \begin{cases} Y_{\hat{L}_t^{-1}}, & t < \hat{L}_\infty \\ \infty, & t = \hat{L}_\infty \end{cases} \tag{2.66}$$

and the range of the process $\hat{H}$ agrees with the range of $\underline{Y}$. Both of the processes H and $\hat{H}$ are subordinators and jumps at the end points of each excursion intervals. Then, for example, the Lévy measure of H is given by $\overline{n}(\epsilon(\zeta) \in dx)$ for any $x > 0$. The processes H and $\hat{H}$ are called as the *ascending ladder height process* and the *descending ladder height process*, correspondingly.
Wiener-Hopf Factorisation: If we denote the Laplace exponents of H and $\hat{H}$ by k and $\hat{k}$, correspondingly, that is

$$k(\lambda) = \frac{1}{t} \log \mathbb{E}[e^{-\lambda H_t}] \quad \text{and} \quad \hat{k}(\lambda) = \frac{1}{t} \log \mathbb{E}[e^{-\lambda \hat{H}_t}] \tag{2.67}$$

for any $\lambda, t \geq 0$, we have the following result called as a *Wiener-Hopf factorisation.*

Theorem 2.5.3. *Let Ψ is the characteristic exponent of a Lévy process Y, k and $\hat{k}$ are the Laplace exponents of the ascending and descending ladder height processes for Y. Then, for any $z \in \mathbb{R}$, we have*

$$\Psi(z) = k(-iz)\hat{k}(iz). \tag{2.68}$$

2.5.4 Radial excursion theory

Based on the results above for the excursions of real-valued Lévy processes and the Lamperti-Kiu representation of self-similar Markov processes in Section 2.4.2, Kyprianou et al [16] introduced Radial Excursion theory for the stable Lévy processes which will be explained in Chapter 4. The main idea of the Radial Excursion theory is to represent d-dimensional stable Lévy process via a MAP consisting of the polar decomposition where its modular part can be represented by a Lévy process through Lamperti-Kiu transform. The Radial Excursion theory serves main tool in our analysis to extend conditioned stable processes to be issued from the boundary in Chapter 4.

2.6 Time reversal duality: Hunt-Nagasawa duality

We are interested in the time-reversal from the first time when the stable process hits some open or closed subset in $\mathbb{R}^d$. Thus, we would like to briefly summarise time-reversal from the particular case of a random time for the general class of temporally homogeneous Markov processes, introduced by Nagasawa in [18].

2.6.1 Temporally homogeneous Markov processes

Let E be a locally compact metric space and $\mathcal{B}$ denote the Borel field of E. Let $\{\partial\}$ be a cemetery state and $E^* = E \cup \{\partial\}$. Define the space of mappings $W := \{w\colon [0,\infty] \to E^*\}$ satisfying:

(w_1) There exists a killing time $\zeta(w) \in [0,\infty]$ such that $w(t) \in E$ for $t < \zeta(w)$ and $w(t) = \partial$ for $t \geq \zeta(w)$,

(w_2) $w(t)$ is a càdlàg function in $[0, \zeta(w))$.

Let $X_t(w) = w(t)$ be the coordinate mapping. The shifted path $(w_t\colon t \geq 0)$ of w is defined by $X_s(w_t) = X_{t+s}(w)$ for any $s \geq 0$. Let $\mathcal{N}$ be the σ-field of W generated by $\{\{X_s \in A\}, s \geq 0, A \in \mathcal{B}\}$. Put $W_t = \{w : w \in W, \zeta(w) > t\}(t \geq 0)$, that is the set of all trajectories survived up to time t, and $N_t = \sigma(\{X_s \in A\}; s \in [0,t], A \in \mathcal{B})$ be the σ-field of W_t generated by $\{\{X_s \in A\}, s \in [0,t], A \in \mathcal{B}\}$,

Definition 2.6.1. *Let $\{\mathbf{P}_a; a \in E\}$ be a system of a probability measures on $(W, \mathcal{N})$ satisfying:*

(p_1) *For every $t \geq 0$ and $A \in \mathcal{B}$, the mapping $a \to \mathbf{P}_a[X_t \in A]$ is $\mathcal{B}$-measurable;*

(p_2) *$\mathbf{P}_a[X_0 = a] = 1$ for each $a \in E$;*

(p_3) *$\mathbf{P}_a[X_{t+s} \in A|\mathcal{N}_t] = \mathbf{P}_{X_t}[X_s \in A]$, $\quad \mathbf{P}_a$-almost everywhere on W_t for all $t, s \geq 0, a \in E, A \in \mathcal{B}$.*

Then, a system $X = (X_t, \zeta, \mathcal{N}_t, \mathbf{P}_a)$ is said to be a temporally homogeneous Markov process.

Let $X = (X_t, \zeta, \mathcal{N}_t, \mathbf{P}_a)$ be a temporally homogeneous Markov process. For a measure ν on $(E, \mathcal{B})$, we set

$$\mathbf{P}_\nu[B] = \int_E \nu(da)\mathbf{P}_a[B], \quad B \in \mathcal{N}.$$

Let $\overline{\mathcal{N}} = \cap_\nu \mathcal{N}(\mathbf{P}_\nu)$ where $\mathcal{N}(\mathbf{P}_\nu)$ is the completion of $\mathcal{N}$ by $\mathbf{P}_\nu$, that is a σ-algebra generated by $\mathcal{N}$ that contains all the $\mathbf{P}_\nu$-null sets. (ν varies over all probability measures on $(E, \mathcal{B})$). Similarly, let $\overline{\mathcal{N}_t} = \cap_\nu \mathcal{N}_t(\mathbf{P}_\nu)$ where $\mathcal{N}_t(\mathbf{P}_\nu)$ is the completion of $\mathcal{N}_t$ by $\mathbf{P}_\nu$.

Let $\zeta^{'}(w) : W \to [0,\infty]$ be an $\overline{\mathcal{N}}$-measurable function on W with values $[0,\infty]$ and $Z_t(w)$ be defined for $w \in W_0 \subset W$ and $t \in [0, \zeta^{'}(w)]$ with values in E and put $W_t^{'} = \{\zeta^{'} > t\} \cap W_0$ for $t \geq 0$, that is the set of trajectories in W_0 and that haven't died before time t. Let $\mathcal{M}_t = \sigma(\{Z_s \in A, \zeta' > t\}; A \in \mathcal{B}, x \in [0,t])$ and $\mathcal{M}$ be a σ-field on W_0 containing all $\mathcal{M}_t, t \geq 0$.

Definition 2.6.2. *Let $\mathbf{P}$ be a measure on $(W_0, \mathcal{M})$, which is σ-finite on $(W_t^{'}, \mathcal{M}_t)$ for every $t \geq 0$. A system $(Z_t, \zeta^{'}, \mathcal{M}_t, \mathbf{P})$, for brevity $(Z_t, \mathbf{P})$, is said to have temporally homogeneous Markov property with a transition probability $P(t, a, A)$, if, for every compact set A,*

$$\mathbf{P}[Z_t \in A|\mathcal{M}_s] = \mathbf{P}[Z_t \in A|Z_s] = P(t-s, Z_s, A), \quad \mathbf{P} - a.e. \text{ on } \quad W_s^{'}, 0 \leq s < t.$$

Further, if Z_0 is defined and

$$\mathbf{P}[Z_o \in A] = \mu(A).$$

Then a system is said to have the initial measure μ.

2.6.2 Time reversal of the temporally homogeneous Markov process

Now, we give definitions of particular types of random times, called an L-time and an almost L-time. Then, we state Nagasawa's time reversal duality concepts from such L-times.

Definition 2.6.3. *A function $\tau(w)\colon W \to \{-\infty\} \cup [0,\infty]$ is called a random time of type L (briefly, L-time) if it has the properties:*

(L_1) $\tau(w)$ *is $\overline{\mathcal{N}}$-measurable and* $\tau(w) \leq \zeta(w)$,

(L_2) $\{s < \tau(w) - t < \infty\} = \{s < \tau(w_t) < \infty\}$, *for any $t, s \geq 0$.*

Also, τ is said to be an almost L-time if it satisfies (L_1) and, instead of (L_2),

(L_2') $\{s < \tau(w) - t < \infty\} = \{s < \tau(w_t) < \infty\}$, $\mathbf{P}_a$*-a.e for any $t, s \geq 0$.*

In [18], it was shown that the killing time $\zeta(w)$ and the last exit time from an open set $D \subset E$ are L-times.

Definition 2.6.4. *Let τ be an almost L-time and $W_0 = \{w\colon 0 < \tau(w) < \infty\}$. Set, for $w \in W_0$,*

$$Z_t(w) = \begin{cases} X_{\tau(w)-t-}, & 0 < t < \tau(w), \\ \partial, & t \geq \tau(w), \end{cases}$$

(if there exists $X_{\tau-}$, we permit $t = 0$). The process $(Z_t, \mathbf{P}_\nu)$ defined on the space $(W_0, \mathcal{N}|w_0)$ is said to be the reversed process of $(X_t, \mathbf{P}_\nu)$ from an almost L-time τ, where ν is a σ-finite measure on $(E, \mathcal{B})$.

Let $\mathbf{B}(E), \mathbf{C}(E)$, and $\mathbf{C}_0(E)$ be the spaces of bounded $\mathcal{B}$-measurable functions, bounded continuous functions, and continuous functions with compact supports, respectively. For any $\gamma > 0$, define

$$G_\gamma(a, A) = \mathbf{E}_a\big[\int_0^\infty e^{-\gamma t}\mathbf{1}_{\{X_t \in A\}} dt\big].$$

In [18], it was shown that, under the following conditions, the reversed process $(Z_t, \mathbf{P}_\nu)$ is a temporally homogeneous Markov process with a transition probability.

A.1 Assume that $G_0(a, \cdot)$ is σ-finite. For a σ-finite measure ν, put

$$\eta(A) = \int_E \nu(da) G_0(a, A), \tag{2.69}$$

for $A \in \mathcal{B}$, then there exists a transition probability $\hat{P}(t, a, A)$ such that

$$\int T_t f(a) g(a) \eta(da) = \int f(a) \hat{T}_t g(a) \eta(da) \tag{2.70}$$

for every $f, g \in B_0(E)$, where $T_t f(a) = \mathbf{E}_a[f(X_t)]$ and $\hat{T}_t f(a) = \int \hat{P}(t, a, db) f(b)$.

A.2 ν is a σ-finite measure on $(E, \mathcal{B})$ satisfying

$$\mathbf{P}_\nu[Z_t \in K] < \infty, \quad t > 0 \tag{2.71}$$

and

$$\int_0^\infty e^{-\gamma t} dt \mathbf{P}_\nu[Z_t \in K] < \infty, \quad \gamma > 0 \tag{2.72}$$

for each compact K.

A.3. Putting

$$\hat{G}_\gamma f(a) = \int_0^\infty e^{-\gamma t} dt \hat{T}_t f(a), \quad \gamma > 0, \tag{2.73}$$

for any $f \in C_0(E)$,

(i) $\hat{T}_t f(a)$ is right continuous in t, and

(ii) $\hat{G}_\gamma f(Z_t)$ is right continuous in $t, \mathbf{P}_a$-a.e. for any $\gamma > 0$.

Theorem 2.6.1 (Nagasawa [18]). *(i) If a Markov process X and a measure ν satisfy A.1, A.2, and A.3, then the reversed process $\{(Z_t, \mathbf{P}_\nu), (t > 0)\}$ of $(X_t, \mathbf{P}_\nu)$ from an almost L-time τ has temporally homogeneous Markov property and its transition probability is $\hat{P}(t, a, A)$, i.e.*

$$\begin{aligned} \mathbf{P}_\nu[Z_t \in A | Z_r, 0 < r < s] &= \mathbf{P}_\nu[Z_t \in A | Z_s] = \hat{P}(t-s, Z_s, A), \mathbf{P}_\nu - a.e.\ on \\ &\quad \{s < \tau < \infty\}, \quad 0 < s < t. \end{aligned} \tag{2.74}$$

(ii) Moreover, if the process $(X_t, \mathbf{P}_\nu)$ satisfies,

A.0.1 $\mathbf{P}_\nu[0 < \tau < \infty$ and $X_{\tau-}$ does not exist$] = 0$

A.0.2 $\mathbf{P}_\nu[Z_0 \in K] < \infty$ for every compact set K, and

A.0.3 $\hat{T}_f(f(a)) \in C(E)$ for each $F \in C_0(E)$,

then (i) is true for the reversed process $\{(Z_t, \mathbf{P}_\nu), (t \geq 0)\}$ and (2.74) is true for $0 \leq s < t$.

We will use Theorem 2.6.1 to establish our results on the time-reversal from the hitting time to show duality relationships between the conditioned stable process and its time-reversed process from the hitting time both in the Chapter 3 and 4. Note that the key relationship to reveal time-reversal duality given in Equation 2.19 and Equation 2.70 are the same for the deterministic or the random cases.

Now, we have finished reviewing existing theories and results used for our analysis.

Bibliography

[1] Alili, L., Chaumont, L., Graczyk, P. and Zak. T. (2017) Inversion, duality and doob h-transforms for self-similar Markov processes. *Electron. J. Probab.*, **22**, no. 20, 18 pp.

[2] Bertoin, J. (1996) *Lévy processes.* Cambridge University Press.

[3] Bertoin, J. and Doney, R.A. (1994) On conditioning a random walk to stay nonnegative *Ann. Probab.* **22**, 2152–2167.

[4] Blumenthal, R.M. (1992) *Excursions of Markov processes.* Birkhäuser, Boston, Basel, Berlin.

[5] Blumenthal, R.M., Getoor, R.K., and Ray, D.B. (1961) On the distribution of first hits for the symmetric stable processes. *Trans. Amer. Math. Soc.* **99**, 540–554.

[6] Bogdan, K. and Żak, T. (2006) On Kelvin transformation. *J. Theoret. Probab.* **19**, 89–120.

[7] Cinlar E. (2011) *Probability and Stochastics.* Springer-Verlag New York.

[8] Chaumont, L. and Doney, R. A. (2005) On Lévy processes conditioned to stay positive. *Electron. J. Probab.* **10**, no. 28, 948–961.

[9] Chung K. L. and Walsh J.B. (2005) *Markov Processes, Brownian Motion, and Time Symmetry.* Springer-Verlag New York.

[10] Döring, L. and Weissman, P. (2020) Stable processes conditioned to hit an interval continuously from the outside. *Bernoulli* **26**, no. 2, 980–1015.

[11] Doob, J.L. (1957) Conditional brownian motion and the boundary limits of harmonic functions *Bulletin de la S. M. F.*, **85**, 431–458.

[12] Kyprianou, A.E. (2014) *Fluctuations of Lévy processes with applications.* Springer.

[13] Kyprianou, A.E. (2016) Deep factorisation of the stable process. *Electron. J. Probab.* **21**, no. 23, 1–28.

[14] Kyprianou, A.E. (2018) Stable processes, self-similarity and the unit ball *ALEA, Lat. Am. J. Probab. Math. Stat.* **15**, 617–690.

[15] Kyprianou, A.E. and Pardo, J.C. (2021) *Stable Lévy processes via Lamperti-type representations*. Cambridge University Press.

[16] Kyprianou, A.E., Rivero V., and Satitkanitkul, W. (2020) Deep factorization of the stable process III: Radial excursion theory and the point of closest reach. *Potential Anal.* **53**, 1347–1375.

[17] Lamperti, J. (1972) Semi-stable Markov processes. I. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **22**, 205–225.

[18] Nagasawa, M. (1964) Time reversions of Markov processes. *Nagoya. Math. J.* **24**, 177–204.

Appendix 6B: Statement of Authorship

This declaration concerns the article entitled:
Oscillatory attraction and repulsion from a subset of the unit sphere or hyperplane for isotropic stable L\'evy processes

Publication status (tick one)

Draft manuscript		Submitted		In review		Accepted	X	Published	

Publication details (reference)	Accepted for Progresses in Probability. arXiv:2011.07402

Copyright status (tick the appropriate statement)

I hold the copyright for this material		Copyright is retained by the publisher, but I have been given permission to replicate the material here	X

Candidate's contribution to the paper (provide details, and also indicate as a percentage)	*All research has been conducted in equal partnership with collaborators and supervisors. It is unwise to try and measure percentages of intellectual contribution as research scholarship lies as much in the escalation of ideas through mathematical discourse as it does with the seed of ideas themselves. That said, the mathematical content of this thesis is, as an entire piece of work, inextricably associated to its author through intellectual ownership. Formulation of ideas: The candidate played an integral and fully collaborative role in the formulation of ideas. Design of methodology: The candidate played an integral and fully collaborative role in the design of methodology. Experimental work: N/A Presentation of data in journal format: N/A *The wording in this box follows the advice and approval of my supervisor, Professor Kyprianou.
Statement from Candidate	This paper reports on original research I conducted during the period of my Higher Degree by Research candidature.

Signed	Tsogzolmaa Saizmaa	**Date**	20/06/2021

Chapter 3

Oscillatory attraction and repulsion from a subset of the unit sphere or hyperplane for isotropic stable Lévy processes

Mateusz Kwaśnicki[1], Andreas E. Kyprianou[2], Sandra Palau[3] Tsogzolmaa Saizmaa[4]

Abstract

Suppose that S is a closed set of the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ in dimension $d \geq 2$, which has positive surface measure. We construct the law of absorption of an isotropic stable Lévy process in dimension $d \geq 2$ conditioned to approach S continuously, allowing for the interior and exterior of $\mathbb{S}^{d-1}$ to be visited infinitely often. Additionally, we show that this process is in duality with the unconditioned stable Lévy process. We can replicate the aforementioned results by similar ones in the setting that S is replaced by D, a closed bounded subset of the hyperplane $\{x \in \mathbb{R}^d : (x, v) = 0\}$ with positive surface measure, where v is the unit orthogonal vector and where $(\cdot, \cdot)$ is the usual Euclidean inner product. Our results complement similar results of the authors [17] in which the stable process was further constrained to attract to and repel from S from either the exterior or the interior of the unit sphere.

[1]Department of Pure Mathematics Wroclaw University of Science and Technology ul. Wybrzeze Wyspiańskiego 27 50-370 Wroclaw, Poland. Email: `Mateusz.Kwasnicki@pwr.edu.pl`

[2]Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, UK. Email: `a.kyprianou@bath.ac.uk` and `t.saizmaa@bath.ac.uk`

[3]Department of Statistics and Probability, Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México, México. E-mail: `sandra@sigma.iimas.unam.mx`

[4]Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, UK and National University of Mongolia, Baga-toiruu, Sukhbaatar district, Ulaanbaatar, Mongolia. Email: `tsogzolmaa@num.edu.mn`

3.1 Introduction

Let $X = (X_t, t \geq 0)$ be a d-dimensional stable Lévy process ($d \geq 2$) with probabilities $(\mathbb{P}_x, x \in \mathbb{R}^d)$. This means that X has càdlàg paths with stationary and independent increments, and there exists an $\alpha > 0$ such that, for $c > 0$, and $x \in \mathbb{R}^d$,

$$\text{under } \mathbb{P}_x \text{ the law of } (cX_{c^{-\alpha}t}, t \geq 0) \text{ is equal to } \mathbb{P}_{cx}.$$

The latter is the property of so-called self-similarity. It turns out that stable Lévy processes necessarily have $\alpha \in (0, 2]$. The case $\alpha = 2$ is that of standard d-dimensional Brownian motion, thus has a continuous path. All other $\alpha \in (0, 2)$ have no Gaussian component and are pure jump processes. In this article we are specifically interested in phenomena that can only occur when jumps are present. We thus restrict ourselves henceforth to the setting $\alpha \in (0, 2)$.

Although Brownian motion is isotropic, this need not be the case in the stable case when $\alpha \in (0, 2)$. *Nonetheless, we will restrict to the isotropic setting.* To be more precise, this means, for all orthogonal transformations $U : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $x \in \mathbb{R}^d$,

$$\textit{the law of} \quad (UX_t, t \geq 0) \ \textit{under} \quad \mathbb{P}_x \ \textit{is equal to} \ (X_t, t \geq 0) \ \textit{under} \ \mathbb{P}_{Ux}.$$

For convenience, we will henceforth refer to X as a *stable process.*

As a Lévy process, our stable process of index $(0, 2)$ has a characteristic triplet $(0, 0, \Pi)$, where the jump measure Π satisfies

$$\Pi(B) = \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|} \int_B \frac{1}{|y|^{\alpha+d}} \ell_d(\mathrm{d}y), \quad B \subseteq \mathcal{B}(\mathbb{R}^d), \tag{3.1}$$

where ℓ_d is d-dimensional Lebesgue measure[5]. This is equivalent to identifying its characteristic exponent as

$$\Psi(\theta) = -\frac{1}{t} \log \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta \cdot X_t}) = |\theta|^\alpha, \quad \theta \in \mathbb{R}^d,$$

where we write $\mathbb{P}$ in preference to $\mathbb{P}_0$.

In this article, we characterise the law of a stable process conditioned to continuously approach a closed subdomain of the surface of a unit sphere, say $\mathsf{S} \subseteq \mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$, which has non-zero surface measure. Moreover, our conditioning will allow the stable process to approach S by visiting the exterior and interior of $\mathbb{S}^{d-1}$ infinitely often. We note that when $\alpha \in (1, 2)$, stable processes will hit the unit sphere with probability 1 and otherwise, when $\alpha \in (0, 1]$ they hit the unit sphere with probability zero; see e.g. [25] or [16]. The aforesaid conditioning is thus only of interest when $\alpha \in (0, 1]$.

In addition to constructing the conditioned process, we develop an expression for the law of the limiting point of contact on S. Moreover, we show that, when time is reversed from the

[5]We will distinguish integrals with respect to one-dimensional Lebesgue measure as taking the form $\int \cdot \, \mathrm{d}x$, where as higher dimensional integrals will always indicate the dimension, for example $\int \cdot \, \ell_d(\mathrm{d}x)$.

strike point on S, the resulting process can be described as nothing more than the stable process itself.

It turns out that the methodology we use here is robust enough to cover a similar suite of results for the case of an isotropic stable process conditioned to a closed subdomain of an arbitrary hyperplane in $\mathbb{R}^d$ that is orthogonal to an arbitrary unit-length vector $v \in \mathbb{R}^d$.

Our results naturally complement those of the recent paper [17], which considers a similar type of conditioning, albeit requiring the stable process to additionally remain either inside or outside of the unit ball. Other related works include [9] and [14], who considered a real valued stable process conditioned to hit 0 continuously and a real valued stable process conditioned to continuously approach the boundary of the interval $[-1, 1]$ from the outside, respectively. In order to make our results pertinent, we restrict ourselves to the case that $d \geq 2$.

3.2 Oscillatory attraction towards the patch S

Let $\mathbb{D}(\mathbb{R}^d)$ denote the space of càdlàg paths $\omega : [0, \infty) \to \mathbb{R}^d \cup \partial$ with lifetime $\zeta(\omega) = \inf\{s > 0 : \omega(s) = \partial\}$, where ∂ is a cemetery point. The space $\mathbb{D}(\mathbb{R}^d)$ will be equipped with the Skorokhod topology, with its closed σ-algebra $\mathcal{F}$ and natural filtration $(\mathcal{F}_t, t \geq 0)$. The reader will note that we will also use a similar notion for $\mathbb{D}(E)$ later on in this text in the obvious way for an E-valued Markov process. We will always work with $X = (X_t, t \geq 0)$ to mean the coordinate process defined on the space $\mathbb{D}(\mathbb{R}^d)$. Hence, the notation of the introduction indicates that $\mathbb{P} = (\mathbb{P}_x, x \in \mathbb{R}^d)$ is such that $(X, \mathbb{P})$ is our stable process.

We want to construct the law of the stable process conditioned to continuously limit to $\mathsf{S} \in \mathbb{S}^{d-1}$ whilst visiting both $\mathbb{B}_d := \{x \in \mathbb{R}^d : |x| < 1\}$ and $\bar{\mathbb{B}}_d^c := \mathbb{R}^d \setminus \bar{\mathbb{B}}_d$ infinitely often at arbitrarily small times prior to striking S. We shall denote the associated probabilities by $\mathbb{P}^{\mathsf{S}} = (\mathbb{P}^{\mathsf{S}}_x, x \in \mathbb{R}^d \backslash \mathsf{S})$. For a more precise definition of what is meant by this form of conditioning, let us introduce the stopping times,

$$\tau_\beta = \inf\{t > 0 : \beta^{-1} < |X_t| < \beta\}, \qquad \text{for } \beta > 1. \tag{3.2}$$

Whenever it is well defined, we write, for $t \geq 0$, $\Lambda \in \mathcal{F}_t$ and $x \notin \mathsf{S}$,

$$\mathbb{P}^{\mathsf{S}}_x(\Lambda,\, t < \zeta) = \lim_{\beta \to 1} \lim_{\varepsilon \to 0} \mathbb{P}_x\left(\Lambda, t < \tau_\beta \middle|\, \tau_{\mathsf{S}_\varepsilon} < \infty\right), \tag{3.3}$$

where

$$\tau_{\mathsf{S}_\varepsilon} = \inf\{t > 0 : X_t \in \mathsf{S}_\varepsilon\} \qquad \text{and} \qquad \mathsf{S}_\varepsilon := \{x \in \mathbb{R}^d : 1 - \varepsilon \leq |x| \leq 1 + \varepsilon \text{ and } \arg(x) \in \mathsf{S}\}.$$

Our first main result clarifies that the process $(X, \mathbb{P}^{\mathsf{S}})$ is well defined. In the theorem below, and thereafter, we will understand σ_1 to mean the Lebesgue surface measure on $\mathbb{S}^{d-1}$ normalised to have unit mass, i.e. $\sigma_1(\mathbb{S}^{d-1}) = 1$.

Theorem 3.2.1. *Suppose that $\alpha \in (0,1]$ and the closed set $\mathsf{S} \subseteq \mathbb{S}^{d-1}$ is such that $\sigma_1(\mathsf{S}) > 0$. For $\alpha \in (0,1]$, the limit (3.3) makes sense. Therefore, the process $(X, \mathbb{P}^{\mathsf{S}})$ is well defined such that*

$$\left.\frac{\mathrm{d}\mathbb{P}_x^{\mathsf{S}}}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{H_{\mathsf{S}}(X_t)}{H_{\mathsf{S}}(x)}, \qquad t \geq 0, x \notin \mathsf{S}, \tag{3.4}$$

where

$$H_{\mathsf{S}}(x) = \int_{\mathsf{S}} |x-\theta|^{\alpha-d}\sigma_1(\mathrm{d}\theta), \qquad x \notin \mathsf{S}.$$

Although excluded from the conclusion of Theorem 3.2.1, it is worth dwelling for a moment on the extreme case $\mathsf{S} = \{\theta\}$, for $\theta \in \mathbb{S}^{d-1}$. It has been shown in [20] that, when $\alpha \in (0,1)$, conditioning a stable process to continuously limit to a point (which, by stationary and independent increments, can always be arranged to be $\theta \in \mathbb{S}^{d-1}$) results in a family of probability measures $(\mathbb{P}_x^{\{\theta\}}, x \neq \theta)$ which can be identified via a Doob h-transform with $h_\theta(x) = |x-\theta|^{\alpha-d}$. Although the sense in which the conditioning is performed cannot be contextualised via (3.3), we see that the resulting h-transformation is consistent with the use of the harmonic function H_{S}.

The way in which we will prove Theorem 3.2.1 will be to prove the following subtle result which establishes the leading order behaviour of the probability of hitting the set S_ε.

Theorem 3.2.2. *Let $\mathsf{S} \subseteq \mathbb{S}^{d-1}$ be a closed subset such that $\sigma_1(\mathsf{S}) > 0$.*

(i) Suppose $\alpha \in (0,1)$. For $x \notin \mathsf{S}$,

$$\lim_{\varepsilon \to 0} \varepsilon^{\alpha-1}\mathbb{P}_x(\tau_{\mathsf{S}_\varepsilon} < \infty) = 2^{1-2\alpha}\frac{\Gamma((d+\alpha-2)/2)}{\pi^{d/2}\Gamma(1-\alpha)}\frac{\Gamma((2-\alpha)/2)}{\Gamma(2-\alpha)}H_{\mathsf{S}}(x). \tag{3.5}$$

(ii) When $\alpha = 1$, we have that, for $x \notin \mathsf{S}$,

$$\lim_{\varepsilon \to 0} \, |\log \varepsilon| \, \mathbb{P}_x(\tau_{\mathsf{S}_\varepsilon} < \infty) = \frac{\Gamma((d-1)/2)}{\pi^{(d-1)/2}}H_{\mathsf{S}}(x). \tag{3.6}$$

Theorem 3.2.2 also gives us the opportunity to understand the strike position of the conditioned stable process. Indeed, let S' be a closed subset of S. Define $\mathsf{S}'_\varepsilon = \{x \in \mathbb{R}^d : 1-\varepsilon \leq |x| \leq 1+\varepsilon \text{ and } \arg(x) \in \mathsf{S}'\}$ and $\tau_{\mathsf{S}'_\varepsilon} := \inf\{t > 0 \colon X_t \in \mathsf{S}'_\varepsilon\}$. Then, $\{\tau_{\mathsf{S}'_\varepsilon} < \infty\} \subseteq \{\tau_{\mathsf{S}_\varepsilon} < \infty\}$ and thanks to Theorem 3.2.2, when $\alpha \in (0,1)$, we have

$$\lim_{\varepsilon \to 0}\mathbb{P}_x(\tau_{\mathsf{S}'_\varepsilon} < \infty | \tau_{\mathsf{S}_\varepsilon} < \infty) = \lim_{\varepsilon \to 0}\frac{\varepsilon^{\alpha-1}\mathbb{P}_x(\tau_{\mathsf{S}'_\varepsilon} < \infty)}{\varepsilon^{\alpha-1}\mathbb{P}_x(\tau_{\mathsf{S}_\varepsilon} < \infty)} = \frac{H_{\mathsf{S}'}(x)}{H_{\mathsf{S}}(x)}, \qquad x \notin \mathsf{S}.$$

A similar statement also holds when $\alpha = 1$ by changing the scaling in ε to $|\log \varepsilon|$. This gives us the following result.

Corollary 3.2.1. *For a closed $\mathsf{S} \subseteq \mathbb{S}^{d-1}$ such that $\sigma_1(\mathsf{S}) > 0$ and $\alpha \in (0,1]$, we have that for all closed $\mathsf{S}' \subseteq \mathsf{S}$,*

$$\mathbb{P}_x^{\mathsf{S}}(X_{\zeta-} \in \mathsf{S}') = \frac{H_{\mathsf{S}'}(x)}{H_{\mathsf{S}}(x)}, \qquad x \notin \mathsf{S}. \tag{3.7}$$

In light of the above Corollary, it is worth remarking that we can also see the probabilities $\mathbb{P}^{\mathsf{S}}$ as the result of first conditioning to continuously hit $\mathbb{S}^{d-1}$ and then conditioning the strike point to be in S. Indeed, we note that, for $A \in \mathcal{F}_t$ and $t \geq 0$,

$$\begin{aligned}
\mathbb{P}_x^{\mathbb{S}^{d-1}}(A|X_{\zeta-} \in \mathsf{S}) &= \mathbb{E}_x^{\mathbb{S}^{d-1}}\left[\mathbf{1}_A \frac{\mathbb{P}_{X_t}^{\mathbb{S}^{d-1}}(X_{\zeta-} \in \mathsf{S})}{\mathbb{P}_x^{\mathbb{S}^{d-1}}(X_{\zeta-} \in \mathsf{S})}\right] \\
&= \mathbb{E}_x\left[\mathbf{1}_A \frac{H_{\mathbb{S}^{d-1}}(X_t)}{H_{\mathbb{S}^{d-1}}(x)} \frac{H_{\mathsf{S}}(X_t)}{H_{\mathbb{S}^{d-1}}(X_t)} \frac{H_{\mathbb{S}^{d-1}}(x)}{H_{\mathsf{S}}(x)}\right] \\
&= \mathbb{E}_x\left[\mathbf{1}_A \frac{H_{\mathsf{S}}(X_t)}{H_{\mathsf{S}}(x)}\right] \\
&= \mathbb{P}_x^{\mathsf{S}}(A).
\end{aligned}$$

Moreover, by shrinking $\mathsf{S}' \subseteq \mathsf{S} \subseteq \mathbb{S}^{d-1}$ to a singleton $\theta \in \mathbb{S}^{d-1}$, one can similarly show that

$$\mathbb{P}_x^{\mathsf{S}}(A|X_{\zeta-} = \theta) = \mathbb{P}_x^{\{\theta\}}(A).$$

This has the flavour of a Williams' type decomposition that was shown for general Lévy processes conditioned to stay positive and subordinators conditioned to remain in an interval; see e.g [11] and [19].

3.3 Oscillatory repulsion from the patch S and duality

Roughly speaking, we want to describe what we see when we time reverse the process $(X, \mathbb{P}^{\mathsf{S}})$ from its strike point on S, i.e. its so-called dual process. Such a process will necessarily avoid visiting S. Recalling that, for $\alpha \in (0,1]$, the stable process hits spherical surfaces with probability zero (cf. [16, 25]), a heuristic guess for the aforesaid dual process is the stable process itself (see Figure 3-1). This turns out to be precisely the case. In order to make this rigorous, we will use the language of Hunt-Nagasawa duality for Markov processes.

Suppose that $Y = (Y_t, t \leq \zeta)$ with probabilities $\mathtt{P}_x$, $x \in E$, is a regular Markov process on an open domain $E \subseteq \mathbb{R}^d$ (or more generally, a locally compact Hausdorff space with countable base), with cemetery state Δ and killing time $\zeta = \inf\{t > 0 : Y_t = \Delta\}$. Let us additionally write $\mathtt{P}_\nu = \int_E \nu(\mathrm{d}a)\mathtt{P}_a$, for any probability measure ν on the state space of Y.

Suppose that $\mathcal{G}$ is the σ-algebra generated by Y and write $\mathcal{G}(\mathtt{P}_\nu)$ for its completion by the null sets of $\mathtt{P}_\nu$. Moreover, write $\overline{\mathcal{G}} = \bigcap_\nu \mathcal{G}(\mathtt{P}_\nu)$, where the intersection is taken over all probability measures on the state space of Y, excluding the cemetery state. A finite random time $\mathtt{k}$ is called an L-time (generalized last exit time) if, given a coordinate process $\omega = (\omega_t, t \geq 0)$ on $\mathbb{D}(E)$,

(i) $\mathtt{k}$ is measurable in $\overline{\mathcal{G}}$, and $\mathtt{k} \leq \zeta$ almost surely with respect to $\mathtt{P}_\nu$, for all ν,

(ii) $\{s < \mathtt{k}(\omega) - t\} = \{s < \theta_t \circ \mathtt{k}\}$ for all $t, s \geq 0$,

where θ_t is the Markov shift of ω to time t. The most important examples of L-times are killing times and last exit times from closed sets.

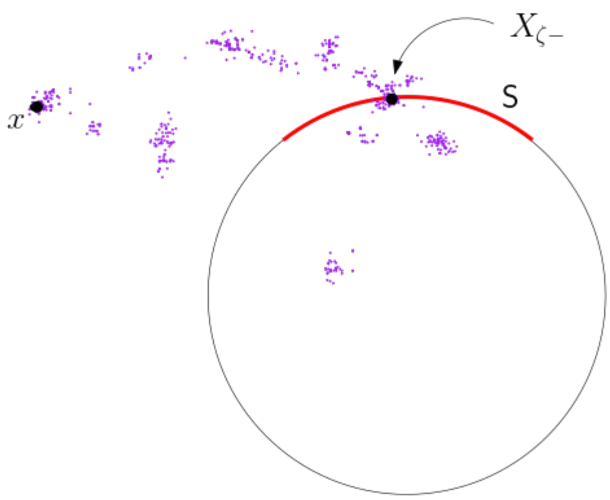

Figure 3-1: The process $(X, \mathbb{P}^{\mathsf{S}})$ when time reversed is stochastically equal in law to $(X, \mathbb{P})$.

Theorem 3.3.1. *Suppose that* $\alpha \in (0,1]$. *For a given closed set* $\mathsf{S} \subset \mathbb{S}^{d-1}$ *with* $\sigma_1(\mathsf{S}) > 0$, *write*

$$\nu(\mathrm{d}a) := \frac{\sigma_1(\mathrm{d}a)}{\sigma_1(\mathsf{S})}, \qquad a \in \mathsf{S}. \tag{3.8}$$

For every L-time $\mathtt{k}$ *of* $(X, \mathbb{P})$, *the process* $(X_{(\mathtt{k}-t)-}, t < \mathtt{k})$ *under* $\mathbb{P}_\nu$ *is a time-homogeneous Markov process whose transition probabilities agree with those of* $(X, \mathbb{P}^{\mathsf{S}})$.

3.4 The setting of a subset in an $\mathbb{R}^{d-1}$ hyperplane

As alluded to in the introduction, the methods used in Sections 3.2 and 3.3 are robust enough to deal with the setting of an arbitrary $(d-1)$-dimensional hyperplane in $\mathbb{R}^d$. Without loss of generality, we can describe such a hyperplane with unit orthogonal vector $v \in \mathbb{S}^{d-1}$ via

$$\mathbb{H}^{d-1} = \{x \in \mathbb{R}^d : (x, v) = 0\},$$

where $(\cdot,\cdot)$ is the usual Euclidean inner product. Henceforth, we will assume that $v \in \mathbb{S}^{d-1}$ is given, as it otherwise plays no role in the forthcoming. We are interested in defining the law of the stable process conditioned to hit $\mathsf{D} \subseteq \mathbb{H}^{d-1}$ in a similar spirit to the discussion in Section 3.2.

To this end, let us define

$$\kappa_\beta = \inf\{t > 0 : -\beta < (v, X_t) < \beta\}, \qquad \text{for } \beta > 0.$$

Whenever it is well defined, we will write, for $t \geq 0$, $\Lambda \in \mathcal{F}_t$ and $x \notin \mathsf{D}$,

$$\mathbb{P}^{\mathsf{D}}_x(\Lambda,\, t < \zeta) = \lim_{\beta\to 0}\lim_{\varepsilon\to 0} \mathbb{P}_x\left(\Lambda, t < \kappa_\beta \middle|\, \tau_{\mathsf{D}_\varepsilon} < \infty\right), \tag{3.9}$$

where

$$\tau_{\mathsf{D}_\varepsilon} = \inf\{t > 0 : X_t \in \mathsf{D}_\varepsilon\} \qquad \text{and} \qquad \mathsf{D}_\varepsilon := \{x \in \mathbb{R}^d : -\varepsilon \le (v, x) \le \varepsilon \text{ and } \hat{x} \in \mathsf{D}\}.$$

Here $\hat{x}$ denotes the orthogonal projection of x onto $\mathbb{H}^{d-1}$; in other words. $\hat{x} = x - v(v, x)$. We can gather the analogous conclusions of Theorems 3.2.1, 3.2.2, 4.4.3 and Corollary 4.3.1 into one theorem.

Theorem 3.4.1. *Suppose that $\alpha \in (0, 1]$ and the closed and bounded set $\mathsf{D} \subseteq \mathbb{H}^{d-1}$ is such that $0 < \ell_{d-1}(\mathsf{D}) < \infty$, where we recall that ℓ_{d-1} is $(d-1)$-dimensional Lebesgue measure.*

(i) Suppose $\alpha \in (0, 1)$. For $x \notin \mathsf{D}$,

$$\lim_{\varepsilon \to 0} \varepsilon^{\alpha-1} \mathbb{P}_x(\tau_{\mathsf{D}_\varepsilon} < \infty) = 2^{1-\alpha} \pi^{-(d-2)/2} \frac{\Gamma(\frac{d-2}{2}) \Gamma(\frac{d-\alpha}{2}) \Gamma(\frac{2-\alpha}{2})^2}{\Gamma(\frac{1-\alpha}{2}) \Gamma(\frac{d-1}{2}) \Gamma(2-\alpha)} M_{\mathsf{D}}(x), \tag{3.10}$$

where

$$M_{\mathsf{D}}(x) = \int_{\mathsf{D}} |x - y|^{\alpha-d} \ell_{d-1}(\mathrm{d}y), \qquad x \notin \mathsf{D}.$$

(ii) Suppose $\alpha = 1$. For $x \notin \mathsf{D}$,

$$\lim_{\varepsilon \to 0} \; |\log \varepsilon| \; \mathbb{P}_x(\tau_{\mathsf{D}_\varepsilon} < \infty) = \frac{\Gamma(\frac{d-2}{2})}{\pi^{(d-2)/2}} M_{\mathsf{D}}(x). \tag{3.11}$$

(iii) The limit (3.9) makes sense, therefore the process $(X, \mathbb{P}^{\mathsf{D}})$ is well defined and

$$\left.\frac{\mathrm{d}\mathbb{P}_x^{\mathsf{D}}}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{M_{\mathsf{D}}(X_t)}{M_{\mathsf{D}}(x)}, \qquad t \ge 0, x \notin \mathsf{D}. \tag{3.12}$$

(iv) We have for all closed $\mathsf{D}' \subseteq \mathsf{D}$,

$$\mathbb{P}_x^{\mathsf{D}}(X_{\zeta-} \in \mathsf{D}') = \frac{M_{\mathsf{D}'}(x)}{M_{\mathsf{D}}(x)}, \qquad x \notin \mathsf{D}. \tag{3.13}$$

(v) Write $\nu(\mathrm{d}a) := \ell_{d-1}(\mathrm{d}a)/\ell_{d-1}(\mathsf{D})$, $a \in \mathsf{D}$. For every L-time $\mathtt{k}$ of $(X, \mathbb{P})$, the process $(X_{(\mathtt{k}-t)-}, t < \mathtt{k})$ under $\mathbb{P}_\nu$ is a time-homogeneous Markov process whose transition probabilities agree with those of $(X, \mathbb{P}^{\mathsf{D}})$.

Roughly speaking, Theorem 3.4.1 are to be expected as, following the ideas of [22] one may map $\mathbb{S}^{d-1}$ onto $\mathbb{H}^{d-1}$ via a standard sphere inversion transformation, which, thanks to the Riesz–Bogdan–Żak transform, also transforms the paths of the stable processes into that of a h-transformed stable processes; see [8]. The proofs we have given below, however, are direct nonetheless, following similar steps to those of Theorems 3.2.1, 3.2.2 and 4.4.3, as well as Corollary 4.3.1.

3.5 Heuristic for the proof of Theorem 3.2.2

Let us begin with a sketch of the proof of Theorem 3.2.2. We start by recalling an identity that is known in quite a general setting from the potential analysis literature; see for example Section 13.11 of [13] and Section VI.2 of [7]. Suppose that A is a bounded closed set and let $\tau_A = \inf\{t > 0 : X_t \in A\}$. Let μ_A be a finite measure supported on A, which is absolutely continuous with respect to Lebesgue measure and define its potential by

$$U\mu_A(x) := \int_A |x-y|^{\alpha-d}\mu_A(\mathrm{d}y), \qquad x \in \mathbb{R}^d.$$

On account of the fact that μ_A is absolutely continuous, recalling that $|x|^{\alpha-d}$ is the potential of the stable process issued from the origin, stationary and independent increments allows us to identify

$$U\mu_A(x) = \int_A |x-y|^{\alpha-d} m_A(y)\ell_d(\mathrm{d}y) = \mathbb{E}_x\left[\int_0^\infty m_A(X_t)\mathrm{d}t\right], \qquad x \notin A,$$

where m_A is the density of μ_A with respect to Lebesgue measure, ℓ_d. As the support of μ_A is precisely A, we must have $m_A(y) = 0$ for all $y \notin A$. As such, the Strong Markov Property tells us that

$$U\mu_A(x) = \mathbb{E}_x\left[\mathbf{1}_{\{\tau_A<\infty\}}\int_{\tau_A}^\infty m_A(X_t)\mathrm{d}t\right] = \mathbb{E}_x\left[U\mu_A(X_{\tau_A})\mathbf{1}_{\{\tau_A<\infty\}}\right], \qquad x \notin A. \tag{3.14}$$

Note, the above equality is also true when $x \in A$ as, in that case, $\tau_A = 0$.

Replacing τ_A by a general stopping time τ in the above calculation changes the first equality in (3.14) to an inequality, thus giving the excessive property

$$U\mu_A(x) \geq \mathbb{E}_x\left[U\mu_A(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right], \qquad x \in \mathbb{R}^d. \tag{3.15}$$

This family of inequalities together with the Strong Markov Property easily gives us the classical result that $(U\mu_A(X_t), t \geq 0)$ is a supermartingale.

Let us now suppose that μ can be constructed in such a away that it is supported on A such that, for all $x \in A$, $U\mu(x) = 1$. We then recover from identity (3.14) the corollary to Theorem 1 in Chapter 5 of [13], see also equation (21) in the same chapter, which states that

$$\mathbb{P}_x(\tau_A < \infty) = U\mu(x), \qquad x \notin A.$$

Returning to the problem at hand, we can use the principals above to develop a 'guess and verify' approach to the proof, in particular, since we are not chasing an exact formula for $\mathbb{P}_x(\tau_{\mathsf{S}_\varepsilon} < \infty)$, but rather the asymptotic leading order behaviour. Indeed, suppose we can 'guess' a measure, say $\mu_\varepsilon^{\mathsf{S}}$, supported on S_ε, such that

$$U\mu_\varepsilon^{\mathsf{S}}(x) = 1 + o(1), \qquad x \in \mathsf{S}_\varepsilon \text{ as } \varepsilon \to 0, \tag{3.16}$$

so that

$$(1+o(1))\mathbb{P}_x(\tau_{\mathsf{S}_\varepsilon}<\infty)=U\mu_\varepsilon^{\mathsf{S}}(x), \qquad x\notin \mathsf{S}_\varepsilon. \tag{3.17}$$

Then, this would be a good basis from which to draw out the leading order decay in ε, especially if our guess of μ_ε is such that $U\mu_\varepsilon$ is tractable.

In one dimension, we know from Lemma 1 of [26], that for a one-dimensional symmetric stable process, the unique measure that satisfies (3.16) has density $(1-y)^{-\alpha/2}(1+y)^{-\alpha/2}$, i.e.

$$\int_{-1}^{1}|x-y|^{\alpha-1}(1-y)^{-\alpha/2}(1+y)^{-\alpha/2}\mathrm{d}y=1, \qquad x\in[-1,1]. \tag{3.18}$$

We can use this to build a reasonable choice of $\mu_\varepsilon^{\mathsf{S}}$. Indeed, writing $X=|X|\arg(X)$, when X begins in the neighbourhood of S, then $|X|$ begins in the neighbourhood of 1 and $\arg(X)$, essentially, from within S. On short-time scales and short-range, the time change $|X|$ behaves similarly to a one-dimensional stable process. Moreover, $\arg(X)$ is an isotropic process. A reasonable guess for $\mu_\varepsilon^{\mathsf{S}}$ would be to base it on the measure

$$\mu_\varepsilon(\mathrm{d}y)=c_{\alpha,d}(|y|-(1-\varepsilon))^{-\alpha/2}(1+\varepsilon-|y|)^{-\alpha/2}\ell_d(\mathrm{d}y), \tag{3.19}$$

restricted to S_ε, where we recall $c_{\alpha,d}$ is a constant to be determined so that (3.16) holds. As we will shortly see, when $\alpha\in(0,1)$, the constant $c_{\alpha,d}$ does not depend on ε, however, when $\alpha=1$, in order to respect (3.16) we need to make it depend on ε.

3.6 Proof of Theorem 3.2.2 (i)

As alluded to in the previous section, we will work with the guess $\mu_\varepsilon^{\mathsf{S}}$ given by the measure μ_ε defined in (3.19) restricted to S_ε. In order to show (3.16), we will take advantage of some of the symmetric features of μ_ε, when seen as a measure over $\mathbb{S}_\varepsilon^{d-1}=\{x\in\mathbb{R}^d: 1-\varepsilon\le|x|\le 1+\varepsilon\}$. For a subset $A\subset\mathbb{S}_\varepsilon^{d-1}$ we define μ_ε^A the restriction of μ_ε to A. In particular, writing $\mu_\varepsilon^{(1)}$ as μ_ε restricted to $\mathbb{S}_\varepsilon^{d-1}$ and $\mu_\varepsilon^{(2)}$ as μ_ε restricted to $\hat{\mathsf{S}}_\varepsilon:=\mathbb{S}_\varepsilon^{d-1}\backslash\mathsf{S}_\varepsilon$, we have the obvious difference

$$U\mu_\varepsilon^{\mathsf{S}}(x)=U\mu_\varepsilon^{(1)}(x)-U\mu_\varepsilon^{(2)}(x), \qquad x\in\mathsf{S}_\varepsilon. \tag{3.20}$$

Moreover, we would like to introduce

$$\mu_{\varepsilon,\delta}^{(2)}:=\mu_\varepsilon|_{\hat{\mathsf{S}}_\varepsilon^\delta}$$

where $\hat{\mathsf{S}}_\varepsilon^\delta=\mathbb{S}_\varepsilon^{d-1}\backslash\mathsf{S}_\varepsilon^\delta$ and

$\mathsf{S}_\varepsilon^\delta:=\{x\in\mathbb{R}^d\colon 1-\varepsilon<|x|<1+\varepsilon \text{ and } \arg(x)\in\mathsf{S}^\delta\}$, where $\mathsf{S}^\delta=\{x\in\mathbb{S}^{d-1}:\mathrm{dist}(\arg(x),\mathsf{S})<\delta\}$, for some small $\delta > 0$, which, in due course, will depend on ε. Note that, since S is closed, S^δ (resp. $\mathsf{S}_\varepsilon^\delta$) shrinks to S (resp. S_ε) when $\delta \to 0$. Then, we also have that

$$U\mu_\varepsilon^{\mathsf{S}^\delta}(x) = U\mu_\varepsilon^{(1)}(x) - U\mu_{\varepsilon,\delta}^{(2)}(x), \qquad x \in \mathsf{S}_\varepsilon. \tag{3.21}$$

The estimate (3.21) will be useful for a certain lower bound that will give us what we need to prove Theorem 3.2.2. We need to prove two technical lemmas first. The first one deals with the term $U\mu_\varepsilon^{(1)}$.

Lemma 3.6.1. *Suppose that we choose*

$$c_{\alpha,d} = \frac{\Gamma((d+\alpha-2)/2)}{2^\alpha \pi^{d/2}\Gamma(1-\alpha)\Gamma((2-\alpha)/2)}.$$

Then,

$$\lim_{\varepsilon\to 0} \sup_{x\in\mathbb{S}_\varepsilon^{d-1}} |U\mu_\varepsilon^{(1)}(x) - 1| = 0.$$

Proof. Appealing to (A.1), we have, for $x \in \mathbb{S}_\varepsilon^{d-1}$,

$$\begin{aligned}
&U\mu_\varepsilon^{(1)}(x)\\
&= c_{\alpha,d}\int_{\mathbb{S}_\varepsilon^{d-1}} |x-y|^{\alpha-d}(|y|-(1-\varepsilon))^{-\alpha/2}(1+\varepsilon-|y|)^{-\alpha/2}\ell_d(\mathrm{d}y)\\
&= \frac{2c_{\alpha,d}\pi^{(d-1)/2}}{\Gamma((d-1)/2)}\int_{1-\varepsilon}^{1+\varepsilon} \frac{r^{d-1}}{(r-(1-\varepsilon))^{\alpha/2}(1+\varepsilon-r)^{\alpha/2}}\mathrm{d}r \int_0^\pi \frac{\sin^{d-2}\theta\,\mathrm{d}\theta}{(|x|^2-2|x|r\cos\theta+r^2)^{(d-\alpha)/2}}\\
&= \frac{2c_{\alpha,d}\pi^{d/2}}{\Gamma(d/2)}|x|^{\alpha-d}\int_{1-\varepsilon}^{|x|} \frac{{}_2F_1\Big(\frac{d-\alpha}{2}, 1-\frac{\alpha}{2}; \frac{d}{2}; (r/|x|)^2\Big) r^{d-1}}{(r-(1-\varepsilon))^{\alpha/2}(1+\varepsilon-r)^{\alpha/2}}\mathrm{d}r\\
&\qquad\qquad + \frac{2c_{\alpha,d}\pi^{d/2}}{\Gamma(d/2)}\int_{|x|}^{1+\varepsilon} \frac{{}_2F_1\Big(\frac{d-\alpha}{2}, 1-\frac{\alpha}{2}; \frac{d}{2}; (|x|/r)^2\Big) r^{\alpha-1}}{(r-(1-\varepsilon))^{\alpha/2}(1+\varepsilon-r)^{\alpha/2}}\mathrm{d}r.
\end{aligned} \tag{3.22}$$

With a simple change of variables we can reduce this more simply to

$$\begin{aligned}
U\mu_\varepsilon^{(1)}(x) &= \frac{2c_{\alpha,d}\pi^{d/2}}{\Gamma(d/2)}\int_{\frac{1-\varepsilon}{|x|}}^{1} \frac{{}_2F_1\Big(\frac{d-\alpha}{2}, 1-\frac{\alpha}{2}; \frac{d}{2}; r^2\Big) r^{d-1}}{\Big(r-\frac{1-\varepsilon}{|x|}\Big)^{\alpha/2}\Big(\frac{1+\varepsilon}{|x|}-r\Big)^{\alpha/2}}\mathrm{d}r\\
&\qquad + \frac{2c_{\alpha,d}\pi^{d/2}}{\Gamma(d/2)}\int_1^{\frac{1+\varepsilon}{|x|}} \frac{{}_2F_1\Big(\frac{d-\alpha}{2}, 1-\frac{\alpha}{2}; \frac{d}{2}; r^{-2}\Big) r^{\alpha-1}}{\Big(r-\frac{1-\varepsilon}{|x|}\Big)^{\alpha/2}\Big(\frac{1+\varepsilon}{|x|}-r\Big)^{\alpha/2}}\mathrm{d}r.
\end{aligned} \tag{3.23}$$

For the first term on the right-hand side of (3.23), we can appeal to (A.1) and (A.2) to deduce that

$$\lim_{\varepsilon\to 0}\sup_{x\in\mathbb{S}^{d-1}_\varepsilon}\left|\frac{2c_{\alpha,d}\pi^{d/2}}{\Gamma(d/2)}\int_{\frac{1-\varepsilon}{|x|}}^1\frac{{}_2F_1\Big(\frac{d-\alpha}{2},1-\frac{\alpha}{2};\frac{d}{2};r^2\Big)r^{d-1}}{\Big(r-\frac{1-\varepsilon}{|x|}\Big)^{\alpha/2}\Big(\frac{1+\varepsilon}{|x|}-r\Big)^{\alpha/2}}\mathrm{d}r\right.$$
$$-\frac{2c_{\alpha,d}\pi^{d/2}\Gamma(1-\alpha)}{\Gamma((d-\alpha)/2)\Gamma((2-\alpha)/2)}\int_{\frac{1-\varepsilon}{|x|}}^1\frac{(1-r^2)^{\alpha-1}r^{d-1}}{\Big(r-\frac{1-\varepsilon}{|x|}\Big)^{\alpha/2}\Big(\frac{1+\varepsilon}{|x|}-r\Big)^{\alpha/2}}\mathrm{d}r$$
$$\left.-\frac{2c_{\alpha,d}\pi^{d/2}\Gamma(1-\alpha)}{\Gamma(\alpha/2)\Gamma((d+\alpha-2)/2)}\int_{\frac{1-\varepsilon}{|x|}}^1\frac{r^{d-1}}{\Big(r-\frac{1-\varepsilon}{|x|}\Big)^{\alpha/2}\Big(\frac{1+\varepsilon}{|x|}-r\Big)^{\alpha/2}}\mathrm{d}r\right|=0. \tag{3.24}$$

Note that, by using the transformation $r=(1-\varepsilon+2\varepsilon u)/|x|$,

$$\begin{aligned}\int_{\frac{1-\varepsilon}{|x|}}^1 & r^{d-1}\Big(r-\frac{1-\varepsilon}{|x|}\Big)^{-\alpha/2}\Big(\frac{1+\varepsilon}{|x|}-r\Big)^{-\alpha/2}\mathrm{d}r\\ &=|x|^{\alpha-d}(2\varepsilon)^{1-\alpha}\int_0^{(|x|-1+\varepsilon)/2\varepsilon}(2\varepsilon u+1-\varepsilon)^{d-1}u^{-\alpha/2}(1-u)^{-\alpha/2}\mathrm{d}u\\ &\le|x|^{\alpha-d}(2\varepsilon)^{1-\alpha}\frac{\Gamma((2-\alpha)/2)^2}{\Gamma(2-\alpha)},\end{aligned}\tag{3.25}$$

which tends to zero uniformly in $x\in\mathbb{S}^{d-1}_\varepsilon$ as $\varepsilon\to 0$.

The asymptotic (3.25) also tells us that the approximating term of interest in (3.24) is the middle term. For that, we can use (A.8) to observe

$$\lim_{\varepsilon\to 0}\sup_{x\in\mathbb{S}^{d-1}_\varepsilon}\left|\int_{\frac{1-\varepsilon}{|x|}}^1(1-r^2)^{\alpha-1}r^{d-1}\Big(r-\frac{1-\varepsilon}{|x|}\Big)^{-\alpha/2}\Big(\frac{1+\varepsilon}{|x|}-r\Big)^{-\alpha/2}\mathrm{d}r\right.$$
$$\left.-2^{\alpha-1}\int_{\frac{1-\varepsilon}{|x|}}^1(1-r)^{\alpha-1}\Big(r-\frac{1-\varepsilon}{|x|}\Big)^{-\alpha/2}\Big(\frac{1+\varepsilon}{|x|}-r\Big)^{-\alpha/2}\mathrm{d}r\right|=0 \tag{3.26}$$

and

$$\begin{aligned}&\frac{2^\alpha c_{\alpha,d}\pi^{d/2}\Gamma(1-\alpha)}{\Gamma(\alpha/2)\Gamma((d+\alpha-2)/2)}\int_{\frac{1-\varepsilon}{|x|}}^1(1-r)^{\alpha-1}\Big(r-\frac{1-\varepsilon}{|x|}\Big)^{-\alpha/2}\Big(\frac{1+\varepsilon}{|x|}-r\Big)^{-\alpha/2}\mathrm{d}r\\ &=\frac{2^\alpha c_{\alpha,d}\pi^{d/2}\Gamma(1-\alpha)}{\Gamma(\alpha/2)\Gamma((d+\alpha-2)/2)}\int_0^{1-\frac{1-\varepsilon}{|x|}}u^{\alpha-1}\Big(1-\frac{1-\varepsilon}{|x|}-u\Big)^{-\alpha/2}\Big(\frac{1+\varepsilon}{|x|}-1+u\Big)^{-\alpha/2}\mathrm{d}r\\ &=\frac{2^\alpha c_{\alpha,d}\pi^{d/2}\Gamma(1-\alpha)\Gamma((2-\alpha)/2)\Gamma(\alpha)}{\Gamma(\alpha/2)\Gamma((d+\alpha-2)/2)\Gamma((2+\alpha)/2)}\left(\frac{|x|-1+\varepsilon}{1+\varepsilon-|x|}\right)^{\alpha/2}\\ &\qquad\qquad{}_2F_1\Big(\alpha/2,\alpha;1+\alpha/2;-\frac{|x|-1+\varepsilon}{1+\varepsilon-|x|}\Big).\end{aligned}\tag{3.27}$$

The second term on the right-hand side of (3.23) can be dealt with similarly. Indeed, using (A.2) we can produce an analogous statement to (3.24), from which, the leading order approximating term is the integral

$$
\begin{aligned}
&\frac{2c_{\alpha,d}\pi^{d/2}\Gamma(1-\alpha)}{\Gamma(\alpha/2)\Gamma((d+\alpha-2)/2)}\int_1^{\frac{1+\varepsilon}{|x|}}(1-r^{-2})^{\alpha-1}r^{d-1}\Big(r-\frac{1-\varepsilon}{|x|}\Big)^{-\alpha/2}\Big(\frac{1+\varepsilon}{|x|}-r\Big)^{-\alpha/2}\mathrm{d}r\\
&\sim\frac{2^{\alpha}c_{\alpha,d}\pi^{d/2}\Gamma(1-\alpha)}{\Gamma(\alpha/2)\Gamma((d+\alpha-2)/2)}\int_1^{\frac{1+\varepsilon}{|x|}}(r-1)^{\alpha-1}\Big(r-\frac{1-\varepsilon}{|x|}\Big)^{-\alpha/2}\Big(\frac{1+\varepsilon}{|x|}-r\Big)^{-\alpha/2}\mathrm{d}r\\
&=\frac{2^{\alpha}c_{\alpha,d}\pi^{d/2}\Gamma(1-\alpha)}{\Gamma(\alpha/2)\Gamma((d+\alpha-2)/2)}\int_0^{\frac{1+\varepsilon}{|x|}-1}u^{\alpha-1}\Big(u+1-\frac{1-\varepsilon}{|x|}\Big)^{-\alpha/2}\Big(\frac{1+\varepsilon}{|x|}-1-u\Big)^{-\alpha/2}\mathrm{d}u\\
&=\frac{2^{\alpha}c_{\alpha,d}\pi^{d/2}\Gamma(1-\alpha)\Gamma((2-\alpha)/2)\Gamma(\alpha)}{\Gamma(\alpha/2)\Gamma((d+\alpha-2)/2)\Gamma((2+\alpha)/2)}\Big(\frac{1+\varepsilon-|x|}{|x|-1+\varepsilon}\Big)^{\alpha/2}\\
&\qquad\qquad {}_2F_1\Big(\alpha/2,\alpha;1+\alpha/2;-\frac{1+\varepsilon-|x|}{|x|-1+\varepsilon}\Big),
\end{aligned}
\tag{3.28}
$$

uniformly for $x\in\mathbb{S}_\varepsilon^{d-1}$ as $\varepsilon\to 0$, where we have again used (A.8) to develop the right-hand side.

Somewhat remarkably, if we add together the right-hand side of (3.27) and (3.28), using the identity in (A.6), we see that the sum is equal to

$$
\frac{2^{\alpha}c_{\alpha,d}\pi^{d/2}\Gamma(1-\alpha)\Gamma((2-\alpha)/2)}{\Gamma((d+\alpha-2)/2)}=1,
\tag{3.29}
$$

where the equality with unity follows from the choice of $c_{\alpha,d}$ in the statement of the lemma.

Piecing together then uniform estimates above as well as the simplification of the two integrals (3.27) and (3.28) as well as the decay of the term (3.25) in (3.24) and the analogous term when dealing with the second term on the right-hand side of (3.23), the statement of the lemma follows. □

Next we deal with the term $U\mu^{(2)}_{\varepsilon,\delta}$.

Lemma 3.6.2. *Recalling that $c_{\alpha,d}$ is the constant given in Lemma 3.6.1, take $\delta(\varepsilon)=\varepsilon^{(1-\alpha)/2(d-\alpha)}$, then*

$$
\lim_{\varepsilon\to 0}\sup\sup_{x\in\mathsf{S}_\varepsilon}\varepsilon^{(\alpha-1)/2}U\mu^{(2)}_{\varepsilon,\delta(\varepsilon)}(x)\le C_{\alpha,d},
$$

where

$$
C_{\alpha,d}=c_{\alpha,d}\frac{2^{2-\alpha}\pi^{(d-1)/2}\Gamma((2-\alpha)/2)^2}{\Gamma(2-\alpha)\Gamma((d-1)/2)}.
$$

In particular,

$$
\lim_{\varepsilon\to 0}\sup_{x\in\mathsf{S}_\varepsilon}U\mu^{(2)}_{\varepsilon,\delta(\varepsilon)}(x)=0.
$$

Proof. Since $x \in \mathsf{S}_\varepsilon$ and $y \in \hat{\mathsf{S}}^\delta_\varepsilon$, i.e. $|x-y| > \delta$, we have,

$$\begin{aligned}\sup_{x\in\mathsf{S}_\varepsilon} U\mu^{(2)}_{\varepsilon,\delta}(x) &= \int_{\hat{\mathsf{S}}^\delta_\varepsilon} \frac{1}{|x-y|^{d-\alpha}}\mu_\varepsilon(\mathrm{d}y) \\ &\leq \frac{1}{\delta^{d-\alpha}} \int_{\hat{\mathsf{S}}^\delta_\varepsilon} \mu_\varepsilon(\mathrm{d}y) \\ &\leq \frac{1}{\delta^{d-\alpha}} \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)} \int_{1-\varepsilon}^{1+\varepsilon} r^{d-1} m_\varepsilon(r)\mathrm{d}r, \end{aligned} \tag{3.30}$$

where $m_\varepsilon(r) = c_{\alpha,d}(r-(1-\varepsilon))^{-\alpha/2}(1+\varepsilon-r)^{-\alpha/2}$. It is easy to see that

$$\begin{aligned}\int_{1-\varepsilon}^{1+\varepsilon} m_\varepsilon(r)\mathrm{d}r &= c_{\alpha,d}\int_{1-\varepsilon}^{1+\varepsilon} (r-(1-\varepsilon))^{-\alpha/2}(1+\varepsilon-r)^{-\alpha/2}\mathrm{d}r \\ &= c_{\alpha,d}\varepsilon^{1-\alpha}2^{1-\alpha}\frac{\Gamma((2-\alpha)/2)^2}{\Gamma(2-\alpha)}. \end{aligned} \tag{3.31}$$

Putting (3.30) and (3.31) together we have

$$\sup_{x\in\mathsf{S}_\varepsilon} U\mu^{(2)}_{\varepsilon,\delta}(x) \leq c_{\alpha,d}\frac{2^{2-\alpha}\pi^{(d-1)/2}\Gamma((2-\alpha)/2)^2}{\Gamma(2-\alpha)\Gamma((d-1)/2)} \times \frac{\varepsilon^{1-\alpha}}{\delta^{d-\alpha}}(1+\varepsilon)^{d-1}. \tag{3.32}$$

By choosing $\delta = \delta(\varepsilon)$, the result follows. □

Let us now return to the proof of Theorem 3.2.2. We show that we can make careful sense of (3.16) and (3.17). Using (3.20) in (3.14) we see that for $x \notin \mathsf{S}$,

$$\begin{aligned} U\mu^{\mathsf{S}}_\varepsilon(x) &= \mathbb{E}_x\left[(U\mu^{(1)}_\varepsilon(X_{\tau_{\mathsf{S}_\varepsilon}})-1);\tau_{\mathsf{S}_\varepsilon}<\infty\right] + \mathbb{P}_x(\tau_{\mathsf{S}_\varepsilon}<\infty) - \mathbb{E}_x\left[U\mu^{(2)}_\varepsilon(X_{\tau_{\mathsf{S}_\varepsilon}});\tau_{\mathsf{S}_\varepsilon}<\infty\right] \\ &\leq \mathbb{E}_x\left[(U\mu^{(1)}_\varepsilon(X_{\tau_{\mathsf{S}_\varepsilon}})-1);\tau_{\mathsf{S}_\varepsilon}<\infty\right] + \mathbb{P}_x(\tau_{\mathsf{S}_\varepsilon}<\infty). \end{aligned} \tag{3.33}$$

Then, due to Lemma 3.6.1, for each $x \notin \mathsf{S}$ and $\upsilon > 0$, we can choose ε sufficiently small such that

$$U\mu^{\mathsf{S}}_\varepsilon(x) \leq (1+\upsilon)\mathbb{P}_x(\tau_{\mathsf{S}_\varepsilon}<\infty). \tag{3.34}$$

Since we can take υ arbitrarily small, we have the lower bound on a liminf version of the statement of Theorem 3.2.2 given by

$$\liminf_{\varepsilon\to 0} \varepsilon^{\alpha-1}U\mu^{\mathsf{S}}_\varepsilon(x) \leq \liminf_{\varepsilon\to 0}\varepsilon^{\alpha-1}\mathbb{P}_x(\tau_{\mathsf{S}_\varepsilon}<\infty), \qquad x\notin\mathsf{S}. \tag{3.35}$$

On the other hand, suppose instead of S, we replace its role by $\mathsf{S}^{\delta(\varepsilon)}$, where $\delta(\varepsilon)$ was given in the statement of Lemma 3.6.2, we have from the excessive property (3.15) associated to $U\mu^{\mathsf{S}^{\delta(\varepsilon)}}_\varepsilon$ that

$$U\mu^{\mathsf{S}^{\delta(\varepsilon)}}_\varepsilon(x) \geq \mathbb{E}_x\left[U\mu^{\mathsf{S}^{\delta(\varepsilon)}}_\varepsilon(X_{\tau_{\mathsf{S}_\varepsilon}});\tau_{\mathsf{S}_\varepsilon}<\infty\right], \qquad x\notin\mathsf{S}. \tag{3.36}$$

Now appealing to (3.21), we get

$$U\mu_\varepsilon^{\mathsf{S}^{\delta(\varepsilon)}}(x) \geq \mathbb{E}_x\left[U\mu_\varepsilon^{(1)}(X_{\tau_{\mathsf{S}_\varepsilon}}) - 1; \tau_{\mathsf{S}_\varepsilon} < \infty\right] + \mathbb{P}_x(\tau_{\mathsf{S}_\varepsilon} < \infty) - \mathbb{E}_x\left[U\mu_{\varepsilon,\delta(\varepsilon)}^{(2)}(X_{\tau_{\mathsf{S}_\varepsilon}}); \tau_{\mathsf{S}_\varepsilon} < \infty\right].$$

Appealing to Lemmas 3.6.1 and 3.6.2, for each $\upsilon > 0$, we can choose ε small enough such that, for each $x \notin \mathsf{S}$,

$$U\mu_\varepsilon^{\mathsf{S}^{\delta(\varepsilon)}}(x) \geq (1-\upsilon)\mathbb{P}_x(\tau_{\mathsf{S}_\varepsilon} < \infty). \tag{3.37}$$

Hence, since we can choose υ as small as we like, we have

$$\limsup_{\varepsilon\to 0} \varepsilon^{\alpha-1} U\mu_\varepsilon^{\mathsf{S}^{\delta(\varepsilon)}}(x) \geq \limsup_{\varepsilon\to 0} \varepsilon^{\alpha-1}\mathbb{P}_x(\tau_{\mathsf{S}_\varepsilon} < \infty), \qquad x \notin \mathsf{S}. \tag{3.38}$$

It follows from (3.35) and (3.38) that, as soon as

$$\limsup_{\varepsilon\to 0} \varepsilon^{\alpha-1} U\mu_\varepsilon^{\mathsf{S}^{\delta(\varepsilon)}}(x) = \liminf_{\varepsilon\to 0} \varepsilon^{\alpha-1} U\mu_\varepsilon^{\mathsf{S}}(x), \qquad x \notin \mathsf{S}, \tag{3.39}$$

and noting that $U\mu_\varepsilon^{\mathsf{S}} \leq U\mu_\varepsilon^{\mathsf{S}^{\delta(\varepsilon)}}$, we have

$$\lim_{\varepsilon\to 0} \varepsilon^{\alpha-1} U\mu_\varepsilon^{\mathsf{S}}(x) = \lim_{\varepsilon\to 0} \varepsilon^{\alpha-1}\mathbb{P}_x(\tau_{\mathsf{S}_\varepsilon} < \infty), \qquad x \notin \mathsf{S}.$$

Let us thus complete the proof by verifying the limit on the equality (3.39) holds and by finding the left-hand side limit in the previous equation.

To this end, using that $|x-y|^{\alpha-d}$ is continuous on S_ε and, when $x \notin \mathsf{S}$, without less of generality, we can take ε small enough so that $x \notin \mathsf{S}_\varepsilon$. For each $x \notin \mathsf{S}$, using the Mean Valued Theorem, there exists a $r_\varepsilon^* \in (1-\varepsilon, 1+\varepsilon)$ such that

$$\begin{aligned} U\mu_\varepsilon^{\mathsf{S}}(x) &= \int_{\mathsf{S}_\varepsilon} |x-y|^{\alpha-d} m_\varepsilon(|y|)\ell_d(\mathrm{d}y) \\ &= (r_\varepsilon^*)^{d-1} \int_{\mathsf{S}} |x - r_\varepsilon^*\theta|^{\alpha-d}\sigma_1(\mathrm{d}\theta) \int_{1-\varepsilon}^{1+\varepsilon} m_\varepsilon(r)\mathrm{d}r, \end{aligned} \tag{3.40}$$

where we recall that $m_\varepsilon(r) = c_{\alpha,d}(r-(1-\varepsilon))^{-\alpha/2}(1+\varepsilon-r)^{-\alpha/2}$. By using (3.31) we get

$$\varepsilon^{\alpha-1}U\mu_\varepsilon^{\mathsf{S}}(x) = (r_\varepsilon^*)^{d-1}2^{1-\alpha}c_{\alpha,d}\frac{\Gamma((2-\alpha)/2)^2}{\Gamma(2-\alpha)}\int_{\mathsf{S}} |x - r_\varepsilon^*\theta|^{\alpha-d}\sigma_1(\mathrm{d}\theta), \qquad x \notin \mathsf{S}. \tag{3.41}$$

Taking limits in (3.41) as $\varepsilon \to 0$ and recalling the value of $c_{\alpha,d}$ from the statement of Lemma 3.6.1, we have, for $x \notin \mathsf{S}$

$$\lim_{\varepsilon\to 0}\varepsilon^{\alpha-1}U\mu_\varepsilon^{\mathsf{S}}(x) = 2^{1-2\alpha}\frac{\Gamma((d+\alpha-2)/2)}{\pi^{d/2}\Gamma(1-\alpha)}\frac{\Gamma((2-\alpha)/2)}{\Gamma(2-\alpha)}\int_{\mathsf{S}} |x-\theta|^{\alpha-d}\sigma_1(\mathrm{d}\theta). \tag{3.42}$$

An application of the recursion formula for gamma functions allows us to identify the right-hand side as equal to that of the right-hand side of (3.5). Very little changes in the above calculation if we replace S by $\mathsf{S}^{\delta(\varepsilon)}$. As such, (3.42) allows us to conclude (3.39), and thus gives the statement of the Theorem 3.2.2. □

3.7 Proof of Theorem 3.2.2 (ii)

The proof needs some adaptation when we deal with the case $\alpha = 1$. Principally, we need to focus on Lemmas 3.6.1 and 3.6.2. What is different in these two lemmas is that the normalisation constant $c_{\alpha,d}$ must now depend on ε. The replacement for Lemma 3.6.1 and Lemma 3.6.2 (combined into one result) now takes the following form.

Lemma 3.7.1. *Suppose that we define, for $0 < \varepsilon < 1$,*

$$\mu_\varepsilon(\mathrm{d}y) = \frac{c_{1,d}}{|\log\varepsilon|}(|y| - (1-\varepsilon))^{-\alpha/2}(1+\varepsilon-|y|)^{-\alpha/2}\ell_d(\mathrm{d}y), \tag{3.43}$$

and

$$c_{1,d} = \frac{\Gamma((d-1)/2)}{\pi^{(d+1)/2}}.$$

(i) We have

$$\lim_{\varepsilon\to 0}\sup_{x\in\mathbb{S}^{d-1}_\varepsilon}|U\mu^{(1)}_\varepsilon(x) - 1| = 0.$$

(ii) take $\delta(\varepsilon) = |\log\varepsilon|^{-1/2(d-1)}$, then

$$\limsup_{\varepsilon\to 0}\sup_{x\in\mathsf{S}_\varepsilon}\sqrt{|\log\varepsilon|}U\mu^{(2)}_{\varepsilon,\delta(\varepsilon)}(x) < \infty,$$

so that

$$\lim_{\varepsilon\to 0}\sup_{x\in\mathsf{S}_\varepsilon}U\mu^{(2)}_{\varepsilon,\delta(\varepsilon)}(x) = 0.$$

Proof. We give only a sketch proof of both parts for the interested reader to use as a guide to reproduce the finer details.

(i) The essence of the proof is an adaptation of the proof of Lemma 3.6.1. We pick up the proof of the latter at the analogue of (3.23), albeit $\alpha = 1$ and $c_{\alpha,d}$ is replaced by $c_{1,d}/|\log\varepsilon|$, i.e.

$$U\mu^{(1)}_\varepsilon(x) = \frac{2c_{1,d}\pi^{d/2}}{|\log\varepsilon|\Gamma(d/2)}\int_{\frac{1-\varepsilon}{|x|}}^{1}\frac{{}_2F_1\left(\frac{d-1}{2},\frac{1}{2};\frac{d}{2};r^2\right)r^{d-1}}{\left(r-\frac{1-\varepsilon}{|x|}\right)^{1/2}\left(\frac{1+\varepsilon}{|x|}-r\right)^{1/2}}\mathrm{d}r$$
$$+\frac{2c_{1,d}\pi^{d/2}}{|\log\varepsilon|\Gamma(d/2)}\int_{1}^{\frac{1+\varepsilon}{|x|}}\frac{{}_2F_1\left(\frac{d-1}{2},\frac{1}{2};\frac{d}{2};r^{-2}\right)r^{1-1}}{\left(r-\frac{1-\varepsilon}{|x|}\right)^{1/2}\left(\frac{1+\varepsilon}{|x|}-r\right)^{1/2}}\mathrm{d}r. \tag{3.44}$$

Appealing to (A.4), noting that $\log(1-r^2) \sim \log(1-r) + \log 2$, as $r \to 1$, we can deduce that there is an unimportant constant, say χ, such that

$$\lim_{\varepsilon\to 0}\sup_{x\in\mathbb{S}^{d-1}_{\varepsilon}}\Bigg|\frac{2c_{1,d}\pi^{d/2}}{|\log\varepsilon|\Gamma(d/2)}\int_{\frac{1-\varepsilon}{|x|}}^{1}\frac{{}_2F_1\left(\frac{d-1}{2},\frac{1}{2};\frac{d}{2};r^2\right)r^{d-1}}{\left(r-\frac{1-\varepsilon}{|x|}\right)^{1/2}\left(\frac{1+\varepsilon}{|x|}-r\right)^{1/2}}\mathrm{d}r$$

$$+\frac{2c_{1,d}\pi^{d/2}}{|\log\varepsilon|\Gamma((d-1)/2)\Gamma(1/2)}\int_{\frac{1-\varepsilon}{|x|}}^{1}\frac{r^{d-1}\log(1-r)}{\left(r-\frac{1-\varepsilon}{|x|}\right)^{1/2}\left(\frac{1+\varepsilon}{|x|}-r\right)^{1/2}}\mathrm{d}r$$

$$-\frac{c_{1,d}\chi}{|\log\varepsilon|}\int_{\frac{1-\varepsilon}{|x|}}^{1}\frac{r^{d-1}}{\left(r-\frac{1-\varepsilon}{|x|}\right)^{1/2}\left(\frac{1+\varepsilon}{|x|}-r\right)^{1/2}}\mathrm{d}r\Bigg|=0. \tag{3.45}$$

A similar uniform limiting control can be undertaken by subtracting off analogous terms from the second integral in (3.44), i.e. the integral

$$\frac{2c_{1,d}\pi^{d/2}}{|\log\varepsilon|\Gamma(d/2)}\int_{1}^{\frac{1+\varepsilon}{|x|}}\frac{{}_2F_1\left(\frac{d-1}{2},\frac{1}{2};\frac{d}{2};r^{-2}\right)}{\left(r-\frac{1-\varepsilon}{|x|}\right)^{1/2}\left(\frac{1+\varepsilon}{|x|}-r\right)^{1/2}}\mathrm{d}r.$$

Using (3.25), again noting $\alpha=1$, we can uniformly control the last term in (3.45) and note that it is $O(1/|\log\varepsilon|)$. Similarly to (3.26), the second term in (3.45) has the same behaviour as

$$-\frac{2c_{1,d}\pi^{d/2}}{|\log\varepsilon|\Gamma((d-1)/2)\Gamma(1/2)}\int_0^{1-\frac{1-\varepsilon}{|x|}}\frac{\log u}{(\frac{1+\varepsilon-|x|}{|x|}+u)^{1/2}(\frac{|x|-(1-\varepsilon)}{|x|}-u)^{1/2}}\mathrm{d}u. \tag{3.46}$$

To evaluate (3.46), using the change of variable $u=a-(a+b)/(t^2+1)$

$$\begin{aligned}\int_0^a\frac{\log u}{\sqrt{(b+u)(a-u)}}\mathrm{d}u &= 2\int_{\sqrt{\frac{b}{a}}}^{\infty}\log\left(a-\frac{a+b}{t^2+1}\right)\frac{\mathrm{d}t}{t^2+1}\\ &= \int_0^{\arctan\sqrt{\frac{a}{b}}}\log(a-(a+b)\sin^2 w)\mathrm{d}w\\ &= \int_0^{\arctan\sqrt{\frac{a}{b}}}\log a+\log\left(1-\frac{\sin^2 w}{\frac{a}{a+b}}\right)\mathrm{d}w\\ &= \arctan\sqrt{\frac{a}{b}}\log(a+b)-L\Big(\frac{\pi}{2}-2\arctan\sqrt{\frac{a}{b}}\Big)-\frac{\pi}{2}\log 2,\end{aligned} \tag{3.47}$$

where we have used formula 4.226(5) of [15], which tells us that

$$\int_0^u\log\left(1-\frac{\sin^2 w}{\sin^2 v}\right)\mathrm{d}w=-u\log\sin^2 v-L(\frac{\pi}{2}-v+u)-L(\frac{\pi}{2}-v-u), \tag{3.48}$$

for any $-\frac{\pi}{2}\le u\le\frac{\pi}{2}$ and $|\sin u|\le|\sin v|$ where $L(x)$ is the Lobachevsky function. Note that, Lobachevsky's function is defined and represented as

$$L(x) = -\int_0^x\log\cos\theta\,\mathrm{d}\theta=x\log 2-\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin 2kx}{k^2}. \tag{3.49}$$

Using (3.47) to evaluate (3.46) as well to evaluate the partner integral to (3.46), which comes from the analogous control of the second integral in (3.44), we get a nice cancellation of terms (as happened at this stage of the argument for $\alpha \in (0,1)$), to give us the controlled feature that

$$\lim_{\varepsilon\to 0}\sup_{x\in\mathbb{S}^{d-1}_\varepsilon}\left|U\mu^{(1)}_\varepsilon(x)+\frac{2c_{1,d}\pi^{d/2}}{|\log\varepsilon|\Gamma((d-1)/2)\Gamma(1/2)}\frac{\pi}{2}\log\varepsilon\right|=0.$$

Noting that with the indicated choice of $c_{1,d}$, we have

$$\frac{2c_{1,d}\pi^{d/2}}{\Gamma((d-1)/2)\Gamma(1/2)}\frac{\pi}{2}=1,$$

which concludes the proof of part (i).

(ii) For the second part, the proof is almost identical to the proof of Lemma 3.6.2. Indeed, following the calculations through to (3.32), recalling that we have replaced $c_{\alpha,d}$ by $c_{1,d}/|\log\varepsilon|$, we get, up to an unimportant constant χ',

$$\sup_{x\in\mathsf{S}_\varepsilon}U\mu^{(2)}_{\varepsilon,\delta}(x)\le\chi'\frac{1}{|\log\varepsilon|\delta^{d-1}}. \tag{3.50}$$

Hence, by taking $\delta=\delta(\varepsilon)=|\log\varepsilon|^{-1/2(d-1)}$ the statement of part (ii) follows. □

With Lemma 3.7.1 in hand, we can now complete the proof of Theorem 3.2.2 (ii). Inequalities (3.34) and (3.37) are still at our disposal for the same reasons as before. The proof thus boils down to the asymptotic treatment of the term $U\mu^{\mathsf{S}}_\varepsilon(x)$ as in (3.40) for $x\notin\mathsf{S}$. Recalling that we have replaced $c_{\alpha,d}$ by $c_{1,d}/|\log\varepsilon|$ we get from (3.31) and the constant $c_{1,d}$ given in the statement of Lemma 3.7.1,

$$\lim_{\varepsilon\to 0}\ |\log\varepsilon|\ \mathbb{P}_x(\tau_{\mathsf{S}_\varepsilon}<\infty)=\frac{\Gamma((d-1)/2)}{\pi^{(d+1)/2}}\Gamma(1/2)^2H_{\mathsf{S}}(x)=\frac{\Gamma((d-1)/2)}{\pi^{(d-1)/2}}H_{\mathsf{S}}(x),$$

where we have used that $\Gamma(1/2)=\sqrt{\pi}$. □

3.8 Proof of Theorem 3.2.1

Recall the definition $\tau_\beta:=\inf\{t>0\colon 1/\beta<|X_t|<\beta\}$ for $\beta>1$ and fix $\varepsilon_0>0$ such that, for all $0<\varepsilon<\varepsilon_0$, $(1-\varepsilon,1+\varepsilon)\subset(1/\beta,\beta)$. Then, by applying the Markov property at time t, we have, for any $\Lambda\in\mathcal{F}_t$,

$$\mathbb{P}_x(\Lambda,t<\tau_\beta|\tau_{\mathsf{S}_\varepsilon}<\infty)=\mathbb{E}_x\left[\mathbf{1}_{\{\Lambda,t<\tau_\beta\}}\frac{\mathbb{P}_{X_t}(\tau_{\mathsf{S}_\varepsilon}<\infty)}{\mathbb{P}_x(\tau_{\mathsf{S}_\varepsilon}<\infty)}\right]. \tag{3.51}$$

The event $\{t<\tau_\beta\}$ implies that either $|X_t|>\beta>1$ or $|X_t|<1/\beta<1$. Hence, for all $0<\varepsilon<\varepsilon_0$ and $y\in\mathbb{S}^{d-1}_\varepsilon$, on $\{t<\tau_\beta\}$,

$$|X_t-y|^{\alpha-d}<\max\{((1-\varepsilon_0)-1/\beta)^{\alpha-d},(\beta-(1+\varepsilon_0))^{\alpha-d}\}.$$

Hence, on $\{t < \tau_\beta\}$, we have from (3.37) and (3.41) that we can choose ε sufficiently small such that

$$\varepsilon^{\alpha-1}\mathbb{P}_{X_t}(\tau_{\mathsf{S}_\varepsilon} < \infty) < K_1,$$

for some constant $K_1 \in (0,\infty)$. In a similar spirit, using (3.34) and (3.41), since $x \notin \mathsf{S}$ and S is closed, it follows similarly that there is another constant $K_2 \in (0,\infty)$ such that, for x given in (3.51), we can choose ε sufficiently small such that

$$\varepsilon^{\alpha-1}\mathbb{P}_{x}(\tau_{\mathsf{S}_\varepsilon} < \infty) > K_2.$$

Theorem 3.2.2, dominated convergence and monotone convergence gives us, for all $\Lambda \in \mathcal{F}_t$, $t \geq 0$,

$$\lim_{\beta\to 1}\lim_{\varepsilon\to 0}\mathbb{P}_x(\Lambda, t < \tau_\beta | \tau_{\mathsf{S}_\varepsilon} < \infty) = \lim_{\beta\to 1}\mathbb{E}_x\left[\mathbf{1}_{\{\Lambda, t<\tau_\beta\}}\lim_{\varepsilon\to 0}\frac{\varepsilon^{\alpha-1}\mathbb{P}_{X_t}(\tau_{\mathsf{S}_\varepsilon} < \infty)}{\varepsilon^{\alpha-1}\mathbb{P}_{x}(\tau_{\mathsf{S}_\varepsilon} < \infty)}\right] = \mathbb{E}_x\left[\mathbf{1}_\Lambda\frac{H_\mathsf{S}(X_t)}{H_\mathsf{S}(x)}\right],$$

as required. □

3.9 Proof of Theorem 4.4.3

Recall the notation for a general Markov process $(Y, \mathtt{P})$ on E preceding the statement of Theorem 4.4.3. We will additionally write $\mathcal{P} := (\mathcal{P}_t, t \geq 0)$ for the semigroup associated to $(Y, \mathtt{P})$.

Theorem 3.5 of Nagasawa [23], shows that, under suitable assumptions on the Markov process, L-times form a natural family of random times at which the pathwise time-reversal

$$\overleftarrow{Y}_t := Y_{(\mathtt{k}-t)-}, \qquad t \in [0, \mathtt{k}],$$

is again a Markov process. Let us state Nagasawa's principle assumptions.

(A) The potential measure $U_Y(a,\cdot)$ associated to $\mathcal{P}$, defined by the relation

$$\int_E f(x)U_Y(a,\mathrm{d}x) = \int_0^\infty \mathcal{P}_t[f](a)\mathrm{d}t = \mathtt{E}_a\left[\int_0^\infty f(X_t)\,\mathrm{d}t\right], \qquad a \in E, \tag{3.52}$$

for bounded and measurable f on E, is σ-finite. Assume that there exists a probability measure, ν, such that, if we put

$$\mu(A) = \int U_Y(a, A)\,\nu(\mathrm{d}a) \quad \text{ for } A \in \mathcal{B}(\mathbb{R}), \tag{3.53}$$

then, there exists a Markov transition semigroup, say $\hat{\mathcal{P}} := (\hat{\mathcal{P}}_t, t \geq 0)$ such that

$$\int_E \mathcal{P}_t[f](x)g(x)\,\mu(\mathrm{d}x) = \int_E f(x)\hat{\mathcal{P}}_t[g](x)\,\mu(\mathrm{d}x), \quad t \geq 0, \tag{3.54}$$

for bounded, measurable and compactly supported test-functions f, g on E.

(B) For any continuous test-function $f \in C_0(E)$, the space of continuous and compactly supported functions, and $a \in E$, assume that $\mathcal{P}_t[f](a)$ is right-continuous in t for all $a \in E$ and, for $q > 0$, $U^{(q)}_{\hat{Y}}[f](\overleftarrow{Y}_t)$ is right-continuous in t, where, for bounded and measurable f on E,

$$U^{(q)}_{\hat{Y}}[f](a) = \int_0^\infty \mathrm{e}^{-qt}\hat{\mathcal{P}}_t[f](a)\mathrm{d}t, \qquad a \in E,$$

is the q-potential associated to $\hat{\mathcal{P}}$.

Nagasawa's duality theorem, Theorem 3.5. of [23], now reads as follows.

Theorem 3.9.1 (Nagasawa's duality theorem)**.** *Suppose that assumptions* **(A)** *and* **(B)** *hold. For the given starting probability distribution ν in* **(A)** *and any L-time* $\mathtt{k}$*, the time-reversed process $\overleftarrow{Y}$ under $\mathtt{P}_\nu$ is a time-homogeneous Markov process with transition probabilities*

$$\mathtt{P}_\nu(\overleftarrow{Y}_t \in A \mid \overleftarrow{Y}_r, 0 < r < s) = \mathtt{P}_\nu(\overleftarrow{Y}_t \in A \mid \overleftarrow{Y}_s) = p_{\hat{Y}}(t-s, \overleftarrow{Y}_s, A), \quad \mathtt{P}_\nu\text{-almost surely}, \tag{3.55}$$

for all $0 < s < t$ and closed A in $\mathbb{R}$, where $p_{\hat{Y}}(u, x, A)$, $u \geq 0$, $x \in \mathbb{R}$, is the transition measure associated to the semigroup $\hat{\mathcal{P}}$.

Completing the proof of Theorem 4.4.3

We will make a direct application of Theorem 4.7.1, with Y taken to be the process $(X, \mathbb{P}_\nu)$ where ν satisfies (3.8). Recall that its potential is written U and we will denote its transition semigroup by $(\mathcal{P}_t, t \geq 0)$. Moreover, the dual process, formerly $\hat{Y}$, is taken to be $(X, \mathbb{P}^{\mathsf{S}})$ and we will, in the obvious way, work with the notation U^{S} in place of $U_{\hat{Y}}$, $\mathcal{P}^{\mathsf{S}}$ in place of $\hat{\mathcal{P}}$ and so on. We need only to verify the two assumptions **(A)** and **(B)**.

In order to verify **(A)**, writing

$$U(x, \mathrm{d}y) = \int_0^\infty \mathbb{P}_x(X_t \in \mathrm{d}y)\mathrm{d}t = \frac{\Gamma((d-\alpha)/2)}{2^\alpha \pi^{d/2}\Gamma(\alpha/2)}|x-y|^{\alpha-d}\ell_d(\mathrm{d}y), \qquad x, y \in \mathbb{R}^d,$$

we have, up to a multiplicative constant,

$$\eta(\mathrm{d}x) \quad = \quad \int_{\mathbb{R}^d} U(a, \mathrm{d}x)\nu(\mathrm{d}a) = \frac{1}{\sigma_1(\mathsf{S})}\int_{\mathsf{S}} |x-a|^{\alpha-d}\sigma_1(\mathrm{d}a) \propto H_{\mathsf{S}}(x)\mathrm{d}x. \tag{3.56}$$

Now, we need to verify that (A.54) holds. Hunt's switching identity (cf. Chapter II.1 of [4]) for $(X, \mathbb{P})$, states that

$$\mathcal{P}_t(y, \mathrm{d}x)\mathrm{d}y = \mathcal{P}_t(x, \mathrm{d}y)\mathrm{d}x, \qquad x, y \in \mathbb{R}^d.$$

Using Hunt's switching identity together with (3.56), we have for $x, y \in \mathbb{R}^d \setminus \mathsf{S}$

$$\mathcal{P}_t(y, \mathrm{d}x)\eta(\mathrm{d}y) = \mathcal{P}_t(y, \mathrm{d}x)H_{\mathsf{S}}(y)\mathrm{d}y = \mathcal{P}_t(x, \mathrm{d}y)\frac{H_{\mathsf{S}}(y)}{H_{\mathsf{S}}(x)}H_{\mathsf{S}}(x)\mathrm{d}x = \mathcal{P}^{\mathsf{S}}_t(x, \mathrm{d}y)\eta(\mathrm{d}x).$$

Let us now turn to the verification of assumption **(B)**. This assumption is immediately satisfied on account of the fact that $\mathcal{P}^{\mathsf{S}}$ is a right-continuous semigroup by virtue of its definition as a Doob h-transform with respect to the Feller semigroup $\mathcal{P}$ of the stable process.

With both **(A)** and **(B)** in hand, we are ready to apply Theorem 4.7.1 and the desired result thus follows. □

3.10 Proof of Theorem 3.4.1

For the proof of Theorem 3.4.1, we focus on just part (i) and (ii) as the proof of parts (iii)–(v) are essentially verbatim the same as for the case of $\mathsf{S} \in \mathbb{S}^{d-1}$. Moreover, for both parts (i) and (ii) we will provide only a sketch proof as the reader will quickly see that the proof is not hugely different form that of Theorem 3.2.2, albeit for a few technical details.

(i) The substance of the proof of part (i) is thus to follow a similar strategy as with Theorem 3.2.2 and build a measure $\rho_\varepsilon^{\mathsf{D}}$ such that the analogue of (3.16) holds, i.e. $U\rho_\varepsilon^{\mathsf{D}}(x) = 1 + o(1)$, for $x \in \mathsf{D}$ so that $(1 + o(1))\mathbb{P}_x(\tau_{\mathsf{D}\varepsilon} < \infty) = U\rho_\varepsilon^{\mathsf{D}}(x)$, $x \notin \mathsf{D}\varepsilon$. More precisely, we develop analogues of Lemmas 3.6.1 and 3.6.2 to help make this precise.

Following what we have learned for $\mu_\varepsilon^{\mathsf{S}}$, our choice of $\rho_\varepsilon^{\mathsf{D}}$ is built from the base measure

$$\rho_\varepsilon(\mathrm{d}y) = k_{\alpha,d}((v, y) + \varepsilon)^{-\alpha/2}(\varepsilon - (v, y))^{-\alpha/2}\ell_d(\mathrm{d}y). \tag{3.57}$$

for an appropriate choice of $k_{\alpha,d}$. As in (3.20) the we can work with the decomposition,

$$U\rho_\varepsilon^{\mathsf{D}}(x) = U\rho_\varepsilon^{(1)}(x) - U\rho_\varepsilon^{(2)}(x), \qquad x \in \mathsf{D}_\varepsilon, \tag{3.58}$$

where $\rho_\varepsilon^{(1)}$ (resp. $\rho_\varepsilon^{(2)}$) is the restriction of ρ_ε to $\mathbb{H}_\varepsilon^{d-1} := \{x \in \mathbb{R}^d : -\varepsilon < (v, x) < \varepsilon\}$ (resp. to $\hat{\mathsf{D}}_\varepsilon := \mathbb{H}_\varepsilon^{d-1} \setminus \mathsf{D}_\varepsilon$). This helps with lower bounding $\liminf_{\varepsilon\to 0} \varepsilon^{\alpha-1}\mathbb{P}_x(\tau_{\mathsf{D}_\varepsilon} < \infty)$ by following steps of (3.33)–(3.35) together with the last paragraph of the Proof of Theorem 3.2.2, for which an analogue of Lemma 3.6.1 is needed.

For each $|u| < \varepsilon$, define the following sets: $\mathsf{D}^\delta = \{x \in \mathbb{H}^{d-1} : \mathrm{dist}(x, \mathsf{D}) < \delta\}$, $\mathsf{D}_\varepsilon^\delta = \{y \in \mathbb{H}_\varepsilon^{d-1} \colon \hat{y} \in \mathsf{D}^\delta\}$ (recalling $\hat{y}$ is the orthogonal projection of y on to $\mathbb{H}^{d-1}$) and $\hat{\mathsf{D}}_\varepsilon^\delta = \mathbb{H}_\varepsilon^{d-1} \setminus \mathsf{D}_\varepsilon^\delta$. Moreover, for any $u \in \mathbb{R}$, we define $\mathbb{H}^{d-1}(u) = \{x \in \mathbb{R}^d : (v, x) = u\}$, $\mathsf{D}(u) := \{y \in \mathbb{H}^{d-1}(u) \colon \hat{y} \in \mathsf{D}\}$, $\mathsf{D}^\delta(u) := \{y \in \mathbb{H}^{d-1}(u) \colon \hat{y} \in \mathsf{D}^\delta\}$, and $\hat{\mathsf{D}}^\delta(u) = \mathbb{H}^{d-1}(u) \setminus \mathsf{D}^\delta(u)$. Similarly, in the spirit of (3.21) we can use the decomposition

$$U\rho_\varepsilon^{\mathsf{D}^\delta}(x) = U\rho_\varepsilon^{(1)}(x) - U\rho_{\varepsilon,\delta}^{(2)}(x), \qquad x \in \mathsf{D}_\varepsilon, \tag{3.59}$$

where $\rho_{\varepsilon,\delta}^{(2)}$ is the restriction of ρ_ε to $\hat{\mathsf{D}}_\varepsilon^\delta$. which helps with $\limsup_{\varepsilon\to 0} \varepsilon^{\alpha-1}\mathbb{P}_x(\tau_{\mathsf{D}_\varepsilon} < \infty)$ by following steps (3.36)–(3.39) together with the last paragraph of the Proof of Theorem 3.2.2, for which an analogue of Lemma 3.6.2 is needed.

Let us address the technical detail that differs from the proof of Theorem 3.2.2 that we alluded to above. For $x \in \mathsf{D}_\varepsilon$,

$$
\begin{aligned}
&U\rho_\varepsilon^{(1)}(x)\\
&= k_{\alpha,d}\int_{\mathbb{H}_\varepsilon^{d-1}} |x-y|^{\alpha-d}((v,y)+\varepsilon)^{-\alpha/2}(\varepsilon-(v,y))^{-\alpha/2}\ell_d(\mathrm{d}y)\\
&= k_{\alpha,d}\int_{-\varepsilon}^{\varepsilon}(u+\varepsilon)^{-\alpha/2}(\varepsilon-u)^{-\alpha/2}\mathrm{d}u\int_{\mathbb{H}^{d-1}(u)}|x-y|^{\alpha-d}\ell_{d-1}(\mathrm{d}y)\\
&= k_{\alpha,d}\int_{-\varepsilon}^{\varepsilon}(u+\varepsilon)^{-\alpha/2}(\varepsilon-u)^{-\alpha/2}\mathrm{d}u\int_{\mathbb{H}^{d-1}((v,x))}(|x-\hat{z}|^2+|(v,x)-u|^2)^{\frac{\alpha-d}{2}}\ell_{d-1}(\mathrm{d}\hat{z}),
\end{aligned}
$$

where $\hat{z}$ is the orthogonal projection of $y \in \mathbb{H}^{d-1}(u)$ onto $\mathbb{H}^{d-1}((v,x))$, which satisfies $|\hat{z}-y| = |(v,x)-u|$ and $\ell_{d-1}(\mathrm{d}\hat{z}) = \ell_{d-1}(\mathrm{d}y)$. Note also that $(v, x-\hat{z}) = 0$, for $\hat{z} \in \mathbb{H}^{d-1}((v,x))$, and hence $x - \mathbb{H}^{d-1}((v,x))$ is equal to $\mathbb{H}^{d-1}(0)$, which, in turn, can otherwise be identified as $\mathbb{R}^{d-1}$. Therefore, if we used generalised polar coordinates to integrate over $\mathbb{H}^{d-1}((v,x))$ identified as $x - \mathbb{R}^{d-1}$, we have

$$
\begin{aligned}
&U\rho_\varepsilon^{(1)}(x)\\
&= k_{\alpha,d}\int_{-\varepsilon}^{\varepsilon}(u+\varepsilon)^{-\alpha/2}(\varepsilon-u)^{-\alpha/2}\mathrm{d}u\int_{\mathbb{H}^{d-1}((v,x))}\Big(|x-\hat{z}|^2+|(v,x)-u|^2\Big)^{\frac{\alpha-d}{2}}\ell_{d-1}(\mathrm{d}\hat{z})\\
&= \frac{2k_{\alpha,d}\pi^{(d-2)/2}}{\Gamma((d-2)/2)}\int_{-\varepsilon}^{\varepsilon}(u+\varepsilon)^{-\alpha/2}(\varepsilon-u)^{-\alpha/2}\mathrm{d}u\int_0^\infty\int_{\mathbb{S}^{d-2}}\Big(r^2+|(v,x)-u|^2\Big)^{\frac{\alpha-d}{2}}r^{d-2}\mathrm{d}r\sigma_1(d\theta)\\
&= \frac{2k_{\alpha,d}\pi^{(d-2)/2}}{\Gamma((d-2)/2)}\int_{-\varepsilon}^{\varepsilon}(u+\varepsilon)^{-\alpha/2}(\varepsilon-u)^{-\alpha/2}\mathrm{d}u\int_0^\infty\Big(r^2+|(v,x)-u|^2\Big)^{\frac{\alpha-d}{2}}r^{d-2}\mathrm{d}r\\
&= \frac{k_{\alpha,d}\pi^{(d-2)/2}}{\Gamma((d-2)/2)}\int_{-\varepsilon}^{\varepsilon}(u+\varepsilon)^{-\alpha/2}(\varepsilon-u)^{-\alpha/2}\mathrm{d}u\int_0^\infty\Big(w+|(v,x)-u|^2\Big)^{\frac{\alpha-d}{2}}w^{\frac{d-3}{2}}\mathrm{d}w \qquad (3.60)\\
&= \frac{k_{\alpha,d}\pi^{(d-2)/2}\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d-2}{2})\Gamma(\frac{d-\alpha}{2})}\int_{-\varepsilon}^{\varepsilon}(u+\varepsilon)^{-\alpha/2}(\varepsilon-u)^{-\alpha/2}|(v,x)-u|^{\alpha-1}\mathrm{d}u\\
&= \frac{k_{\alpha,d}\pi^{(d-2)/2}\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d-2}{2})\Gamma(\frac{d-\alpha}{2})}\int_{-1}^{1}(1+w)^{-\alpha/2}(1-w)^{-\alpha/2}|\varepsilon^{-1}(v,x)-w|^{\alpha-1}\mathrm{d}w, \qquad (3.61)
\end{aligned}
$$

where, in the penultimate equality, we used a classical representation of the Beta function (see formula 3.191.2 in [15]), which tells us that, for any $\mathrm{Re}(\nu) > \mathrm{Re}(\gamma) > 0$ and $z > 0$,

$$
\int_0^\infty (y+z)^{-\nu}y^{\gamma-1}\mathrm{d}y = z^{\gamma-\nu}\frac{\Gamma(\nu-\gamma)\Gamma(\gamma)}{\Gamma(\nu)},
$$

and in the final equality, we have changed variables using $w = \varepsilon u$. Next, we observe that $|\varepsilon^{-1}(v,x)| \leq 1$ on account of the fact that $x \in \mathsf{D}_\varepsilon \subseteq \mathbb{H}_\varepsilon^{d-1}$. Now choose $k_{\alpha,d}$, so that

$$
\frac{k_{\alpha,d}\pi^{(d-2)/2}\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d-2}{2})\Gamma(\frac{d-\alpha}{2})} = 1. \qquad (3.62)
$$

We can now appeal directly to (3.18) to deduce that, for $x \in \mathsf{D}_\varepsilon$

$$U\rho_\varepsilon^{(1)}(x) = 1. \tag{3.63}$$

In the spirit of (3.33)–(3.35), it now follows that, for $x \notin \mathsf{D}$ and ε sufficiently small,

$$U\rho_\varepsilon^{\mathsf{D}}(x) \leq \mathbb{P}_x(\tau_{\mathsf{D}_\varepsilon} < \infty).$$

So that

$$\liminf_{\varepsilon\to 0} U\rho_\varepsilon^{\mathsf{D}}(x) \leq \liminf_{\varepsilon\to 0} \mathbb{P}_x(\tau_{\mathsf{D}_\varepsilon} < \infty), \qquad x \notin \mathsf{D}. \tag{3.64}$$

Now we turn our attention to (3.59). Noting that when $x \in \mathsf{D}_\varepsilon$, $|x-y| > \delta$ for $y \in \hat{\mathsf{D}}_\varepsilon^\delta$, we have, for all $x \in \mathsf{D}_\varepsilon$,

$$\begin{aligned}
U\rho_{\varepsilon,\delta}^{(2)}(x) &= k_{\alpha,d}\int_{\hat{\mathsf{D}}_\varepsilon^\delta} |x-y|^{\alpha-d}((v,y)+\varepsilon)^{-\alpha/2}(\varepsilon-(v,y))^{-\alpha/2}\ell_d(\mathrm{d}y)\\
&\leq k_{\alpha,d}\delta^{\alpha-d}\int_{-\varepsilon}^{\varepsilon}(u+\varepsilon)^{-\alpha/2}(\varepsilon-u)^{-\alpha/2}\mathrm{d}u\int_{\mathsf{D}^\delta((v,x))}\ell_{d-1}(\mathrm{d}\hat{y})\\
&\leq \delta^{\alpha-d}k_{\alpha,d}\ell_{d-1}(\mathsf{D}^\delta)\int_{-\varepsilon}^{\varepsilon}(u+\varepsilon)^{-\alpha/2}(\varepsilon-u)^{-\alpha/2}\mathrm{d}u\\
&= \delta^{\alpha-d}\varepsilon^{1-\alpha}k_{\alpha,d}\ell_{d-1}(\mathsf{D}^\delta)2^{1-\alpha}\frac{\Gamma((2-\alpha)/2)^2}{\Gamma(2-\alpha)},
\end{aligned}$$

where we have used the calculation in (3.31) in the final equality. Choosing $\delta = \delta(\varepsilon) = \varepsilon^{(1-\alpha)/2(d-\alpha)}$, and noting that $\ell_{d-1}(\mathsf{D}^\delta)$ is uniformly bounded from above by an unimportant constant for e.g. all $\delta < 1$ (thanks to the assumption that $\ell_{d-1}(\mathsf{D}) < \infty$), we see that

$$\lim_{\varepsilon\to 0}\sup_{x\in\mathsf{D}_\varepsilon} U\rho_{\varepsilon,\delta(\varepsilon)}^{(2)}(x) = 0.$$

In a similar spirit to (3.36)–(3.38), we now have that

$$\limsup_{\varepsilon\to 0}\varepsilon^{\alpha-1}U\rho_\varepsilon^{\mathsf{D}^{\delta(\varepsilon)}}(x) \geq \limsup_{\varepsilon\to 0}\varepsilon^{\alpha-1}\mathbb{P}_x(\tau_{\mathsf{D}_\varepsilon} < \infty), \qquad x \notin \mathsf{D}. \tag{3.65}$$

Matching up the left-hand side of (3.64) with that of (3.65), we can proceed in a similar fashion to (3.41) – (3.42), leading to the statement of Theorem 3.4.1 (i) as promised. The calculation is based around the fact that

$$
\begin{aligned}
\lim_{\varepsilon\to0}\varepsilon^{\alpha-1}U\rho_\varepsilon^{\mathsf{D}}(x) &= \lim_{\varepsilon\to0}k_{\alpha,d}\varepsilon^{\alpha-1}\int_{\mathsf{D}_\varepsilon}|x-y|^{\alpha-d}((v,y)+\varepsilon)^{-\alpha/2}(\varepsilon-(v,y))^{-\alpha/2}\ell_d(\mathrm{d}y)\\
&= \lim_{\varepsilon\to0}k_{\alpha,d}\varepsilon^{\alpha-1}\int_{-\varepsilon}^{\varepsilon}(u+\varepsilon)^{-\alpha/2}(\varepsilon-u)^{-\alpha/2}\mathrm{d}u\int_{\mathsf{D}(u)}|x-\hat y|^{\alpha-d}\ell_{d-1}(\mathrm{d}\hat y)\\
&= k_{\alpha,d}2^{1-\alpha}\frac{\Gamma((2-\alpha)/2)^2}{\Gamma(2-\alpha)}\int_{\mathsf{D}}|x-y|^{\alpha-d}\ell_{d-1}(\mathrm{d}y)\\
&= 2^{1-\alpha}\pi^{-(d-2)/2}\frac{\Gamma(\frac{d-2}{2})\Gamma(\frac{d-\alpha}{2})\Gamma(\frac{2-\alpha}{2})^2}{\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{d-1}{2})\Gamma(2-\alpha)}\int_{\mathsf{D}}|x-y|^{\alpha-d}\ell_{d-1}(\mathrm{d}y),
\end{aligned}
\tag{3.66}
$$

where we have used the calculation in (3.31) and (3.62) in the third equality.

(ii) The setting $\alpha=1$ requires yet another delicate handing of the associated potentials. Given that all the main ideas are now present in the paper, we simply lay out the key points of the proof, leaving the remaining detail for the reader.

Our calculations begin in the same way as in part (i), in particular, we work with the core measure ρ_ε as in (3.57), albeit (as with Theorem 3.2.2 (ii)) replacing $k_{1,d}$ by $k_{1,d}/|\log\varepsilon|$, to be used in the constructions (3.58) and (3.59). An immediate complication we have is in evaluating $U\rho_\varepsilon^{(1)}(x)$, for $x\in\mathsf{D}_\varepsilon$, can be seen when we pick up the computations for part (i) at (3.60). Indeed, at that point, we are confronted with the integral

$$
\int_0^\infty\left(w+|(v,x)-u|^2\right)^{\frac{1-d}{2}}w^{\frac{d-3}{2}}\mathrm{d}w=\infty.
$$

The solution to this is to adjust the core measure ρ_ε as follows. Since D is bounded, we can choose an $R>0$ sufficiently large that, $\mathsf{D}\subset\mathbb{S}^{d-2}(0,R):=\{y\in\mathbb{H}^{d-1}:|y|\le R\}$ strictly contains D. Denote $\mathbb{S}^{d-2}_\varepsilon(0,R)=\{x\in\mathbb{H}^{d-1}_\varepsilon:\hat x\in\mathbb{S}^{d-2}(0,R)\}$, where $\hat x$ is the orthogonal projection of x on to $\mathbb{H}^{d-1}$. Suppose we now make a slight adjustment and replace ρ_ε by

$$
\rho_\varepsilon(\mathrm{d}y)=\frac{k_{1,d,R}}{|\log\varepsilon|}((v,y)+\varepsilon)^{-\alpha/2}(\varepsilon-(v,y))^{-\alpha/2}\mathbf{1}_{(y\in\mathbb{S}^{d-2}_\varepsilon(0,R))}\ell_d(\mathrm{d}y),
$$

for an appropriate choice of $k_{1,d,R}$. We may now continue the argument from (3.60) with the calculation

$$
|\log\varepsilon|U\rho_\varepsilon^{(1)}(x)=\frac{k_{\alpha,d,R}\pi^{(d-2)/2}}{\Gamma((d-2)/2)}\int_{-\varepsilon}^{\varepsilon}(u+\varepsilon)^{-1/2}(\varepsilon-u)^{-1/2}\mathrm{d}u\int_0^R\left(w+|(v,x)-u|^2\right)^{\frac{1-d}{2}}w^{\frac{d-3}{2}}\mathrm{d}w.
\tag{3.67}
$$

Denote by $I(R,\varepsilon,x)$ the right-hand side of 3.67, ensuring that ε is small enough that $\varepsilon\ll R$.

Appealing to (A.8),

$$
\begin{aligned}
I(R,\varepsilon,x) &= \frac{k_{\alpha,d,R}\pi^{(d-2)/2}}{\Gamma((d-2)/2)} \int_{-\varepsilon}^{\varepsilon} (u+\varepsilon)^{-1/2}(\varepsilon-u)^{-1/2}\mathrm{d}u \int_0^R \left(w+|(v,x)-u|^2\right)^{\frac{1-d}{2}} w^{\frac{d-3}{2}}\mathrm{d}w \\
&= \frac{k_{\alpha,d,R}\pi^{(d-2)/2}}{\Gamma((d-2)/2)} \int_{-\varepsilon}^{\varepsilon} (\varepsilon^2-u^2)^{-1/2}|(v,x)-u|^{1-d}\mathrm{d}u \int_0^R \left(\frac{w}{|(v,x)-u|^2}+1\right)^{\frac{1-d}{2}} w^{\frac{d-3}{2}}\mathrm{d}w \\
&= \frac{k_{\alpha,d,R}\pi^{(d-2)/2}}{\Gamma((d-2)/2)} \int_{-\varepsilon}^{\varepsilon} (\varepsilon^2-u^2)^{-1/2}|(v,x)-u|^{1-d} \\
&\qquad\qquad \frac{R^{(d-1)/2}}{(d-1)/2}\, {}_2F_1\left(\frac{d-1}{2},\frac{d-1}{2};\frac{d+1}{2};-\frac{R}{|(v,x)-u|^2}\right)\mathrm{d}u,
\end{aligned}
$$

where we have used the identity in (A.9). One of the many identities for hypergeometric functions, see [2], offers us the growth condition, for $c-a\in\mathbb{N}$, as $|z|\to\infty$,

$$
{}_2F_1(a,a;c;z) \sim \frac{\Gamma(c)(\log(-z)-\psi(c-a)-\psi(a)-2\gamma)(-z)^{-a}}{\Gamma(a)(c-a-1)!} + \frac{\Gamma(c)2(-z)^{-c}}{\Gamma(a)^2((c-a)!)^2}, \tag{3.68}
$$

where γ is an unimportant constant and $\psi(z)=\Gamma'(z)/\Gamma(z)$ is the di-gamma function. In the spirit of previous calculations, we can thus find to leading order, uniformly over $x\in\mathsf{D}_\varepsilon$,

$$
U\rho_\varepsilon^{(1)}(x) \sim 2\frac{\pi^{d/2}k_{\alpha,d,R}}{\Gamma((d-2)/2)}, \tag{3.69}
$$

which remarkably does not depend on R. This means we should choose the constant

$$
k_{\alpha,d,R} = \frac{\Gamma((d-2)/2)}{2\pi^{d/2}}
$$

for this asymptotic to serve our purpose.

At this point in the proof, recalling the fundamental decomposition (3.58), it is worth bringing in the term $U\mu_\varepsilon^{(2)}$ and noting that one can compute with relatively coarse estimates that

$$
\sup_{x\in\mathsf{D}_\varepsilon}\left|U\rho_\varepsilon^{(2)}(x)\right| \le \frac{C}{|\log\varepsilon|},
$$

for some unimportant constant $C>0$. Together with (3.69), in a calculation similar to (3.66) we can put the pieces together to get the asymptotic, for $x\notin\mathsf{D}$ and ε sufficiently small,

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\ |\log\varepsilon|\ \mathbb{P}_x(\tau_{\mathsf{D}_\varepsilon}<\infty) &= \lim_{\varepsilon\to 0}\ |\log\varepsilon|\ U\rho_\varepsilon^{\mathsf{D}}(x) \\
&= \lim_{\varepsilon\to 0}\frac{\Gamma((d-2)/2)}{2\pi^{d/2}}\int_{\mathsf{D}_\varepsilon}|x-y|^{1-d}(\varepsilon^2-(v,y)^2)^{-1/2}\ell_d(\mathrm{d}y) \\
&= \lim_{\varepsilon\to 0}\frac{\Gamma((d-2)/2)}{\pi^{d/2}}\int_{-\varepsilon}^{\varepsilon}(\varepsilon^2-u^2)^{-1/2}\mathrm{d}u\int_{\mathsf{D}(u)}|x-\hat{y}|^{1-d}\ell_{d-1}(\mathrm{d}\hat{y}) \\
&= \frac{\Gamma((d-2)/2)}{\pi^{(d-2)/2}}M_{\mathsf{D}}(x).
\end{aligned} \tag{3.70}
$$

The proof is complete. □

Appendix: Hypergeometric identities

We work with the standard definition for the hypergeometric function,

$$ {}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \qquad |z| < 1. $$

Of the many identities for hypergeometric functions, we need the following:

$$ \begin{aligned} {}_2F_1(a,b,c;z) = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b} {}_2F_1(c-a,c-b,1+c-a-b;1-z) \\ + \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a,b,a+b-c+1;1-z), \end{aligned} \tag{A.1} $$

for $c - a - b \notin \mathbb{Z}$. Hence, thanks to continuity,

$$ \begin{aligned} \lim_{\varepsilon \to 0} \sup_{r \in [1-\varepsilon, 1]} \Bigg| {}_2F_1\Big(\frac{d-\alpha}{2}, 1 - \frac{\alpha}{2}; \frac{d}{2}; r^2\Big) \\ - \frac{\Gamma(d/2)\Gamma(1-\alpha)}{\Gamma((d-\alpha)/2)\Gamma((2-\alpha)/2)}(1-r^2)^{\alpha-1} - \frac{\Gamma(d/2)\Gamma(\alpha-1)}{\Gamma(\alpha/2)\Gamma((d+\alpha-2)/2)} \Bigg| = 0. \end{aligned} \tag{A.2} $$

We will need to apply a similar identity to (A.1) but for the setting that $c - a - b = 0$, which violates the assumption behind (A.1). In that case, we need to appeal to the formula

$$ \begin{aligned} {}_2F_1(a,b,a+b,z) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \Big(\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(k!)^2} (2\psi(k+1) - \psi(a+k) - \psi(b+k))(1-z)^k \\ &- \log(1-z)\, {}_2F_1(a,b,1,1-z) \Big), \end{aligned} \tag{A.3} $$

for $|1 - z| < 1$ where the di-gamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$ is defined for all $z \neq -n, n \in \mathbb{N}$.

Again, thanks to continuity, we can write

$$ \begin{aligned} \lim_{\varepsilon \to 0} \sup_{r \in [1-\varepsilon, 1]} \Bigg| {}_2F_1\Big(\frac{d-1}{2}, \frac{1}{2}; \frac{d}{2}; r^2\Big) + \frac{\Gamma(d/2)}{\Gamma((d-1)/2)\Gamma(1/2)} \log(1-r^2) \\ - \frac{2\Gamma(d/2)(\psi(1) - \psi((d-1)/2) - \psi(1/2))}{\Gamma((d-1)/2)\Gamma(1/2)} \Bigg| = 0. \end{aligned} \tag{A.4} $$

A second identity that is needed is the following combination formula, which states that for any $|z| < 1$, we have

$$ \begin{aligned} {}_2F_1(a,b;c;z) = \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(c-a)\Gamma(b)}(-z)^{-a} {}_2F_1\left(a, a-c+1; a-b+1; \frac{1}{z}\right) \\ + \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(c-b)\Gamma(a)}(-z)^{-b} {}_2F_1\left(b-c+1, b; -a+b+1; \frac{1}{z}\right). \end{aligned} \tag{A.5} $$

Its proof can be found, for example at [1]. In the main body of the text, we use this identity for the setting that $a = \alpha/2$, $b = \alpha$ and $c = 1 + \alpha/2$. This gives us the identity

$$\begin{aligned}
{}_2F_1\left(\frac{\alpha}{2}, \alpha; 1 + \frac{\alpha}{2}; z\right) &= \frac{\Gamma(\alpha/2)\Gamma((2+\alpha)/2)}{\Gamma(\alpha)}(-z)^{-\alpha/2}{}_2F_1\left(\alpha/2, 0; 1 - \alpha/2; \frac{1}{z}\right) \\
&\qquad + \frac{\Gamma(-\alpha/2)\Gamma((2+\alpha)/2)}{\Gamma((2-\alpha)/2)\Gamma(\alpha/2)}(-z)^{-\alpha}{}_2F_1\left(\alpha/2, \alpha; 1 + \alpha/2; \frac{1}{z}\right) \\
&= \frac{\Gamma(\alpha/2)\Gamma((2+\alpha)/2)}{\Gamma(\alpha)}(-z)^{-\alpha/2} \\
&\qquad - (-z)^{-\alpha}{}_2F_1\left(\alpha/2, \alpha; 1 + \alpha/2; \frac{1}{z}\right),
\end{aligned}$$

where we have used the recursion formula for gamma functions twice in the final equality. This allows us to come to rest at the following useful identity

$$(-z)^{-\alpha/2}{}_2F_1\left(\alpha/2, \alpha; 1 + \alpha/2; \frac{1}{z}\right) + (-z)^{\alpha/2}{}_2F_1\left(\frac{\alpha}{2}, \alpha; 1 + \frac{\alpha}{2}; z\right) = \frac{\Gamma(\alpha/2)\Gamma((2+\alpha)/2)}{\Gamma(\alpha)}. \tag{A.6}$$

We are also interested in integral formulae, for which the hypergeometric function is used to evaluate an integral. The first is aversion of formula 3.665(2) in [15] which states that, for any $0 < |a| < r$ and $\nu > 0$, as

$$\int_0^\pi \frac{\sin^{d-2}\phi}{(a^2 + 2ar\cos\phi + r^2)^\nu}\mathrm{d}\phi = \frac{1}{r^{2\nu}}B\Big(\frac{d-1}{2}, \frac{1}{2}\Big)\,{}_2F_1\Big(\nu, \nu - \frac{d}{2} + 1; \frac{d}{2}; \frac{a^2}{r^2}\Big), \tag{A.7}$$

where $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ is the Beta function. The second is formula 3.197.8 in [15], which states that, for $Re(\mu) > 0, Re(\nu) > 0$ and $|\arg(u/\beta)| < \pi$,

$$\int_0^u x^{\nu-1}(u - x)^{\mu-1}(x + \beta)^\lambda \mathrm{d}x = \beta^\lambda u^{\mu+\nu-1}B(\mu, \nu){}_2F_1\left(-\lambda, \nu; \mu + \nu; -\frac{u}{\beta}\right). \tag{A.8}$$

The third is 3.194.1 of [15] and states that, for $|\arg(1 + \beta u)| > \pi$ and $Re(\mu) > 0$, $Re(\nu) > 0$,

$$\int_0^u x^{\mu-1}(1 + \beta x)^{-\nu}\mathrm{d}x = \frac{u^\mu}{\mu}{}_2F_1(\nu, \nu - \mu; 1 + \mu; -\beta u), \tag{A.9}$$

where ${}_2F_1$ in the above identity is understood as its analytic extension in the event that $|\beta u| > 1$.

Acknowledgements

TS acknowledges support from a Schlumberger Faculty of the Future award. SP acknowledges support from the Royal Society as a Newton International Fellow Alumnus (AL201023) and UNAM-DGAPA-PAPIIT grant no. IA103220.

Bibliography

[1] https://functions.wolfram.com/HypergeometricFunctions/Hypergeometric2F1/17/02/09/

[2] https://functions.wolfram.com/HypergeometricFunctions/Hypergeometric2F1/06/01/05/01/05/0001/

[3] Alili, L., Chaumont, L., Graczyk, P. and Zak. T. (2017) Inversion, duality and Doob h-transforms for self-similar Markov processes. *Electron. J. Probab.*, **22**, no. 20, 18 pp.

[4] Bertoin, J. (1996) *Lévy processes.* Cambridge University Press.

[5] Bliedtner, J. and Hansen, W. (1986) *Potential theory.* Universitext. Springer-Verlag, Berlin.

[6] Blumenson, L. (1960) A derivation of n-dimensional spherical coordinates *The American Mathematical Monthly*, **67**, 63–66.

[7] Blumenthal, R.M. and Getoor, R.K. (1968) *Markov processes and potential theory.* Pure and Applied Mathematics, Vol. 29 Academic Press, New York-London.

[8] Bogdan, K. and Żak, T. (2006) On Kelvin transformation. *J. Theoret. Probab.* **19**, 89–120.

[9] Caballero, M.E. and Chaumont, L. (2006) Conditioned stable L évy processes and Lamperti representation. *Journal of Applied Probability,* **43**, 4, 967–983.

[10] Caballero, M.E., Pardo, J.C. and Pérez, J.L. (2011) Explicit identities for Lévy processes associated to symmetric stable processes. *Bernoulli,* **17**, 1, 34–59.

[11] Chaumont, L. (1996) Conditionings and path decompositions for Lévy processes. *Stoch. Proc. Appl.* **64**, 39–54.

[12] Chaumont, L.; Pantí, H. and Rivero, V. (2013) The Lamperti representation of real-valued self-similar Markov processes. *Bernoulli,* **19**, 5B, 2494–2523.

[13] Chung, K. L. and Walsh, J. B. (2005) *Markov processes, Brownian motion, and time symmetry.* Second edition. Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), 249. Springer, New York.

[14] Döring, L. and Weissman, P. (2020) Stable processes conditioned to hit an interval continuously from the outside. *Bernoulli*, **26**, 2, 980–1015.

[15] Gradshteyn, I.S. and Ryzhik, I.M. (1965) *Table of integrals, series and products.* Academic Press, New York and London.

[16] Kyprianou, A.E. (2018) Stable processes, self-similarity and the unit ball *ALEA, Lat. Am. J. Probab. Math. Stat.* 15, 617–690.

[17] Kyprianou, A.E., Palau, S. and Saizmaa, T. (2020) Attraction to and repulsion from a subset of the unit sphere for isotropic stable Lévy processes *arXiv:1911.05867 [math.PR]*

[18] Kyprianou, A.E. and Pardo, J.C. (2021) *Stable Lévy processes via Lamperti-type representations.* Cambridge University Press.

[19] Kyprianou, A.E., Rivero V., and Satitkanitkul, W. (2019) Conditioned real self-similar Markov processes. *Stoch. Proc. Appl.*, **129**, 954–977.

[20] Kyprianou, A.E., Rivero V., and Sengul, B. (2017) Conditioning subordinators embedded in Markov processes. *Stoch. Proc. Appl.*, **127**, 1234–1254.

[21] Lebedev, N.N. (1965) *Special functions and their applications.* Prentice-Hall, New Jersey.

[22] Luks, T. (2013) Boundary Behavior of α-Harmonic Functions on the Complement of the Sphere and Hyperplane. *Potential Anal.* **39**, 29–67.

[23] Nagasawa, M. (1964) Time reversions of Markov processes. *Nagoya. Math. J.* **24**, 177–204.

[24] Osypchuk, M.M. and Portenko, M.I. (2018) On the distribution of a rotationally invariant α-stable process at the hitting time of a given hyperplane *Dopov. Nac. akad. nauk Ukr.* **12**, 14–20.

[25] Port, S. C. (1969) First hitting distribution of a sphere for symmetric stable processes. *Trans. Amer. Math. Soc.* **135**, 115–125.

[26] Profeta, C. and Simon, T. (2016) On the harmonic measure of stable processes. *Séminaire de Probabilités* **XLVIII**, 325–345, Lecture Notes in Math., 2168, Springer.

Appendix 6B: Statement of Authorship

This declaration concerns the article entitled:	
Attraction to and repulsion from a subset of the unit sphere for isotropic stable L\'evy processes	
Publication status (tick one)	
Draft manuscript [ ] **Submitted** [ ] **In review** [ ] **Accepted** [ ] **Published** [X]	
Publication details (reference)	Published in Stochastic Processes and their Applications, SPA 0304-4149. Volume 137, Pages 272-293 https://doi.org/10.1016/j.spa.2021.04.004
Copyright status (tick the appropriate statement)	
I hold the copyright for this material [ ]	Copyright is retained by the publisher, but I have been given permission to replicate the material here [X]
Candidate's contribution to the paper (provide details, and also indicate as a percentage)	*All research has been conducted in equal partnership with collaborators and supervisors. It is unwise to try and measure percentages of intellectual contribution as research scholarship lies as much in the escalation of ideas through mathematical discourse as it does with the seed of ideas themselves. That said, the mathematical content of this thesis is, as an entire piece of work, inextricably associated to its author through intellectual ownership. Formulation of ideas: The candidate played an integral and fully collaborative role in the formulation of ideas. Design of methodology: The candidate played an integral and fully collaborative role in the design of methodology. Experimental work: N/A Presentation of data in journal format: N/A *The wording in this box follows the advice and approval of my supervisor, Professor Kyprianou.
Statement from Candidate	This paper reports on original research I conducted during the period of my Higher Degree by Research candidature.
Signed	Tsogzolmaa Saizmaa — **Date** 20/06/2021

Chapter 4

Attraction to and repulsion from a subset of the unit sphere for isotropic stable Lévy processes

Andreas E. Kyprianou[1], Sandra Palau[2] Tsogzolmaa Saizmaa[3]

Abstract

Taking account of recent developments in the representation of d-dimensional isotropic stable Lévy processes as self-similar Markov processes, e.g. deep factorisation or radial excursions of stable processes cf. [17, 20, 22], we consider a number of new ways to condition its path. Suppose that S is a region of the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$. We construct the aforesaid stable Lévy process conditioned to approach S continuously from either inside or outside of the sphere. Additionally, we show that these processes are in duality with the stable process conditioned to remain inside the sphere and absorb continuously at the origin and to remain outside of the sphere, respectively. Our results extend the recent contributions of [12], where similar conditioning is considered, albeit in one dimension as well as providing analogues of the same and very classical results for Brownian motion, [13]. As in [12], we appeal to recent fluctuation identities related to the deep factorisation of stable processes, cf. [17, 20, 22].

[1]Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, UK. Email: a.kyprianou@bath.ac.uk

[2]Department of Statistics and Probability, Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México, México. E-mail: sandra@sigma.iimas.unam.mx

[3]Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, UK and National University of Mongolia, Baga-toiruu, Sukhbaatar district, Ulaanbaatar, Mongolia. Email: t.saizmaa@bath.ac.uk

4.1 Introduction

Let $X = (X_t, t \geq 0)$ be a d-dimensional stable Lévy process with probabilities $(\mathbb{P}_x, x \in \mathbb{R}^d)$. This means that X has càdlàg paths with stationary and independent increments as well as respecting a property of self-similarity: There is an $\alpha > 0$ such that, for $c > 0$, and $x \in \mathbb{R}^d$, under $\mathbb{P}_x$, the law of $(cX_{c^{-\alpha}t}, t \geq 0)$ is equal to $\mathbb{P}_{cx}$. It turns out that stable Lévy processes necessarily have the scaling index $\alpha \in (0, 2]$. The case $\alpha = 2$ pertains to a standard d-dimensional Brownian motion, thus has a continuous path. The processes we construct are arguably less interesting in the diffusive setting and thus we restrict ourselves to the isotropic pure jump setting of $\alpha \in (0, 2)$ in dimension $d \geq 2$.

To be more precise, this means, for all orthogonal transformations $U : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $x \in \mathbb{R}^d$,

$$\textit{the law of} \quad (UX_t, t \geq 0) \ \textit{under} \quad \mathbb{P}_x \ \textit{is equal to} \ (X_t, t \geq 0) \ \textit{under} \ \mathbb{P}_{Ux}.$$

For convenience, we will henceforth refer to X just as a *stable process.*

The stable Lévy process has a the jump measure Π that satisfies

$$\Pi(B) = \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|} \int_B \frac{1}{|y|^{\alpha+d}} \mathrm{d}y, \quad B \subseteq \mathcal{B}(\mathbb{R}^d).$$

The constant in the definition of $\Pi(B)$ can be arbitrary, however, our choice corresponds to the one one that allows us to identify the characteristic exponent Lévy process as

$$\Psi(\theta) = -\frac{1}{t} \log \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta \cdot X_t}) = |\theta|^\alpha, \quad \theta \in \mathbb{R}^d,$$

where we write $\mathbb{P}$ in preference to $\mathbb{P}_0$; more precisely, the coefficient of $|\theta|^\alpha$ is one.

In this article, we characterise the law of a stable process conditioned to hit continuously a part of the surface, say $\mathsf{S} \subseteq \mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$, either from the inside or from the outside of the unit sphere. We develop an expression for the law of the limiting point of contact on S. Moreover, we show that, when time reversed from the strike point on S, the resulting process can also be seen as a conditioned stable process. The extreme cases that $\mathsf{S} = \mathbb{S}^{d-1}$ (the whole unit sphere) and $\mathsf{S} = \{\vartheta\} \in \mathbb{S}^{d-1}$ (a single point on the unit sphere) are included in our analysis, however, we will otherwise insist that the Lebesgue surface measure of S is strictly positive.

Our results relate to the recent work of [12], who considered a real valued Lévy process conditioned to continuously approach the boundary of the interval $[-1, 1]$ from the outside. *In order to avoid repetition, we always remain in two or more dimensions.* As in [12], we rely heavily on recent fluctuation identities that are connected to the deep factorisation of the stable process; cf. [17, 20, 22]. The results here are also related the classical results of Doob [13], who deals with similar conclusions for Brownian motion and as well as echoing the general theory of conditioned stochastic processes in the potential-analytic sense (via a Doob h-transform), see e.g. Chapter 14 of [11].

4.2 Attraction towards the patch S

For convenience, we will work with the definition $\mathbb{B}_d = \{x \in \mathbb{R}^d : |x| < 1\}$. Let $\mathbb{D}(\mathbb{R}^d)$ denote the space of càdlàg paths $\omega : [0,\infty) \to \mathbb{R}^d \cup \partial$ with lifetime $\mathtt{k}(\omega) = \inf\{s > 0 : \omega(s) = \partial\}$, where ∂ is a cemetery point. The space $\mathbb{D}(\mathbb{R}^d)$ will be equipped with the Skorokhod topology, with its Borel σ-algebra $\mathcal{F}$ and natural filtration $(\mathcal{F}_t, t \geq 0)$. The reader will note that we will also use a similar notion for $\mathbb{D}(\mathbb{R} \times \mathbb{S}^{d-1})$ later on in this text in the obvious way. We will always work with $X = (X_t, t \geq 0)$ to mean the coordinate process defined on the space $\mathbb{D}(\mathbb{R}^d)$. Hence, the notation of the introduction indicates that $\mathbb{P} = (\mathbb{P}_x, x \in \mathbb{R}^d)$ is such that $(X, \mathbb{P})$ is our stable process.

Consider a subset $\mathsf{S} \subseteq \mathbb{S}^{d-1}$ such that it has strictly positive Lebesgue surface measure or it is a point. We want to construct the law of X conditioned to approach S continuously from within $\bar{\mathbb{B}}_d^c := \mathbb{R}^d \setminus \bar{\mathbb{B}}_d$. From a potential-theoretic perspective, this law can be obtained as a Doob h-transform of the killed stable process in $\overline{\mathbb{B}}_d^c$, provided that h is a positive harmonic function in $\overline{\mathbb{B}}_d^c$ which is equal to zero in $\overline{\mathbb{B}}_d$ and which goes to zero at infinity and at $\mathbb{S}^{d-1} \setminus \mathsf{S}$; cf. [11, Chapter 14]. In this paper, we want to give a probabilistic construction, which identifies a more physical meaning to the conditioning in terms of the paths of the stable process; see e.g. the classical work of [3, 9]. Similarly, we want the law of X conditioned to approach S continuously from within $\mathbb{B}_d$. More precisely, via an appropriate limiting procedure, we want to build a new family of probabilities $\mathbb{P}^\vee = (\mathbb{P}^\vee_x, x \in \bar{\mathbb{B}}_d^c)$ such that

$$\mathbb{P}^\vee_x(X_s \in \bar{\mathbb{B}}_d^c, s < \mathtt{k} \text{ and } X_{\mathtt{k}-} \in \mathsf{S}) = 1, \qquad x \in \bar{\mathbb{B}}_d^c,$$

with a similar statement holding when the conditioning is undertaken from within $\mathbb{B}_d$.

As we are considering two or higher dimensions, the process $(X, \mathbb{P})$ is transient in the sense that $\lim_{t\to\infty} |X_t| = \infty$ almost surely. Defining

$$\underline{G}(t) := \sup\{s \leq t \colon |X_s| = \inf_{u \leq s} |X_u|\}, \qquad t \geq 0,$$

we thus have by monotonicity and the transience of $(X, \mathbb{P})$ that $\underline{G}(\infty) := \lim_{t\to\infty} \underline{G}(t)$ exists and, moreover, $X_{\underline{G}(\infty)}$ describes the point of closest reach to the origin in the range of X.

We can similarly define $\overline{G}(t) = \sup\{s \leq t \colon |X_s| = \sup_{u \leq s} |X_u|\}$, $t \geq 0$, so that $\overline{G}(\tau_1^\ominus -)$ is the point of furthest reach from the origin prior to exiting $\mathbb{B}_d$, where

$$\tau_1^\ominus = \inf\{t > 0 : |X_t| > 1\}.$$

Let us turn to what we mean by conditioning to attract to the set S from either the interior or the exterior of the sphere. If S is not a point, we define $A_\varepsilon = \{r\theta : r \in (1, 1+\varepsilon), \theta \in \mathsf{S}\}$ and $B_\varepsilon = \{r\theta : r \in (1-\varepsilon, 1), \theta \in \mathsf{S}\}$, for $0 < \varepsilon < 1$ and define the corresponding events $C_\varepsilon^\vee := \{X_{\underline{G}(\infty)} \in A_\varepsilon\}$, and $C_\varepsilon^\wedge := \{X_{\overline{G}(\tau_1^\ominus -)} \in B_\varepsilon\}$. Let

$$\tau_1^\oplus = \inf\{t > 0 : |X_t| < 1\}.$$

We are interested in the asymptotic conditioning

$$\mathbb{P}^{\vee}_x(A,\, t < \mathtt{k}) = \lim_{\varepsilon\to 0}\mathbb{P}_x(A,\, t<\tau_1^{\oplus}|C_\varepsilon^{\vee}), \tag{A.1}$$

when $x \in \bar{\mathbb{B}}_d^c$ and

$$\mathbb{P}^{\wedge}_x(A,\, t < \mathtt{k}) = \lim_{\varepsilon\to 0}\mathbb{P}_x(A,\, t<\tau_1^{\ominus}|C_\varepsilon^{\wedge}), \tag{A.2}$$

when $x \in \mathbb{B}_d$, for all $A \in \mathcal{F}_t$.

When $\mathsf{S} = \{\vartheta\} \in \mathbb{S}^{d-1}$, we need to adapt slightly the sets A_ε and B_ε so that $A_\varepsilon = \{r\phi\colon r \in (1, 1+\varepsilon), \phi \in \mathbb{S}^{d-1}, |\phi - \vartheta| < \varepsilon\}$ and $B_\varepsilon = \{r\phi\colon r \in (1-\varepsilon, 1), \phi \in \mathbb{S}^{d-1}, |\phi-\vartheta| < \varepsilon\}$.

We will go a little further in due course and give a fuller description of these two conditioned processes by including the cases that X is issued from the unit sphere itself but not within S, i.e. $\mathbb{S}^{d-1} \setminus \mathsf{S}$. For now, we have our first main result, given immediately below, for which we define the function

$$H_{\mathsf{S}}(x) = \begin{cases} ||x|^2-1|^{\alpha/2}\displaystyle\int_{\mathsf{S}} |\theta - x|^{-d}\sigma_1(\mathrm{d}\theta) & \text{if } \sigma_1(\mathsf{S}) > 0, \\ \\ ||x|^2-1|^{\alpha/2}|\vartheta - x|^{-d} & \text{if } \mathsf{S} = \{\vartheta\}, \end{cases} \tag{A.3}$$

for $|x| \neq 1$, where $\sigma_1(\mathrm{d}\theta)$ is the Lebesgue surface measure on $\mathbb{S}^{d-1}$ normalised to have unit mass. It is worthy of note that, when $\mathsf{S} = \mathbb{S}^{d-1}$, the integral in (A.3) can be computed precisely. Indeed, up to an unimportant (for our purposes) multiplicative constant, $C > 0$, which may change from line to line, we note that, for $|x| > 1$,

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} |x-\theta|^{-d}\sigma_1(\mathrm{d}\theta) &= C\int_0^\pi \frac{(\sin\phi)^{d-2}}{(|x|^2 - 2|x|\cos\phi + 1)^{d/2}}\mathrm{d}\phi \\ &= C|x|^{-d}{}_2F_1(d/2, 1; d/2, |x|^{-2}) \\ &= C|x|^{-d}\left(1 - \frac{1}{|x|^2}\right)^{-1}, \end{aligned}$$

where we have used the hypergeometric identity in (A.1) of the Appendix. We can perform a similar calculation when $|x| < 1$ and, obtain, up to a multiplicative constant, $C > 0$, that

$$\int_{\mathbb{S}^{d-1}} |x-\theta|^{-d}\sigma_1(\mathrm{d}\theta) = C\left(1 - |x|^2\right)^{-1}. \tag{A.4}$$

All together, noting that we may ignore multiplicative constants, we have

$$H_{\mathbb{S}^{d-1}}(x) = \begin{cases} |x|^{\alpha-d}\left(1-|x|^{-2}\right)^{\frac{\alpha}{2}-1} & \text{if } |x| > 1 \\ \\ (1-|x|^2)^{\frac{\alpha}{2}-1} & \text{if } |x| < 1. \end{cases} \tag{A.5}$$

As the next result will make clear, H_{S} is a positive harmonic function for both $(X_t, t < \tau_1^{\ominus})$ and $(X_t, t<\tau_1^{\oplus})$. From the potential-theoretic perspective, it can be described as an integral of the Martin kernel over S. Then, by the Martin boundary theory, the h-conditioned process will approach S with probability one, see [11, Chapter 14] as well as the classical results of Doob for Brownian motion, cf. Theorem 7.1 [13].

Theorem 4.2.1 (Stable process conditioned to attract to S continuously from one side)**.** *Let $\mathsf{S} \subseteq \mathbb{S}^{d-1}$ be an closed set with $\sigma_1(\mathsf{S}) > 0$ or $\mathsf{S} = \{\vartheta\}$ for a fixed point $\vartheta \in \mathbb{S}^{d-1}$. Then for all points of issue $x \in \mathbb{R}^d \setminus \mathbb{S}^{d-1}$ we have*

$$\left.\frac{d\mathbb{P}^{\vee}_x}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = \mathbf{1}_{(t<\tau_1^{\oplus})}\frac{H_{\mathsf{S}}(X_t)}{H_{\mathsf{S}}(x)}, \qquad \text{if } x \in \bar{\mathbb{B}}_d^c \tag{A.6}$$

and otherwise

$$\left.\frac{d\mathbb{P}^{\wedge}_x}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = \mathbf{1}_{(t<\tau_1^{\ominus})}\frac{H_{\mathsf{S}}(X_t)}{H_{\mathsf{S}}(x)}, \qquad \text{if } x \in \mathbb{B}_d. \tag{A.7}$$

In particular, $(\mathbb{P}^{\vee}_x, x \in \bar{\mathbb{B}}_d^c)$ and $(\mathbb{P}^{\wedge}_x, x \in \bar{\mathbb{B}}_d^c)$ are Markovian families.

Remark 4.2.1. The choice of limiting conditioning procedure that we have used reflects a similar approach taken in [12] in one dimension. It is worth noting at this point that the choice of $C^{\vee}_\varepsilon$ and $C^{\wedge}_\varepsilon$ are by no means the only possibilities as far as performing a limiting conditioning that results in (A.6) and (A.7). For example, once the reader is familiar with the proof of Theorem 4.2.1, it will quickly become clear that, when S is not a singleton, by defining e.g. $C^{\vee}_\varepsilon = \{X_{\tau_1^{\oplus}} \in B_\varepsilon\}$, or indeed $C^{\vee}_\varepsilon = \{X_{\tau_1^{\oplus}-} \in A_\varepsilon\}$, the limit (A.1) will still produce the change of measure (A.6). Once the reader is familiar with the proof of Theorem 4.2.1, it is a worthwhile exercise to verify the two proposed alternative definitions of $C^{\vee}_\varepsilon$ for the limiting process by appealing to the fluctuation identities in e.g. [20]. Other definitions of $C^{\vee}_\varepsilon$ giving a consistent limit may indeed also be possible.

Whilst the above theorem deals with the construction of the conditioned process up to but not including its terminal position, we characterise the latter in the next result, which resonates with Theorem 14.8 of [11].

Proposition 4.2.1 (Distribution of the hitting location)**.** *Suppose that $\mathsf{S} \subseteq \mathbb{S}^{d-1}$ be a closed set with $\sigma_1(\mathsf{S}) > 0$. Let S' be an closed subset of S. Then for any $x \in \mathbb{R}^d \setminus \bar{\mathbb{B}}_d$, we have*

$$\mathbb{P}^{\vee}_x(X_{\mathtt{k}-} \in \mathsf{S}') = \frac{\int_{\mathsf{S}'} |\theta - x|^{-d}\sigma_1(\mathrm{d}\theta)}{\int_{\mathsf{S}} |\theta - x|^{-d}\sigma_1(\mathrm{d}\theta)}, \tag{A.8}$$

with an identical result holding for $X_{\mathtt{k}-}$ under $\mathbb{P}^{\wedge}_x$, with $x \in \mathbb{B}_d$.

4.3 Lamperti–Kiu representation and radial excursions

The basic definition of the stable process conditioned to attract continuously to S from one side is not quite complete. Strictly speaking, we could think about defining the process to include the points of issue in $\mathbb{S}^{d-1} \setminus \mathsf{S}$. It turns out that this is possible. However, we first need to remind the reader of the recently described radial excursion theory, see [20, 21]. The starting point for the aforementioned is the Lamperti–Kiu transform which identifies the stable process as a self-similar Markov process.

To describe it, we need to introduce the notion of a Markov Additive Process, henceforth written MAP for short. Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$. With an abuse of previous notation, we say that $(\Xi, \Upsilon) = ((\Xi_t, \Upsilon_t), t \geq 0)$ is a MAP if it is Feller process on $\mathbb{R}^n \times \mathbb{S}^{d-1}$, with probabilities $\mathtt{P}_{x,\theta}$, $x \in \mathbb{R}^n$, $\theta \in \mathbb{S}^{d-1}$, such that, for any $t \geq 0$, the conditional law of the process $((\Xi_{s+t} - \Xi_t, \Upsilon_{s+t}) : s \geq 0)$, given $((\Xi_u, \Upsilon_u), u \leq t)$, is that of (Ξ, Υ) under $\mathtt{P}_{0,\theta}$, with $\theta = \Upsilon_t$. For a MAP pair $((\Xi_t, \Upsilon_t), t \geq 0)$, we call Ξ the *ordinate* and Υ the *modulator*.

According to one of the main results in [1], there exists a MAP on $\mathbb{R} \times \mathbb{S}^{d-1}$, which we will henceforth write as (ξ, Θ), with probabilities $\mathbf{P} = (\mathbf{P}_{x,\theta}, x \in \mathbb{R}, \theta \in \mathbb{S}^{d-1})$ such that the d-dimensional stable process can be written

$$X_t = \exp\{\xi_{\varphi(t)}\}\Theta_{\varphi(t)} \qquad t \geq 0, \tag{A.9}$$

where

$$\varphi(t) = \inf\left\{s > 0 : \int_0^s \mathrm{e}^{\alpha\xi_u}\mathrm{d}u > t\right\}. \tag{A.10}$$

Whilst Θ alone is a Feller process, it is not necessarily true that ξ alone is. However, it is a consequence of isotropy that this is the case here. Moreover, ξ alone is a Lévy process whose characteristic exponent is known (but not important in the current context); see for example [8]. What is important for our purposes is to note for now that it has paths of unbounded variation, and therefore is regular for the upper and lower half line (in the sense of Definition 6.4 of [16]).

It is not difficult to show that the pair $((\xi_t - \underline{\xi}_t, \Theta_t), t \geq 0)$, forms a strong Markov process, where $\underline{\xi}_t := \inf_{s \leq t} \xi_s$, $t \geq 0$ is the running minimum of ξ. On account of the fact that ξ, alone, is a Lévy process, $(\xi_t - \underline{\xi}_t,\ t \geq 0)$ is also a strong Markov process. Suppose we denote by $\ell = (\ell_t, t \geq 0)$ the local time at zero of $\xi - \underline{\xi}$, then we can introduce the following processes

$$H^-_t = -\xi_{\ell^{-1}_t} \text{ and } \Theta^-_t = \Theta_{\ell^{-1}_t}, \qquad t \geq 0,$$

and define $(H^-_{\ell^{-1}_t}, \Theta^-_{\ell^{-1}_t}) = (\partial, \dagger)$, a cemetery state, if $\ell^{-1}_t = \infty$. Then, the pair (ℓ^{-1}, H^-), without reference to the associated modulation Θ^-, are Markovian and play the role of the descending ladder time and height subordinators of ξ. Moreover, the strong Markov property tells us that $(\ell^{-1}_t, H^-_t, \Theta^-_t)$, $t \geq 0$, defines a Markov Additive Process on $\mathbb{R}^2 \times \mathbb{S}^{d-1}$, whose first two elements are ordinates that are non-decreasing. In this sense, ℓ also serves as an adequate choice for the local time of the Markov process $(\xi - \underline{\xi}, \Theta)$ on the set $\{0\} \times \mathbb{S}^{d-1}$. (See [20]).

Suppose we define $\mathtt{g}_t = \sup\{s < t : \xi_s = \underline{\xi}_s\}$, and recall that the regularity of ξ for $(-\infty, 0)$ and $(0, \infty)$ ensures that it is well defined, as is $\mathtt{g}_\infty = \lim_{t\to\infty} \mathtt{g}_t$. Set

$$\mathtt{d}_t = \inf\{s > t : \xi_s = \underline{\xi}_s\}.$$

For all $t > 0$ such that $\mathtt{d}_t > \mathtt{g}_t$ the process

$$(\epsilon_{\mathtt{g}_t}(s), \Theta^{\epsilon}_{\mathtt{g}_t}(s)) := (\xi_{\mathtt{g}_t+s} - \xi_{\mathtt{g}_t}, \Theta_{\mathtt{g}_t+s}), \qquad s \leq \zeta_{\mathtt{g}_t} := \mathtt{d}_t - \mathtt{g}_t,$$

codes the excursion of $(\xi - \underline{\xi}, \Theta)$ from the set $(0, \mathbb{S}^{d-1})$ which straddles time t. Such excursions live in the space $\underline{\mathbb{U}}(\mathbb{R} \times \mathbb{S}^{d-1})$, the space of càdlàg paths in $\mathbb{R} \times \mathbb{S}^{d-1}$, written in canonical form

$$(\epsilon, \Theta^{\epsilon}) = ((\epsilon(t), \Theta^{\epsilon}(t)) : t \leq \zeta) \text{ with lifetime } \zeta = \inf\{s > 0 : \epsilon(s) < 0\},$$

such that $(\epsilon(0), \Theta^{\epsilon}(0)) \in \{0\} \times \mathbb{S}^{d-1}$, $(\epsilon(s), \Theta^{\epsilon}(s)) \in (0, \infty) \times \mathbb{S}^{d-1}$, for $0 < s < \zeta$, and $\epsilon(\zeta) \in (-\infty, 0]$.

Taking account of the Lamperti–Kiu transform (A.9), it is natural to consider how the excursion of $(\xi - \underline{\xi}, \Theta)$ from $\{0\} \times \mathbb{S}^{d-1}$ translates into a radial excursion theory for the process

$$Y_t := \mathrm{e}^{\xi_t} \Theta_t, \qquad t \geq 0.$$

Ignoring the time change in (A.9), we see that the radial minima of the process Y agree with the radial minima of the stable process X. Indeed, each excursion of $(\xi - \underline{\xi}, \Theta)$ from $\{0\} \times \mathbb{S}^{d-1}$ is uniquely associated to exactly one excursion of $(Y_t / \inf_{s \leq t} |Y_s|, t \geq 0)$, from $\mathbb{S}^{d-1}$, or equivalently an excursion of Y from its running radial infimum. Moreover, we see that, for all $t > 0$ such that $\mathtt{d}_t > \mathtt{g}_t$,

$$Y_{\mathtt{g}_t+s} = \mathrm{e}^{\xi_{\mathtt{g}_t}} \mathrm{e}^{\epsilon_{\mathtt{g}_t}(s)} \Theta^{\epsilon}_{\mathtt{g}_t}(s) = |Y_{\mathtt{g}_t}| \mathrm{e}^{\epsilon_{\mathtt{g}_t}(s)} \Theta^{\epsilon}_{\mathtt{g}_t}(s), \qquad s \leq \zeta_{\mathtt{g}_t}.$$

This will be useful to keep in mind for the forthcoming excursion computations.

For $t > 0$, let $R_t = \mathtt{d}_t - t$, and define the set $\mathbb{G} = \{t > 0 : R_{t-} = 0, R_t > 0\} = \{\mathtt{g}_s : s \geq 0\}$. The classical theory of exit systems in [23] (see Theorems (4.1) and (6.3) therein) now implies that there exists an additive functional $(\Lambda_t, t \geq 0)$ and a family of *excursion measures*, $(\underline{\mathbb{N}}_{\theta}, \theta \in \mathbb{S}^{d-1})$ such that:

(i) Λ is an additive functional of (ξ, Θ), has a bounded 1-potential and is carried by the set of times $\{t \geq 0 : (\xi_t - \underline{\xi}_t, \Theta_t) \in \{0\} \times \mathbb{S}^{d-1}\}$,

(ii) the map $\theta \mapsto \underline{\mathbb{N}}_{\theta}$ is an $\mathbb{S}^{d-1}$-indexed kernel on $\underline{\mathbb{U}}(\mathbb{R} \times \mathbb{S}^{d-1})$ such that $\underline{\mathbb{N}}_{\theta}(1 - \mathrm{e}^{-\zeta}) < \infty$;

(iii) we have the *exit formula*

$$\mathbf{E}_{x,\theta}\left[\sum_{g \in \mathbb{G}} F((\xi_s, \Theta_s) : s < g) H((\epsilon_g, \Theta^{\epsilon}_g))\right] \\ = \mathbf{E}_{x,\theta}\left[\int_0^{\infty} F((\xi_s, \Theta_s) : s < t) \underline{\mathbb{N}}_{\Theta_t}(H(\epsilon, \Theta^{\epsilon})) \mathrm{d}\Lambda_t\right], \qquad \text{(A.11)}$$

for $x \neq 0$, where F is continuous on the space of càdlàg paths $\mathbb{D}(\mathbb{R} \times \mathbb{S}^{d-1})$ and H is measurable on the space of càdlàg paths $\underline{\mathbb{U}}(\mathbb{R} \times \mathbb{S}^{d-1})$;

(iv) under any measure $\underline{\mathbb{N}}_\theta$ the process $((\epsilon(s), \Theta^\epsilon(s)), s < \zeta)$ is a strong Markov process with the same semigroup as (ξ, Θ) killed at its first hitting time of $(-\infty, 0] \times \mathbb{S}^{d-1}$.

The couple $(\Lambda, \underline{\mathbb{N}}_\cdot)$ is called an exit system. Note that in Maisonneuve's original formulation, the pair Λ, $\underline{\mathbb{N}}_\cdot := (\underline{\mathbb{N}}_\theta, \theta \in \mathbb{S}^{d-1})$ is not unique, but once Λ is chosen the measures $(\underline{\mathbb{N}}_\theta, \theta \in \mathbb{S}^{d-1})$ exist however, are only unique up to Λ-*neglectable* sets, i.e. sets $\mathcal{A}$ such that $\mathbf{E}_{x,\theta}(\int_0^\infty \mathbf{1}_{\{\Theta_s \in \mathcal{A}\}} \mathrm{d}\Lambda_s) = 0$. Another example of where this theory has been used is in the construction of excursions from a set is that of Brownian motion away from a hyperplane; see [7].

Now referring back to ℓ, the local time of $\xi - \underline{\xi}$ at 0, since it is an additive functional with a bounded 1-potential, there is an exit system which corresponds to $(\ell, \underline{\mathbb{N}}_\cdot)$. With this choice of ℓ we assume that the choice of $\underline{\mathbb{N}}_\cdot$ is fixed despite the fact that we can induce subtle variations in $\underline{\mathbb{N}}_\cdot$ on a Λ-negligible set of $\theta \in \mathbb{S}^{d-1}$ e.g. by setting $\underline{\mathbb{N}}_\theta \equiv 0$ there. The reader is referred to Chapter VII of [4] for further discussion on this matter. Note that $\underline{\mathbb{N}}_\theta$ is not isotropic in θ. For example, excursions that begin at the 'North Pole', say $\mathtt{1}$, are, with high frequency, arbitrarily small and hence will end near to $\mathtt{1}$. That said, depending on the event A, it is possible that $\underline{\mathbb{N}}_\theta(A)$ does not depend on $\theta \in \mathbb{S}^{d-1}$; for example, $\underline{\mathbb{N}}_\theta(\zeta = \infty)$. The reason for this is that it must agree with the rate at which the infinite excursion of $\xi - \underline{\xi}$ occurs, according to the local time ℓ. More generally, we have that, for all orthogonal transformations $U : \mathbb{R}^d \mapsto \mathbb{R}^d$ and f such that $\underline{\mathbb{N}}_\theta(f(\epsilon, \Theta^\epsilon)) < \infty$, $\theta \in \mathbb{S}^{d-1}$, isotropy implies that $\underline{\mathbb{N}}_\theta(f(\epsilon, U\Theta^\epsilon)) = \underline{\mathbb{N}}_{U\theta}(f(\epsilon, \Theta^\epsilon))$. On account of the fact that ℓ is only defined up to a multiplicative constant, we can use the common value of $\underline{\mathbb{N}}_\theta(\zeta = \infty)$ to fix a normalisation the local time, or equivalently, of the excursion measures $(\underline{\mathbb{N}}_\theta, \theta \in \mathbb{S}^{d-1})$. We thus fix it to take the value of unity. The place at which this choice of normalisation becomes relevant is when we cite certain identities from (cf. (A.39) below) from [20], in which this assumption was also made. Henceforth, this is the exit system we will work with and the system of excursion associated to it is what we call our *radial excursion theory.*

Later in our proofs we will use a variant of the above excursion theory based on the MAP $(\overline{\xi} - \xi, \Theta)$, where $\overline{\xi}$ is the process $\overline{\xi}_t = \sup_{s \le t} \xi_s$, $t \ge 0$. We leave the details until that point in the text. With our excursion theory in hand, we can now proceed to identify the completion of Theorem 4.2.1.

Theorem 4.3.1. *Let $\mathsf{S} \subseteq \mathbb{S}^{d-1}$ be an closed set with $\sigma_1(\mathsf{S}) > 0$ or $\mathsf{S} = \{\vartheta\}$ for a fixed point $\vartheta \in \mathbb{S}^{d-1}$. The processes $(X, \mathbb{P}^\vee)$ and $(X, \mathbb{P}^\wedge)$ can be extended in a consistent way to include points of issue $x \in \mathbb{S}^{d-1} \setminus \mathsf{S}$ with pathwise continuous entry via*

$$\mathbb{P}^\vee_x(X_t \in \mathrm{d}y,\, t < \mathtt{k}) := \frac{\Gamma(d/2)}{\Gamma(\alpha/2+1)\Gamma((d-\alpha)/2)} \frac{H_{\mathsf{S}}(y)}{h(x)} \underline{\mathbb{N}}_x \left(X^\epsilon(t) \in \mathrm{d}y,\, t < \varsigma\right), \qquad |y| > 1 \tag{A.12}$$

and

$$\mathbb{P}^\wedge_x(X_t \in \mathrm{d}y,\, t < \mathtt{k}) := \frac{\Gamma(d/2)}{\Gamma(\alpha/2+1)\Gamma((d-\alpha)/2)} \frac{H_{\mathsf{S}}(y)}{h(x)} \overline{\mathbb{N}}_x \left(X^\epsilon(t) \in \mathrm{d}y,\, t < \varsigma\right), \qquad |y| < 1, \tag{A.13}$$

where,

$$h(x) = \int_{\mathsf{S}} |x-\theta|^{-d}\sigma_1(\mathrm{d}\theta)$$

and, for $(\epsilon, \Theta^\epsilon)$ selected from $\underline{\mathbb{U}}(\mathbb{R}\times\mathbb{S}^{d-1})$ or $\overline{\mathbb{U}}(\mathbb{R}\times\mathbb{S}^{d-1})$, respectively,

$$X^\epsilon(t) = \mathrm{e}^{\epsilon(\varphi(t))}\Theta^\epsilon(\varphi(t)) \text{ and } \varsigma = \varphi^{-1}(\zeta) = \int_0^\zeta |X^\epsilon(u)|^\alpha \mathrm{d}u. \tag{A.14}$$

Here, pathwise continuous entry means that

$$\mathbb{P}^\vee_x(\lim_{t\to 0} X_t = x) = \mathbb{P}^\wedge_x(\lim_{t\to 0} X_t = x) = 1 \tag{A.15}$$

for all $x \in \mathbb{S}^{d-1}\setminus\mathsf{S}$.

Note, referring to the discussion preceding Theorem 4.3.1 that pertains to the choice of excursion measures and local time, given the choice of local time ℓ leaves a free choice of multiplicative constant in the definition of local time, which may depend on $x \in \mathbb{S}^{d-1}\setminus\mathsf{S}$. In the proof of Theorem 4.3.1, we use a method of continuity of resolvents to pin down the aforesaid constants. We also note that extending the notion of a Doob h-transformed process to include certain 'boundary points' in the way we have seen in Theorem 4.3.1 can be seen in e.g. [25, 9] as well as the classical work of Doob [13].

4.4 Repulsion and duality

In this section, we want to introduce two new processes, which will turn out to be dual to $(X, \mathbb{P}^\vee)$ and $(X, \mathbb{P}^\wedge)$ in the sense of time reversal. The two processes we are interested give meaning to the stable process conditioned to remain in $\bar{\mathbb{B}}_d^c$ and $\mathbb{B}_d$, respectively, in an appropriate sense.

An important tool that we will make use of in analysing the aforesaid time reversed processes comes through the so-called Riesz–Bogdan–Żak transform, which relates path behaviour of the stable process outside of the unit sphere to its behaviour inside the unit sphere. In order to state it, we need to introduce the process $(X, \mathbb{P}^\circ)$, where the probabilities $\mathbb{P}^\circ = (\mathbb{P}^\circ_x, x \neq 0)$ are given by

$$\left.\frac{\mathrm{d}\mathbb{P}^\circ_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \qquad \text{on } t < \tau_\varepsilon := \inf\{t>0 : |X_t| < \varepsilon\} \tag{A.16}$$

for all $\varepsilon > 0$. Since $\alpha < 2 \leq d$, we note that the change of measure rewards paths that approach the origin and punishes paths that wander far from the origin. Intuitively, it is clear that $(X, \mathbb{P}^\circ)$ describes the stable process conditioned to continuously approach the origin. Nonetheless, this heuristic can be made into a rigorous statement, see for example [18, 20, 21, 22]. The reader will also note from these references (and it is easy to prove that) that $(X, \mathbb{P}^\circ)$ is also a self-similar Markov process with the same index of self-similarity as $(X, \mathbb{P})$.

Theorem 4.4.1 (Riesz–Bogdan–Żak transform)**.** *Suppose we write $Kx = x/|x|^2$, $x \in \mathbb{R}^d$ for the classical inversion of space through the sphere $\mathbb{S}^{d-1}$. Then, in dimension $d \geq 2$, for $x \neq 0$,*

$(KX_{\eta(t)}, t \geq 0)$ *under* $\mathbb{P}_x$ *is equal in law to* $(X, \mathbb{P}^\circ_{Kx})$*, where* $\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha}\mathrm{d}u > t\}$*.*

Let us return to our duality concerns. To this end, let us introduce the probabilities

$$H^\ominus(x) = \mathbb{P}_x(\tau_1^\oplus = \infty) = \frac{\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)} \int_0^{|x|^2-1} (u+1)^{-d/2} u^{\alpha/2-1} \mathrm{d}u, \tag{A.17}$$

for $|x| > 1$, where the second inequality is lifted from [5], and,

$$H^\oplus(x) = |x|^{\alpha-d} H^\ominus(Kx),$$

for $|x| < 1$.

These two functions are positive harmonic for X and can be used to define the two families of probabilities $\mathbb{P}^\ominus = (\mathbb{P}^\ominus_x, |x| > 1)$ and $\mathbb{P}^\oplus = (\mathbb{P}^\oplus_x, |x| < 1)$ via the Doob h-transforms,

$$\left.\frac{\mathrm{d}\mathbb{P}^\ominus_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{H^\ominus(X_t)}{H^\ominus(x)} \mathbf{1}_{(t<\tau^\oplus)}, \qquad t \geq 0, |x| > 1 \tag{A.18}$$

and,

$$\left.\frac{\mathrm{d}\mathbb{P}^\oplus_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{H^\oplus(X_t)}{H^\oplus(x)} \mathbf{1}_{(t<\tau^\ominus)}, \qquad t \geq 0, |x| < 1. \tag{A.19}$$

The first of these two changes of measure corresponds to the stable process conditioned to avoid entering $\mathbb{B}_d$ by a simple restriction on the probability space (remembering that $\lim_{t\to\infty} |X_t| = \infty$). Note from Theorem 4.4.1 that

$$H^\ominus(Kx) = \mathbb{P}_{Kx}(\tau_1^\oplus = \infty) = \mathbb{P}^\circ_x(\tau^{\{0\}} < \tau_1^\ominus),$$

where $\tau^{\{0\}} = \inf\{t > 0 : |X_{t-}| = 0\}$. The second change of measure, (A.19), is a composition of conditioning the stable process to be absorbed continuously at the origin, followed by conditioning it not to exit $\mathbb{B}_d$ via a simple restriction on the probability space (noting that $\lim_{t\to\infty} |X_t| = 0$ under $\mathbb{P}^\circ$).

The reader will also note that the Riesz-Bogdan-Żak transform also implies a similar spatial inversion and time change must hold for the pair $(X, \mathbb{P}^\ominus)$ and $(X, \mathbb{P}^\oplus)$.

Corollary 4.4.1. *For* $|x| > 1$*,* $(KX_{\eta(t)}, t \geq 0)$ *under* $\mathbb{P}^\ominus_x$ *is equal in law to* $(X, \mathbb{P}^\oplus_{Kx})$*, where* $\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha}\mathrm{d}u > t\}$*. Similarly, for* $|x| < 1$*,* $(KX_{\eta(t)}, t \geq 0)$ *under* $\mathbb{P}^\oplus_x$ *is equal in law to* $(X, \mathbb{P}^\ominus_{Kx})$*.*

Proof. Suppose that $F(X_s, s \leq t)$ is a bounded $\mathcal{F}_t$-measurable function for each $t \geq 0$. Then, for $|x| > 1$, appealing to Theorem 4.4.1, we have

$$\begin{aligned}
\mathbb{E}^{\ominus}_x \left[F(KX_{\eta(s)}, s \le t)\right] &= \mathbb{E}_x \left[F(KX_{\eta(s)}, s \le t)\frac{H^{\ominus}(K(KX_{\eta(t)}))}{H^{\ominus}(x)}\mathbf{1}_{(\eta(t)<\tau^{\oplus})}\right] \\
&= \mathbb{E}^{\circ}_{Kx}\left[F(X_s, s \le t)\frac{H^{\ominus}(KX_t)}{H^{\ominus}(x)}\mathbf{1}_{(t<\tau^{\ominus})}\right] \\
&= \mathbb{E}_{Kx}\left[F(X_s, s \le t)\frac{|X_t|^{\alpha-d}}{|Kx|^{\alpha-d}}\frac{H^{\ominus}(KX_t)}{H^{\ominus}(K(Kx))}\mathbf{1}_{(t<\tau^{\ominus})}\right] \\
&= \mathbb{E}^{\oplus}_{Kx}\left[F(X_s, s \le t)\right].
\end{aligned}$$

This shows the first half of the claim. The second part of the claim is proved using the same technique and the details are omitted for brevity given how straightforward they are. $\square$

In the spirit of other cases of conditionings from an extreme boundary point (e.g. conditioning a Lévy process to avoid the origin, cf. [25], or to stay positive, cf. [9]), we can extend the definitions given in (A.18) and (A.19) by appealing to the Markov property of the excursion measures $\underline{\mathbb{N}}_x$ and $\overline{\mathbb{N}}_x$, $x \in \mathbb{S}^{d-1}$.

Theorem 4.4.2. *The processes $(X, \mathbb{P}^{\ominus})$ and $(X, \mathbb{P}^{\oplus})$ can be extended in a consistent way to include points of issue on $\mathbb{S}^{d-1}$. Specifically,*

$$\mathbb{P}^{\ominus}_x(X_t \in \mathrm{d}y) = H^{\ominus}(y)\underline{\mathbb{N}}_x\left(X^{\epsilon}(t) \in \mathrm{d}y,\, t < \varsigma\right), \qquad x \in \mathbb{S}^{d-1}, |y| > 1 \tag{A.20}$$

and similarly

$$\mathbb{P}^{\oplus}_x(X_t \in \mathrm{d}y) = H^{\oplus}(y)\overline{\mathbb{N}}_x\left(X^{\epsilon}(t) \in \mathrm{d}y,\, t < \varsigma\right), \qquad x \in \mathbb{S}^{d-1}, |y| < 1, \tag{A.21}$$

(specifically, the normalisation of the excursion measure is unity in both cases) where we have used the notation given in (A.14). As in Theorem 4.3.1, there is pathwise continuous entry.

Our objective is to pair up $(X, \mathbb{P}^{\vee})$, $(X, \mathbb{P}^{\ominus})$ and $(X, \mathbb{P}^{\wedge})$, $(X, \mathbb{P}^{\oplus})$ via Nagasawa's duality theorem for time reversal; cf [24]. To this end we need to introduce the notion of L-times.

Suppose that $Y = (Y_t, t \le \zeta)$ with probabilities $\mathtt{P}_x$, $x \in E$, is a regular Markov process on an open domain $E \subseteq \mathbb{R}^d$ (or more generally, a locally compact Hausdorff space with countable base), with cemetery state Δ and killing time $\zeta = \inf\{t > 0 : Y_t = \Delta\}$. Let us additionally write $\mathtt{P}_\nu = \int_E \nu(\mathrm{d}a)\mathtt{P}_a$, for any probability measure ν on the state space of Y.

Suppose that $\mathcal{G}$ is the σ-algebra generated by Y and write $\mathcal{G}(\mathtt{P}_\nu)$ for its completion by the null sets of $\mathtt{P}_\nu$. Moreover, write $\overline{\mathcal{G}} = \bigcap_\nu \mathcal{G}(\mathtt{P}_\nu)$, where the intersection is taken over all probability measures on the state space of Y, excluding the cemetery state. A finite random time $\mathtt{k}$ is called an L-time (generalized last exit time) if

(i) $\mathtt{k}$ is measurable in $\overline{\mathcal{G}}$, and $\mathtt{k} \le \zeta$ almost surely with respect to $\mathtt{P}_\nu$, for all ν,

(ii) $\{s < \mathtt{k}(\omega) - t\} = \{s < \mathtt{k}(\omega_t)\}$ for all $t, s \ge 0$, where ω_t is the Markov shift of ω to time t. The most important examples of L-times are killing times and last exit times.

Theorem 4.4.3. *In what follows, we work with the probability distribution*

$$\nu(\mathrm{d}a) := \frac{\sigma_1(\mathrm{d}a)|_{\mathsf{S}}}{\sigma_1(\mathsf{S})}, \qquad a \in \mathbb{R}^d, \tag{A.22}$$

if S *is closed and* $\sigma_1(\mathsf{S}) > 0$ *and, otherwise, if* $\mathsf{S} = \{\vartheta\}$, $\vartheta \in \mathbb{S}^{d-1}$, *we understand*

$$\nu(\mathrm{d}a) = \delta_{\{\vartheta\}}(\mathrm{d}a), \qquad a \in \mathbb{R}^d. \tag{A.23}$$

(i) For every L-time $\mathtt{k}$ *of* $(X, \mathbb{P}^{\ominus})$*, the time reversed process* $(X_{(\mathtt{k}-t)-}, t < \mathtt{k})$ *under* $\mathbb{P}^{\ominus}_{\nu}$ *is a time-homogeneous Markov process whose transition probabilities agree with those of* $(X, \mathbb{P}^{\vee})$*.*

(ii) Similarly, for every L-time $\mathtt{k}$ *of* $(X, \mathbb{P}^{\oplus})$*, the time reversed process* $(X_{(\mathtt{k}-t)-}, t < \mathtt{k})$ *under* $\mathbb{P}^{\oplus}_{\nu}$ *is a time-homogeneous Markov process whose transition probabilities agree with those of* $(X, \mathbb{P}^{\wedge})$*.*

Nagasawa's result, [24, Theorem 3.5], allows the definition of the time reversed process only for $t > 0$, however we can extend it for $t = 0$. Indeed, in (i), $\mathtt{k} < \zeta = \infty$ with probability $\mathbb{P}^{\ominus}$ one, and the time-reversal can include $t = 0$; in (ii), we may have $\mathtt{k} = \zeta < \infty$ with positive probability $\mathbb{P}^{\oplus}$, but in this case $X_{\zeta-} = 0$ with probability $\mathbb{P}^{\oplus}$ one, and therefore again $t = 0$ can be included in the time reversed process. That means, if the duality is true for $t > 0$, it must be true for all $t \geq 0$.

4.5 Proof of Theorem 4.2.1

We start by recalling two useful identities. In Theorem 1.1 in [20], the law of $X_{\underline{G}(\infty)}$ is given by

$$\mathbb{P}_x(X_{\underline{G}(\infty)} \in \mathrm{d}z) = c_{\alpha,d} \frac{(|x|^2 - |z|^2)^{\alpha/2}}{|z|^{\alpha}} |x - z|^{-d} \mathrm{d}z, \qquad |x| > |z| > 0, \tag{A.24}$$

where

$$c_{\alpha,d} = \pi^{-d/2} \frac{\Gamma(d/2)^2}{\Gamma((d-\alpha)/2)\,\Gamma(\alpha/2)}.$$

Similarly, from Corollary 1.1 of [20], it was also shown that

$$\mathbb{P}_x(X_{\overline{G}(\tau_1^{\ominus})} \in \mathrm{d}z, X_{\tau_1^{\ominus}} \in \mathrm{d}v) = C_{\alpha,d} \frac{(|z|^2 - |x|^2)^{\alpha/2}}{(|v|^2 - |z|^2)^{\alpha/2} |z - v|^d |z - x|^d} \mathrm{d}z \mathrm{d}v, \tag{A.25}$$

for $|x| < |z| < 1$ and $|v| > 1$, where

$$C_{\alpha,d} = \frac{\Gamma(d/2)^2}{\pi^d |\Gamma(-\alpha/2)| \Gamma(\alpha/2)}.$$

First take $x \in \bar{\mathbb{B}}_d^c$. Let $\tau_\beta^\oplus := \inf\{t > 0\colon |X_t| < \beta\}$ for any $\beta > 1$. For any $A \in \mathcal{F}_t$, define

$$\mathbb{P}_x^\vee(A, t < \tau_\beta^\oplus) = \lim_{\varepsilon \to 0} \mathbb{P}_x(A, t < \tau_\beta^\oplus | C_\varepsilon^\vee). \tag{A.26}$$

The Markov property gives us

$$\mathbb{P}_x(A, t < \tau_\beta^\oplus | C_\varepsilon^\vee) \quad = \mathbb{E}_x\left[\mathbf{1}_{\{A, t<\tau_\beta^\oplus\}} \frac{\mathbb{P}_{X_t}(C_\varepsilon^\vee)}{\mathbb{P}_x(C_\varepsilon^\vee)}\right]. \tag{A.27}$$

In order to prove the Theorem 4.2.1, it is enough to prove that, for all $\beta > 1$, (A.6) is true for sets of the form $A \cap \{t < \tau_\beta^\oplus\} \in \mathcal{F}_t$, in which case the full statement (A.6) follows by the Monotone Convergence Theorem as we take $\beta \downarrow 1$. Next note from (A.24) that

$$\begin{aligned}
\mathbb{P}_x(X_{\underline{G}(\infty)} \in A_\varepsilon) &= c_{\alpha,d} \int_{z \in A_\varepsilon} \frac{(|x|^2 - |z|^2)^{\alpha/2}}{|z|^\alpha} |x - z|^{-d} \mathrm{d}z \\
&= c'_{\alpha,d} \int_1^{1+\varepsilon} \int_{\mathbb{S}} \frac{(|x|^2 - r^2)^{\alpha/2}}{r^\alpha} |x - r\theta|^{-d} r^{d-1} \mathrm{d}r \sigma_1(\mathrm{d}\theta),
\end{aligned}$$

where $c'_{\alpha,d}$ is an unimportant constant.

Since $(|x|^2 - r^2)^{\alpha/2}|x - r\theta|^{-d}$ is continuous at $r = 1$ with fixed $|x| > 1$, for any $\delta > 0$, there exists $\varepsilon > 0$ such that for all $1 < r < 1 + \varepsilon$,

$$(1 - \delta)(|x|^2 - 1)^{\alpha/2}|x - \theta|^{-d} < (|x|^2 - r^2)^{\alpha/2}|x - r\theta|^{-d} < (1 + \delta)(|x|^2 - 1)^{\alpha/2}|x - \theta|^{-d}$$

and

$$\int_1^{1+\varepsilon} r^{d-\alpha+1} \mathrm{d}r = c\varepsilon^{d-\alpha} + o(\varepsilon^{d-\alpha}),$$

where c is an unimportant constant. Hence, we have

$$\lim_{\varepsilon \to 0} \varepsilon^{\alpha-d} \mathbb{P}_x(X_{\underline{G}(\infty)} \in A_\varepsilon) = c'_{\alpha,d} \int_{\mathbb{S}} (|x|^2 - 1)^{\alpha/2} |x - \theta|^{-d} \sigma_1(\mathrm{d}\theta),$$

where $c'_{\alpha,d}$ does not depend on x and may change from the previous one. Note, moreover, that for all fixed $\beta > 1$

$$\sup_{|x|>\beta} \frac{\int_{\mathbb{S}} (|x|^2 - 1)^{\alpha/2} |x - \theta|^{-d} \sigma_1(\mathrm{d}\theta)}{|x|^{\alpha-d}} < \infty. \tag{A.28}$$

We can both make use of the limit

$$\lim_{\varepsilon \to 0} \frac{\mathbb{P}_{X_t}(X_{\underline{G}(\infty)} \in A_\varepsilon)}{\mathbb{P}_x(X_{\underline{G}(\infty)} \in A_\varepsilon)} = \frac{\int_{\mathbb{S}} |\theta - X_t|^{-d} (|X_t|^2 - 1)^{\alpha/2} \sigma_1(\mathrm{d}\theta)}{\int_{\mathbb{S}} |\theta - x|^{-d} (|x|^2 - 1)^{\alpha/2} \sigma_1(\mathrm{d}\theta)}, \qquad t < \tau_\beta^\oplus. \tag{A.29}$$

as well as (A.28) and the Dominated Convergence Theorem to ensure the limit may be passed through the expectation in (A.27) to give (A.6) on $\{t < \tau_\beta^\oplus\}$, thus giving the desired result.

Next we look at the proof of (A.7). In a similar way, it is enough to work with sets of the form $A \cap \{t < \tau_\beta^\ominus\} \in \mathcal{F}_t$, with $\beta < 1$. From (A.25), recalling $C_\varepsilon^\wedge := \{X_{\overline{G}(\tau_1^\ominus-)} \in B_\varepsilon\}$, we have

$$\begin{aligned}\mathbb{P}_x(C_\varepsilon^\wedge) &= \mathbb{P}_x(X_{\overline{G}(\tau_1^\ominus-)} \in B_\varepsilon)\\ &= C_{\alpha,d}\int_{z\in B_\varepsilon}\int_{v\in\mathbb{B}_d^c}\frac{(|z|^2-|x|^2)^{\alpha/2}}{(|v|^2-|z|^2)^{\alpha/2}|z-v|^d|z-x|^d}\mathrm{d}z\mathrm{d}v\\ &= C'_{\alpha,d}\int_{z\in B_\varepsilon}\frac{(|z|^2-|x|^2)^{\alpha/2}}{|z-x|^d}\mathrm{d}z\int_1^\infty\frac{r^{d-1}\mathrm{d}r}{(r^2-|z|^2)^{\alpha/2}}\int_{\mathbb{S}^{d-1}(0,r)}\frac{1}{|z-\theta|^d}\sigma_r(\mathrm{d}\theta),\end{aligned}\tag{A.30}$$

where $\sigma_r(\mathrm{d}\theta)$ is the surface measure on $\mathbb{S}^{d-1}(0,r)$, the sphere centred at 0 of radius r, normalised to have unit mass and $C'_{\alpha,d}$ is henceforth a constant whose value may change from line to line, which depends only on α and d. The Poisson formula (giving the probability that a d-dimensional Brownian motion issued from z (with $|z| < 1$) will hit the sphere $\mathbb{S}^{d-1}(0,r)$) tells us that

$$\int_{\mathbb{S}^{d-1}(0,r)}\frac{r^{d-2}(r^2-|z|^2)}{|z-\theta|^d}\sigma_r(\mathrm{d}\theta) = 1, \qquad |z| < 1 < r, \tag{A.31}$$

see for example Remark III.2.5 in [18]. Putting (A.31) in (A.30) gives us

$$\begin{aligned}\mathbb{P}_x(C_\varepsilon^\wedge) &= C'_{\alpha,d}\int_{z\in B_\varepsilon}\frac{(|z|^2-|x|^2)^{\alpha/2}}{|z-x|^d}\mathrm{d}z\int_1^\infty\frac{r^{d-1}}{(r^2-|z|^2)^{\alpha/2}}\frac{1}{r^{d-2}(r^2-|z|^2)}\mathrm{d}r\\ &= C'_{\alpha,d}\int_{z\in B_\varepsilon}\frac{(|z|^2-|x|^2)^{\alpha/2}}{|z-x|^d}\frac{1}{(1-|z|^2)^{\alpha/2}}\mathrm{d}z\\ &= C'_{\alpha,d}\int_{1-\varepsilon}^1\int_{\mathsf{S}}\frac{(u^2-|x|^2)^{\alpha/2}}{(1-u^2)^{\alpha/2}|u\theta-x|^d}u^{d-1}\mathrm{d}u\,\sigma_1(\mathrm{d}\theta).\end{aligned}$$

Since $(u^2-|x|^2)^{\alpha/2}|x-u\theta|^{-d}$ is continuous at $u = 1$ with fixed $0 < |x| < 1$, for any $\delta > 0$, there exists $\varepsilon > 0$ such that for all $1-\varepsilon < u < 1$,

$$(1-\delta)(1-|x|^2)^{\alpha/2}|x-\theta|^{-d} < (u^2-|x|^2)^{\alpha/2}|x-u\theta|^{-d} < (1+\delta)(1-|x|^2)^{\alpha/2}|x-\theta|^{-d}$$

and

$$\int_{1-\varepsilon}^1\frac{u^{d-1}}{(1-u^2)^{\alpha/2}}\mathrm{d}u = \int_0^\varepsilon\frac{(1-r)^{d-1}}{r^{\alpha/2}(2-r)^{\alpha/2}}\mathrm{d}r = c\varepsilon^{1-\alpha/2} + o(\varepsilon^{1-\alpha/2}),$$

for an unimportant constant $c > 0$.

It is now clear that

$$\lim_{\varepsilon\to 0}\varepsilon^{\alpha/2-1}\mathbb{P}_x(X_{\overline{G}(\tau_1^\ominus-)} \in B_\varepsilon) = C'_{\alpha,d}\int_{\mathsf{S}}(1-|x|^2)^{\alpha/2}|x-\theta|^{-d}\sigma_1(\mathrm{d}\theta).$$

Finally, we get again

$$\lim_{\varepsilon\to 0}\frac{\mathbb{P}_{X_t}(X_{\overline{G}(\tau_1^\ominus-)} \in B_\varepsilon)}{\mathbb{P}_x(X_{\overline{G}(\tau_1^\ominus-)} \in B_\varepsilon)} = \frac{\int_{\mathsf{S}}|\theta-X_t|^{-d}(1-|X_t|^2)^{\alpha/2}\sigma_1(\mathrm{d}\theta)}{\int_{\mathsf{S}}|\theta-x|^{-d}(1-|x|^2)^{\alpha/2}\sigma_1(\mathrm{d}\theta)}, \qquad t < \tau_\beta^\ominus. \tag{A.32}$$

and we can proceed as in (A.26), noting the application of dominated convergence and that for every fixed $\beta < 1$,

$$\sup_{|x|<\beta} \frac{\int_{\mathsf{S}} (|x|^2-1)^{\alpha/2}|x-\theta|^{-d}\sigma_1(\mathrm{d}\theta)}{|x|^{\alpha-d}} < \infty.$$

In a similar manner, when $\mathsf{S} = \{\vartheta\}$, we work with sets of the form $A \cap \{t < \tau_\beta^\oplus\} \in \mathcal{F}_t$ or $A \cap \{t < \tau_{\beta'}^\ominus\} \in \mathcal{F}_t$, with $\beta' < 1 < \beta$, respectively. In this case, $A_\varepsilon = \{r\phi\colon r \in (1, 1+\varepsilon), \phi \in \mathbb{S}^{d-1}, |\phi - \vartheta| < \varepsilon\}$ and $B_\varepsilon = \{r\phi\colon r \in (1-\varepsilon, 1), \phi \in \mathbb{S}^{d-1}, |\phi - \vartheta| < \varepsilon\}$, thus it is clear by similar analysis that

$$\lim_{\varepsilon\to 0} \frac{\mathbb{P}_{X_t}(X_{\underline{G}(\infty)} \in A_\varepsilon)}{\mathbb{P}_x(X_{\underline{G}(\infty)} \in A_\varepsilon)} = \lim_{\varepsilon\to 0} \frac{\mathbb{P}_{X_t}(X_{\overline{G}(\tau_1^\ominus-)} \in B_\varepsilon)}{\mathbb{P}_x(X_{\overline{G}(\tau_1^\ominus-)} \in B_\varepsilon)} = \frac{|\theta - X_t|^{-d}||X_t|^2-1|^{\alpha/2}}{|\theta - x|^{-d}||x|^2-1|^{\alpha/2}}. \tag{A.33}$$

The rest of the proof is otherwise a minor adjustment of what we have seen previously, now taking account of the continuity of $(u,\theta) \mapsto |u^2 - |x|^2|^{\alpha/2}|x - u\theta|^{-d}$ as well as the fact that $\sup_{|x|>\beta}((|x|^2-1)^{\alpha/2}|x-\theta|^{-d})/|x|^{\alpha-d} < \infty$ and $\sup_{|x|<\beta'}((1-|x|^2)^{\alpha/2}|x-\theta|^{-d})/|x|^{\alpha-d} < \infty$, in order to use dominated convergence. $\square$

4.5.1 Proof of Proposition 4.2.1

To calculate the hitting distribution, recall that $\mathbb{P}^\vee$ is the law of a stable process conditioned to attract to S continuously from the outside, and $A'_\varepsilon = \{r\theta\colon r \in (1, 1+\varepsilon), \theta \in \mathsf{S}'\}$, that is the restriction of A_ε from the set S to its subset $\mathsf{S}' \subset \mathsf{S}$. Then, due to Theorem 1.3 in [20], we have $\mathbb{P}^\vee_x(X_{\mathtt{k}-} \in \mathsf{S}') = \lim_{\varepsilon\to 0} \mathbb{P}_x(X_{\underline{G}(\infty)} \in A'_\varepsilon | C^\vee_\varepsilon)$. Then

$$\begin{aligned}
\lim_{\varepsilon\to 0} \mathbb{P}_x(X_{\underline{G}(\infty)} \in A'_\varepsilon | C^\vee_\varepsilon) &= \lim_{\varepsilon\to 0} \mathbb{P}_x(X_{\underline{G}(\infty)} \in A'_\varepsilon | X_{\underline{G}(\infty)} \in A_\varepsilon) \\
&= \lim_{\varepsilon\to 0} \frac{\mathbb{P}_x(X_{\underline{G}(\infty)} \in A'_\varepsilon)}{\mathbb{P}_x(X_{\underline{G}(\infty)} \in A_\varepsilon)} \\
&= \frac{\int_{\mathsf{S}'} |\theta - x|^{-d}\sigma_1(\mathrm{d}\theta)}{\int_{\mathsf{S}} |\theta - x|^{-d}\sigma_1(\mathrm{d}\theta)},
\end{aligned} \tag{A.34}$$

which concludes the statement in the Proposition 4.2.1 for the case when X is issued from outside. Similar computations give the result when X is issued from inside $\mathbb{B}_d$. $\square$

4.6 Proof of Theorems 4.3.1 and 4.4.2

Proof of Theorem 4.3.1: Let us restrict our attention to the extension of $(X, \mathbb{P}^\vee)$ to include $\mathbb{S}^{d-1} \setminus \mathsf{S}$. We need to prove that the proposed definition of $\mathbb{P}^\vee_\theta$, for any $\theta \in \mathbb{S}^{d-1} \setminus \mathsf{S}$, is is well defined as a finite entity, conforms to the correct normalisation to represent a probability measure and is consistent with the definition of $(X, \mathbb{P}^\vee)$ given in Theorem 4.2.1 on $\bar{\mathbb{B}}^c_d$, as well as offering continuous entry from the boundary $\mathbb{S}^{d-1} \setminus \mathsf{S}$.

We start with finiteness. To this end, we must show that, for $\theta \in \mathbb{S}^{d-1} \setminus \mathsf{S}$

$$\underline{\mathbb{N}}_\theta(H_{\mathsf{S}}(X^\epsilon(t)); t < \varsigma) < \infty, \qquad t > 0. \tag{A.35}$$

Noting from (A.5) that $H_{\mathsf{S}}(x) \leq H_{\mathbb{S}^{d-1}}(x) = |x|^{\alpha-d}(1-|x|^{-2})^{\frac{\alpha}{2}-1}$, which tends to 0 as $|x| \to \infty$, it suffices to prove that, for any $R > 1$,

$$\underline{\mathbb{N}}_\theta((|X^\epsilon(t)| \geq R, t < \varsigma) + \underline{\mathbb{N}}_\theta(H_{\mathsf{S}}(X^\epsilon(t)); |X^\epsilon(t)| < R, t < \varsigma) < \infty, \qquad t > 0. \tag{A.36}$$

Abusing notation and using $\tau_R^\ominus = \inf\{t > 0 : |X_t^\epsilon| > R\}$ in the canonical sense,

$$\underline{\mathbb{N}}_\theta((|X^\epsilon(t)| \geq R, t < \varsigma) \leq \underline{\mathbb{N}}_\theta(t < \varsigma \wedge \tau_R^\ominus) \leq \underline{n}(t < \kappa_{\log R} \wedge \zeta) < \infty \tag{A.37}$$

where $\kappa_{\log R} = \inf\{t > 0 : \epsilon(t) > \log R\}$ and $\underline{n}$ is the excursion measure of $\xi - \underline{\xi}$. (The fact that the second expression in (A.37) is finite is a well known fact from the theory of Lévy processes; otherwise there would be an infinite rate of having arbitrary large excursions, which occurs with probability zero.)

Our objective now will be to show that in fact the resolvent

$$I_{\mathsf{S}}(\theta) := \int_0^\infty \underline{\mathbb{N}}_\theta(H_{\mathsf{S}}(X^\epsilon(t)); |X^\epsilon(t)| < R, t < \varsigma)\mathrm{d}t < \infty, \tag{A.38}$$

which ensures that (A.35) is finite for Lebesgue almost all $t > 0$ and hence, thanks to stochastic continuity of the excursion measure, for all $t > 0$.

To prove (A.38) consider $|y| \geq 1$ and $\theta \in \mathbb{S}^{d-1} \setminus \mathsf{S}$, we can appeal to Proposition 5.2 of [20], which identifies, for $x \in \mathbb{R}^d \backslash \{0\}$, and continuous $g : \mathbb{R}^d \mapsto \mathbb{R}$ whose support is compactly embedded in the exterior of the ball of radius $|x|$,

$$\underline{\mathbb{N}}_{\arg(x)}\left(\int_0^\zeta g(|x|\mathrm{e}^{\epsilon(u)}\Theta^\epsilon(u))\mathrm{d}u\right) = \frac{\Gamma((d-\alpha)/2)}{2^\alpha \pi^{d/2}\Gamma(\alpha/2)} \int_{|x|<|z|} g(z)\frac{(|z|^2-|x|^2)^{\alpha/2}}{|z|^\alpha |x-z|^d}\mathrm{d}z. \tag{A.39}$$

This gives us

$$\begin{aligned}
\rho^\vee(\theta, \mathrm{d}y) &:= \int_0^\infty \mathbb{P}_\theta^\vee(X_t \in \mathrm{d}y, t < \mathtt{k})\mathrm{d}t \\
&= \frac{\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2+1)} \frac{H_{\mathsf{S}}(y)}{h(\theta)} \int_0^\infty \underline{\mathbb{N}}_\theta\left(X^\epsilon(t) \in \mathrm{d}y,\, t < \varsigma\right)\mathrm{d}t \\
&= \frac{\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2+1)} \frac{H_{\mathsf{S}}(y)}{h(\theta)}|y|^\alpha \int_0^\infty \underline{\mathbb{N}}_\theta\left(\mathrm{e}^{\epsilon(u)}\Theta^\epsilon(u) \in \mathrm{d}y,\, u < \zeta\right)\mathrm{d}u \\
&= \frac{\Gamma(d/2)}{2^\alpha\pi^{d/2}\Gamma(\alpha/2)\Gamma(\alpha/2+1)} \frac{||y|^2-1|^{\alpha/2}}{|\theta-y|^d}\frac{H_{\mathsf{S}}(y)}{h(\theta)}\mathrm{d}y \\
&= \frac{\Gamma(d/2)}{2^\alpha\pi^{d/2}\Gamma(\alpha/2)\Gamma(\alpha/2+1)} \frac{H_{\{\theta\}}(y)H_{\mathsf{S}}(y)}{h(\theta)}\mathrm{d}y
\end{aligned}$$

where we recall $h(\theta) = \int_{\mathsf{S}} |\theta - \vartheta|^{-d}\sigma_1(\mathrm{d}\vartheta)$, the representation of X^ϵ is given in (A.14) and the fact that $\mathrm{e}^{\alpha\epsilon(\varphi(t))}\mathrm{d}\varphi(t) = |X_t^\epsilon|^\alpha \mathrm{d}\varphi(t) = \mathrm{d}t$ on $t < \varsigma$.

It now follows that, up to a multiplicative constant C (which in the following calculations will play the role of different constants that may change from line to line)

$$
\begin{aligned}
I_{\mathsf{S}}(\theta) &= \int_{1<|y|<R} \rho^{\vee}(\theta, \mathrm{d}y) \\
&= \frac{C}{h(\theta)} \int_{1<|y|<R} H_{\{\theta\}}(y) H_{\mathsf{S}}(y) \mathrm{d}y \\
&= \frac{C}{h(\theta)} \int_{1<|y|<R} \int_{\phi \in \mathsf{S}} \frac{||y|^2-1|^{\alpha}}{|\theta-y|^d |\phi-y|^d} \mathrm{d}y \sigma_1(\mathrm{d}\phi).
\end{aligned} \tag{A.40}
$$

Since $\mathbb{S}^{d-1} \backslash \mathsf{S}$ is open, it is easy to see that we can choose ε small enough such that, for $\theta \in \mathbb{S}^{d-1} \backslash \mathsf{S}$,

$$
\begin{aligned}
\{y \in \mathbb{R}^d : |y| > 1\} = \{y \in \mathbb{R}^d : |y| > 1 \text{ and } |y-\theta| > \varepsilon\} \\
\cup \{y \in \mathbb{R}^d : |y| > 1 \text{ and } |y-\phi| > \varepsilon, \text{ for all } \phi \in \mathsf{S}\},
\end{aligned} \tag{A.41}
$$

such that

$$
\begin{aligned}
\{y \in \mathbb{R}^d : |y| > 1 \text{ and } |y-\theta| \leq \varepsilon\} \\
\cap \{y \in \mathbb{R}^d : |y| > 1 \text{ and } |y-\phi| \leq \varepsilon, \text{ for some } \phi \in \mathsf{S}\} = \emptyset.
\end{aligned}
$$

Making use of (A.4), (A.5) and (A.1), and that, for $r>1$, ${}_2F_1(d/2, 1; d/2, r^{-2}) = (1-r^{-2})^{-1}$, allowing C to again play the role of a strictly positive constant that may change from line to line, we have, for $\theta \notin \mathsf{S}$,

$$
\begin{aligned}
I_{\mathsf{S}}(\theta) &= \frac{C}{h(\theta)} \int_{1<|y|<R} \int_{\phi \in \mathsf{S}} \frac{||y|^2-1|^{\alpha}}{|\theta-y|^d |\phi-y|^d} \mathrm{d}y \sigma_1(\mathrm{d}\phi) \\
&\leq \frac{C}{h(\theta)} \int_{1<|y|<R, |\theta-y|\geq\varepsilon} \int_{\phi \in \mathsf{S}} \frac{||y|^2-1|^{\alpha}}{|\theta-y|^d |\phi-y|^d} \mathrm{d}y \sigma_1(\mathrm{d}\phi) \\
&\qquad + \frac{C}{h(\theta)} \int_{1<|y|<R, |\phi-y|\geq\varepsilon} \int_{\phi \in \mathsf{S}} \frac{||y|^2-1|^{\alpha}}{|\theta-y|^d |\phi-y|^d} \mathrm{d}y \sigma_1(\mathrm{d}\phi) \\
&\leq \varepsilon^{-d} \frac{C}{h(\theta)} \Big(\int_{1<|y|<R} \int_{\phi \in \mathsf{S}} \frac{||y|^2-1|^{\alpha}}{|\phi-y|^d} \mathrm{d}y \sigma_1(\mathrm{d}\phi) + \int_{1<|y|<R} \int_{\phi \in \mathsf{S}} \frac{||y|^2-1|^{\alpha}}{|\theta-y|^d} \mathrm{d}y \sigma_1(\mathrm{d}\phi) \Big) \\
&\leq \varepsilon^{-d} \frac{C}{h(\theta)} \Big(\int_{1<|y|<R} \int_{\phi \in \mathbb{S}^{d-1}} \frac{||y|^2-1|^{\alpha}}{|\phi-y|^d} \mathrm{d}y \sigma_1(\mathrm{d}\phi) + \sigma_1(\mathsf{S}) \int_{1<|y|<R} \frac{||y|^2-1|^{\alpha}}{|\theta-y|^d} \mathrm{d}y \Big) \\
&= \varepsilon^{-d} \frac{C}{h(\theta)} \Big(\int_{1<|y|<R} ||y|^2-1|^{\alpha} |y|^{\alpha-d} (1-|y|^{-2})^{\frac{\alpha}{2}-1} \mathrm{d}y \\
&\qquad + \int_1^R \int_0^{\pi} \frac{r^{d-1}(r^2-1)^{\alpha}(\sin\vartheta)^{d-2}}{(r^2-2r\cos(\vartheta)+1)^{d/2}} \mathrm{d}r\mathrm{d}\vartheta \Big) \\
&= \varepsilon^{-d} \frac{C}{h(\theta)} \left(\int_1^R (r^2-1)^{\frac{3\alpha}{2}-1} r\mathrm{d}r + \int_1^R (r^2-1)^{\alpha-1} r\mathrm{d}r \right) \\
&= \varepsilon^{-d} \frac{C}{h(\theta)} \left(\int_1^{R^2} (u-1)^{\frac{3\alpha}{2}-1} \mathrm{d}u + \int_1^{R^2} (u-1)^{\alpha-1} \mathrm{d}u \right) < \infty.
\end{aligned}
$$

Now let us turn to the issue of consistency. Recall that $(\Lambda, \underline{\mathbb{N}}_\cdot)$ is an exit system for the process (ξ, Θ). In particular, under any measure $\underline{\mathbb{N}}_\theta$ the process $((\epsilon(s), \Theta^\epsilon(s)), s < \zeta)$ is a strong Markov process with the same semigroup as (ξ, Θ) killed at its first hitting time of $(-\infty, 0] \times \mathbb{S}^{d-1}$, see [23, Theorem 6.3]. As a consequence, for $\theta \in \mathbb{S}^{d-1} \setminus \mathsf{S}$,

$$
\begin{aligned}
\mathbb{E}^\vee_\theta[g(X_{t+s})] &= \frac{C}{h(\theta)} \underline{\mathbb{N}}_\theta (H_\mathsf{S}(X^\epsilon_{t+s}) g(X^\epsilon_{t+s}) \mathbf{1}_{(s+t<\varsigma)}) \\
&= \frac{C}{h(\theta)} \underline{\mathbb{N}}_\theta \left(H_\mathsf{S}(X^\epsilon_t) \mathbf{1}_{(t<\varsigma)} \mathbb{E}_{X^\epsilon_t} \left[\frac{H_\mathsf{S}(X_s)}{H_\mathsf{S}(X^\epsilon_t)} g(X_s) \mathbf{1}_{(s<\tau_1^\ominus)} \right] \right) \\
&= \frac{C}{h(\theta)} \underline{\mathbb{N}}_\theta \left(H_\mathsf{S}(X^\epsilon_t) \mathbf{1}_{(t<\varsigma)} \mathbb{E}^\vee_{X^\epsilon_t}[g(X_s)] \right) \\
&= \mathbb{E}^\vee_\theta \left[H_\mathsf{S}(X^\epsilon_t) \mathbf{1}_{(t<\varsigma)} \mathbb{E}^\vee_{X^\epsilon_t}[g(X_s)] \right], \qquad (A.42)
\end{aligned}
$$

where $C = \Gamma(d/2)/\Gamma(\alpha/2+1)\Gamma((d-\alpha)/2)$. Hence, using the notation $\mathcal{P}^\vee_t[g](x) := \mathbb{E}^\vee_x[g(X_t)]$, we have $\mathcal{P}^\vee_{t+s}[g](x) = \mathcal{P}^\vee_t[\mathcal{P}^\vee_s[g]](x)$ for any $x \in \mathbb{R}^d \setminus (\mathbb{B}_d \cup \mathsf{S})$, and the required consistency follows.

To demonstrate the consistent choice of normalisation in our definition of $\mathbb{P}^\vee$, we will reconsider a different derivation of the resolvent $\rho^\vee$. To this end, suppose that $x \in \bar{\mathbb{B}}^c$ and we can similarly consider the resolvent of $(X, \mathbb{P}^\vee_x)$. This calculation can be developed using the nature of the Doob h-transform (A.6) and Theorem III.3.4 in [18] and takes the form

$$
\rho^\vee(x, \mathrm{d}y) = \frac{H_\mathsf{S}(y)}{H_\mathsf{S}(x)} \rho^\oplus(x, \mathrm{d}y), \qquad |x|, |y| > 1, \qquad (A.43)
$$

where

$$
\rho^\oplus(x, \mathrm{d}y) = \frac{\Gamma(d/2)}{2^\alpha \pi^{d/2} \Gamma(\alpha/2)^2} |x-y|^{\alpha-d} \int_0^{\zeta^\oplus(x,y)} (u+1)^{-d/2} u^{\alpha/2-1} \mathrm{d}u \, \mathrm{d}y \qquad (A.44)
$$

and $\zeta^\oplus(x,y) = (|x|^2-1)(|y|^2-1)/|x-y|^2$. To show continuity as $x \to \theta \in \mathbb{S}^{d-1} \setminus \mathsf{S}$, and hence that the choice of normalisation in (A.12) is correct, we note that, as $r \to 1$,

$$
\begin{aligned}
\frac{H_\mathsf{S}(y)}{H_\mathsf{S}(x)} \rho^\oplus(r\theta, \mathrm{d}y) &\sim \frac{\Gamma(d/2) |r\theta - y|^{\alpha-d} H_\mathsf{S}(y)}{2^\alpha \pi^{d/2} \Gamma(\alpha/2)^2 h(\theta)} \\
&\qquad \times \frac{2r(|y|^2-1)|r\theta - y|^{-2} \zeta(r\theta, y)^{\alpha/2-1} (1 + \zeta(r\theta, y))^{-d/2}}{2r(\alpha/2)(r^2-1)^{\alpha/2-1}} \mathrm{d}y \\
&\sim \frac{\Gamma(d/2)(|y|^2-1)^{\alpha/2} |\theta - y|^{-d} H_\mathsf{S}(y)}{2^\alpha \pi^{d/2} \Gamma(\alpha/2) \Gamma(\alpha/2+1) h(\theta)} \mathrm{d}y \\
&= \frac{\Gamma(d/2)}{2^\alpha \pi^{d/2} \Gamma(\alpha/2) \Gamma(\alpha/2+1)} \frac{H_{\{\theta\}}(y) H_\mathsf{S}(y)}{h(\theta)} \mathrm{d}y \qquad (A.45)
\end{aligned}
$$

Now, we need to show that $\mathbb{P}^\vee_\theta(X_{0+} = \theta) = 1$ for any $\theta \in \mathbb{S}^{d-1} \setminus \mathsf{S}$. Since $\lim_{t \downarrow 0} \varphi(t) = 0$, it suffices to show that

$$
\mathbb{P}^\vee_\theta(X_0 \neq \theta) = \underline{\mathbb{N}}_\theta \left(\{ \lim_{t\downarrow 0} \epsilon(t) = 0, \, \lim_{t \downarrow 0} \Theta^\epsilon(t) = \theta \}^c \right) = 0. \qquad (A.46)
$$

Let us first observe ϵ is an excursion of ξ from its running minimum and ξ is a hypergeometric Lévy process with unbounded variation, hence 0 is regular for $(0, \infty)$, that is

$$\mathbf{P}_{0,\theta}(\tau_0^+ = 0) = 1, \qquad \theta \in \mathbb{S}^{d-1},$$

where $\tau_0^+ = \inf\{t > 0 : \xi_t > 0\}$. Classical excursion theory for Lévy processes implies that the excursions of ξ from its infimum begin continuously. Thanks to isotropy, this is equivalent to saying

$$\underline{\mathbb{N}}_\theta\left(\{\lim_{t\downarrow 0} \epsilon(t) = 0\}^c\right) = 0. \tag{A.47}$$

Since the jump measure of X in radial form is

$$\Pi(\mathrm{d}r, \mathrm{d}\theta) = \frac{1}{r^{1+\alpha}}\sigma_1(\mathrm{d}\theta)\mathrm{d}r, \qquad r > 0, \theta \in \mathbb{S}^{d-1},$$

as a consequence, the process (ξ, Θ) has the property that both the modulator and the ordinate must jump simultaneously (the precise jump rate was explored in [18]). If it were the case that $\underline{\mathbb{N}}_\theta\left(\{\lim_{t\downarrow 0} \Theta^\epsilon(t) = \theta\}^c\right) > 0$ (and hence for all $\theta \in \mathbb{S}^{d-1}$ by rotational symmetry), this would be tantamount to a discontinuity in Θ but not in ξ, which is a contradiction since $((\epsilon(s), \Theta^\epsilon(s)), s < \zeta)$ under $\underline{\mathbb{N}}_\theta$ has the same semigroup as the isotropic process (ξ, Θ) killed at its first hitting time of $(-\infty, 0] \times \mathbb{S}^{d-1}$. The requirement (A.46) now follows. This completes the proof of Theorem 4.3.1 as far as $\mathbb{P}^\vee$ is concerned.

The proof of Theorem 4.3.1 for $(X, \mathbb{P}^\wedge)$ is essentially the same as soon as we have an analogous identity for (A.39), but for $\overline{\mathbb{N}}_\theta$. Unfortunately this does not seem to be available in the literature, and so we spend a little time developing it here. However the remaining details of the proof of Theorem 4.3.1 we leave to the reader.

The main idea behind the derivation of an analogue to (A.39) for $\overline{\mathbb{N}}_\theta$ lies with the use of the Riesz–Bogdan-Żak transform in Theorem 4.4.1. Let us consider a variant of the radial excursion process which is based on the MAP $(\overline{\xi} - \xi, \Theta)$, that is associated to X but now under the change of measure (A.16). The reader will recall that the probabilities $\mathbb{P}^\circ = (\mathbb{P}^\circ_x, x \neq 0)$ correspond to conditioning the process X to be continuously absorbed at the origin. It turns out that if (ξ, Θ), with probabilities $\mathbf{P}^\circ = (\mathbf{P}^\circ_{x,\theta}, x \in \mathbb{R}, \theta \in \mathbb{S}^{d-1})$, is the MAP whose Lamperti transform gives $(X, \mathbb{P}^\circ)$, then $(-\xi, \Theta)$, is the MAP whose Lamperti transform gives $(X, \mathbb{P})$; see Theorem 1.3.6 and Corollary 1.3.17 of [18].

In the spirit of (A.11) we can write down the exit system for the radial excursion process of $(\overline{\xi} - \xi, \Theta)$ from $\{0\} \times \mathbb{S}^{d-1}$ under $\mathbb{P}^\circ$. Suppose that ℓ°, and $(\overline{\mathbb{N}}^\circ_\theta, \theta \in \mathbb{S}^{d-1}))$ denote the associated local time and system of excursion measures. As with excursion theory from the radial minimum of X, isotropy allows us to conclude that we may choose ℓ° to be the local time at 0 of $\overline{\xi} - \xi$, and that ξ (without its modulator Θ) is necessarily a Lévy process under $\mathbf{P}$. Since $\lim_{t\to\infty} \xi_t = -\infty$ under $\mathbf{P}^\circ$, we can also appeal to isotropy again to normalise ℓ° in such a way that $\overline{\mathbb{N}}^\circ(\zeta = \infty) = 1$.

With this set up we can follow the reasoning in [20] and deduce that, for positive, bounded and measurable g on $\mathbb{R}^d$,

$$\overline{\mathbb{N}}^{\circ}_{\arg(x)}\left(\int_0^{\zeta} g(\mathrm{e}^{\epsilon(s)}\Theta^{\epsilon}(s))\mathrm{e}^{\alpha\epsilon(s)}\mathrm{d}s\right) = \lim_{|x|\uparrow 1}\frac{\mathbb{E}^{\circ}_x\left(\int_0^{\tau_1^{\ominus}} g(X_s)\mathrm{d}s\right)}{\mathbb{P}^{\circ}_x(\tau_1^{\ominus}=\infty)}, \tag{A.48}$$

where we recall that $\tau_1^{\ominus} = \inf\{t > 0 : |X_t| > 1\}$. Note that the choice of normalisation of ℓ° is implicit in the aforementioned limiting equality. Appealing to numerous calculations involving the Riesz–Bogdan–Żak transformation e.g. in [18] or indeed [19], we can rewrite the limit

$$\lim_{|x|\uparrow 1}\frac{\mathbb{E}^{\circ}_x\left(\int_0^{\tau_1^{\ominus}} g(X_s)\mathrm{d}s\right)}{\mathbb{P}^{\circ}_x(\tau_1^{\ominus}=\infty)} = \lim_{|x|\uparrow 1}\frac{\mathbb{E}_{Kx}\left(\int_0^{\tau_1^{\oplus}} g(KX_s)|X_s|^{-2\alpha}\mathrm{d}s\right)}{\mathbb{P}_{Kx}(\tau_1^{\oplus}=\infty),}$$

where $Kx = x/|x|^2$. Appealing to the identities provided in (A.17) and (A.44), the limiting ratio is computable directly giving us in (A.48)

$$\overline{\mathbb{N}}^{\circ}_{\arg(x)}\left(\int_0^{\zeta} g(\mathrm{e}^{\epsilon(s)}\Theta^{\epsilon}(s))\mathrm{e}^{\alpha\epsilon(s)}\mathrm{d}s\right) = \int_{|z|>1} g(Kz)|z|^{-2\alpha}\frac{(|z|^2-1)^{\alpha/2}}{|\arg(x)-z|^d}\mathrm{d}z.$$

An easy change of variables $y = Kz$, noting the classical analytical facts that $\mathrm{d}z = |y|^{-2d}\mathrm{d}y$ and $|\theta - Ky| = |\theta - y|/|y|$, for $\theta \in \mathbb{S}^{d-1}$,

$$\overline{\mathbb{N}}^{\circ}_{\theta}\left(\int_0^{\zeta} g(\mathrm{e}^{\epsilon(s)}\Theta^{\epsilon}(s))\mathrm{e}^{\alpha\epsilon(s)}\mathrm{d}s\right) = \int_{|y|<1} g(y)|y|^{\alpha-d}\frac{(1-|y|^2)^{\alpha/2}}{|\theta-y|^d}\mathrm{d}z. \tag{A.49}$$

As noted in [18], the change of measure (A.16) when understood as a change of measure affecting (ξ, Θ), is equivalent to the martingale change of measure,

$$\left.\frac{\mathrm{d}\mathbf{P}^{\circ}_{x,\theta}}{\mathrm{d}\mathbf{P}_{x,\theta}}\right|_{\sigma((\xi_s,\Theta_s),s\le t)} = \mathrm{e}^{(\alpha-d)(\xi_t-x)}. \tag{A.50}$$

We can use this to compare the left-hand side of (A.49) with an analogous object albeit for $\overline{\mathbb{N}}_{\theta}$, the excursion measure of $(\overline{\xi}-\xi, \Theta)$ from $\{0\}\times\mathbb{S}^{d-1}$ under $\mathbf{P}$, by studying the effect of (A.50) on the exit formula (A.11). It is straightforward to show that, for $\theta \in \mathbb{S}^{d-1}$ and positive, bounded and measurable g,

$$\overline{\mathbb{N}}^{\circ}_{\theta}\left(\int_0^{\zeta} g(\mathrm{e}^{\epsilon(s)}\Theta^{\epsilon}(s))\mathrm{d}s\right) = \overline{\mathbb{N}}_{\theta}\left(\int_0^{\zeta} g(\mathrm{e}^{\epsilon(s)}\Theta^{\epsilon}(s))\mathrm{e}^{(\alpha-d)\epsilon(s)}\mathrm{d}s\right).$$

Note that the normalisation of local time for (ξ, Θ) under $\mathbf{P}$ is, in effect, chosen by the above equality. It follows that

$$\overline{\mathbb{N}}_{\theta}\left(\int_0^{\zeta} g(\mathrm{e}^{\epsilon(s)}\Theta^{\epsilon}(s))\mathrm{d}s\right) = \int_{|y|<1} g(y)\frac{(1-|y|^2)^{\alpha/2}}{|y|^{\alpha}|\theta-y|^d}\mathrm{d}y. \tag{A.51}$$

The reader will note that, aside from the domain of integration on the right-hand side, this agrees with (A.39).

With (A.51) in hand, as alluded to above, we can now leave the reader to verify that the proof of Theorem 4.3.1 for $(X, \mathbb{P}^{\wedge})$ is essentially verbatim the same as for $(X, \mathbb{P}^{\vee})$. $\square$

Proof of Theorem 4.4.2: Given the proof of Theorem 4.3.1 above, we refrain from giving the proof of Theorem 4.4.2, noting only that it is a variant of the arguments given there. The details are, once again, left to the reader. We additionally note that e.g. in this case of $\mathbb{P}^{\ominus}$, the excursion may begin anywhere on $\mathbb{S}^{d-1}$ and, when proving that e.g. $\underline{\mathbb{N}}_\theta(H^{\ominus}(X^\epsilon(t)); t < \varsigma) < \infty$, it is much easier to show that the analogue of (A.38) is finite without needing to split space up as in (A.41). $\square$

4.7 Proof of Theorem 4.4.3

Recall the notation for a general Markov process $(Y, \mathtt{P})$ on E preceding the statement of Theorem 4.4.3. We will additionally write $\mathcal{P} := (\mathcal{P}_t, t \geq 0)$ for the semigroup associated to $(Y, \mathtt{P})$.

Theorem 3.5 of Nagasawa [24], shows that, under suitable assumptions on the Markov process, L-times form a natural family of random times at which the pathwise time-reversal

$$\overleftarrow{Y}_t := Y_{(\mathtt{k}-t)-}, \qquad t \in (0, \mathtt{k}),$$

is again a Markov process. Let us state Nagasawa's principle assumptions.

(A) The potential measure $U_Y(a, \cdot)$ associated to $\mathcal{P}$, defined by the relation

$$\int_E f(x) U_Y(a, \mathrm{d}x) = \int_0^\infty \mathcal{P}_t[f](a)\mathrm{d}t = \mathtt{E}_a\left[\int_0^\infty f(X_t)\,\mathrm{d}t\right], \qquad a \in E, \tag{A.52}$$

for bounded and measurable f on E, is σ-finite. Assume that there exists a probability measure, ν, such that, if we put

$$\mu(A) = \int U_Y(a, A)\,\nu(\mathrm{d}a) \qquad \text{for } A \in \mathcal{B}(\mathbb{R}), \tag{A.53}$$

then there exists a Markov transition semigroup, say $\hat{\mathcal{P}} := (\hat{\mathcal{P}}_t, t \geq 0)$ such that

$$\int_E \mathcal{P}_t[f](x) g(x)\,\mu(\mathrm{d}x) = \int_E f(x) \hat{\mathcal{P}}_t[g](x)\,\mu(\mathrm{d}x), \quad t \geq 0, \tag{A.54}$$

for bounded, measurable and compactly supported test-functions f, g on E.

(B) For any continuous test-function $f \in C_0(E)$, the space of continuous and compactly supported functions, and $a \in E$, assume that $\mathcal{P}_t[f](a)$ is right-continuous in t for all $a \in E$ and, for $q > 0$, $U^{(q)}_{\hat{Y}}[f](\overleftarrow{Y}_t)$ is right-continuous in t, where, for bounded and measurable f on E,

$$U^{(q)}_{\hat{Y}}[f](a) = \int_0^\infty \mathrm{e}^{-qt} \hat{\mathcal{P}}_t[f](a) dt, \qquad a \in E,$$

is the q-potential associated to $\hat{\mathcal{P}}$.

Nagasawa's duality theorem, Theorem 3.5. of [24], now reads as follows.

Theorem 4.7.1 (Nagasawa's duality theorem)**.** *Suppose that assumptions* **(A)** *and* **(B)** *hold. For the given starting probability distribution ν in* **(A)** *and any L-time $\mathtt{k}$, the time-reversed process $\overleftarrow{Y}$ under $\mathtt{P}_\nu$ is a time-homogeneous Markov process with transition probabilities*

$$\mathtt{P}_\nu(\overleftarrow{Y}_t \in A \mid \overleftarrow{Y}_r, 0 < r < s) = \mathtt{P}_\nu(\overleftarrow{Y}_t \in A \mid \overleftarrow{Y}_s) = p_{\hat{Y}}(t - s, \overleftarrow{Y}_s, A), \quad \mathtt{P}_\nu\text{-almost surely}, \tag{A.55}$$

for all $0 < s < t$ and Borel A in $\mathbb{R}$, where $p_{\hat{Y}}(u, x, A)$, $u \geq 0$, $x \in \mathbb{R}$, is the transition measure associated to the semigroup $\hat{\mathcal{P}}$.

Proof of Theorem 4.4.3. We give the proof of (i), the proof of (ii) is almost identical albeit requiring some straightforward adjustments. Once again, we leave the details to the reader. When $t > 0$, we use Nagasawa's duality theorem. However, since the process is conditioned to hit continuously, its dual processes from the hitting time must leave the sphere continuously. That means, if the duality is true for $t > 0$, it must be true for all $t \geq 0$.

We will make a direct application of Theorem 4.7.1, with Y taken to be the process $(X, \mathbb{P}^{\ominus}_\nu)$ where ν satisfies (A.22) or (A.23) according to the nature of S. Accordingly, we will write $U^{\ominus}$ in place of U_Y, $\mathcal{P}^{\ominus}$ in place of $\mathcal{P}$ etc. Moreover, the dual process, formerly $\hat{Y}$, is taken to be $(X, \mathbb{P}^{\vee})$ and we will, in the obvious way, work with the notation $U^{\vee}$ in place of $U_{\hat{Y}}$, $\mathcal{P}^{\vee}$ in place of $\hat{\mathcal{P}}$ and so on. In essence we need only to verify the two assumptions **(A)** and **(B)**. Let us momentarily take the former of these two cases.

In order to verify **(A)** we will make use of (A.39). Noting that $\mathrm{e}^{\alpha\epsilon_{\varphi(t)}}\mathrm{d}\varphi(t) = \mathrm{d}t$, we have for $a \in \mathbb{S}^{d-1} \setminus \mathsf{S}$ and bounded measurable $f : \mathbb{R}^d \setminus (\mathbb{B}_d \cup \mathsf{S}) \to [0, \infty)$,

$$\begin{aligned} U^{\ominus}[f](a) &= \mathbb{E}^{\ominus}_a\left[\int_0^\infty f(X_t)\mathrm{d}t\right] \\ &= \underline{\mathbb{N}}_a\left(\int_0^\varsigma H^{\ominus}(X^\epsilon_t) f(X^\epsilon_t)\mathrm{d}t\right) \\ &= \underline{\mathbb{N}}_a\left(\int_0^\varsigma H^{\ominus}(\mathrm{e}^{\epsilon(u)}\Theta^\epsilon(u)) f(\mathrm{e}^{\epsilon(u)}\Theta^\epsilon(u))\mathrm{e}^{\alpha\epsilon_u}\mathrm{d}u\right) \\ &= C\int_{\mathbb{R}^d\setminus(\mathbb{B}_d\cup\mathsf{S})} H^{\ominus}(y) f(y)(|y|^2 - 1)^{\alpha/2}|a - y|^{-d}\mathrm{d}y, \end{aligned} \tag{A.56}$$

where $U^{\ominus}[f](a) = \int_{\mathbb{R}^d\setminus(\mathbb{B}_d\cup\mathsf{S})} f(y)U^{\ominus}(a, \mathrm{d}y)$, $C > 0$ is an unimportant constant and we have used (A.20) in the second equality.

Next, we need to develop an expression for the reference measure μ. This only needs to be identified up to a multiplicative constant. As such, in the setting that $\sigma_1(\mathsf{S}) > 0$, recalling

(A.53), (A.22) and (A.3), we can take (ignoring multiplicative constants in each line)

$$\begin{aligned}\mu(\mathrm{d}y) &= \int_{\mathsf{S}} \nu(\mathrm{d}a) U^{\ominus}(a, \mathrm{d}y) \\ &= \int_{\mathsf{S}} \sigma_1(\mathrm{d}a) H^{\ominus}(y)(|y|^2 - 1)^{\alpha/2}|a - y|^{-d}\mathrm{d}y \\ &= H_{\mathsf{S}}(y) H^{\ominus}(y)\mathrm{d}y, \qquad y \in \mathbb{R}^d \setminus (\mathbb{B}_d \cup \mathsf{S}).\end{aligned} \tag{A.57}$$

When $\mathsf{S} = \{\vartheta\}$, we replace the use of (A.22) by (A.23) in the above calculation and the same answer comes out (up to a multiplicative constant).

Next, we need to verify that (A.54) holds. Indeed, using Hunt's switching identity (cf. Chapter II.1 of [2]) for the process $(X_t, t < \tau_1^{\oplus})$, we have for $x, y \in \mathbb{R}^d \setminus \bar{\mathbb{B}}_d$

$$\begin{aligned}\mu(\mathrm{d}y)\mathcal{P}_t^{\ominus}(y, \mathrm{d}x) &= \mathcal{P}_t^{\ominus}(y, \mathrm{d}x) H_{\mathsf{S}}(y) H^{\ominus}(y)\mathrm{d}y \\ &= \frac{H^{\ominus}(x)}{H^{\ominus}(y)} \mathcal{P}_t^{\mathbb{B}_d}(y, \mathrm{d}x) H_{\mathsf{S}}(y) H^{\ominus}(y)\mathrm{d}y \\ &= \mathcal{P}_t^{\mathbb{B}_d}(x, \mathrm{d}y) H_{\mathsf{S}}(y) H^{\ominus}(x)\mathrm{d}x \\ &= \mathcal{P}_t^{\vee}(x, \mathrm{d}y)\mu(\mathrm{d}x),\end{aligned}$$

where $\mathcal{P}_t^{\mathbb{B}_d}(x, \mathrm{d}y) = \mathbb{P}_x(X_t \in \mathrm{d}y,\, t < \tau_1^{\oplus})$. Note, as the measure μ is absolutely continuous with respect to Lebesgue measure, we do not need to deal with the case that x or y belong to $\mathbb{S}^{d-1} \setminus \mathsf{S}$.

Let us now turn to the verification of assumption **(B)**. This assumption is immediately satisfied on account of the fact that both $\mathcal{P}^{\ominus}$ and $\mathcal{P}^{\vee}$ are right-continuous semigroups by virtue of their definition as a Doob h-transform with respect to the Feller semigroup $\mathcal{P}^{\mathbb{B}_d}$ of the stable process killed on entry to $\mathbb{B}_d$. With both **(A)** and **(B)** in hand, we can invoke Theorem 4.7.1 and the desired result follows. □

4.8 Concluding remarks

The results in this paper have considered the setting of conditioning a relatively special class of Markov process to continuously hit a subset of the unit sphere with a one-sided approach. Taking a step back, one would ideally like to drop a number of the specialisms specific to our approach e.g. moving to a general Markov process and conditioning it continuously hit a suitably general domain. The current proofs rely on too many particular features of stable Lévy processes for the results to directly generalise in this respect. For example, suppose that we drop the assumption that the stable process continuously approaches S from just one side, but instead we allow it to continuously approach without radial confinement. This is a topic that has been addressed in follow-on work [15], for which a mixture of features that are specific to stable Lévy processes together with general potential-analytic considerations are used. The classical work of Doob [13] for the setting of Brownian motion also gives insight in how one may go about dealing with greater generality.

Appendix: Hypergeometric identity

An identity for the hypergeometric function that has been used twice in the main body of the text is taken from formula 3.665(2) in [14]. It states that, for any $0 < |a| < r$ and $\nu > 0$, as

$$\int_0^\pi \frac{\sin^{d-2}\phi}{(a^2 + 2ar\cos\phi + r^2)^\nu} \mathrm{d}\phi = \frac{1}{r^{2\nu}} B\Big(\frac{d-1}{2}, \frac{1}{2}\Big)\, {}_2F_1\Big(\nu, \nu - \frac{d}{2} + 1; \frac{d}{2}; \frac{a^2}{r^2}\Big), \tag{A.1}$$

Acknowledgements

All three authors are grateful to two anonymous referees for careful reading of an earlier version of this article and asking challenging, insightful questions, which resulted in a significant improvements to this article. TS acknowledges support from a Schlumberger Faculty of the Future award. SP acknowledges support from the Royal Society as a Newton International Fellow Alumnus (AL191032) and UNAM-DGAPA-PAPIIT grant no. IA103220.

Bibliography

[1] Alili, L., Chaumont, L., Graczyk, P. and Zak. T. (2017) Inversion, duality and doob h-transforms for self-similar Markov processes. *Electron. J. Probab.*, **22**, no. 20, 18 pp.

[2] Bertoin, J. (1996) *Lévy processes.* Cambridge University Press.

[3] Bertoin, J. and Doney, R.A. (1994) On conditioning a random walk to stay nonnegative *Ann. Probab.* **22**, 2152–2167.

[4] Blumenthal, R.M. (1992) *Excursions of Markov processes.* Birkhäuser, Boston, Basel, Berlin.

[5] Blumenthal, R.M., Getoor, R.K., and Ray, D.B. (1961) On the distribution of first hits for the symmetric stable processes. *Trans. Amer. Math. Soc.* **99**, 540–554.

[6] Bogdan, K. and Żak, T. (2006) On Kelvin transformation. *J. Theoret. Probab.* **19**, 89–120.

[7] Burdzy, K. (1986) Brownian excursions from hyperplanes and smooth surfaces. *Trans. Amer. Math. Soc.* **295**, no. 1, 35–57.

[8] Caballero, M.E., Pardo, J.C. and Pérez, J.L. (2011) Explicit identities for Lévy processes associated to symmetric stable processes. *Bernoulli* **17**, no. 1, 34–59.

[9] Chaumont, L. and Doney, R. A. (2005) On Lévy processes conditioned to stay positive. *Electron. J. Probab.* **10**, no. 28, 948–961.

[10] Chen, Z.-Q., Fukushima,M. and Ying, J. (2007) *Extending Markov processes in weak duality by Poisson point processes of excursions*, volume 2 of *Abel Symp.*, pages 153–196. Springer, Berlin.

[11] Chung K. L. and Walsh J.B. (2005) *Markov Processes, Brownian Motion, and Time Symmetry.* Springer-Verlag New York.

[12] Döring, L. and Weissman, P. (2020) Stable processes conditioned to hit an interval continuously from the outside. *Bernoulli* **26**, no. 2, 980–1015.

[13] Doob, J.L. (1957) Conditional brownian motion and the boundary limits of harmonic functions *Bulletin de la S. M. F.*, **85**, 431–458.

[14] Gradshteyn, I.S. and Ryzhik, I.M. (1965) *Table of integrals, series and products.* Academic Press, New York and London.

[15] Kwaśniki, M., Kyprianou, A.E., Palau, S. and Saizmaa, Ts. Oscillatory attraction and repulsion from a subset of the unit sphere or hyperplane for isotropic stable Lévy processes. *Preprint.*

[16] Kyprianou, A.E. (2014) *Fluctuations of Lévy processes with applications.* Springer.

[17] Kyprianou, A.E. (2016) Deep factorisation of the stable process. *Electron. J. Probab.* **21**, no. 23, 1–28.

[18] Kyprianou, A.E. (2018) Stable processes, self-similarity and the unit ball *ALEA, Lat. Am. J. Probab. Math. Stat.* **15**, 617–690.

[19] Kyprianou, A.E. and Pardo, J.C. (2021) *Stable Lévy processes via Lamperti-type representations.* Cambridge University Press.

[20] Kyprianou, A.E., Rivero V., and Satitkanitkul, W. (2020) Deep factorization of the stable process III: Radial excursion theory and the point of closest reach. *Potential Anal.* **53**, 1347–1375.

[21] Kyprianou, A.E., Rivero V., and Satitkanitkul, W. (2018) Stable Lévy processes in a cone. *arXiv:1804.08393v2 [math.PR]*

[22] Kyprianou, A.E., Rivero V., and Sengul, B. (2017) Conditioning subordinators embedded in Markov processes. *Stoch. Proc. Appl.* **127**, 1234–1254.

[23] Maisonneuve, B. (1975) Exit systems. *Ann. Probability*, **3**, no. 3, 399–411.

[24] Nagasawa, M. (1964) Time reversions of Markov processes. *Nagoya. Math. J.* **24**, 177–204.

[25] Panti, H. (2017) On Lévy Processes conditioned to avoid zero. *ALEA, Lat. Am. J. Prob. Math. Stat.* **14**, 657–690.

[26] Skorohod,. A. V. (1956) Limit theorems for stochastic processes. *Teor. Veroyatnost. i Primenen.*, 1:289–319.

[27] Skorohod, A. V. (1957) Limit theorems for stochastic processes with independent increments. *Teor. Veroyatnost. i Primenen.*, 2:145–177.

[28] Whitt, W. (2002) *Stochastic-Process Limits. An Introduction to Stochastic-Process Limits and their Application to Queues*, Springer.

[29] Yano, K, Yano, Y. and Yor, M. (2009) Penalising symmetric stable Lévy paths. *J. Math. Soc. Japan*, 61 (3): 757–798.

Chapter 5

Conclusions

In this thesis, we studied new ways of path conditioning of the stable Lévy processes taking account of the classic potential theoretic approach as well as the recent developments in the representation of a d-dimensional isotropic stable Lévy process as a self-similar Markov process.

First, in Chapter 3, we characterised a stable Lévy process conditioned to hit subset of the unit sphere or hyperplane in $\mathbb{R}^d, d \geq 2$ using classical potential theory approach. The distribution of the hitting position is also characterised. Then, we reveal time-reversal duality from any $L-$time between the conditioned stable Lévy process and underlying stable Lévy process.

Second, in Chapter 4, we characterised a stable Lévy process conditioned to hit subset of the unit sphere in $\mathbb{R}^d, d \geq 2$ with the constraint that the stable Lévy process remains one side of the sphere. The distribution of the hitting position is also characterised. Moreover, we extend the conditioned process to be issued from the boundary, e.g. sphere. Finally, we characterise the time-reversed process of the conditioned processes from any L-time.

In Chapter 4, the question to arise in the conditioning procedure was whether the different definition of the conditioning could lead to the same result. The choice of the conditioning events $C_\varepsilon^\vee$ and $C_\varepsilon^\wedge$ were relied on the point of closest reach to the origin in the range of X and the point of furthest reach from the origin prior to exiting $\mathbb{B}_d$ correspondingly. Although we do not produce the calculations here, it turns out that, by defining e.g. $C_\varepsilon^\vee = \{X_{\tau_1^\oplus} \in B_\varepsilon\}$, or indeed $C_\varepsilon^\vee = \{X_{\tau_1^\oplus -} \in A_\varepsilon\}$, the limiting conditioning will still produce the same change of measure. Other definitions of $C_\varepsilon^\vee$ giving a consistent limit are also possible. For example, we can choose $C_\varepsilon^\vee = \{X_{\tau_1^\oplus -} \in A_\varepsilon, X_{\tau_1^\oplus} \in B_\varepsilon\}$. The analogue choices for $C_\varepsilon^\wedge$ also produce consistent results. In short, we showed that the choice of $C_\varepsilon^\vee$ and $C_\varepsilon^\wedge$ are by no means the only possibilities as far as performing a limiting conditioning that leads to the same result.

We finish by discussing some of the open questions that have resulted from this research. In Chapter 3, we were able to condition stable Lévy processes to be attracted to the hyper-plane in a similar manner to the case where it is attracted to the sphere. However, in Chapter 4, it was not possible to replicate the result for the sphere to the hyperplane. The reason is that we used existing characterisation of the point of closest/furthest reach to the sphere for the stable Lévy process while no such characterisation for closest/furthest reach to the hyperplane exists. Moreover, when we extended the conditioned process from the boundary, we used radial excursions. For the hyperplane setting, there is no analogous excursion theory; this remains an open problem.

Generally, it remains open whether we can perform analogous conditionings in relation to the hyperplane as in Chapter 4 and what the associated excessive function looks like.