Fragmentation-Coalescence Processes: Theory and Applications

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for the degree of Doctor of Philosophy

of the

University of Bath

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January 20, 2017

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SUMMARY

The main objects of study in this thesis are fragmentation-coalescence processes, where particles are grouped into clusters and evolve by either joining together, to form larger clusters, or splitting apart, to form smaller clusters. The focus is on the number of these clusters and the distribution of their sizes.

In particular, we show for a certain class of processes defined on a finite system that there is convergence in the thermodynamic limit to an infinite system. For a second class of processes we show there is a phase transition between regimes where the number of clusters has an entrance law from ∞ or not.

ACKNOWLEDGEMENTS

First of all, I would like to thank my superivsors Dr. Tim Rogers and Prof. Andreas Kyprianou for all their guidance, support and patience over the last three years, without which this thesis simply would not exist. Our meetings have always been interesting and helpful.

I would also like to thank my other coauthors who have been great to work with, and their many good ideas. Also, I thank all the members of the Prob-L@b and CNCB seminar groups for providing a friendly working environment and a safe, inviting place to discuss ideas and problems. Many thanks must also go to the University of Bath for awarding me a University Research Studentship to support my studies.

The department has an excellent social atmosphere in which I have taken part. I'd like to thank everyone who ever went to the Parade on a Friday, the two pub quiz teams that I've been a part of, those who played board games in the evening, and everyone who didn't run away when I asked them to bake a cake for Wednesday afternoons. In particular, I thank everyone who has shared 4W 1.15 with me these past three years whose friendship and support have made my time here so enjoyable.

Finally, I would like to thank my family, in particular my parents, for their love and support as I continued in my endeavour to become, in their words, an "enternal student".

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CHAPTER 1

INTRODUCTION

Coalescence processes and their converse, fragmentation, have been widely studied since the seminal work of Smoluchowski [70, 71] 100 years ago. They have been used in a broad range of applications from the traditional ones in physical chemistry [37, 47, 69] and genealogy [75] to more modern ones in group dynamics in the social sciences and biology [23, 39]. An excellent background to such processes has been given by Aldous [2]. This thesis, however, considers models which combine the effects of fragmentation and coalescence, which have been less studied.

Informally, fragmentation-coalescence processes give rules which govern how clusters (collections of identical elements) join together to form larger clusters, or split apart into a number of smaller clusters. As a general rule, the key questions concern the distribution of the number of clusters, the distribution of the sizes of these clusters, and how they both change through time.

This thesis will focus on two different areas of fragmentation-coalescence processes: the processes defined on a finite system of particles grouped into clusters, and *exchangeable* fragmentation-coalescence processes which take place on the space of partitions on $\mathbb{N}$ (and so have infinitely many particles) where the blocks of the partition represent the clusters. However, while these are inherently different types of processes, one of the key themes of this thesis will be transferring the main features of both types of process to the other and discovering whether this is possible, and how it changes the behaviour. For example, we will import the multiple-block coalescence from the exchangeable class of processes (known as a Λ-coalescent) to the finite class to see what differences there are compared to the binary coalescence case, which has been the focus of most of the study in the past. This will be one of the main focuses of Chapter 2.

This thesis will be organised as follows. The remainder of Chapter 1 presents the background to fragmentation-coalescence processes from its origins with Smoluchowski a century ago. Chapter 2 looks at the questions asked above for a certain class of

fragmentation-coalescence processes. Chapter 3 takes this work, adapts it into the field of exchangeable fragmentation-coalescence processes and asks further questions. Finally, Chapter 4 looks at questions that are still open and conjectures that have arisen.

1.1   Classical fragmentation-coalescence processes

There are two main models or scales on which classical fragmentation-coalescence processes are studied:

- the microscopic scale, henceforth known as the finite volume model, which is a stochastic process acting at the level of particles [54, 52, 53].

- the macroscopic scale, henceforth known as the infinite volume model, which is a deterministic model, where such stochastic interactions between clusters are averaged out [70, 71, 57].

The latter can make the system somewhat easier to analyse and can be a good approximation, both for use in some applications, and as an approximation to the finite volume case, when that volume is large.

The infinite volume model assumes infinitely many particles arranged into infinitely many clusters with the main quantities of interest being the density of clusters of each size. It averages out the stochastic mechanisms underlying fragmentation and coalescence of individual clusters, which gives continuous rate kernels. This leads to an infinite system of differential equations, which describe the evolution of the density of clusters of each size, based on these kernels. The main questions here concern what conditions on the kernels are required in order for this system of ODEs to have solutions which are unique and have properties which you would expect physical systems to have (for example to conserve mass).

The aforementioned finite volume model has finitely many particles arranged into clusters, where the coalescence and fragmentation of clusters occurs randomly. These models are often concerned with whether a process can be defined on a finite system which when you take this system size to infinity, converges to the infinite volume equivalent. That is, do the density of clusters of each size (which is random in the finite case) converge to solutions to a certain system of ODEs?

This section will outline the known models and results in both the infinite and finite volume case, starting with coalescence processes before generalising to fragmentation-coalescence processes.

1.1.1 Coalescent processes

Here the two main models of classical coalescent processes will be introduced. First, the infinite volume model which was introduced by Smoluchowski and lead to the famous Smoluchowski coagulation equations which have been the subject of much study. Then, the finite volume model, which was introduced by Marcus and Lushnikov some fifty years later.

Smoluchowski coagulation equations

As was previously mentioned, coalescent processes have their roots in Smoluchowski's work beginning in 1916 where he introduced his model describing systems which exhibit binary coalescence (e.g. aerosols). This thesis will focus on the case where mass is discrete (versions with continuous mass are available) so clusters will be made up of a number of particles, each of mass 1. The main quantities of interest will be the density of clusters of size j at time t, which will be denoted $w_j(t)$. The model evolves by pairs of clusters coalescing, the local dynamics of which may be similar to ones found in the finite volume case, but this is all averaged into what is known as the rate kernel, K. That is, the average rate at which clusters of size i and size j coalesce is given by $K(i, j)$. Hence the density $w_j(t)$ changes in two ways:

(i) it increases as smaller clusters coalesce into ones of size j,

(ii) it decreases as clusters of size j coalesce with other clusters.
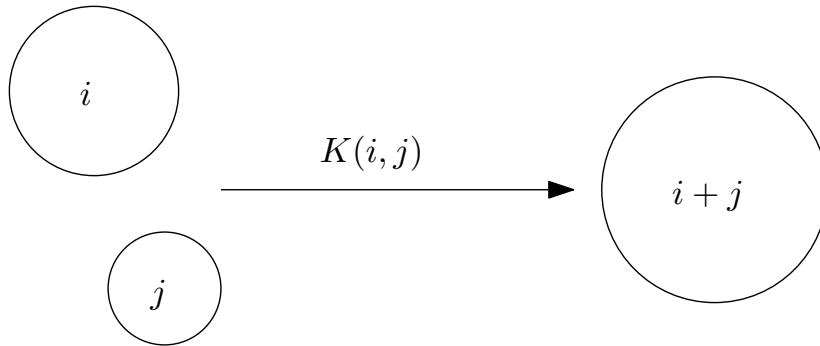
Figure 1-1: *A figure showing two clusters coalescing in the Smoluchowski model.*

Therefore, putting these two dynamics together with the rate kernel gives us the Smoluchowski coagulation equations first seen in [70, 71], which have been the subject of much study.

$$\frac{d}{dt} w_j(t) = \frac{1}{2} \sum_{l=1}^{j-1} K(l, j-l) w_l(t) w_{j-l}(t) - w_j(t) \sum_{l=1}^{\infty} K(j, l) w_l(t), \quad j \geq 1. \qquad (1.1)$$

The first term on the right-hand side is how you gain clusters of size j, by coalescing two smaller clusters of various sizes to make one of size j, each possible pairing both coalescing at the rate given by $K(l, j-l)$ and with a term proportional to the densitities of both sizes involved. The second term is how you lose clusters of size j, by coalescing them with other clusters, and summing over all the possibilities.

Some specific kernels, such as the cases $K(i, j) = 1, K(i, j) = ij$ and $K(i, j) = i+j$, have been studied in detail and the solutions to the equations in these cases found. For a more general look at these equations the reader is directed to Drake [27], White [77] and Dubovskiĭ [29]. The first two cases mentioned are particularly relevant to this chapter and the next.
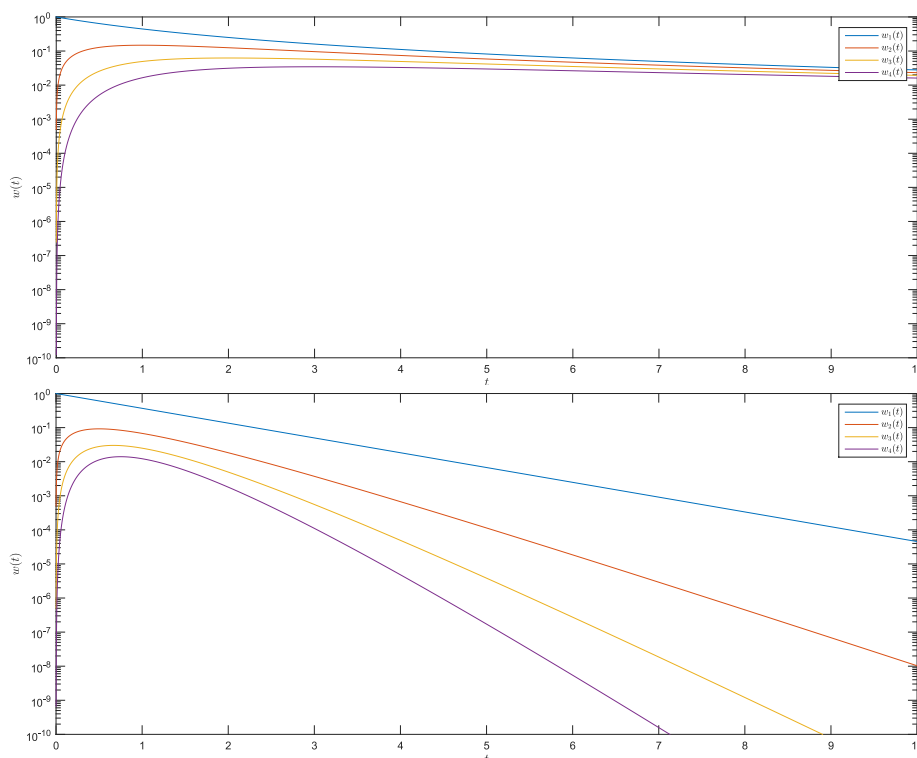
Figure 1-2: *Top: The evolution of the density of clusters for the case $K \equiv 1$. Bottom: The evolution of the density of clusters for the case $K(i, j) = ij$.*

The case where K is the unit kernel was the first to be studied by Smoluchowski [71] who found unique solutions to the system (1.1)

$$w_j(t) = \left(\frac{2}{2+t}\right)^2 \left(\frac{t}{2+t}\right)^{j-1}, \quad j \in \mathbb{N}, \, t \geq 0, \tag{1.2}$$

and the case where $K(i, j) = ij$, known as the multiplicative coalescent, was studied by McLeod [55, 56] who found unique solutions to the system (1.1), which in this case

are

$$w_j(t) = \frac{1}{j!}t^{j-1}j^{j-2}e^{-jt}, \quad j \in \mathbb{N}, \ t \geq 0. \tag{1.3}$$

These two cases highlight how large differences in behaviour can arise when the rate kernel is changed, as shown below. An important quantity for the model is the mass-density defined as

$$m_1(t) = \sum_{j=1}^{\infty} jw_j(t),$$

which can be thought of as a measure of what proportion of the particles are in a finite sized cluster at time t. One would expect that for all finite t, that $m_1(t) = 1$, that is mass-density is a conserved quantity. However, if this quantity falls below 1 then some proportion of the mass would seem to have disappeared. In actuality, what is said to occur is *gelation* and at least one infinite mass cluster (gel) is formed.

It is straightforward to show that in the unit kernel case mass-density is conserved for all $t \geq 0$, and therefore no gelation occurs, simply by evaluating the sum. However, in the multiplicative case, mass-density is conserved up until time $t = 1$ because for $t \leq 1$ we have the following relationship [55]

$$\sum_{j=1}^{\infty} \frac{1}{j!}j^{j-1}\left(te^{-t}\right)^{j-1} \equiv e^t, \tag{1.4}$$

which can be seen using the Taylor series expansion of the Lambert-W function around zero

$$W(x) = \sum_{j=1}^{\infty} \frac{(-j)^{j-1}}{j!}x^j.$$

Hence, multiplying (1.4) through by e^{-t} gives $\sum_j jw_j(t) \equiv 1$, for $t \leq 1$. For $t > 1$, we note that 1 is where xe^{-x} attains its maximum and hence there exists $t' < 1$ such that $te^{-t} = t'e^{-t'}$ and thus using (1.4) gives that $\sum_j jw_j(t) = e^{-t}e^{t'} < 1$.

This gelation time occurs when the system is in a critical state, just as the gel forms, where small perturbations can cause large-scale effects, and then transitions to another non-critical state. With the addition of fragmentation, it can be thought that this critical point might be visited more frequently, or perhaps even sustained indefinitely in which case we will say the system self-organises into its critical state. We will see more on in the section on forest fire models.

Marcus-Lushnikov processes

The natural question to ask was whether there exists a finite volume model equivalent to the infinite volume model and the Smoluchowski coagulation equations. This question was somewhat answered by Marcus [54] and Lushnikov [52, 53] who introduced and

studied a stochastic finite volume model, which has become known as the Marcus-Lushnikov process. Again, the discrete mass version is the one that is presented here.

Start with total mass n, some positive integer, split up into clusters of various numbers of particles. Again assume that the system is spatially stationary and the clusters are well-mixed. The system evolves with every pair of clusters coalescing according to some rate kernel K. More formally, every pair of clusters of sizes i and j coalesces to form a cluster of mass $i + j$ at rate $n^{-1}K(i, j)$. The n^{-1} is to control the rate at which coalescence events occur, without this the process would be much faster than the infinite volume equivalent.

With the assumptions made above, we need only know the number of clusters to each size to know everything about the current state of the system. To that end define the following random variables

$$w_{n,j}(t) := \#\{\text{clusters of size } j \text{ at time } t\}, \quad 1 \leq j \leq n.$$

Note: often in the literature this is labelled $ML^{(n)}(j, t)$. Then,

$$w_n(t) = (w_{n,1}(t), \ldots, w_{n,n}(t)),$$

specifies the state of the system.

Definition 1.1. *Let $e_{n,i} \in \mathbb{Z}^n$ be the vector with 1 in position i and 0 everywhere else. A Markov process $w_n = (w_n(t), t \geq 0)$ on state space $\{w \in \mathbb{Z}^n : w_i \geq 0, \sum_{i=1}^{n} iw_i = n\}$ is an $ML^{(n)}$ process if the only transitions allowed are*

1. *$w \mapsto w - e_{n,i} - e_{n,j} + e_{n,i+j}$, which happens at rate $K(i, j)w_iw_j/n$, for $1 \leq i \neq j \leq n$,*

2. *$w \mapsto w - 2e_{n,i} + e_{n,2i}$, which happens at rate $K(i, i)w_i(w_i - 1)/(2n)$, for $1 \leq i \leq n/2$.*
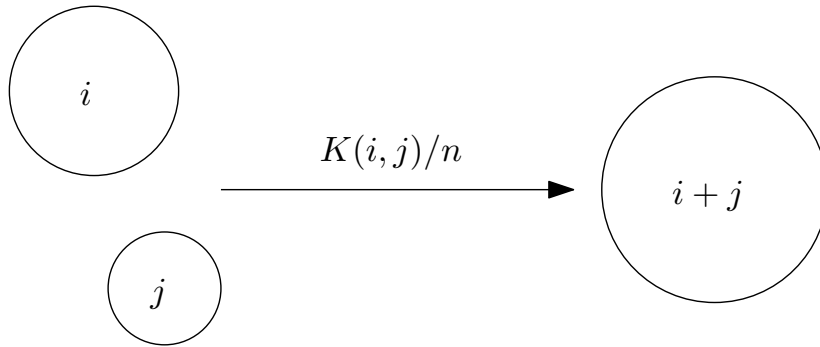
Figure 1-3: *A figure showing two clusters coalescing in the Marcus-Lushnikov process.*

Naïvely, if the possible changes that can happen to $w_{n,j}(t)$ in a small time $h \ll 1$ are considered, we see that

$$\frac{1}{h}\mathbb{E}[w_{n,j}(t+h) - w_{n,j}(t)|\mathcal{F}_t] \approx$$

$$\frac{1}{2n}\sum_{i=1}^{j-1} K(i, j-i)w_{n,i}(t)w_{n,j-i}(t) - \frac{w_{n,j}(t)}{n}\sum_{i=1}^{n} K(i,j)w_{n,i}(t)$$

$$- \mathbb{1}_{\{j \text{ even}\}}\frac{K(j/2, j/2)}{2n}w_{n,j/2}(t) + \frac{K(j,j)}{2n}w_{n,j}(t)(w_{n,j} + 1)$$

where the first term is the expected gain from coalescence events which result in a cluster of size j, the second term is the expected loss from coalescence events that involve clusters of size j and the remaining two terms are corrections for when the clusters involved are the same size. These are similar to the Smoluchowski coagulation equations, with some additional, perhaps small, error terms. In the large n limit does this approximation become exact? By which we mean, is there a weak law of large numbers for the $w_{n,j}(t)$? Conjecture 5.3 in Aldous [2] gives some suggestion of when this may be true, which is reproduced here.

Conjecture 1.2. *Let K be a coagulation kernel and $w_n(t)$ be the $ML^{(n)}$ process with kernel K. Then, for fixed t,*

$$\frac{1}{n}w_{n,j}(t) \xrightarrow{\mathbb{P}} w_j(t),$$

as $n \to \infty$, where $w_j(t)$ is the solution to the corresponding Smoluchowski coagulation equation, if either

(i) $K(i,j) = o(ij)$

(ii) $t < T$

where T is the gelation time for the Smoluchowski coagulation equation and a function $f(i,j) = o(ij)$, if $f(i,j)/(ij) \to 0$ as $i, j \to \infty$.

For certain K, this result has already been shown. For example, the case where $K(i,j) = a + b(i + j)$ has been shown by Hendricks et. al [35] by first looking at the evolution of the number of clusters through time (which has a closed form that does not depend on the sizes of the clusters), then studying the evolution of the sizes of the clusters given the number of clusters. This question about convergence of the finite volume model to the infinite volume model comes up again repeatedly throughout this chapter and the next.

1.1.2 Fragmentation-coalescence processes

As a general rule, most natural systems do not allow for unlimited growth of objects by further and further aggregation of mass. It is natural to believe that if there is a

mechanism whereby objects can cluster together into larger structures, then there is also a mechanism whereby they can fragment into smaller ones also. There are many examples of systems which undergo fission and fusion simultaneously, for example nucleation in liquids [1], protein and RNA chains [25] and stellar formation [66]. Hence, generalising coalescence models to also allow for fragmentation to occur was an obvious next step. The models considered throughout this thesis shall assume that the coalescence and fragmentation mechanisms are independent of each other.

This section starts with the original model, known as the Becker-Döring model, which allows for coalescence and fragmentation by single particles splitting off or joining to clusters. Then moves to the more general model, which allows clusters to join together in pairs or split into two. Finally, it will look at some of the work on the finite volume version of the latter model, often known as the $CF^{(n)}$ process.

Becker-Döring model

Arguably the first fragmentation-coalescence model analysed was the Becker-Döring model in 1935 [8] in regards to homogeneous nucleation theory [1]. This is an infinite volume model with discrete masses, so the clusters are made up of particles of mass one. Again, the system is assumed to be spatially stationary and well-mixed. This process evolves by binary coalescence between clusters of sizes j and 1, and fragmenting clusters of size j into two clusters of sizes $j-1$ and 1, $j \geq 1$. So, it can be thought of as individual particles joining clusters and individual particles splitting off clusters.
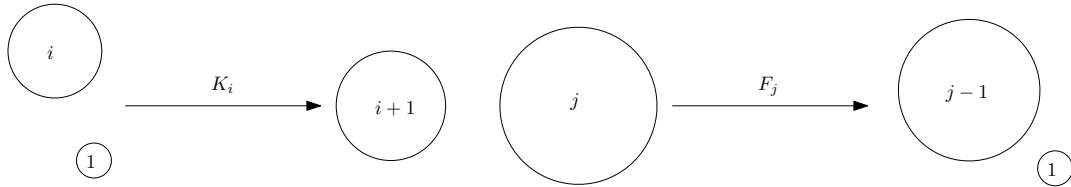
Figure 1-4: *A figure showing how coalescing and fragmenting works in the Becker-Döring model.*

Thus, there are two rate kernels, often denoted $K = \{K_j\}_{j \geq 1}$ and $F = \{F_j\}_{j \geq 2}$, which specify the average rate rate at which a cluster of size j gains or loses a particle respectively. Hence, following a similar procedure to the Smoluchowski coagulation equations, one arrives at the Becker-Döring equations

$$\frac{d}{dt}w_j(t) = K_{j-1}(t)w_{j-1}(t)w_1(t) - F_j w_j(t) - K_j w_j(t)w_1(t) + F_{j+1}w_{j+1}(t), \quad j \geq 2,$$

(1.5)

$$\frac{d}{dt}w_1(t) = \sum_{i=3}^{\infty} F_i w_i(t) - w_1(t)\sum_{i=2}^{\infty} K_i w_i(t) + F_2 w_2(t) - 2K_1 w_1(t)^2,$$ (1.6)

where the $w_i(t)$ are the same as defined earlier, the density of clusters of size i. Often these equations are rewritten as

$$\frac{d}{dt}w_j(t) = H_{j-1}(w(t)) - H_j(w(t)), \quad j \geq 2,$$

$$\frac{d}{dt}w_1(t) = -H_1(w(t)) - \sum_{j=1}^{\infty} H_j(w(t)),$$

where

$$H_j(w(t)) = K_j w_j(t) w_1(t) - F_{j+1} w_{j+1}(t).$$

Note: the original model fixed a density of clusters of size 1, and did not allow this to change in time, more modern adaptations of this model by Lebowitz and Penrose [51] allow this to evolve in time.

A thorough analysis of these equations comes courtesy of Ball et. al. [6] and we will condense some of their results (Theorems 2.2 and 3.5, Corollary 2.6) into the following.

Theorem 1.3. *Assume that $K_j = O(j)$, that is there exists a constant C such that $K_j \leq Cj$ for all $j \in \mathbb{N}$. Assume also that $\sum_j j w_j(0) < \infty$. Then*

(i) there exists a unique solution to the Becker-Döring equations (1.5), on $[0, \infty)$

(ii) mass density is preserved

$$\sum_{j=1}^{\infty} j w_j(t) = \sum_{j=1}^{\infty} j w_j(0), \quad t \in [0, \infty).$$

Remark 1.4. *(i) These conditions are sufficient but not necessary.*

(ii) This result is independent of $\{F_j\}_{j\geq 2}$, because in the proof it is shown that the necessary bounds can be attained using assumptions only on $\{K_j\}_{j\geq 1}$.

(iii) There are a set of more technical conditions that can be made which assume slightly less of $\{K_j\}_{j\geq 1}$ but for brevity we do not state these. They can be found in Theorem 2.2 of [6].

(iv) Part (ii) of this theorem is actually true for all solutions of the Becker-Döring equations and doesn't require the restrictions on $\{K_j\}_{j\geq 1}$ that part (i) does.

The proof of the existence of these solutions uses a method often used in showing the existence of solutions to an infinite system of ODEs. For example, it was also used by McLeod in his proof for the existence of solutions to certain Smoluchowski coagulation equations [55, 56]. First, truncate the system of ODEs to some finite $n \in \mathbb{N}$ which is straightforward to show has a unique solution. Then, by the Arzela-Ascoli theorem for $j \geq 2$ there is a uniformly convergent subsequence of these finite solutions to some w_j

which can be shown to solve the Becker-Döring equations. The main complication is the convergence of these finite solutions in the case $j = 1$ due to the lack of conditions on $\{F_j\}_{j \geq 2}$. However, bounds on the solutions to and uniform convergence for the cases $j \geq 2$, and finding a weak* convergent subsequence in the case $j = 1$ allows them to bypass this issue.

Further analysis of these equations, including special cases where no solution exists for all possible time horizons (where K_j grows super-linearly, and F_j is bounded above by K_j) and the asymptotics of solutions as $t \to \infty$ can be found in Sections 2, 4 and 5 of [6].

General binary fragmentation-coalescence processes

The Becker-Döring model was a very specific case of binary fragmentation-coalescence, but what about the more general class? A more general model was introduced by Melzak [57] in 1957, who added spontaneous binary fragmentation to the Smoluchowski coagulation equations in the continuous mass setting. Under conditions on the rate kernels of both coalescence and fragmentation (continuous, bounded, and some integrability conditions on fragmentation) that the solution to the equations existed globally and was unique, continuous, bounded and non-negative.

The first in the discrete case would appear to be Spouge [72] although the equations will look more familiar in Ball and Carr [5] who generalised the work they had previously done on the Becker-Döring equations with Penrose, which we will come back to in more detail.

This model allows any pair of clusters to coalesce according to a rate kernel K (like in the general coalescent process) but now there is also the idea of fission or binary fragmentation. Binary fragmentation lets a cluster split into two parts whose sizes sum to make the size of the original cluster. Hence, this too has a rate kernel, $F(i, j - i)$, which specifies the rate at which a cluster of size j fragments into clusters of sizes i and $j - i$. So now the coagulation-fragmentation equations in the infinite volume model are written
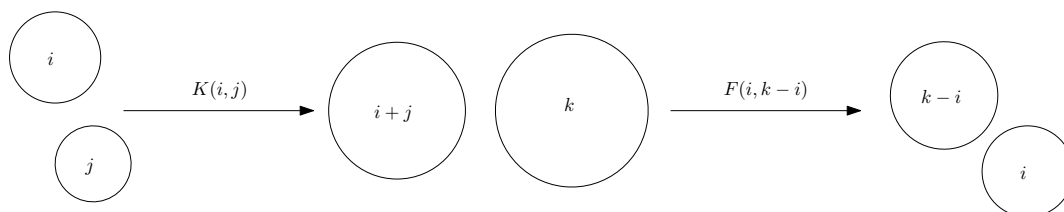
Figure 1-5: *A figure showing how coalescence and fragmentation works in the binary fragmentation-coalescence process.*

$$\frac{d}{dt}w_j(t) = \frac{1}{2}\sum_{i=1}^{j-1}(K(i,j-i)w_i(t)w_{j-i}(t) - F(i,j-i)w_j(t))$$

$$- w_j(t)\sum_{i=1}^{\infty}K(i,j)w_i(t) + \sum_{i=1}^{\infty}F(i,j)w_{i+j}(t), \quad j \geq 1. \qquad (1.7)$$

As often is the case, the main interest in these equations is in conditions needed on K and F such that

(i) solutions exist,

(ii) they are unique,

(iii) mass density is preserved i.e. $\sum_j jw_j(t) = \sum_j jw_j(0)$.

In the Becker-Döring case the only conditions found were those on K without any needed on F. In addition, any solution to the Becker-Döring equations conserved mass. However, in this more general case this is not true; if the fragmentation is too quick, for example, some (not all) solutions to the coagulation-fragmentation equations actually have mass-density that grows exponentially [5]. Hence, with these additional difficulties, it is not straightforward to combine the results of Ball and Carr [5] as was done for the Becker-Döring case, so we will state them separately. First, a theorem giving sufficient conditions such that solutions exist.

Theorem 1.5. *If $K(i,j) \leq c(i+j)$ for all $i,j \geq 1$ for some $c \geq 0$, then there exists a solution to the system* (1.7).

This looks fairly similar to the Becker-Döring case as there are no conditions on F, and the proof follows a similar pattern of looking at a truncated system of the ODEs. The only real difference is that Helly's Selection Theorem simplifies the extraction of a convergent subsequence of solutions to the truncated systems, as they have uniformly bounded total variation. The proof also gives a construction of one of the solutions, and later in the paper Ball and Carr show that this constructed solution conserves mass-density. However, they also show cases where there are other solutions which do not do so. To guarantee mass-density conservation further conditions are required.

Theorem 1.6. *Suppose that*

(i) *$K(i,j) = r_i + r_j + \alpha_{i,j}$ where $\{r_j\}$ is a non-negative sequence and for some $c \geq 0$, $\alpha_{i,j} \leq c(i+j)$ for all $i,j \geq 1$,*

(ii)

$$\sum_{i=1}^{m/2} iF(m-i,i) \leq cm$$

Then all solutions to (1.7) *have the property that mass-density is preserved.*

Unsurprisingly, as there are cases of pure fragmentation processes which don't conserve mass-density, restrictions on how fast fragmentation can occur were required. Finally, we note some conditions when solutions are unique.

Theorem 1.7. *Let $c > 0$, $0 \leq \alpha \leq 1/2$ and suppose that*

(i) $K(i, j) \leq c(ij)^{\alpha}$,

(ii)

$$\sum_{i=1}^{m/2} i^{1-\alpha} F(m - i, i) \leq cm^{1-\alpha}, \quad \text{for all } m \geq 2.$$

Then there is a unique solution to (1.7).

The proof of this is very similar to that of the same result for the Becker-Döring equations. For a more thorough analysis of cases where mass-density is not preserved because gelation occurs, the reader is directed to Jeon [43]. More on the continuous mass case can be found in Barrow [7], Stewart [73, 74] and Dubovskiĭ et. al. [30, 31].

Finite volume fragmentation-coalescence models

In a similar vein to the Marcus-Lushnikov process with regards to the infinite-volume coalescence model, there is a finite-volume stochastic fragmentation-coalescence model. The first recognisable model comes from Gueron [38] in his work on animal grouping in 1997 which was later expanded on in conjunction with Durrett and Granovsky [34]. The definition below is taken from [38]

Start with mass n split into n particles in clusters. The same assumptions as before are made: that the system is spatially stationary and the clusters are well-mixed. The model evolves with every pair of clusters coalescing and fragmenting according to rate kernels K and F respectively, with the usual requirements on them (symmetric, non-negative).

More formally, every pair of clusters of sizes i and j coalesces to form a cluster of mass $i + j$ at rate $K(i, j)$. Every cluster of size j fragments into 2 clusters of sizes i and $j - i$, $1 \leq i \leq j - 1$, at rate $F(i, j - i)$. Again, with the assumptions about mixing, we need only know the number of clusters of each size to know everything about the current state of the system. So the same random variables are used as before

$$w_{n,j}(t) := \#\{\text{clusters of size } j \text{ at time } t\}, \quad 1 \leq j \leq n.$$

Then, the state of the system is specified by

$$w_n(t) = (w_{n,1}(t), \ldots, w_{n,n}(t)).$$

Definition 1.8. *A Markov process* $w_n = (w_n(t), t \geq 0)$ *on state space* $\Omega_n := \{w \in \mathbb{Z}^n : w_i \geq 0, \sum_{i=1}^{n} iw_i = n\}$ *is a* $CF^{(n)}$ *process if the only transitions allowed are*

1. $w \mapsto w - e_{n,i} - e_{n,j} + e_{n,i+j}$, *which happens at rate* $K(i,j)w_i w_j$, *for* $1 \leq i \neq j \leq n$,

2. $w \mapsto w - 2e_{n,i} + e_{n,2i}$, *which happens at rate* $K(i,i)w_i(w_i - 1)/2$, *for* $1 \leq i \leq n/2$, *and finally*

3. $w \mapsto w + e_{n,i} + e_{n,j-i} - e_{n,j}$, *which happens at rate* $F(i, j-i)w_j$, *for* $1 \leq i \leq j \leq n$.

Remark 1.9. *This is very similar to the Marcus-Lushnikov process on the same state space,* Ω_n, *where the state specifies the number of clusters of each size. However, the reader may note that there is no scaling of the coalescence rates by the number of particles. This is largely due to Gueron not having the same motivations as Marcus and Lushnikov. He was not looking for a finite-volume model that converges to the previously defined infinite-volume model in the thermodynamic limit and hence was not concerned with coalescence occurring 'too quickly'.*

The main focus of the work on $CF^{(n)}$ processes is concerned with the stationary distribution of this finite system and the asymptotics of the system under stationarity as $n \to \infty$. Gueron [38] indeed did this in his paper focussing on the expected cluster size distribution, in particular he considered truncating the system to only allow clusters up to some certain size. In his work with Durrett and Granovsky [34], criteria were found under which the process was time-reversible, using detailed balance. Finally, more recently, Han, Zhang and Zheng [40] looked at these reversible processes in the thermodynamic limit in a bit more detail, their main focus being the distributions of the number of clusters of size in different scales compared with n.

There is a similar open problem here to that of Conjecture 1.2 about Marcus-Lushnikov processes. That is, when do the random variables $n^{-1}w_{n,k}(t)$ converge to the solutions to the equivalent infinite-volume differential equations, $w_k(t)$? Little has been done here, but this is perhaps unsurprising as it is still an open question in pure coalescence, so adding complexity, by introducing fragmentation, is unlikely to help. However, some progress in this vein has been made in a related field which will be expanded on in the next section.

1.2 Forest fire models

In this section a set of models will be considered which are similar to the fragmentation-coalescence models elaborated on earlier, but tend to have more geometry attached to them. These are the "forest fire" models first introduced by Bak et. al. [3, 24] and studied with more mathematical rigour by Drossel and Schwabl [28]. The model is defined on a finite d-dimensional lattice in discrete time where sites are either occupied (by a tree), burning, or empty.

Definition 1.10. *Let L be a finite d-dimensional lattice and define the set $w_L(t) = \{w_l(t) : l \in L, w_l \in \{0, 1, 2\}\}$ where 0 represents empty, 1 represents occupied and 2 represents burning. The dynamics are as follows*

   (i) empty sites become occupied with probability p in the next time step,

   (ii) burning sites turn empty in the next time step,

  (iii) occupied sites who are neighbours of burning sites, switch to burning in the next time step.

It was conjectured that this model exhibited *self-organised criticality* when $p \to 0$, however this was shown to be false. Self-organised criticality is an important phenomenon in statistical physics and biology as an explanation of how complexity evolves naturally from a closed system. It's the idea that some dynamical systems naturally reach their critical point where they display scale-invariant behaviour without outside influence or particular tuning. The term was first put forward by Bak, Tang & Wiesenfeld [4] in their work on critical phenomena and 1/f noise. The example they provided was the famous Abelian sandpile model on a square lattice where each site has a number of grains piled on it. If this number is at least 4 the pile is said to be unstable and topples; giving one particle to each of its neighbours. This may cause another pile to topple, and so forth, causing an avalanche. Once the avalanche ends a site is chosen at random and a grain is added there and the process repeats. Such a system is believed to self-organise into a state with a power-law probability distribution for the size of an avalanche, although this remains an open problem to be proven rigorously. Once in this state the addition of another grain can cause avalanches of any size: some small, some spanning the entire lattice; the so-called critical behaviour, that small perturbations can cause large-scale effects.

Whilst the original forest fire model didn't exhibit self-organised criticality, Drossel and Schwabl introduced a variant of Definition 1.10 which included an additional dynamic, 'lightning', where sites that were occupied turn burning with probability f in the next time step if they have no burning neighbours (i.e. they spontaneously started burning). This modified model was shown to have the conjectured self-organised criticality that the original model was supposed to have, but now this is seen when both $p \to 0$ and $f/p \to 0$. There is a variation of this model where the lightning comes at rate $\lambda(d)$ to each occupied vertex but now instantaneously burns the connected component that vertex belongs to (instead of propagating through the component over time). Such models and their behaviour as the lattice size tends to infinity have been studied by (amongst others) Dürre [32, 33] who proved that when $\lambda(d)$ is a positive constant that the infinite system exists and is unique.

A variation of the forest fire model that is of particular interest was introduced and analysed by Ráth and Tóth [63] which adds lightning to the Erdős-Rényi coagulation

on the complete graph on n vertices. This coagulation is the continuous time version of that studied by Erdős and Rényi [36] where, starting from the empty graph, in the next time step one of the empty edges is chosen at random and becomes occupied. Therefore two vertices are in the same cluster if there is a path of occupied edges between them. The dynamics of the continuous time version are as follows

(i) empty edges become occupied at rate $1/n$,

(ii) sites are hit by lightning at rate $\lambda(n)$, when a site is hit all edges in the cluster it belongs to turn instantaneously empty.

Both of these occur independent of everything else that happens in the system. Hence, the first rule defines pairwise coagulation between clusters which happens at rate proportional to the product of the sizes of the clusters; the latter defines a rather extreme form of fragmentation where each cluster fragments into singletons at a rate proportional to the size of the cluster.

As was the case in previous forest fire models, Ráth and Tóth were largely concerned with the asymptotics of the process as the system size n tended to infinity, with questions regarding whether such a process makes sense and its behaviour if it does. As this process takes place on the complete graph the current state of the system is entirely characterised by the collection of random variables

$$w_{n,k}(t) := \#\{\text{clusters of size } k \text{ at time } t\}, \quad k = 1, \ldots, n$$

as the 'location' of each cluster need not be considered, as all clusters can coalesce with all other blocks.

The case without fragmentation had previously been covered [36] and it had been shown that

$$w_{n,k}(t) \xrightarrow{\mathbb{P}} w_k(t)$$

where the $w_k(t)$ solve the Smoluchowski coagulation equations with kernel $K(x,y) = xy$. Ráth and Tóth discovered four regimes for $\lambda(n)$ with differing behaviours in the thermodynamic limit:

(i) $\lambda(n) \ll n^{-1}$, (ii) $\lambda(n) = n^{-1}\lambda$, (iii) $n^{-1} \ll \lambda(n) \ll 1$, (iv) $\lambda(n) = \lambda$,

where $\lambda \in (0, \infty)$. The focus will be on the results for the last two regimes as these are the ones of most interest.

Theorem 1.11. *As $n \to \infty$, $w_{n,k}(t) \to w_k(t)$ where*

(iii) in this regime the $w_k(t)$ solve the a system of ODEs

$$\frac{d}{dt}w_k(t) = \frac{1}{2}\sum_{l=1}^{k-1} l(k-l)w_l(t)w_{k-l}(t) - kw_k(t), \quad k \geq 2,$$

$$\sum_{k=1}^{\infty} kw_k(t) = 1,$$

which are the same as the Smoluchowski coagulation equations for the pure coalescent process until gelation (critical) time but after it remains in a critical state forever with the $w_k(t)$ following a power law stationary distribution

$$w_k = 2\binom{2k-2}{k-1}\frac{1}{k^2}4^{-k} \sim k^{-5/2}$$

(iv) in this regime the $w_k(t)$ now solve a system of ODEs which are similar to the Smoluchowski coagulation equations but now with a fragmentation term

$$\frac{d}{dt}w_k(t) = \frac{1}{2}\sum_{l=1}^{k-1} l(k-l)w_l(t)w_{k-l}(t) - kw_k(t) - \lambda kw_k(t), \quad k \geq 2,$$

$$\frac{d}{dt}w_1(t) = -w_1(t) + \lambda\sum_{l=1}^{\infty} l^2 w_l(t),$$

which has a unique solution which remains subcritical and stationary solution

$$w_k \sim \left(1 - \frac{\lambda^2}{(1+\lambda)^2}\right)^k k^{-5/2}.$$

In both of these cases the random variables $w_{n,k}(t)$ converge to deterministic solutions of ODEs, hence there is no randomness in the infinite limit (with regards to the proportion of clusters of each size).

The questions this thesis will address revolve around the robustness of these results and what changes occur if the dynamics are changed. In particular, what changes if you drop the size-biased nature of the dynamics? That is, if the clusters coalesce or fragment at rates independent of their size. In addition, what if you allow multiple clusters to converge into a single cluster at once? Does the thermodynamic limit still make sense? Do any regimes display the self-organised criticality that regime (iii) does here? These questions will be answered in Chapter 2.

1.3 Exchangeable fragmentation-coalescence processes

After introducing the classical models of fragmentation-coalescence processes, we move on now to discuss the second class of fragmentation-coalescence process we are inter-

ested in. These are the more recently studied exchangeable fragmentation-coalescence processes which have their roots in Kingman's work on partitions and the coalescent around 40 years ago.

1.3.1 Exchangeable random partitions on $\mathbb{N}$

To introduce exchangeable fragmentation-coalescence processes, first the theory behind exchangeable random partitions is needed. Exchangeability and random partitions were first studied by Kingman [44, 45] in relation to his work on population genetics. This section starts with some basic definitions and introduces some of the notation that this thesis will use.

Definitions and notation

Definition 1.12.   *1. A partition of $\mathbb{N}$, $\pi = (\pi_1, \pi_2, \ldots)$, is a collection of subsets of $\mathbb{N}$ such that*

$$\bigcup_{i=1}^{\infty} \pi_i = \mathbb{N}, \quad and \quad \pi_i \cap \pi_j = \varnothing, \quad \forall i \neq j.$$

where the subsets, π_i, (to be referred to as the blocks of the partition) are ordered by least element, that is

$$\min \pi_1 \leq \min \pi_2 \leq \cdots$$

where we take $\min \varnothing = +\infty$.

2. The collection of all partitions of $\mathbb{N}$ will be denoted $\mathcal{P}$.

3. A random partition is a $\mathcal{P}$-valued random variable.

4. The number of non-empty blocks of π shall be denoted

$$\#\pi = \max\{i \in \mathbb{N} : \pi_i \neq \varnothing\}.$$

The ordering by least element is a convention to ensure a unique representative for each partition of $\mathbb{N}$. This has the useful property that $1 \in \pi_1$ for all $\pi \in \mathcal{P}$. There are two partitions which we will give special notation: the trivial partition, $\mathbf{1} = (\mathbb{N}, \varnothing, \varnothing, \ldots)$, and the singleton partition, $\mathbf{0} = (\{1\}, \{2\}, \ldots)$.

In addition, one can define partitions, $\pi^{(n)}$, of the first n natural numbers, which for brevity's sake shall be represented by $[n] := \{1, \ldots, n\}$, in a similar way. The space of all such partitions will be denoted $\mathcal{P}_n$. With this in mind compatibility can now be defined, which gives a vital representation of all elements of $\mathcal{P}$ in terms of elements of $\mathcal{P}_1, \mathcal{P}_2, \ldots$.

Definition 1.13. *A sequence of partitions,* $(\pi^{(1)}, \pi^{(2)}, \ldots)$ *of* $[1], [2], \ldots$ *respectively, is said to be compatible if for all* $i \leq j$

$$\pi^{(j)}\big|_{[i]} = \pi^{(i)},$$

where $\pi\big|_{[n]} = (\pi_1 \cap [n], \pi_2 \cap [n], \ldots)$ *is the restriction of a partition* π *to* $[n]$.

Thus, there exists a representation for all partitions $\pi \in \mathcal{P}$, which will be very useful in later parts.

Lemma 1.14. *A sequence of partitions,* $(\pi^{(1)}, \pi^{(2)}, \ldots)$ *of* $[1], [2], \ldots$ *respectively, is compatible if, and only if, there exists a partition* $\pi \in \mathcal{P}$ *such that*

$$\pi\big|_{[n]} = \pi^{(n)}.$$

Hence, $\mathcal{P}$ can be equipped with a metric

$$d(\pi, \pi') = \frac{1}{\max\left\{n \in \mathbb{N} : \pi\big|_{[n]} = \pi'\big|_{[n]}\right\}},$$

and with this metric $(\mathcal{P}, d)$ is compact [11].

Exchangeability

Now the idea of exchangeability can be introduced. First, note that any partition $\pi \in \mathcal{P}$ defines an equivalence relation on $\mathbb{N}$ in the following way

$$i \overset{\pi}{\sim} j \quad \text{if, and only if,} \quad i, j \in \pi_k \text{ for some } k \in \mathbb{N}.$$

In other words, i and j are in the same block of π. Using this relation, define new partitions using permutations on $\mathbb{N}$, which will be the basis of exchangeability. More formally, for a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ with finite support and a partition $\pi \in \mathcal{P}$, define $\sigma(\pi) \in \mathcal{P}$ by the following relation

$$i \overset{\sigma(\pi)}{\sim} j \quad \text{if, and only if,} \quad \sigma(i) \overset{\pi}{\sim} \sigma(j).$$

Put in other words, the blocks of $\sigma(\pi)$ are given by applying σ^{-1} to the blocks of π, then reordering by least element. The definition of an exchangeable random partition can now be stated.

Definition 1.15. *A random partition,* Π, *is said to be exchangeable if*

$$\sigma(\Pi) =^d \Pi$$

for all permutations $\sigma : \mathbb{N} \to \mathbb{N}$ *with finite support.*

Therefore, **1** and **0** are the only deterministic exchangeable partitions, as they are the only partitions unaffected by all permutations.

A concept of the size of blocks is needed in order to compare them with each other, and to compare partitions with the same number of blocks. This will also provide an important link between exchangeable random partitions and mass partitions which will be introduced in a moment. A count of the number of integers in the block is going to be infinite for most of the blocks, so is not particularly useful, instead we use compatibility of exchangeable random partitions to provide a scaling of this count. This is known as the asymptotic frequency.

Definition 1.16. *The asymptotic frequency of a block, π_k, of a partition, π, is*

$$|\pi_k| = \lim_{n \to \infty} \frac{1}{n} \#\{\pi_k \cap [n]\},$$

when this limit exists.

There are several important properties of exchangeable random partitions which will be needed. This will involve introducing Kingman's paintbox construction of exchangeable random partitions using mass partitions [46].

Definition 1.17. *A mass partition is a sequence $\boldsymbol{s} = (s_1, s_2, \ldots)$ such that*

$$s_1 \geq s_2 \geq \cdots \quad and \quad \sum_{k=1}^{\infty} s_k \leq 1,$$

and the space of all such mass partitions will be denoted $S^{\downarrow}$.

Suppose a mass partition **s** is given, this can be used to partition the unit interval into the following tiling

$$[0, s_1), [s_1, s_1 + s_2), \ldots, \left[\sum_{i=1}^{k-1} s_i, \sum_{i=1}^{k} s_i\right), \ldots$$

then sample $U_1, U_2, \ldots$ IID $U[0, 1]$ random variables and construct an exchangeable partition $\pi \in \mathcal{P}$ as follows

$i \overset{\pi}{\sim} j \iff i = j$, or U_i and U_j landed in the same tile in the unit interval partition.

The first part of the right-hand side of the construction is required in case $\sum_{k=1}^{\infty} s_k < 1$, and so part of the unit interval is not covered by the partition. In this case $\{j\}$ is a singleton block of π. Partitions constructed in this way are exchangeable because for all permutations σ with finite support, $U_{\sigma(1)}, U_{\sigma(2)}, \ldots$ are still an iid sequence of $U[0, 1]$ random variables. All partitions constructed in this way have the following important properties
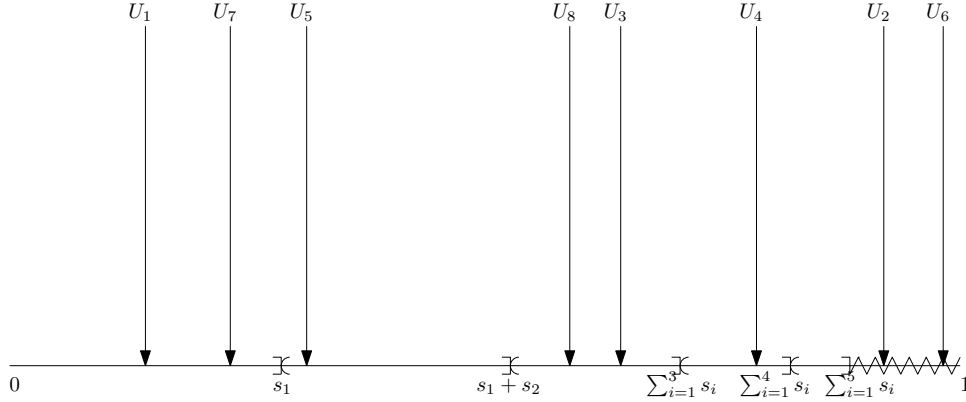
Figure 1-6: *A paintbox construction of an exchangeable partition using $\boldsymbol{s} = (s_1, \ldots, s_5, 0, \ldots)$ restricted to the first eight integers. It gives the partition $\pi^{(8)} = \{\{1,7\}, \{2\}, \{3,8\}, \{4\}, \{5\}, \{6\}\}$.*

Lemma 1.18. *If π is constructed as a paintbox, using a mass partition $\boldsymbol{s}$, then we have that*

1. *$|\pi_i|$ exists for all $i \in \mathbb{N}$, and if you put the sequence $(|\pi_1|, |\pi_2|, \ldots)$ in decreasing order, denoted $|\pi|^{\downarrow}$, this is the mass partition $\boldsymbol{s}$ that was used to generate π,*

2. *$|\pi_i| = 0$ implies that π_i is empty or π_i is a singleton,*

3. *if we have π_i is a singleton for some $i \in \mathbb{N}$, then the asymptotic frequency of the union of all such π_i is positive and equal to $s_0 = 1 - \sum_{k=1}^{\infty} s_k$.*

In particular, blocks are either infinite in size or are singletons, and if there are any singletons, then there are infinitely many of them. Finally, the main result for exchangeable random partitions from [46]: that there is a bijection between exchangeable probability measures on $\mathcal{P}$ and probability measures on $S^{\downarrow}$ is stated below.

Theorem 1.19 (Kingman, 1982)**.** *If π is an exchangeable random partition, then there exists a measure, ν_s, on $S^{\downarrow}$ such that for any bounded measurable function $f : \mathcal{P} \to \mathbb{R}$,*

$$\mathbb{P}(\pi \in \cdot) = \int_{S^{\downarrow}} \mathbb{P}(|\pi|^{\downarrow} \in d\boldsymbol{s}) \mu_{\nu_s}(\cdot),$$

where μ_{ν_s} is the law of the paintbox based on ν_s.

Hence, all the conclusions from Lemma 1.18 apply to all exchangeable random partitions. With the basis behind exchangeable random partitions in place, the next section can now introduce random processes on $\mathcal{P}$.

1.3.2   Exchangeable coalescence processes

Kingman's coalescent

The beginnings of exchangeable coalescent theory can be found in Kingman's work on genealogical structures [46]. These coalescent processes are concerned with observing n such individuals in a haploid population and modelling the family relationships that they have. These processes are defined on partitions of $\{1, \ldots, n\}$, with the integers representing the individuals. A partition, π, represents the genealogical structure when $\#\pi$ of the most recent common ancestors were alive, with a block of elements representing that the common ancestor of those individuals was alive at that time. A figure picturing this is shown below.
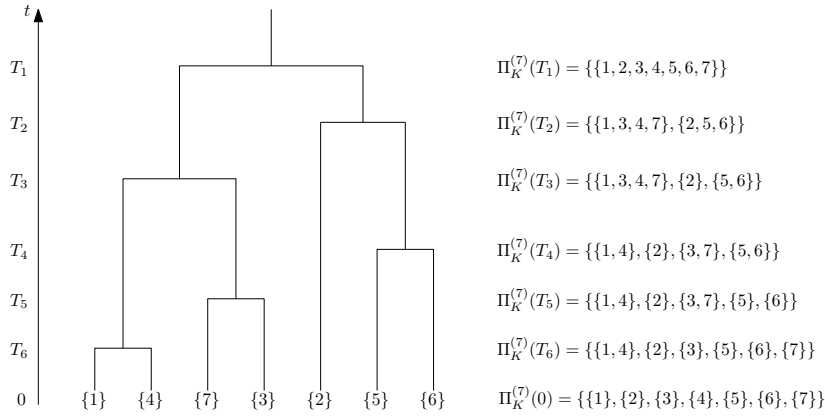
Figure 1-7: *A realisation of Kingman's coalescent restricted to the first seven integers. T_i is the hitting time of a state with $N_t = i$.*

Kingman's coalescent models this structure. Let $\Pi_K^{(n)}$ be the process starting in the state $\{\{1\}, \ldots, \{n\}\}$, and then evolving by each pair of blocks coalescing at rate 1 until it reaches the partition $\{\{1, \ldots, n\}\}$. So, if the number of blocks in $\Pi_K^{(n)}(t)$ is k, it jumps to $k - 1$ at rate $\binom{k}{2}$. An important result of Kingman's, Theorem 3 in [46], is that for all $n \in \mathbb{N}$ these processes can be embedded into a process on partitions of $\mathbb{N}$, using compatibility i.e. Π_K is Kingman's coalescent on $\mathcal{P}$ if for all $n \in \mathbb{N}$, $\Pi_K^{(n)}$, the restriction of Π_K to $\{1, \ldots, n\}$, is distributed like Kingman's coalescent on $\mathcal{P}_n$. It is the properties of these extensions to $\mathcal{P}$ in general that are of most interest.

A rather interesting result for the process Π_K defined on $\mathcal{P}$ is that, while it starts in a state with an infinite number of blocks $\{\{1\}, \{2\}, \ldots\}$, it immediately transitions to a state with finitely many blocks almost surely, as stated in Theorem 4 of [46].

Theorem 1.20. *Let Π_K be Kingman's coalescent, and $\pi \in \mathcal{P}$, then*

$$\mathbb{P}_\pi(\inf\{t \geq 0 : \#\Pi_K(t) < \infty\} = 0) = 1.$$
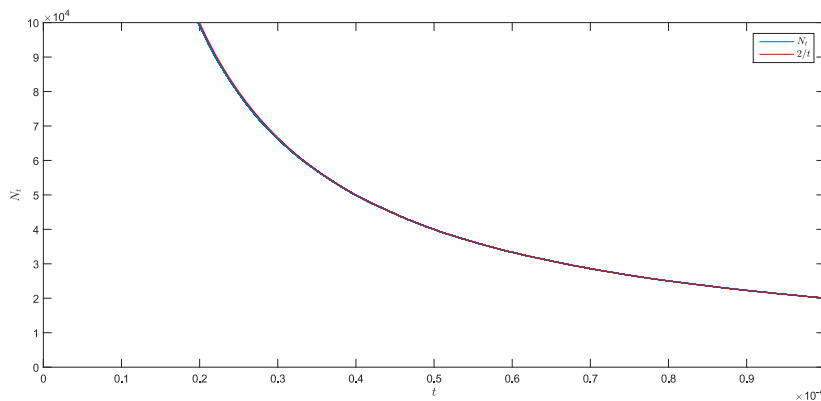
Figure 1-8: *A simulation of the block counting process of Kingman's coalescent. The blue line is the realisation of the simulation; the red line is the theoretical speed of coming down from infinity. Note: N_t does not look random as it is for $t \in (0, 10^{-4})$ when N_t is very large, hence the steps down are small in comparison.*

This property for an exchangeable coalescent process is known as "coming down from infinity" and it is one that will be of interest for exchangeable fragmentation-coalescence processes in general, and is the basis for later results.

Λ-coalescents

In 1999, Pitman [62] and independently Sagitov [65] generalised Kingman's coalescent to exchangeable coalescents that allowed multiple blocks to coalesce into a single block, from here on referred to as simple, as opposed to just binary coalescence. As these processes need to be exchangeable, they need to be completely characterised by the rate at which any k blocks coalesce, when there are b blocks in total, normally denoted $\lambda_{b,k}$, because the rates cannot depend on the sizes of the blocks nor the elements contained in each block. Therefore, the collection

$$(\lambda_{b,k} : b \in \{2, 3, \ldots\}, k \in \{2, \ldots, b\}),$$

should characterise the process Π in this case.

There are restrictions on the form that this collection can take, as the process Π needs to be compatible, that is $\Pi^{(n+1)}|_{[n]}$ needs to be distributed like $\Pi^{(n)}$. Here is a brief heuristic on these restrictions: suppose $\Pi^{(n)}$ has b blocks and look at the rate that k of them coalesce, $\lambda_{b,k}$. As $\Pi^{(n+1)}|_{[n]}$ needs to be distributed like $\Pi^{(n)}$, we look at $\Pi^{(n+1)}$, which has b or $b+1$ blocks. If it has b blocks then k of the merge at rate $\lambda_{b,k}$ and thus $\Pi^{(n+1)}|_{[n]}$ is identical to $\Pi^{(n)}$. If it has $b+1$ blocks, then $\{n+1\}$ is a singleton set and so there are two ways k blocks of $\Pi^{(n+1)}|_{[n]}$ can merge in $\Pi^{(n+1)}$. Either the k blocks out of $b+1$ merge on their own, or the k blocks merge with $\{n+1\}$. Hence, for

compatibility to hold it is required that

$$\lambda_{b+1,k+1} + \lambda_{b+1,k} = \lambda_{b,k}, \quad \forall b \in \{2, 3, \ldots\}, k \in \{2, \ldots, b\},$$

with a more rigorous proof of the result found in Lemma 18 [62]. Pitman's main result in [62], however, goes a further step in classifying these processes; showing that there is a one-to-one correspondence between simple exchangeable coalescent processes and finite measures on [0,1]. The full result is stated below.

Theorem 1.21 (Pitman, 1999). *Let $(\lambda_{b,k} : b \in \{2, 3, \ldots\}, k \in \{2, \ldots, b\})$ be an array of non-negative real numbers. There exists for each $\pi \in \mathcal{P}$ a $\mathcal{P}$-valued coalescent Π with $\Pi(0) = \pi$, whose restriction $\Pi^{(n)}$ is for each n a Markov chain such that, when $\Pi^{(n)}(t)$ has b blocks, each k-tuple of blocks of $\Pi^{(n)}(t)$ is merging to form a single block at rate $\lambda_{b,k}$, if and only if,*

$$\lambda_{b,k} = \int_0^1 x^{k-2}(1-x)^{b-k}\Lambda(dx)$$

for some non-negative finite measure Λ on $[0, 1]$.

There are a few observations to be made about this result.

Remark 1.22. 1. *We may assume that $\Lambda([0,1]) = 1$ as any other possibility, $\widehat{\Lambda}([0,1]) = C > 0$, gives rates which are just a multiple of the equivalent measure scaled by C.*

2. *If $\Lambda = \delta_0$ the process is Kingman's coalescent. If $\Lambda = \delta_1$, the process waits an Exp(1) amount of time, then coalesces into a single block.*

3. *The above two examples are the only coalescents that have one single style of coalescing (only binary, or only everything at once). If $\lambda_{b,k} > 0$ for some $k \neq 2, b$, then $\lambda_{b,k} > 0$ for all $b \in \{2, 3, \ldots\}$, $k \in \{2, \ldots, b\}$. For example, there is no Λ-coalescent that allows only coalescence of three blocks at a time.*

There are similar questions for Λ-coalescents to compare them to Kingman's coalescent. For example, Kingman's coalescent has this property of coming down from infinity, what conditions on Λ mean that the Λ-coalescent comes down from infinity? If Λ has no atom at one, which prevents the possibility of all blocks coalescing at once after an exponential amount of time, then it turns out that there are only two options possible. Suppose $N_t = \#\Pi(t)$, which for coalescent processes is a death process, then either

$$\mathbb{P}(\inf\{t \geq 0 : N_t < \infty\} = 0) = 1, \quad \text{or} \quad \mathbb{P}(\forall t \geq 0, N_t = \infty) = 1,$$

by Proposition 23 and Theorem 4 in [62]. Hence, for Λ-coalescents with no atom at one the coalescent either comes down from infinity or stays infinite. Therefore, a Λ-

coalescent can spend positive time at ∞, then transition into a state with finitely many blocks only if Λ has an atom at one. Proposition 23 in [62] also gives necessary and sufficient conditions for the Λ-coalescent to come down from infinity.

Proposition 1.23. *Let $\Lambda(1) = 0$, and $T_n = \inf\{t \geq 0 : \#\Pi^{(n)} = 1\}$, then Π comes down from infinity if, and only if,*

$$\lim_{n \to \infty} \mathbb{E}[\exp(-\theta T_n)] > 0, \quad \forall \theta > 0.$$

Otherwise, it stays infinite.

This limit can be somewhat challenging to calculate; Schweinsberg [67] gives a different necessary and sufficient condition such that a Λ-coalescent comes down from infinity.

Theorem 1.24 (Schweinsberg, 2000)**.** *Let $\Lambda(1) = 0$ and $\gamma_b = \sum_{k=2}^{b} (k-1)\binom{b}{k}\lambda_{b,k}$. Then Π comes down from infinity if, and only if,*

$$\sum_{b=2}^{\infty} \gamma_b^{-1} < \infty.$$

Otherwise, it stays infinite.

Heuristically, γ_b is the rate at which the number of blocks decreases when the coalescent Π has b blocks. Hence, one would expect that if this was large enough for all b, then the process would come down from infinity. One final result here, which will be important for the next section, from Bertoin and Le Gall in 2006 [21] showed a relationship between Λ-coalescents and continuous-state branching processes.

Theorem 1.25 (Bertoin & Le Gall, 2006)**.** *Let $\Lambda(1) = 0$. Define a function ψ : $(0,\infty) \to (0,\infty)$ as follows*

$$\psi(q) := \int_0^1 (e^{-qx} - 1 + qx)x^{-2}\Lambda(dx).$$

Then, the Λ-coalescent Π comes down from infinity if, and only if, the corresponding continuous-state branching process with branching mechanism ψ goes extinct in finite time almost surely.

Speed of coming down from infinity

As there is a whole class of simple exchangeable coalescent processes that come down from infinity, being able to compare how quickly they do so is an interesting question. This concept is the "speed" of coming down from infinity and is very much related to the small time behaviour of N_t, and is formally defined below.

Definition 1.26. *Suppose Π is an exchangeable coalescent process that comes down from infinity. A function $\nu : (0, \infty) \to (0, \infty)$ is the speed of coming down from infinity if*

$$\frac{N_t}{\nu(t)} \to 1,$$

almost surely, as $t \searrow 0$.

It should be noted that ν is not unique, only its dominant behaviour as $t \searrow 0$ is. The speed of coming down from infinity for the Beta coalescent, where Λ is the Beta$(2 - \alpha, \alpha)$ distribution was found by Berestycki, Berestycki and Schweinsberg in 2008 [13] by comparing them with a continuous state branching process. The answer for general Λ was discovered by Berestycki, Berestycki and Limic in 2010 [12] using a martingale argument, but inspired by the work of Bertoin and Le Gall, and is stated below.

Theorem 1.27 (Berestycki, Berestycki & Limic, 2010)**.** *Let Π be a Λ-coalescent that comes down from infinity and*

$$\psi(q) := \int_0^1 (e^{-qx} - 1 + qx)x^{-2}\Lambda(dx).$$

Then

$$\nu(t) := \inf \left\{ s > 0 : \int_s^\infty \frac{1}{\psi(q)}dq < t \right\}, \quad t > 0,$$

is the speed of coming down from infinity for Π.

Example 1.28. *1. In the case of Kingman's coalescent this result gives $\nu(t) = 2/t$.*

2. If Λ is regularly varying near 0 with index $\alpha \in (1, 2)$, that is

$$\lim_{x \to 0} \frac{\Lambda((\alpha x, 1])}{\Lambda((x, 1])} = d$$

where d is some positive finite constant, then $\nu(t) \sim Ct^{-1/(\alpha-1)}$ as $t \searrow 0$.

A corollary of this result is that Kingman's coalescent is the "quickest" Λ-coalescent that comes down from infinity in that for all Λ-coalescents, Π,

$$N_t \geq \frac{2}{t}(1 - \epsilon),$$

for all $\epsilon > 0$ and t sufficiently small.

Ξ-coalescents

This section will make a few brief comments about coalescents with simultaneous multiple collisions, where many blocks can coalesce into several blocks at the same time,

as opposed to just a single block. They were first studied independently by Schweinsberg [67], and Möhle and Sagitov [59]. These form the coalescent part of the general exchangeable fragmentation-coalescence process and contain Λ-coalescents as a subset. Although not much use will be made of them here, it is still useful to see the most general exchangeable coalescent as it helps with the Poissonian construction of EFC processes. Similar to the Λ-coalescent, the rates of coalescence can conly depend on how many blocks will coalesce, as opposed to the sizes of the blocks, as it needs to be exchangeable. Therefore, Π should be characterised by the collection

$$\left(\lambda_{b;k_1,\ldots,k_r;s} : b \in \{2,3,\ldots\}, r \in \{1,\ldots,b\}, k_1,\ldots,k_r \in \{2,\ldots,b\}, b = s + \sum_{j=1}^{r} k_j \right),$$

where $\lambda_{b;k_1,\ldots,k_r;s}$ is the rate $k_1,\ldots,k_r$ number of blocks coalesce into r separate blocks and s blocks are left unchanged when Π has b blocks.

Theorem 2 in [67] shows that there is a one-to-one correspondence between these coalescents on $\mathcal{P}$ and finite measures Ξ on $\mathcal{S}^{\downarrow}$, which gives the rates above by a similar formula for Λ-coalescents. Also, like Λ-coalescents, one can assume $\Xi(\mathcal{S}^{\downarrow}) = 1$ as otherwise it just affects the rates by a multiplicative constant. This Theorem also concluded that you could decompose Ξ into a Kingman part, and the rest, i.e. that

$$\Xi = c\mathbb{1}_{\{r=1,k_1=2\}} + \nu_{\text{Coag}}$$

where $\nu_{\text{Coag}}(\{0,0,\ldots\}) = 0$ and

$$\int_{\mathcal{S}^{\downarrow}} \left(\sum_{i=1}^{\infty} x_i^2 \right) \nu_{\text{Coag}}(d\mathbf{x}) < \infty.$$

Finally, Proposition 12 in [67] gave a Poissonian construction for Ξ-coalescents that is perhaps simpler to visualise than calculating all the rates in the formulae given for the rates. Let $\epsilon_{i,j}$ be the partition with only one block not a singleton, which is $\{i,j\}$. Using Ξ, you produce a measure on $\mathcal{P}$, C, where

$$C = c \sum_{i<j} \delta_{\epsilon_{i,j}} + \mu_{\nu_{\text{Coag}}}$$

where $\mu_{\nu_{\text{Coag}}}$ is the paintbox measure introduced in Theorem 1.19. Using this measure to choose π', you then define a binary operation $\mathbf{Coag}(\pi,\pi')$ on $\mathcal{P}$ as follows

$$\mathbf{Coag}(\pi,\pi') = \pi'' := \{\pi_1'', \pi_2'', \ldots\}$$

where

$$\pi_i'' = \bigcup_{j \in \pi_i'} \pi_j.$$

Define a Poisson process PPP_C on $[0, \infty) \times \mathcal{P}$ with intensity $\mathrm{d}t \times C(d\pi)$, then if t is an atom time of PPP_C set $\Pi(t) := \mathbf{Coag}(\Pi(t-), \pi(t))$, where $\pi(t)$ is the accompanying mark in $\mathcal{P}$ at the atom time t. Defining a process in such a way gives an exchangeable coalescent with rates given by the measure Ξ as explained above.

1.3.3 Homogeneous fragmentation processes

We move on now to discuss homogeneous fragmentation processes; a subset of exchangeable fragmentation processes. This section will give a brief background and construction of such processes. For a more in-depth look at such processes the reader is directed to Bertoin [16, 18], Bertoin and Rouault [22], Berestycki [10] and Berestycki et. al [14]. Homogeneous processes are a special case of self-similar fragmentations [17, 20] where the index of self-similarity, α, is zero. Roughly speaking, this means that all blocks fragment at the same rate, in the same manner, independent of the sizes of the blocks. This allows such processes to represented by an exchangeable partition valued process on $\mathcal{P}$.

Definition & construction

The definition for homogeneous fragmentations stated here differs slightly from that that Bertoin initially used, but in subsequent papers [22] the following definition is used, and is easier to picture. In addition, everything here is in the same setting as in the previous two subsections about exchangeable random partitions and exchangeable coalescents.

Definition 1.29. *An exchangeable $\mathcal{P}$-valued process $\Pi(\cdot)$ is called a homogeneous fragmentation if for every t_0, t, conditionally on $\Pi(t_0) = (\Pi_1(t_0), \ldots) \in \mathcal{P}$, the partition $\Pi(t + t_0)$ is independent of the partitions $(\Pi(s), 0 \leq s \leq t_0)$ and has the same distribution as the partition obtained by the family of the blocks of the induced partitions $\Gamma_1 \cap \Pi_1(t_0), \Gamma_2 \cap \Pi_2(t_0), \ldots$, where Γ_i are independent copies of $\Pi(t)$.*

Such processes are Feller processes (see Proposition 1 [16]) and like exchangeable coalescents have a Poissonian construction (see Theorem 1 [16]). This will be vital for exchangeable fragmentation-coalescence processes.

First, similarly to results for exchangeable coalescents, there exists a one-to-one correspondence between homogeneous fragmentations, Π, and certain exchangeable measures, F, on $\mathcal{P}$, known as the characteristic measures of homogeneous fragmentations. Such measures have the following properties (see Lemma 1 [16])

1. F is exchangeable,

2. $F(\mathbf{1}) = 0$,

3. $F(\mathcal{P}_n^*) < \infty$,

where $\mathcal{P}_n^*$ is the set of partitions whose restriction to $[n]$ is non-trivial. This third condition is to ensure that the restrictions $\Pi^{(n)}(\cdot)$ are finite rate Markov chains. In Section 1.3.2 it was noted that exchangeable coalescents can be decomposed into a Kingman (continuous) part and a paintbox part. In a similar manner, such measures F can be decomposed into two fundamental fragmentations, erosion, which is a continuous loss of mass from the blocks, and dislocation. Let ϵ_n be the partition with two non-empty blocks, $\mathbb{N}\backslash\{n\}$ and $\{n\}$, then

$$\mathbf{e}(\cdot) = \sum_{n\in\mathbb{N}} \delta_{\epsilon_n}(\cdot)$$

is an exchangeable measure known as the erosion measure. Dislocations are constructed using a Kingman paintbox measure. Let ν_{Disl} be a measure on $\mathcal{S}^\downarrow$ satisfying the conditions $\nu_{\text{Disl}}(\{(1,0,\ldots)\}) = 0$ and

$$\int_{\mathcal{S}^\downarrow} (1 - x_1)\nu_{\text{Disl}}(d\mathbf{x}) < \infty.$$

If you have such a measure, then the paintbox measure on $\mathcal{P}$, $\mu_{\nu_{\text{Disl}}}$, defined in Section 1.3.1 using Kingman's Theorem, is an exchangeable measure that satisfies the three requirements. Note: the integrability condition above on ν_{Disl} is required to ensure that the paintbox measure has the third required property of such measures. Therefore there exists the following decomposition (see Theorem 3 [16]). For all exchangeable measures F on $\mathcal{P}$ with the three required properties, there exists a constant $c_e \geq 0$ and a measure ν_{Disl} on $\mathcal{S}^\downarrow$ such that

$$F = c_e \mathbf{e} + \mu_{\nu_{\text{Disl}}}.$$

Therefore, because of this one-to-one correspondence, a homogeneous fragmentation is entirely characterised by the law of $\Pi(0)$, c_e and ν_{Disl}. Now such processes, Π, can be constructed using a Poisson process. Define an operator which will be the basis of a fragmentation event

$$\mathbf{Frag} : \mathcal{P} \times \mathcal{P} \times \mathbb{N} \to \mathcal{P}$$
$$(\pi, \pi', k) \mapsto \pi''$$

where the blocks of π'' are the blocks π_i where $i \neq k$ and the blocks $\pi_k \cap \pi'_j$ for $j \in \mathbb{N}$, reordered by least element. Let PPP_F be a Poisson point process on $\mathbb{R}^+ \times \mathcal{P} \times \mathbb{N}$ with

intensity measure $dt \otimes F \otimes \#(dk)$ where $\#$ is the counting measure on $\mathbb{N}$. Then define a $\mathcal{P}$-valued process using the atoms $(t, \pi(t), k(t))$ as follows. Let $\Pi(0) = \pi$, where π is some exchangeable random variable in $\mathcal{P}$ then

- if t is not an atom time of PPP_F, then $\Pi(t) = \Pi(t-)$,

- if t is an atom time of PPP_F, then $\Pi(t) = \mathbf{Frag}(\Pi(t-), \pi(t), k(t))$.

Then Theorem 1 in [16] tells us that $\Pi(\cdot)$ is a homogeneous fragmentation with characteristic measure F.

1.3.4 Exchangeable fragmentation-coalescence processes

Exchangeable fragmentation-coalescence (EFC) processes were first introduced by Julien Berestycki in 2004 [11], and can be thought of as the combination of exchangeable coalescents and homogeneous fragmentation processes that were previously covered in this chapter. It is a little explored area compared to what has been achieved when coalescence and fragmentation processes have been considered separately. This is partly because a lot of the nice properties those processes have on their own appear to break down once you put the two together. For example, the previously discussed link between coalescent processes and continuous-state branching processes, which is the basis behind a lot of the results, is lost.

Definition & construction

Definition 1.30 (Exchangeable Fragmentation-Coalescence Process). *A $\mathcal{P}$-valued Markov process $(\Pi(t), t \geq 0)$, is an exchangeable fragmentation-coalescence process if it has the following properties:*

- *it is exchangeable,*

- *its restrictions $\Pi^{(n)}$ are càdlàg finite state Markov chains which can only evolve by fragmentation of one block or by coagulation as defined by the $\mathbf{Frag}$ and $\mathbf{Coag}$ operators.*

There are a few important properties that coalescent and fragmentation processes have when separate, which need to be checked for EFC processes. First, both processes have Poissonian constructions using the exchangeable measures

$$ C = c_k \sum_{i<j} \delta_{\epsilon_{i,j}} + \mu_{\nu_{\mathrm{Coag}}}, \quad \text{and} \quad F = c_e \sum_i \delta_{\epsilon_i} + \mu_{\nu_{\mathrm{Disl}}}, $$

on $\mathcal{P}$. In addition, such processes are uniquely determined by the law of $\Pi(0)$, constant c_k (resp. c_e) and measure on $\mathcal{S}^{\downarrow}$, ν_{Coag} (resp. ν_{Disl}). Proposition 4 of [11] states that combining these five objects completely characterises an EFC process.

The Poissonian construction is as follows. Take two independent Poisson point processes, PPP_C and PPP_F, on $\mathbb{R}^+ \times \mathcal{P}$ and $\mathbb{R}^+ \times \mathcal{P} \times \mathbb{N}$ respectively, with respective intensities $dt \otimes C$ and $dt \otimes F \otimes \#(dk)$ where $\#$ is the counting measure on $\mathbb{N}$. Atom times for PPP_C and PPP_F refer to coalescence and fragmentation actions on Π as explained in the previous sections. Then Proposition 5 of [11] states that this Poissonian construction gives an EFC process, with characteristics c_e, c_k, ν_{Frag}, and ν_{Coag}.

Another important property that coalescent and fragmentation processes have is that the asymptotic frequencies of the blocks of $\Pi(t)$ exist simultaneously for all t almost surely. In addition, if you define X to be the decreasing reordering of the sizes of the blocks, i.e.

$$X(t) = (|\Pi_1(t)|, |\Pi_2(t)|, \ldots)^{\downarrow}$$

then X is a Feller process on $\mathcal{S}^{\downarrow}$. Theorem 7 in [11] shows that this is still the case for EFC processes.

Stationary measure

One difference that EFC processes have is that while the separate processes have trivial stationary measures, namely $\delta_{\mathbf{0}}$ and $\delta_{\mathbf{1}}$, a non-trivial EFC process has a non-trivial stationary measure.

Theorem 1.31 (Berestycki [11])**.** *Given an EFC process Π, there exists a unique exchangeable stationary probability measure ρ on $\mathcal{P}$ and one has*

$$\rho = \delta_{\mathbf{0}} \iff c_k = 0 \text{ and } \nu_{Coag} \equiv 0 \text{ (pure fragmentation)}$$

and

$$\rho = \delta_{\mathbf{1}} \iff c_e = 0 \text{ and } \nu_{Disl} \equiv 0 \text{ (pure coalescence).}$$

Furthermore, $\Pi(\cdot)$ converges in distribution to ρ as $t \to \infty$.

Unfortunately, there is no known way of calculating ρ from $c_e, c_k, \nu_{\text{Disl}}$, and ν_{Coag}. In fact, there is only one known class of non-trivial EFC processes with known stationary measure. This was studied by Bertoin [19], and has a particular simple coalescence and ν_{Disl} is a Poisson-Dirichlet measure, which results in a Poisson-Dirichlet distribution for the stationary measure for $X = \Pi^{\downarrow}$.

However, one can find characteristics of the partitions which ρ charges mass to, in particular the number of blocks those partitions have. For example, Theorems 9 & 10 in [11] give conditions under which ρ charges only partitions with finitely many blocks or not, and conditions under which ρ charges partitions without dust. A little more discussion on these results can be found in Chapter 4.

Path properties

In both fragmentation and coalescence processes the number of blocks, $N_t := \#\Pi(t)$, was a quantity of particular interest, and this remains the same with EFC processes. This process takes values on $\mathbb{N} \cup \{+\infty\}$, but is now not monotonic. It does, however, inherit right-continuity from Π as Proposition 1 of [49] shows. There are some results about N_t that are of particular interest to us. Namely, there is a certain class of EFC processes that are known to come down from infinity. Define

$$\Delta_f := \{x \in \mathcal{S}^\downarrow : \exists k \in \mathbb{N} \text{ s.t. } \sum_{i=1}^{k} x_i = 1\}$$

i.e. all sequences that have mass 1 and finitely many non-zero values, and

$$p_k = \nu_{\text{Disl}}(\{x \in \mathcal{S}^\downarrow : \sum_{i=1}^{k+1} x_i = 1\}).$$

Berestycki, by linking the following class of EFC processes to logistic branching processes studied by Lambert [50], showed the following Proposition.

Proposition 1.32. *If coalescence is purely Kingman, and the fragmentation part has no erosion and satisfies*

$$\nu_{Disl}(\mathcal{S}^\downarrow \backslash \Delta_f) = 0, \quad and \quad \sum_{k=1}^{\infty} p_k \log(k) < \infty,$$

then Π comes down from infinity.

Interestingly, there is no known monotonicity result with regards to such properties. This means "adding more coalescence", letting $\nu_{\text{Coag}}(\mathcal{S}^\downarrow) > 0$, is not known to guarantee that the process still comes down from infinity. This result also raises some questions about Kingman's coalescent, which shall be addressed in detail in Chapter 3. Namely, does there exist a finite rate fragmentation mechanism that prevents Kingman's coalescent from coming down from infinity? The answer to this question is yes, with the details left to Chapter 3.

1.4 This thesis

This thesis details new results in both classical and exchangeable fragmentation-coalescence processes. First, we define a fragmentation-coalescence process on n particles which allows for multiple clusters to coalesce at once (much like the Λ-coalescent) and has a fragmentation mechanism which fragment clusters into clusters of size one. We show that, under mild conditions on the coalescence rates, the number of clusters

of each size converges meaningfully, as $n \to \infty$, to a deterministic object in a similar manner to that of Conjecture 1.2. Then we further study this infinite process and show that, under stationarity, as the fragmentation rate tends to zero this cluster size distribution converges to a universal power law.

Second, we study an EFC process which combines Kingman's coalescent and the same fragmentation mechanism as above. We show that there exists a phase transition between regimes whereby this EFC process comes down from infinity or not. In the case where it does come down from infinity we develop an excursion theory for the process where the excursions are the periods of time spent with finitely many blocks. Using this excursion theory we can compute a number of interesting quantities, such as the speed of coming down from infinity.

1.5 Publication details

The second and third chapters are based on two submitted papers.

CHAPTER 2
[48] *Universality in a class of fragmentation-coalescence processes*, with Andreas E. Kyprianou and Tim Rogers
Preprint arXiv:1504.03196, *under review.*

CHAPTER 3
[49] *A phase transition in excursions from infinity of the 'fast' fragmentation-coalescence process* with Andreas E. Kyprianou, Tim Rogers and Jason Schweinsberg.
Preprint arXiv:1602.05241, *to appear in the Annals of Probability.*

CHAPTER 2

FRAGMENTATION-COALESCENCE PROCESSES ON FINITE SYSTEMS

In this chapter we will introduce and analyse a class of fragmentation-coalescence processes defined on a finite system of particles. As mentioned in the introduction, when such processes are used in applied fields, it is common to use mean-field equations to provide an approximation for the behaviour of the system when the system size is large, see Aldous [2]. An important question therefore arises: are these processes self-averaging so that the mean-field calculations are relevant? In other words, as the system size tends to infinity does the behaviour of the system converge to that of the deterministic mean-field equations?

Another key concept in the understanding of large-scale interacting systems is that of universality – that certain important macroscopic properties often do not depend on the detailed features of the particles and dynamics involved, but rather a much smaller set of properties determine how these processes behave in the thermodynamic limit. In fragmentation-coalescence processes, the antagonistic nature of the driving mechanisms can give rise to self-organised criticality in certain limits, whereby the system evolves to a stationary distribution that exhibits scale-invariant behaviour normally characteristic of a phase transition [63]. It is therefore natural to ask if this behaviour is universal. A distribution, f, is scale-invariant if there exists $x_0 > 0$ and $d \in \mathbb{R}$ such that

$$f(\alpha x) = \alpha^d f(x),$$

for all $\alpha \geq 0, x \geq x_0$, such that $\alpha x \geq x_0$. These distributions are exactly those that have "power-law tails", i.e. distributions, f, such that there exists $x_0 > 0$, $d > 0$ and a constant $c > 0$ such that

$$f(x) = cx^{-d}, \quad \text{for all } x \geq x_0.$$

Here we will show that the answer to the above two questions is yes for a broad class of fragmentation-coalescence processes. The class we study allows for multiple clusters to coalesce simultaneously and varying rates, combined with fragmentation events that shatter clusters to singletons. These events occur at rates independent of cluster size, and some mild technical assumptions are made to control error terms. We show that such processes of this class approach their deterministic mean-field equations as the system size grows, and moreover, that in the limit of small fragmentation rate we observe a universal cluster size distribution with a power-law tail exponent of 3/2.

Finally, we explore this deterministic limit in more detail for certain examples in the class. First, we find a scaling limit for the number of clusters that the system has in the thermodynamic limit of system size. Then we look at the particular case where only binary coalescence is allowed and recover the Smoluchowski coagulation equations under certain fragmentation regimes.

2.1   Introduction and main results

2.1.1   Finite fragmentation-coalescence processes

We now formally introduce the class of processes that we will be working with. They are similar to the one introduced by Ráth and Tóth [63], except we have removed the sized-biased nature by making the fragmentation-coalescence mechanisms act on the clusters rather than the particles in the cluster. In addition, we have generalised the coalescence mechanism.

Consider a collection of n identical particles, grouped together into some number of clusters. We define a stochastic dynamical process as follows:

1. Every subset of k clusters coalesces at rate $\alpha(k)n^{1-k}$, independently of everything else that happens in the system. The coalescing clusters are merged to form a single cluster with size equal to the sum of the sizes of the merged clusters.

2. Clusters fragment at rate $\lambda_n > 0$, independently of everything else that happens in the system. Fragmentation of a cluster of size k results in k 'singleton' clusters of size one.

The standard choice of initial condition is the state with n singleton clusters and this will be the case throughout this chapter. The factor of n^{1-k} appearing in the coalescence rates is included to compensate for the combinatorial explosion in the number of k-tuples as n gets larger. In this way, when there are order n clusters, the global rates of fragmentation and coalescence of any number of clusters are all order n, as long as $\sum_{k=1}^{\infty} \alpha(k)/k! < \infty$. Note that this choice is necessary to ensure that there is a single dominant timescale for the dynamics.
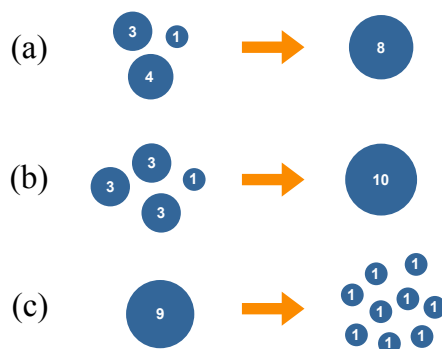
Figure 2-1: *An illustration of possible events (a) coalescence of three clusters, (b) coalescence of four clusters, (c) fragmentation of a cluster.*

For each $n \in \mathbb{N}$, and $j \in \{1, \ldots, n\}$, the state of the system is specified by the number of clusters of size j at time t. To that end we introduce the random variables

$$w_{n,j}(t) := \frac{1}{n} \#\{\text{clusters of size } j \text{ at time } t\}, \quad 1 \leq j \leq n,$$

which we take to be continuous from the left, with right limits. Another natural quantity is the empirical cluster size distribution, defined by

$$p_{n,j}(t) := \frac{\#\{\text{clusters of size } j \text{ at time } t\}}{\#\{\text{clusters at time } t\}}, \quad 1 \leq j \leq n. \tag{2.1}$$

Rather than working with these quantities directly, considerable simplification is possible using the empirical generating functions $G_n : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$

$$G_n(x,t) = \sum_{j=1}^{n} x^j w_{n,j}(t), \qquad n \geq 1, \tag{2.2}$$

and $g_n : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$

$$g_n(x,t) := \sum_{j=1}^{n} x^j p_{n,j}(t) = \frac{G_n(x,t)}{G_n(1,t)}, \qquad n \geq 1. \tag{2.3}$$

2.1.2 Main results

The first main result shows that processes in the class outlined in 2.1.1 are self-averaging with respect to the distribution of cluster sizes as $n \to \infty$.

Theorem 2.1. *Suppose that the coalescence rates $\alpha : \mathbb{N} \to \mathbb{R}^+$ satisfy*

$$\alpha(k) \leq C \exp(\gamma k \ln \ln(k)), \qquad \forall k, \tag{2.4}$$

for some $C > 0$ and $\gamma < 1$. Suppose $\lambda = \lim_{n \to \infty} \lambda_n$ exists, and that $G : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ is the unique solution of the deterministic initial value problem

$$G(x,0) = x\,,$$

$$\frac{\partial G}{\partial t}(x,t) = \lambda(x - G(x,t)) + \sum_{k=2}^{\infty} \frac{\alpha(k)}{k!} \left(G(x,t)^k - kG(1,t)^{k-1}G(x,t) \right). \tag{2.5}$$

(i) *If $\lambda > 0$, then in the limit $n \to \infty$ the empirical generating function $G_n(x,t)$ defined in (2.2) converges to $G(x,t)$ in L^2, uniformly in x and t. That is,*

$$\sup_{x \in [0,1], t \in \mathbb{R}^+} \mathbb{E}\left[(G(x,t) - G_n(x,t))^2 \right] \to 0, \quad as \quad n \to \infty.$$

(ii) *If $\lambda = 0$, then the above result holds for $t \in [0,T]$, for any $T < \infty$.*

Remark 2.2. (i) *It should be noted that condition (2.4) is a sufficient technical condition, but may not be minimal.*

(ii) *There is an intuitive interpretation to the equation that G solves. The λx term is the gain of clusters of size one arising from fragmentation events, with $-\lambda G(x,t)$ being the loss of larger clusters due to fragmentation events. For the second part of the equation, note that the density of k-tuples of blocks is roughly $G(1,t)^k$. Then the $G(x,t)^k$ term denotes gaining larger sized clusters due to coalescence of k smaller clusters (larger powers of x involved), with $-kG(1,t)^{k-1}G(x,t)$ being the loss of k small clusters due to those events (one term with x in the argument).*

2.2 Thermodynamic limit

We will focus our proof on the case of fixed fragmentation rate $\lambda_n \equiv \lambda \in (0, \infty)$; the extension to the joint limit differs only in a few places that we discuss at the end. Besides standard generating function technology, our methods rely on Gronwall's Inequality [76] which we reproduce below for convenience.

Lemma 2.3 (Gronwall's Inequality). *Suppose $f : [0, \infty) \to \mathbb{R}$ is differentiable and satisfies the following differential inequality*

$$\frac{d}{dt} f(t) \le a f(t) + b, \qquad t > 0,$$

where $a, b \in \mathbb{R}$. Then, we have that for all $t \ge 0$

$$f(t) \le \begin{cases} f(0)e^{at} + \frac{b}{a}\left(e^{at} - 1\right) & \text{if } a \neq 0 \\ f(0) + bt & \text{if } a = 0 \end{cases}$$

The idea of the proof of Theorem 2.1 is to bound the derivative of the expected squared difference between $G(x,t)$ and $G_n(x,t)$ in such a way so as we may apply Lemma 2.3, where, here, a will be negative and $b = b_n$ will decay to zero as $n \to \infty$. To achieve this, we will need to prove that the derivative exists and certain error quantities (specified in the next section) converge to zero as n tends to infinity.

2.2.1 Mean-field calculation

We first undertake a mean-field calculation to determine a viable candidate for $\lim_{n\to\infty} G_n$ under self-averaging.

Lemma 2.4. *The expectation of $G_n(x,t)$ is differentiable with respect to t, and satisfies*

$$\frac{\partial}{\partial t}\mathbb{E}[G_n(x,t)]$$
$$= \mathbb{E}\left[\lambda(x - G_n(x,t)) + \sum_{k=2}^{n}\frac{\alpha(k)}{k!}(G_n(x,t)^k - kG_n(1,t)^{k-1}G_n(x,t)) + \beta_n(x,t)\right],$$

such that

$$\mathbb{E}[\beta_n(x,t)] \leq \frac{A}{n}$$

where A is a constant independent of n, x and t.

Proof. We need to consider the left and right derivatives separately. We show the details for the right derivative, the details for the left derivative are almost identical. From now on, let $(\mathcal{F}_n(t))_{t\geq 0}$ be the natural filtration of $(w_{n,j}(t), j = 1, \ldots, n)$, $t \geq 0$, and label the right rate of change of G_n by

$$A_n^+ : [0,1] \times [0,\infty) \times (0,\infty) \to \mathbb{R},$$
$$(x,t,h) \mapsto \frac{1}{h}\mathbb{E}\left[G_n(x,t+h) - G_n(x,t)\big|\mathcal{F}_n(t)\right].$$

Then it is clear that, if the limit exists, $\lim_{h\searrow 0}\mathbb{E}[A_n^+(x,t,h)]$ is the right derivative.

In a small interval of time of length $h \ll 1$, we can expect either nothing to occur, at most one fragmentation to occur, or at most one coagulation to occur; all other possibilities have probability of order $o(h)$ of occurring.

1. Fragmentation of a cluster of size j in time $(t, t+h]$ occurs with probability $h\lambda n w_{n,j}(t) + o(h)$, as $n w_{n,j}(t)$ is the number of clusters of size j. If this happens, we lose one cluster of size j and gain j clusters of size 1.

2. Coagulation of k clusters in time $(t, t+h]$ occurs with probability $h(\alpha(k)/n^{k-1})N_n(l_1, \ldots, l_k, t) + o(h)$, where $l_1, \ldots, l_k$ are the cluster sizes and $N_n(l_1, \ldots, l_k, t)$ is the number of different combinations of clusters sized $l_1, \ldots, l_k$

at time t. In this event we lose clusters of size $l_1, \dots, l_k$ and gain a single cluster of size $l_1 + \cdots + l_k$

Summing over all possibilities we find that

$$A_n^+(x,t,h) = \lambda \sum_{j=1}^{n} jx w_{n,j}(t) - \lambda \sum_{j=1}^{n} x^j w_{n,j}(t)$$

$$+ \sum_{k=2}^{n} \frac{\alpha(k)}{k! n^k} \sum_{l_1=1}^{n} \cdots \sum_{l_k=1}^{n} \left(\prod_{i=1}^{k} x^{l_i} - \sum_{i=1}^{k} x^{l_i} \right) N_n(l_1, \dots, l_k, t) + \frac{o(h)}{h}$$

where the contribution on the first line is the result of possible fragmentation events, and the second line contains the contribution from possible coalescence events.

As n grows large, we claim that the dominant contribution to the combinatorial factor $N_n(l_1, \dots, l_k, t)$ is simply $n^k w_{n,l_1}(t) \cdots w_{n,l_k}(t)$. If the cluster sizes are distinct then this is the only term, otherwise, there is a subdominant correction resulting from the fact that clusters cannot coalesce with themselves. Labelling this correction as $\beta_n(x,t)$, which we will later bound, we obtain

$$A_n^+(x,t,h) = \lambda \sum_{j=1}^{n} jx w_{n,j}(t) - \lambda \sum_{j=1}^{n} x^j w_{n,j}(t)$$

$$+ \sum_{k=2}^{n} \frac{\alpha(k)}{k!} \sum_{l_1=1}^{n} \cdots \sum_{l_k=1}^{n} \left(\prod_{i=1}^{k} x^{l_i} - \sum_{i=1}^{k} x^{l_i} \right) w_{n,l_1}(t) \cdots w_{n,l_k}(t)$$

$$+ \beta_n(x,t) + \frac{o(h)}{h},$$

$$= \lambda(x - G_n(x,t)) + \sum_{k=2}^{n} \frac{\alpha(k)}{k!} \left(G_n(x,t)^k - k G_n(1,t)^{k-1} G_n(x,t) \right)$$

$$+ \beta_n(x,t) + \frac{o(h)}{h}. \tag{2.6}$$

The result is identical for the left derivative.

We must now prove that $\beta_n(x,t) \to 0$ as $n \to \infty$, uniformly in $x \in [0,1]$ and $t \geq 0$. That is, for all possible configurations of $l_1, \dots, l_k$ we must bound the difference between $N_n(l_1, \dots, l_k, t)$ and $n^k w_{n,l_1}(t) \cdots w_{n,l_k}(t)$. To each such configuration $l_1, \dots, l_k$ we associate a partition π of $\{1, \dots, k\}$ by $l_u = l_v$ if and only if $u, v \in \pi_i$ for some i. In this way, the set $\mathcal{P}_k$ of partitions $\{1, \dots, k\}$ enumerates all the ways we could have chosen k different cluster sizes with multiplicity. Summing over this set we obtain

$$\beta_n(x,t) = \sum_{k=2}^{n} \frac{\alpha(k)}{k!} \sum_{\pi \in \mathcal{P}_k} \sum_{l_1=1}^{n} \cdots \sum_{l_{|\pi|}=1}^{n} \gamma_n(\pi, x, t), \tag{2.7}$$

where

$$\gamma_n(\pi, x, t) = \left(\prod_{i=1}^{|\pi|} x^{|\pi_i| l_i} - \sum_{i=1}^{|\pi|} |\pi_i| x^{l_i} \right) \left(\prod_{i=1}^{|\pi|} \frac{(n w_{n,l_i}(t))_{|\pi_i|}}{n^{|\pi_i|}} - \prod_{i=1}^{|\pi|} w_{n,l_i}(t)^{|\pi_i|} \right).$$

Recall that $|\pi|$ denotes the number of clusters of $\pi \in \mathcal{P}_k$, $|\pi_i|$ denotes the cardinality of the ith cluster of π, and $(y)_z$ is the Pochhammer symbol. The first bracket of γ_n is the change to $G_n(x, t)$ when the coalescence occurs and is less than k, as $x \in [0, 1]$. The second bracket is the correction to $N_n(l_1, \ldots, l_k, t)$ and is equal to the value of $N_n(l_1, \ldots, l_k, t)$ minus the dominant term we have already taken out.

Each multinomial in the second bracket of γ_n, when expanded, has at most 2^k terms with varying powers of $1/n$ larger than or equal to one. Also, we can see that all the terms are divisible by $w_{n,l_1}(t) \cdots w_{n,l_{|\pi|}}(t)$ and have a coefficient that is less than or equal to $(k-1)^j$, where j is the power of $1/n$ in that term. Hence,

$$|\beta_n(x, t)| \leq \sum_{k=2}^{n} \frac{\alpha(k)}{(k-1)!} \sum_{\pi \in \mathcal{P}_k} \sum_{l_1=1}^{n} \cdots \sum_{l_{|\pi|}=1}^{n} \frac{2^k(k-1)}{n} w_{n,l_1}(t) \cdots w_{n,l_{|\pi|}}(t).$$

Summing over $l_1, \ldots, l_{|\pi|}$ we have that,

$$|\beta_n(x, t)| \leq \frac{1}{n} \sum_{k=2}^{n} \frac{\alpha(k) 2^k}{(k-2)!} \sum_{\pi \in \mathcal{P}_k} G_n(1, t)^{|\pi|}.$$

Then noting that $G_n(1, t)^{|\pi|} \leq 1$ for all π, we have

$$|\beta_n(x, t)| \leq \frac{1}{n} \sum_{k=2}^{n} \frac{\alpha(k) 2^k}{(k-2)!} B_k,$$

where B_k denotes the kth Bell number and is the size of the set $\mathcal{P}_k$. We bound B_k using a recent result from [9], who extended Dobinksi's formula for the Bell numbers to a Bell function on $\mathbb{R}^+$, then bounding this function using a version of Stirling's formula. Therefore we obtain

$$|\beta_n(x, t)| \leq \frac{1}{n} \left| \sum_{k=2}^{n} \frac{\alpha(k)}{(k-2)!} \left(\frac{1.584k}{\ln(k+1)} \right)^k \right|.$$

If we look at the behaviour of the summands for large k, we see that

$$\frac{1}{(k-2)!} \left(\frac{1.584k}{\ln(k+1)} \right)^k \sim k^{3/2} \exp\left(-k \ln\ln(k) + O(k) \right).$$

And so we can conclude that,

$$|\beta_n(x,t)| \leq \frac{A}{n},$$

where

$$A = \sum_{k=2}^{\infty} \frac{\alpha(k)}{(k-2)!} \left(\frac{1.584k}{\ln(k+1)} \right)^k,$$

as claimed, thus motivating our assumption (2.4) on α.

All that remains to be shown is that

$$\lim_{h \to 0} \mathbb{E}\left[A_n^+(x,t,h) \right] = \mathbb{E}\left[\lim_{h \to 0} A_n^+(x,t,h) \right] \tag{2.8}$$

and then this, combined with what has already been proved, will give us the desired result. We will again look at left and right limits for (2.8) and know that they are equal. We only give the proof of the right limit as the proof of the left limit is almost identical. From equation (2.6) we know that, for all x, t, and h,

$$A_n^+(x,t,h) = \lambda(x - G_n(x,t)) + \sum_{k=2}^{n} \frac{\alpha(k)}{k!} (G_n(x,t)^k - kG_n(1,t)^{k-1}G_n(x,t))$$
$$+ \beta_n(x,t) + \frac{o(h)}{h},$$

which, as $G_n(x,t)^k \leq 1$ and $G_n(x^k,t) \leq 1$ for all $k \in \mathbb{N}$, means that

$$A_n^+(x,t,h) \leq 2\lambda + \sum_{k=2}^{\infty} \frac{\alpha(k)}{(k-1)!} + \frac{A}{n} + \frac{o(h)}{h}.$$

We conclude that, for suitably small h,

$$A_n^+(x,t,h) \leq 2\lambda + 1 + \sum_{k=2}^{\infty} \frac{\alpha(k)}{(k-1)!} + \frac{A}{n}, \tag{2.9}$$

which is finite by assumption (2.4). By Dominated Convergence the desired result holds. $\qquad\square$

2.2.2 Solution of the PDE

Before we can prove Theorem 2.1, we must first check it is well-posed. That is, the PDE (2.5) has a unique solution. The key tool for proving this result is the Picard-Lindehöf Theorem.

Theorem 2.5. *(Picard-Lindehöf) Let $F : \mathbb{R} \times \mathbb{R}^+$ be locally Lipschitz continuous in*

the first argmuent at x_0, with Lipschitz constant L. Then solutions of the initial value problem

$$x'(t) = F(x(t), t), \qquad x(0) = x_0 \tag{2.10}$$

exist and are unique up to any $T < 1/L$.

Lemma 2.6. *The PDE* (2.5) *has a unique solution.*

Proof. First look at the equation for $G(1, t)$. We have

$$\frac{\partial G(1, t)}{\partial t} = F_1(G(1, t)), \tag{2.11}$$

where

$$F_1(g) = \lambda(1 - g) + \sum_{k \geq 2} \frac{\alpha(k)}{k!}(1 - k)g^k. \tag{2.12}$$

The derivative of F_1 is uniformly bounded on $g \in [0, 1]$ since

$$F_1'(g) = -\lambda - \sum_{k \geq 2} \frac{\alpha(k)}{(k - 2)!}g^{k-1}, \tag{2.13}$$

and thus

$$\sup_{g \in [0,1]} \left| F_1'(g) \right| = \lambda + \sum_{k \geq 2} \frac{\alpha(k)}{(k - 2)!} = L_1 < \infty. \tag{2.14}$$

So F is uniformly Lipschitz with constant L_1, and solutions for $G(1, t)$ exist and are unique up to $T = 1/L_1$. In fact, because the Lipschitz constant is the same for all g, we can stitch together as many intervals of length $1/L_1$ as we like to obtain existence and uniqueness for all T.

Now consider any other x,

$$\frac{\partial G(x, t)}{\partial t} = F_x(G(x, t), t), \tag{2.15}$$

where

$$F_x(g, t) = \lambda(x - g) + \sum_{k \geq 2} \frac{\alpha(k)}{k!}\left(g^k - kG(1, t)^{k-1}g\right). \tag{2.16}$$

The derivative is again bounded:

$$F_x'(g, t) = -\lambda + \sum_{k \geq 2} \frac{\alpha(k)}{(k - 1)!}\left(g^{k-1} - G(1, t)^{k-1}\right), \tag{2.17}$$

and $G(1, t) \in [0, 1]$ so an easy bound is

$$\sup_{g \in [0,1], t \geq 0} \left| F_x'(g, t) \right| = \lambda + 2\sum_{k \geq 2} \frac{\alpha(k)}{(k - 1)!} = L < \infty. \tag{2.18}$$

Since this is the same for all x, g and t we get existence and uniqueness globally. $\qquad \square$

2.2.3 Self-averaging

With the mean-field behaviour determined, we proceed to the proof of L^2 convergence for sample paths. The following two lemmas will help us bound the expected squared difference of G_n, in a small interval of time h.

Lemma 2.7. *Define the function $C_n : [0, \infty) \to \mathbb{R}$ as follows*

$$C_n(t) = \mathbb{E}\left[\sum_{j=1}^{n} j^2 w_{n,j}(t)\right],$$

for $n \geq 1$ and $t \geq 0$. Then we have that

$$\sup_{n \in \mathbb{N}} C_n(t) \leq D,$$

where D is a constant independent of x and t.

Proof. Again, this proof will rely on Gronwall's inequality. Hence, we look at the derivative of C_n,

$$\frac{d}{dt}C_n(t) = \lim_{h \to 0} \frac{1}{h}\left(\mathbb{E}\left[\sum_{j=1}^{n} j^2 w_{n,j}(t+h)\right] - \mathbb{E}\left[\sum_{j=1}^{n} j^2 w_{n,j}(t)\right]\right),$$

$$= \lim_{h \to 0} \frac{1}{h}\mathbb{E}\left[\mathbb{E}\left[\sum_{j=1}^{n} j^2 w_{n,j}(t+h) - \sum_{j=1}^{n} j^2 w_{n,j}(t)\bigg| \mathcal{F}_n(t)\right]\right], \quad t \geq 0.$$

Since the internal expectation is bounded above by $n+1$, a similar dominated convergence argument to the one involved in the proof of Lemma 2.4 gives us

$$\frac{d}{dt}C_n(t) = \mathbb{E}\left[\lim_{h \to 0} \frac{1}{h}\mathbb{E}\left[\sum_{j=1}^{n} j^2 w_{n,j}(t+h) - \sum_{j=1}^{n} j^2 w_{n,j}(t)\bigg| \mathcal{F}_n(t)\right]\right], \quad t \geq 0.$$

We proceed by again considering possible changes in a small time period $[t, t+h)$. This

time we obtain

$$\lim_{h \to 0} \frac{1}{h} \mathbb{E}\left[\sum_{j=1}^{n} j^2 (w_{n,j}(t+h) - w_{n,j}(t)) \Big| \mathcal{F}_n(t)\right]$$

$$= -\lambda \sum_{j=1}^{n} (j^2 - j) w_{n,j}(t)$$

$$+ \sum_{k=2}^{n} \frac{\alpha(k)}{k!} \sum_{l_1=1}^{n} \cdots \sum_{l_k=1}^{n} \left[\left(\sum_{i=1}^{k} l_i\right)^2 - \sum_{i=1}^{k} l_i^2\right] N_n(l_1, \ldots, l_k, t)$$

which, using a similar argument to that used when bounding $\beta_n(x,t)$, gives that

$$\lim_{h \to 0} \frac{1}{h} \mathbb{E}\left[\sum_{j=1}^{n} j^2 (w_{n,j}(t+h) - w_{n,j}(t)) \Big| \mathcal{F}_n(t)\right]$$

$$\leq \lambda - \lambda \sum_{j=1}^{n} j^2 w_{n,j}(t) + \sum_{k=2}^{n} \frac{\alpha(k)}{k!} \sum_{l_1=1}^{n} \cdots \sum_{l_k=1}^{n} \left(\sum_{i<j} l_i l_j\right) w_{n,l_1}(t) \cdots w_{n,l_k}(t)$$

$$+ 2\sum_{k=2}^{n} \frac{\alpha(k)}{k!} \frac{2^k (k-1)}{n} \sum_{\pi \in \mathcal{P}_k} \sum_{l_1=1}^{n} \cdots \sum_{l_{|\pi|}=1}^{n} \left(\sum_{i<j} |\pi_i||\pi_j| l_i l_j\right) w_{n,l_1}(t) \cdots w_{n,l_{|\pi|}}(t)$$

$$\leq \lambda - \lambda \sum_{j=1}^{n} j^2 w_{n,j}(t) + 2\sum_{k=2}^{n} \frac{\alpha(k)}{k!} \binom{k}{2} G_n(1,t)^{k-2}$$

$$+ \frac{2}{n} \sum_{k=2}^{n} \frac{\alpha(k)k}{k!} \sum_{\pi \in \mathcal{P}_k} \binom{k}{2} G_n(1,t)^{|\pi|-2}$$

$$\leq \lambda - \lambda \sum_{j=1}^{n} j^2 w_{n,j}(t) + \sum_{k=2}^{n} \frac{\alpha(k)}{(k-2)!} G_n(1,t)^{k-2} + \frac{1}{n} \sum_{k=2}^{n} \frac{\alpha(k)k}{(k-2)!} \left(\frac{1.584k}{\ln(k+1)}\right)^k.$$

Hence we have

$$\frac{d}{dt} C_n(t) = -\lambda C_n(t) + \lambda + \sum_{k=2}^{n} \frac{\alpha(k)}{(k-2)!} \mathbb{E}[G_n(1,t)^{k-2}] + \frac{B}{n},$$

$$\leq -\lambda C_n(t) + \lambda + \sum_{k=2}^{n} \frac{\alpha(k)}{(k-2)!} + \frac{B}{n},$$

where

$$B = \sum_{k=2}^{\infty} \frac{\alpha(k)k}{(k-2)!} \left(\frac{1.584k}{\ln(k+1)}\right)^k. \tag{2.19}$$

Then using Gronwall's Inequality we see that

$$C_n(t) \leq \left(\lambda + \sum_{k=2}^{n} \frac{\alpha(k)}{(k-2)!} + \frac{B}{n} \right) \left(\frac{1}{\lambda} - \frac{1}{\lambda} e^{-\lambda t} \right), \tag{2.20}$$

$$\leq \frac{1}{\lambda} \left(\lambda + \sum_{k=2}^{n} \frac{\alpha(k)}{(k-2)!} + \frac{B}{n} \right).$$

Hence,

$$\sup_{n \in \mathbb{N}} C_n(t) \leq D,$$

where

$$D = \frac{1}{\lambda} \left(\lambda + \sum_{k=2}^{\infty} \frac{\alpha(k)}{(k-2)!} + B \right),$$

as required. $\qquad \square$

Lemma 2.8. *Define $D_n : [0,1] \times [0,\infty) \to \mathbb{R}$ as follows,*

$$D_n(x,t) := \mathbb{E}\left[\lim_{h \to 0} \frac{1}{h} \mathbb{E}\left[(G_n(x,t+h) - G_n(x,t))^2 \big| \mathcal{F}_n(t) \right] \right], \quad x \in [0,1], \ t \geq 0,$$

Then

$$D_n(x,t) \leq \frac{E}{n},$$

where E is a constant independent of n, x and t. Thus D_n converges to zero as n tends to infinity, uniformly in x and t.

Proof. Once again we use the analysis of small time periods discussed in the proof of Lemma 2.4. It is a simple matter of algebra to show that

$$D_n(x,t) = \mathbb{E}\Bigg[\frac{\lambda}{n} \sum_{j=1}^{n} (jx - x^j)^2 w_{n,j}(t)$$

$$+ \frac{1}{n} \sum_{k=2}^{\infty} \frac{\alpha(k)}{k!} \sum_{l_1=1}^{n} \cdots \sum_{l_k=1}^{n} \left(\prod_{i=1}^{k} x^{l_i} - \sum_{i=1}^{k} x^{l_i} \right)^2 N_n(l_1, \ldots, l_k, t) \Bigg],$$

hence, using the same techniques as were used to bound $\beta_n(x,t)$, we see that

$$D_n(x,t) \leq \mathbb{E}\left[\frac{\lambda}{n}x^2C_n(t) - \frac{2\lambda}{n}x^2\frac{\partial}{\partial x}G_n(x,t) + \frac{\lambda}{n}G_n(x^2,t) + \frac{1}{n}\sum_{k=2}^{n}\frac{\alpha(k)}{k!}G_n(x^2,t)^k\right.$$

$$-\frac{2}{n}\sum_{k=2}^{n}\frac{\alpha(k)}{k!}kG_n(x^2,t)G_n(x,t)^{k-1} + \frac{1}{n}\sum_{k=2}^{n}\frac{\alpha(k)}{k!}kG_n(x^2,t)G_n(1,t)^{k-1}$$

$$\left.+\frac{1}{n}\sum_{k=2}^{n}\frac{\alpha(k)}{(k-2)!}G_n(x,t)^2G_n(1,t)^{k-2}\right] + \frac{1}{n^2}\sum_{k=2}^{n}\frac{\alpha(k)k}{(k-2)!}\left(\frac{1.584k}{\ln(k+1)}\right)^k.$$

Applying the same techniques as in the proof of Lemmas 2.4 and 2.7, along with the bound $G_n(x^j,t)^l \leq 1$ for all $j, l \geq 1$, we obtain

$$D_n(x,t) \leq \frac{1}{n}\left(2\lambda + \sum_{k=2}^{n}\alpha(k)\left(\frac{1}{k!} + \frac{3}{(k-1)!} + \frac{2}{(k-2)!}\right) + \frac{B}{n}\right),$$

$$\leq \frac{E}{n},$$

where

$$E = 2\lambda + \sum_{k=2}^{\infty}\alpha(k)\left(\frac{1}{k!} + \frac{3}{(k-1)!} + \frac{2}{(k-2)!}\right) + B,$$

as required. $\qquad\square$

With the help of the above three lemmas we are now ready to prove Theorem 2.1.

Proof of Theorem 1. (i) Recall from (2.5) that we write $G(x,t)$ for the solution of the differential equation

$$\frac{\partial G}{\partial t}(x,t) = \lambda(x - G(x,t)) + \sum_{k=2}^{\infty}\frac{\alpha(k)}{k!}\left(G(x,t)^k - kG(1,t)^{k-1}G(x,t)\right),$$

$$G(x,0) = x.$$

We start noting that in the trivial case $x = 0$, we have for all $t \geq 0$ that

$$G_n(0,t) = G(0,t) = 0.$$

Now, for strictly positive x we define the function $Y_{n,x}(t) : [0,\infty) \to \mathbb{R}$ as follows:

$$Y_{n,x}(t) = \mathbb{E}\left[(G(x,t) - G_n(x,t))^2\right], \quad t \geq 0.$$

We will use Gronwall's Inequality, so we look at the derivative of $Y_{n,x}(t)$ with respect

to t.

$$\frac{dY_{n,x}}{dt}(t) = \lim_{h \to 0} \frac{1}{h} \mathbb{E}\big[(G(x,t+h) - G_n(x,t+h))^2 - (G(x,t) - G_n(x,t))^2\big],$$

$$= \frac{\partial}{\partial t}\big(G(x,t)^2\big) + \lim_{h \to 0} \frac{1}{h} \mathbb{E}\big[G_n(x,t+h)^2 - G_n(x,t)^2\big]$$

$$- 2\lim_{h \to 0} \frac{1}{h}\big(G(x,t+h)\mathbb{E}\big[G_n(x,t+h)\big] - G(x,t)\mathbb{E}\big[G_n(x,t)\big]\big),$$

$$= 2G(x,t)\frac{\partial}{\partial t}G(x,t) + \lim_{h \to 0} \frac{1}{h} \mathbb{E}\big[(G_n(x,t+h) - G_n(x,t))^2\big]$$

$$+ 2\lim_{h \to 0} \frac{1}{h} \mathbb{E}\big[G_n(x,t)(G_n(x,t+h) - G_n(x,t))\big]$$

$$- 2\lim_{h \to 0} \frac{1}{h}\big(G(x,t+h)\mathbb{E}\big[G_n(x,t+h)\big] - G(x,t+h)\mathbb{E}\big[G_n(x,t)\big]\big)$$

$$- 2\lim_{h \to 0} \frac{1}{h}\big(G(x,t+h)\mathbb{E}\big[G_n(x,t)\big] - G(x,t)\mathbb{E}\big[G_n(x,t)\big]\big)$$

$$= 2G(x,t)\frac{\partial}{\partial t}G(x,t) + \lim_{h \to 0} \frac{1}{h} \mathbb{E}\big[\mathbb{E}\big[(G_n(x,t+h) - G_n(x,t))^2\big|\mathcal{F}_n(t)\big]\big]$$

$$+ 2\lim_{h \to 0} \frac{1}{h} \mathbb{E}\big[G_n(x,t)\mathbb{E}\big[G_n(x,t+h) - G_n(x,t)\big|\mathcal{F}_n(t)\big]\big]$$

$$- 2G(x,t)\lim_{h \to 0} \frac{1}{h} \mathbb{E}\big[\mathbb{E}\big[G_n(x,t+h) - G_n(x,t)\big|\mathcal{F}_n(t)\big]\big]$$

$$- 2\frac{\partial}{\partial t}G(x,t)\mathbb{E}\big[G_n(x,t)\big].$$

Using dominated convergence, via (2.9), we can take the limits inside the expectation to get

$$\frac{dY_{n,x}}{dt}(t) = 2G(x,t)\frac{\partial}{\partial t}G(x,t) + \mathbb{E}\left[\lim_{h \to 0} \frac{1}{h} \mathbb{E}\big[(G_n(x,t+h) - G_n(x,t))^2\big|\mathcal{F}_n(t)\big]\right]$$

$$+ 2\mathbb{E}\left[G_n(x,t)\lim_{h \to 0} \frac{1}{h} \mathbb{E}\big[G_n(x,t+h) - G_n(x,t)\big|\mathcal{F}_n(t)\big]\right]$$

$$- 2G(x,t)\mathbb{E}\left[\lim_{h \to 0} \frac{1}{h} \mathbb{E}\big[G_n(x,t+h) - G_n(x,t)\big|\mathcal{F}_n(t)\big]\right]$$

$$- 2\frac{\partial}{\partial t}G(x,t)\mathbb{E}\big[G_n(x,t)\big]. \tag{2.21}$$

Using Lemma 2.4, Lemma 2.8, and the definition of G in (2.5) we see that

$$
\begin{aligned}
\frac{d}{dt} Y_{n,x}(t) &= 2G(x,t)\left(\lambda(x - G(x,t)) + \sum_{k=2}^{\infty} \frac{\alpha(k)}{k!}\left(G(x,t)^k - kG(1,t)^{k-1}G(x,t) \right) \right) \\
&\quad + 2\mathbb{E}\Bigg[G_n(x,t)\Big(\lambda(x - G_n(x,t)) + \beta_n(x,t) \\
&\qquad\qquad + \sum_{k=2}^{n} \frac{\alpha(k)}{k!}(G_n(x,t)^k - kG_n(1,t)^{k-1}G_n(x,t)) \Big)\Bigg] \\
&\quad - 2G(x,t)\mathbb{E}\Bigg[\lambda(x - G_n(x,t)) + \beta_n(x,t) \\
&\qquad\qquad + \sum_{k=2}^{n} \frac{\alpha(k)}{k!}(G_n(x,t)^k - kG_n(1,t)^{k-1}G_n(x,t)) \Bigg] \\
&\quad - 2\mathbb{E}[G_n(x,t)]\Big(\lambda(x - G(x,t)) \\
&\qquad\qquad + \sum_{k=2}^{\infty} \frac{\alpha(k)}{k!}\left(G(x,t)^k - kG(1,t)^{k-1}G(x,t) \right) \Big) \\
&\quad + D_n(x,t), \\
&= -2\lambda Y_{n,x}(t) + D_n(x,t) + 2\mathbb{E}[(G_n(x,t) - G(x,t))\beta_n(x,t)] \\
&\quad + 2\sum_{k=2}^{n} \frac{\alpha(k)}{k!}\mathbb{E}\Bigg[(G(x,t) - G_n(x,t))\Big((G(x,t)^k - G_n(x,t)^k) \\
&\qquad\qquad - k(G(1,t)^{k-1}G(x,t) - G_n(1,t)^{k-1}G_n(x,t)) \Big)\Bigg] \\
&\quad + 2\sum_{k=n+1}^{\infty} \frac{\alpha(k)}{k!}\mathbb{E}\left[(G(x,t) - G_n(x,t))G(x,t)^k \right] \\
&\quad - 2\sum_{k=n+1}^{\infty} \frac{\alpha(k)}{k!}k\mathbb{E}\left[(G(x,t) - G_n(x,t))G(1,t)^{k-1}G(x,t) \right]. \qquad (2.22)
\end{aligned}
$$

Note that the right hand side of (2.22) depends on $G_n(1,t)$. For the boundary case $x = 1$, however, we have a closed expression in $G_n(1,t)$. The plan is thus to first show the Theorem holds for $x = 1$ and use this to complete the proof for general x.

Lemma 2.9. *Define* $X_n(t) := Y_{n,1}(t)$. *Then*

$$
\sup_{t \geq 0} X_n(t) \leq \frac{H_n}{2\lambda}, \qquad (2.23)
$$

where

$$
H_n := \frac{1}{n}\left(2A + E + \sum_{k=2}^{\infty} \frac{2\alpha(k)}{(k-2)!} \right),
$$

In particular, $H_n \to 0$ as $n \to \infty$ because of assumption (2.4).

Proof. Substituting $x = 1$ into (2.22) we see that

$$\frac{dX_n}{dt}(t) = -2\lambda X_n(t) + D_n(1,t) + 2\mathbb{E}[(G_n(1,t) - G(1,t))\beta_n(1,t)]$$

$$+ 2\sum_{k=2}^{n} \frac{\alpha(k)}{k!}\mathbb{E}\left[(G(1,t) - G_n(1,t))(G(1,t)^k - G_n(1,t)^k)\right](1-k)$$

$$+ 2\sum_{k=n+1}^{\infty} \frac{\alpha(k)}{k!}\mathbb{E}[(G(1,t) - G_n(1,t))G(1,t)^k(1-k).$$

Now, $(G(1,t) - G_n(1,t))(G(1,t)^k - G_n(1,t)^k) \geq 0$, which means the second line above is negative. Hence, together with Lemma 2.8, we have

$$\frac{dX_n}{dt}(t) \leq -2\lambda X_n(t) + \frac{1}{n}(E + 2A) + \sum_{k=n+1}^{\infty} \frac{2\alpha(k)}{(k-1)!} \leq -2\lambda X_n(t) + H_n \qquad (2.24)$$

so that, by Gronwall's Inequality, we have

$$X_n(t) \leq \frac{H_n}{2\lambda}\left(1 - e^{-2\lambda t}\right),$$
$$\leq \frac{H_n}{2\lambda},$$

as required. $\qquad \qquad \square$

We can now use this bound to complete the proof of Theorem 2.1.

Proof of Theorem 1 (cont.) (i) Continuing from (2.22), we have that

$$\frac{d}{dt}Y_{n,x}(t) \leq -2\lambda Y_{n,x}(t) + 2\sum_{k=2}^{\infty} S_k + H_n, \qquad (2.25)$$

where

$$S_k = \frac{\alpha(k)}{k!}\mathbb{E}\Bigg[G(x,t)^{k-1}\left(G(x,t)^2 - G(x,t)G_n(x,t)\right)$$

$$+ G_n(x,t)^{k-1}\left(G_n(x,t)^2 - G(x,t)G_n(x,t)\right)$$

$$- kG(1,t)^{k-1}\left(G(x,t)^2 - G(x,t)G_n(x,t)\right)$$

$$- kG_n(1,t)^{k-1}\left(G_n(x,t)^2 - G(x,t)G_n(x,t)\right)\Bigg].$$

We will apply Lemma 2.9 to bound the sum of the S_k. First, it is necessary to bound the sum in terms of $\mathbb{E}[|G(1,t) - G_n(1,t)|]$, and remove any terms involving just $G(x,t)$ and $G_n(x,t)$. To do this we create terms that contain the positive term $(G(x,t) - G_n(x,t))^2$,

so that if we pre-multiply them by something negative we can discard it for an upper bound. For example, looking at the first term above we can do the following by adding a zero

$$
\begin{aligned}
G(x,t)^{k-1}\mathbb{E}\left[G(x,t)^2 - G(x,t)G_n(x,t)\right] &= \\
&= G(x,t)^{k-1}\mathbb{E}\left[G(x,t)^2 - G(x,t)G_n(x,t)\right] \\
&\quad -G(x,t)^{k-1}\mathbb{E}\left[G(x,t)G_n(x,t) - G_n(x,t)^2\right] \\
&\quad +G(x,t)^{k-1}\mathbb{E}\left[G(x,t)G_n(x,t) - G_n(x,t)^2\right] \\
&= G(x,t)^{k-1}\mathbb{E}\left[(G(x,t) - G_n(x,t))^2\right] \\
&\quad +G(x,t)^{k-1}\mathbb{E}\left[G(x,t)G_n(x,t) - G_n(x,t)^2\right], \\
&= G(x,t)^{k-1}\mathbb{E}\left[(G(x,t) - G_n(x,t))^2\right] \\
&\quad +\mathbb{E}\left[G(x,t)^{k-2}G_n(x,t)\left(G(x,t)^2 - G(x,t)G_n(x,t)\right)\right].
\end{aligned}
$$

In creating the square term above, we get a similar term to what we started with but with the exponent of $G(x,t)$ decreased by one and the exponent of $G_n(x,t)$ increased by one. We repeat this process until the exponent of $G_n(x,t)$ is $k-1$. We then subtract and add back in the same terms but with x replaced by 1. Specifically,

$$
\begin{aligned}
S_k =\ &\frac{\alpha(k)}{k!}\mathbb{E}\Big[G(x,t)^{k-1}(G(x,t) - G_n(x,t))^2 + G(x,t)^{k-2}G_n(x,t)(G(x,t) - G_n(x,t))^2 \\
&\quad + \ldots + G(x,t)G_n(x,t)^{k-2}(G(x,t) - G_n(x,t))^2 \\
&\quad + G_n(x,t)^{k-1}(G(x,t) - G_n(x,t))^2\Big] \\
&-\frac{\alpha(k)}{k!}\mathbb{E}\Big[G(1,t)^{k-1}(G(x,t) - G_n(x,t))^2 + G(1,t)^{k-2}G_n(1,t)(G(x,t) - G_n(x,t))^2 \\
&\quad + \ldots + G(1,t)G_n(1,t)^{k-2}(G(x,t) - G_n(x,t))^2 \\
&\quad + G_n(1,t)^{k-1}(G(x,t) - G_n(x,t))^2\Big] \\
&+\frac{\alpha(k)}{k!}\mathbb{E}\Big[G(1,t)^{k-1}(G(x,t) - G_n(x,t))^2 + G(1,t)^{k-2}G_n(1,t)(G(x,t) - G_n(x,t))^2 \\
&\quad + \ldots + G(1,t)G_n(1,t)^{k-2}(G(x,t) - G_n(x,t))^2 \\
&\quad + G_n(1,t)^{k-1}(G(x,t) - G_n(x,t))^2\Big] \\
&-\frac{\alpha(k)}{k!}\Big(kG(1,t)^{k-1}G(x,t)^2 - kG(1,t)^{k-1}G(x,t)\mathbb{E}[G_n(x,t)] \\
&\quad + k\mathbb{E}\left[G_n(1,t)^{k-1}G_n(x,t)^2\right] - kG(x,t)\mathbb{E}\left[G_n(1,t)^{k-1}G_n(x,t)\right]\Big),
\end{aligned}
$$

noting that the first six lines combined give something negative. As $G(x,t) \le G(1,t)$

and $G_n(x,t) \le G_n(1,t)$, we have

$$
\begin{aligned}
S_k \le{} & \frac{\alpha(k)}{k!}\mathbb{E}\left[G(1,t)^{k-1}(G(x,t)^2 - 2G(x,t)G_n(x,t) + G_n(x,t)^2)\right] \\
& + \frac{\alpha(k)}{k!}\mathbb{E}\left[G(1,t)^{k-2}G_n(1,t)(G(x,t)^2 - 2G(x,t)G_n(x,t) + G_n(x,t)^2)\right] \\
& + \ldots + \frac{\alpha(k)}{k!}\mathbb{E}\left[G(1,t)G_n(1,t)^{k-2}(G(x,t)^2 - 2G(x,t)G_n(x,t) + G_n(x,t)^2)\right] \\
& + \frac{\alpha(k)}{k!}\mathbb{E}\left[G_n(1,t)^{k-1}(G(x,t)^2 - 2G(x,t)G_n(x,t) + G_n(x,t)^2)\right] \\
& - \frac{\alpha(k)}{k!}\bigg(kG(1,t)^{k-1}G(x,t)^2 - kG(1,t)^{k-1}G(x,t)\mathbb{E}[G_n(x,t)] \\
& \qquad\qquad + k\mathbb{E}\left[G_n(1,t)^{k-1}G_n(x,t)^2\right] - kG(x,t)\mathbb{E}\left[G_n(1,t)^{k-1}G_n(x,t)\right]\bigg).
\end{aligned}
$$

We gather terms in $G(x,t)^2$, $G_n(x,t)^2$ and $G(x,t)G_n(x,t)$, each multiplied by $|G(1,t)^j - G_n(1,t)^j|$ for some j. Here we have taken absolute values in order to not worry about signs. We do this by matching each of the terms from the first four lines of the above with the corresponding term from the last two lines. Doing this we see that

$$
\begin{aligned}
S_k \le{} & \frac{\alpha(k)}{k!}G(x,t)^2\mathbb{E}\Big[G(1,t)^{k-2}\,|G(1,t) - G_n(1,t)| + \ldots \\
& \qquad\qquad\qquad\qquad + \left|G(1,t)^{k-1} - G_n(1,t)^{k-1}\right|\Big] \\
& + \frac{\alpha(k)}{k!}\mathbb{E}\Big[G_n(x,t)^2\Big(\left|G(1,t)^{k-1} - G_n(1,t)^{k-1}\right| + \ldots \\
& \qquad\qquad\qquad\qquad + G_n(1,t)^{k-2}\,|G(1,t) - G_n(1,t)|\Big)\Big] \\
& + \frac{\alpha(k)}{k!}G(x,t)\mathbb{E}\Big[G_n(x,t)\Big(G(1,t)^{k-2}\,|G(1,t) - G_n(1,t)| + \ldots \\
& \qquad\qquad\qquad\qquad + G(1,t)\left|G(1,t)^{k-2} - G_n(1,t)^{k-2}\right|\Big)\Big] \\
& + \frac{\alpha(k)}{k!}G(x,t)\mathbb{E}\Big[G_n(x,t)\Big(G_n(1,t)\left|G(1,t)^{k-2} - G_n(1,t)^{k-2}\right| + \ldots \\
& \qquad\qquad\qquad\qquad + G_n(1,t)^{k-2}\,|G(1,t) - G_n(1,t)|\Big)\Big].
\end{aligned}
$$

Hence, using the fact that $G(x,t)$, $G(1,t)$, $G_n(x,t)$, $G_n(1,t) \le 1$, we see that

$$
\begin{aligned}
S_k \le{} & \frac{\alpha(k)}{k!}\mathbb{E}\left[\left|G(1,t)^{k-1} - G_n(1,t)^{k-1}\right| + \ldots + |G(1,t) - G_n(1,t)|\right] \\
& + \frac{\alpha(k)}{k!}\mathbb{E}\left[\left(\left|G(1,t)^{k-1} - G_n(1,t)^{k-1}\right| + \ldots + |G(1,t) - G_n(1,t)|\right)\right] \\
& + \frac{\alpha(k)}{k!}\mathbb{E}\left[\left|G(1,t)^{k-2} - G_n(1,t)^{k-2}\right| + \ldots + |G(1,t) - G_n(1,t)|\right] \\
& + \frac{\alpha(k)}{k!}\mathbb{E}\left[\left|G(1,t)^{k-2} - G_n(1,t)^{k-2}\right| + \ldots + |G(1,t) - G_n(1,t)|\right],
\end{aligned}
$$

and so gathering terms gives us

$$S_k \leq 2\frac{\alpha(k)}{k!}\sum_{j=1}^{k-1}\mathbb{E}\left[\left|G(1,t)^j - G_n(1,t)^j\right|\right] + 2\frac{\alpha(k)}{k!}\sum_{j=1}^{k-2}\mathbb{E}\left[\left|G(1,t)^j - G_n(1,t)^j\right|\right],$$

$$S_k \leq 2\frac{\alpha(k)}{k!}\sum_{j=1}^{k-1}j\mathbb{E}\left[|G(1,t) - G_n(1,t)|\right] + 2\frac{\alpha(k)}{k!}\sum_{j=1}^{k-2}j\mathbb{E}\left[|G(1,t) - G_n(1,t)|\right],$$

where the above inequality uses $|y^j - z^j| \leq j|y - z|$ for $y, z \leq 1$. Thus

$$S_k \leq \frac{2\alpha(k)(k-1)}{k(k-2)!}\mathbb{E}\left[|G(1,t) - G_n(1,t)|\right]$$

We can plug this result back into (2.25) and see that

$$\frac{d}{dt}Y_{n,x}(t) \leq -2\lambda Y_{n,x}(t) + 4\mathbb{E}\left[|G(1,t) - G_n(1,t)|\right]\sum_{k=2}^{n}\frac{\alpha(k)(k-1)}{k(k-2)!} + H_n.$$

Now using Lemma 2.9 and the Cauchy-Schwarz inequality, we have that

$$\frac{d}{dt}Y_{n,x}(t) \leq -2\lambda Y_{n,x}(t) + 4\frac{\sqrt{H_n}}{\sqrt{2\lambda}}\sum_{k=2}^{n}\frac{\alpha(k)(k-1)}{k(k-2)!} + H_n. \tag{2.26}$$

Then, we can use Gronwall's Inequality to see that

$$Y_{n,x}(t) \leq \left(2\frac{\sqrt{H_n}}{\sqrt{2\lambda}}\sum_{k=2}^{n}\frac{\alpha(k)(k-1)}{k(k-2)!} + H_n\right)\left(1 - e^{-2\lambda t}\right),$$

$$\leq 2\frac{\sqrt{H_n}}{\sqrt{2\lambda}}\sum_{k=2}^{n}\frac{\alpha(k)(k-1)}{k(k-2)!} + H_n$$

$$\to 0$$

as $n \to \infty$, thanks to (2.4). This convergence is uniform in both x and t, as required.

(ii) For the case $\lambda_n \to 0$, a few small changes need to be made to the proof. First, the proof of Lemma 2.7 is identical up to (2.20), and then

$$C_n(t) \leq \left(\lambda_n + \sum_{k=2}^{n}\frac{\alpha(k)}{(k-2)!} + \frac{B}{n}\right)\left(\frac{1}{\lambda_n} - \frac{1}{\lambda_n}e^{-\lambda_n t}\right),$$

$$\to \left(\sum_{k=2}^{\infty}\frac{\alpha(k)}{(k-2)!}\right)t$$

as $n \to \infty$, and so $(1/n)C_n(t) \to 0$ as $n \to \infty$, but now the convergence is only uniform in $t \in [0,T]$, for any finite T. Hence, the convergence of $D_n(x,t)$ in Lemma 2.8 is only uniform in $t \in [0,T]$.

The main proof of Theorem 2.1 is identical up to (2.21), except any further point which has a $-2\lambda Y_{n,x}(t)$ term should instead be replaced by a different term, as G in this case contributes no terms that involve λ_n. This different term is

$$2\lambda_n \mathbb{E}[G_n(x,t)(x - G_n(x,t))] - 2\lambda_n G(x,t)\mathbb{E}[x - G_n(x,t)]$$
$$= 2\lambda_n \mathbb{E}[(x - G_n(x,t))(G_n(x,t) - G(x,t))]$$
$$\leq 2\lambda_n$$

as both terms inside the expectation have absolute value less than or equal to 1.

It should be noted that only the fragmentation part of any equation changes, anything which is to do with the coalescent part is unchanged. Hence, combining this with the above means we can conclude that in this regime equation (2.24) will read

$$\frac{d}{dt} X_n(t) \leq 2\lambda_n + H_n,$$

and hence, by Gronwall's inequality,

$$X_n(t) \leq (2\lambda_n + H_n)t \leq (2\lambda_n + H_n)T \to 0,$$

as $n \to \infty$, uniformly in $x \in [0,1]$ and $t \in [0,T]$. Finally, we can conclude that in this regime (2.26) will read

$$\frac{d}{dt} Y_{n,x}(t) \leq 2\lambda_n + 4\sqrt{(2\lambda_n + H_n)t} \sum_{k=2}^{n} \frac{\alpha(k)(k-1)}{k(k-2)!} + H_n.$$

Hence, by Gronwall's inequality,

$$Y_{n,x}(t) \leq (2\lambda_n + H_n)t + \frac{8}{3}\left(\sqrt{2\lambda_n + H_n} \sum_{k=2}^{n} \frac{\alpha(k)(k-1)}{k(k-2)!}\right) t^{3/2},$$
$$\leq (2\lambda_n + H_n)T + \frac{8}{3}\left(\sqrt{2\lambda_n + H_n} \sum_{k=2}^{n} \frac{\alpha(k)(k-1)}{k(k-2)!}\right) T^{3/2},$$
$$\to 0,$$

as $n \to \infty$, uniformly in $x \in [0,1]$ and $t \in [0,T]$, as required. $\qquad\square$

2.3 Further results

In the first chapter, we mentioned an open conjecture of Aldous (and others) concerning when a finite volume stochastic coalescent converges to solutions to the equivalent Smoluchowski coagulation equations. The method used in the previous section can be used to provide an alternative proof of the weak law of large numbers results of Norris

[60] in the specific case of the unit kernel, $K \equiv 1$.

Proposition 2.10. *Let $w_n(t)$ be the $ML^{(n)}$ process with kernel $K \equiv 1$. Then, for any $T \geq 0$,*

$$\frac{1}{n} w_{n,j} \xrightarrow{\mathcal{L}^2} w_j,$$

as $n \to \infty$, uniformly on $[0, T]$, where w_j is the solution to the corresponding Smoluchowski coagulation equation

$$\frac{d}{dt} w_j(t) = \frac{1}{2} \sum_{l=1}^{j-1} w_l(t) w_{j-l}(t) - w_j(t) \sum_{l=1}^{\infty} w_l(t), \quad j \geq 1.$$

Note: the $ML^{(n)}$ process in this case can be thought of as the case with $\alpha(2) = 1$ and $\alpha(k) = 0$ otherwise and no fragmentation.

Proof. We use the same generating function G as in the previous section and make a small modification to the proof of Theorem 2.1 part (ii). By removing all mention of λ_n, and therefore fragmentation, we change nothing and the result still holds that

$$G_n(x, t) \to G(x, t)$$

in $\mathcal{L}^2$, as $n \to \infty$ uniformly on $[0, T]$, where G solves the equation

$$G(x, 0) = x$$

$$\frac{\partial G}{\partial t}(x, t) = \frac{1}{2} \left(G(x, t)^2 - 2G(1, t) G(x, t) \right).$$

Now, as

$$G(x, t) = \sum_{k=1}^{\infty} x^k w_{\infty, k}(t),$$

we just need to solve the differential equation and expand it as a power series in x to get the coefficients $w_{\infty, k}(t)$ and show these match the solutions to the Smoluchowski coagulation equations with unit kernel. First solve for $G(1, t)$, which reduces the problem to

$$G(1, 0) = 1$$

$$\frac{dG}{dt}(1, t) = -\frac{1}{2} G(1, t)^2,$$

which has unique solution given by

$$G(1, t) = \frac{2}{2 + t}.$$

Now we can solve the family of ODEs

$$G(x, 0) = x$$

$$\frac{\partial G}{\partial t}(x, t) = \frac{1}{2}\left(G(x, t)^2 - \frac{4}{2+t}G(x, t)\right),$$

which has a unique solution

$$G(x, t) = \frac{4x}{(2+t)(2+t-tx)} = \frac{4x}{(2+t)^2}\frac{1}{1 - \frac{t}{2+t}x}.$$

Hence, expanding this as a power series in x gives

$$G(x, t) = \frac{4x}{(2+t)^2}\sum_{j=0}^{\infty} x^j \left(\frac{t}{2+t}\right)^j = \frac{4}{(2+t)^2}\sum_{j=1}^{\infty} x^j \left(\frac{t}{2+t}\right)^{j-1},$$

and thus

$$w_{\infty,j}(t) = \frac{4}{(2+t)^2}\left(\frac{t}{2+t}\right)^{j-1},$$

which matches the solutions to the Smoluchowski coagulation equations, as required.

$\square$

In fact, this result extends to a new result involving the same process but adding fragmentation at rate $\lambda_n \to 0$, as noted in the following corollary.

Corollary 2.11. *Suppose $\lambda_n \to 0$, $\alpha(k) = 1$ if $k = 2$ and is zero otherwise. Let $w_j(t) := \lim_{n\to\infty} w_{n,j}(t)$, for $j \in \mathbb{N}$. Then*

$$w_j(t) = \frac{4}{(2+t)^2}\left(\frac{t}{2+t}\right)^{j-1},$$

which are the solutions to the Smoluchowski coagulation equations with unit kernel.

Proof. By Theorem 2.1 we know that $G_n(x, t) \to G(x, t)$ in $\mathcal{L}^2$ as $n \to \infty$, where $G(x, t)$ solves the differential equation

$$G(x, 0) = x$$

$$\frac{\partial G}{\partial t}(x, t) = \frac{1}{2}\left(G(x, t)^2 - 2G(1, t)G(x, t)\right).$$

This is the same as in Proposition 2.10 above. The result follows in exactly the same way as before. $\square$

Remark 2.12. *Note here the difference between this result and the results of Ráth and Tóth in [63]. When the coalescence and fragmentation mechanisms were size-biased there were three regimes with differing behaviour depending on how fast λ_n converged*

to zero, including the regime which exhibited self-organised criticality. Whereas, in this case the speed of convergence is irrelevant and they all behave in the same way with no strict self-organised criticality.

We can show a few more interesting results about specific processes in this class, using Theorem 2.1. The first of these, is that we have a scaling limit for the cluster counting process as the system size tends to infinity.

Corollary 2.13. *Let α satisfy the conditions for Theorem 2.1. Define $N_n(t)$ to be the number of clusters at time t in a system with n particles. Then*

$$\frac{N_n(t)}{n} \to G(1,t),$$

in $\mathcal{L}^2$ as $n \to \infty$, where G is defined in Theorem 2.1.

Also, in the special case that we only allow binary coalescence events, we can solve the differential equations to get an explicit formula for this scaling limit.

Corollary 2.14. *Let $\alpha(2) = c$ and $\alpha(k) = 0$ otherwise. Define $N_n(t)$ to be the number of clusters at time t in a system with n particles. Then*
(i) If $\lambda_n \to \lambda \in (0,\infty)$, then

$$\frac{N_n(t)}{n} \to \sqrt{\frac{2\lambda}{c} + \left(\frac{\lambda}{c}\right)^2} \tanh\left(\frac{1}{2}t\sqrt{2\lambda c + \lambda^2} + \operatorname{artanh}\left(\frac{c+\lambda}{\sqrt{2\lambda c + \lambda^2}}\right)\right) - \frac{\lambda}{c},$$

in $\mathcal{L}^2$ as $n \to \infty$, and therefore,

$$\lim_{t\to\infty} \lim_{n\to\infty} \frac{N_n(t)}{n} = \sqrt{\frac{2c}{\lambda} + \left(\frac{\lambda}{c}\right)^2} - \frac{\lambda}{c}.$$

(ii) If $\lambda_n \to 0$ as $n \to \infty$, then

$$\frac{N_n(t)}{n} \to \frac{2}{ct+2},$$

in $\mathcal{L}^2$ as $n \to \infty$.

The first part of Corollary 2.14 can be difficult to visualise so Figure 2-2 shows how this function looks for some values of λ and c.

Proof. Both of these corollaries can be proved at once by noticing that

$$\frac{N_n(t)}{n} = G_n(1,t).$$
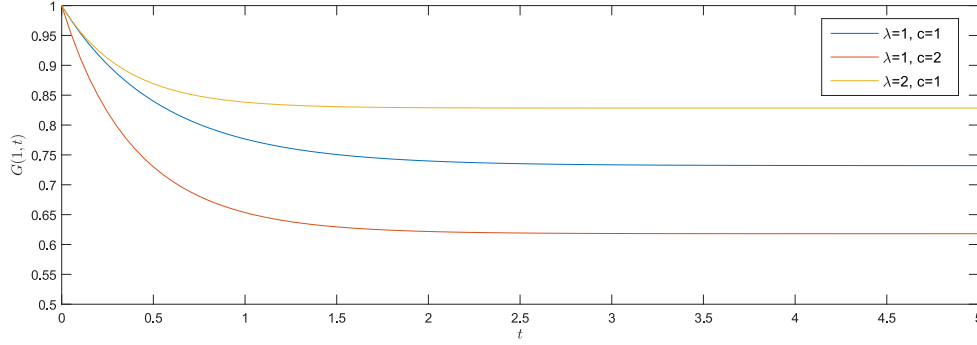
Figure 2-2: *A plot of $G(1,t)$ for various values of λ and c, where only binary coalescence is allowed. Convergence to the steady state appears rather rapid.*

Therefore, Theorem 2.1 states that

$$\frac{N_n(t)}{n} \to G(1,t),$$

in $\mathcal{L}^2$, as $n \to \infty$, which proves Corollary 2.13. For Corollary 2.14, we can just solve the initial value problem for $G(1,t)$ using separation of variables, which in this case is

$$\frac{dG}{dt}(1,t) = \lambda - \lambda G(1,t) - \frac{c}{2}G(1,t)^2$$

$$G(1,0) = 1$$

where $\lambda = 0$ in case (ii). Hence, $G(1,t)$ solves

$$\int \frac{1}{\lambda - \lambda G - (c/2)G^2}dG = t + C, \quad \text{and} \quad G = 1 \text{ if } t = 0$$

where C is a constant, thus in case (i), where $\lambda > 0$, putting (2.3) into the form

$$\int \frac{1}{a^2 - (d + by)^2}dy = \frac{1}{ab}\operatorname{artanh}\left(\frac{by + d}{a}\right) + C_1,$$

and rearranging gives the result. For case (ii), where $\lambda = 0$, using

$$\int -\frac{1}{by^2}dy = \frac{1}{by} + C_2,$$

and rearranging gives the result. $\qquad \square$

2.4   Universal stationary distribution

Now that we know how the empirical generating function $G_n(x,t)$ converges in the large n limit, we can extract a prediction for the stationary distribution of cluster sizes. Recalling from (2.3) that the generating function of the cluster size distribution is obtained via

$$g_n(x,t) = \frac{G_n(x,t)}{G_n(1,t)},$$

we introduce the quantities

$$g(x,t) = \frac{G(x,t)}{G(1,t)}, \qquad\qquad G(x) = \lim_{t\to\infty} G(x,t),$$

$$g(x) = \lim_{t\to\infty} g(x,t) = \frac{G(x)}{G(1)}, \qquad\qquad p_j = \lim_{t\to\infty}\lim_{n\to\infty} p_{n,j}(t).$$

Under self-averaging, the stationary cluster size distribution is found simply by determining the fixed point $G(x)$ of the differential equation (2.5) and developing $g(x)$ as a power series. In practice, this procedure is only tractable in the limit $\lambda \searrow 0$, where we find a particular limiting distribution.
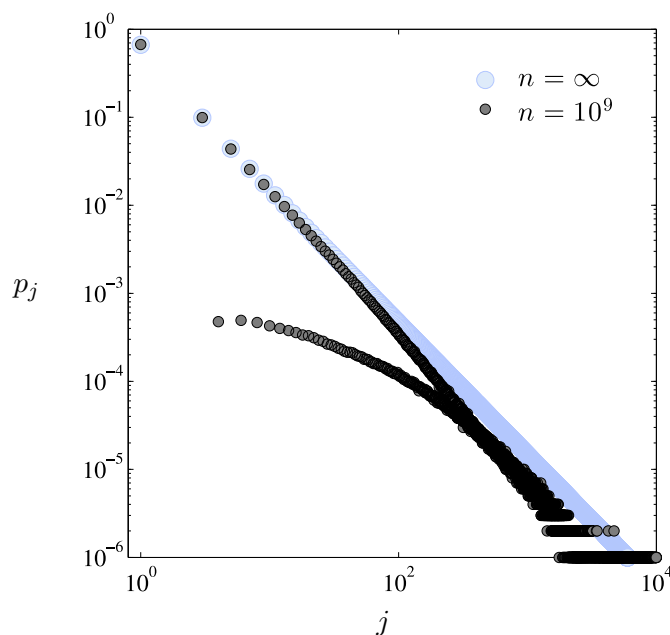
Figure 2-3: *Example of a fragmentation-coalescence process with fragmentation rate $\lambda = 10^{-7}$ and coalescence rates $\alpha(3) = 1$, $\alpha(4) = 2$ and $\alpha(k) = 0$ otherwise. A comparison between a single simulation run with $n = 10^9$ particles, stopped at time $T = 10^6$ (black) and the $n \to \infty$ limit given by Theorem 2.15 (pale blue). The higher points correspond to the odd sized clusters, the lower ones correspond to even sized clusters which are asymptotically vanishing in the limit $n \to \infty$, $\lambda \to 0$.*

Theorem 2.15. *If α satisfies (2.4) and m is the smallest integer such that $\alpha(m) > 0$, then the stationary cluster size distribution obeys*

$$\lim_{\lambda \searrow 0} p_j = \begin{cases} \frac{1}{j} \left(\frac{m-1}{m}\right)^j \left(\frac{1}{m}\right)^{\frac{j-1}{m-1}} \binom{m\left(\frac{j-1}{m-1}\right)}{\frac{j-1}{m-1}} & \text{if } m-1 \text{ divides } j-1, \text{ or } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

and in particular

$$\lim_{\lambda \searrow 0} p_j \sim \begin{cases} j^{-3/2} & \text{if } m-1 \text{ divides } j-1, \text{ or } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $f \sim g$ means $f(j)/g(j) \to c$ as $j \to \infty$ for some positive constant c, regardless of α.

The power-law tail of cluster size distribution is reminiscent of critical behaviour occurring at the gellation time in some pure coalescent processes [2], although here it is not a transient phenomenon but rather is approached in the long time limit. This is the essential characteristic of self-organised criticality, however, there is an unusual complication here that the result depends on the strict order of the $n \to \infty$ and $\lambda \to 0$ limits; taking $\lambda_n \to 0$ we recover a pure coalescent process that does not posses this characteristic power-law.

While it is quite typical for critical processes to exhibit universality in the scaling exponents, our result says something stronger that in fact the limiting cluster size distribution is almost completely independent of the coagulation rates, depending only on the identity of the first non-zero rate. A surprising consequence is that if, for example, the model coalesces clusters in groups of three and four (but not pairs) then in the large n and small λ limit we will see no clusters of even size whatsoever in the stationary distribution. Figure 2-3 shows an example of this phenomenon for the model with rates $\alpha(k) = \delta_{k,3} + 2\delta_{k,4}$. The model has the apparently paradoxical feature that clusters of even size are vanishingly rare, despite the fact that some two-thirds of clusters are singletons, and $\alpha(4) > \alpha(3)$.

The classical technique of series inversion that dates back to Newton, Lagrange, Bürmann and Puiseux will be vital in proving this result. In particular, the following theorem is a sub-case of results of those authors (see [41], p. 183):

Theorem 2.16 (Series Inversion)**.** *Let $f : [0, \infty) \to [0, \infty)$ be defined by the power series $f(x) = \sum_{k \geq m} a_k x^k$ for integer $m \geq 1$ and non-negative real constants $\{a_k\}$. The inverse is expressed via the Puiseux series*

$$f^{-1}(y) = \sum_{j=1}^{\infty} b_j y^{j/m},$$

where

$$b_j = \frac{1}{j!} \lim_{z \to 0} \left[\frac{d^{j-1}}{dz^{j-1}} \left(\frac{z}{f(z)^{1/m}} \right)^j \right].$$

In particular, note that $b_1 = (1/a_1)^{1/m}$.

Proof of Theorem 2.15. Recall that the limit generating function $G(x,t)$ obeys

$$\frac{\partial G}{\partial t}(x,t) = \lambda(x - G(x,t)) + \sum_{k=2}^{\infty} \frac{\alpha(k)}{k!} \left(G(x,t)^k - kG(1,t)^{k-1} G(x,t) \right). \qquad (2.27)$$

We also know that $G(\cdot, t) : [0,1] \to [0,1]$ is a monotonically increasing function. It follows that the RHS of (2.27) is monotonically decreasing as G increases, is positive for $G = 0$ and negative for $G = 1$. Therefore in long times we have $G(x,t) \to G(x)$, where

$$0 = \lambda(x - G(x)) + \sum_{k=2}^{\infty} \frac{\alpha(k)}{k!} \left(G(x)^k - kG(1)^{k-1} G(x) \right). \qquad (2.28)$$

At the point $x = 1$ we rearrange to find λ as a power series in $G(1)$:

$$\lambda = \frac{1}{1 - G(1)} \sum_{k \geq m} \frac{\alpha(k)}{k!} (k-1) G(1)^k = \left(\sum_{k=1}^{\infty} G(1)^k \right) \sum_{k \geq m} \frac{\alpha(k)}{k!} (k-1) G(1)^k$$

$$= \frac{\alpha(m)}{m!} (m-1) G(1)^m + \sum_{k > m} c_k G(1)^k, \qquad (2.29)$$

where m is the least integer such that $\alpha(m) > 0$, and c_k are the appropriate constants. From Theorem 2.16 we thus determine that

$$G(1) = \lambda^{1/m} \left(\frac{m!}{(m-1)\alpha(m)} \right)^{1/m} + o(\lambda^{1/m}).$$

For general x, equation (2.28) implies the same leading order behaviour of $G(1)$ in λ. Recalling the definition $g(x) = G(x)/G(1)$ at the start of this section, we see that

$$0 = \lambda (x - G(1)g(x)) + \sum_{k=m}^{\infty} \frac{\alpha(k)}{k!} G(1)^k \left(g(x)^k - kg(x) \right),$$

$$= \lambda x - \left(\frac{m!}{(m-1)\alpha(m)} \right)^{1/m} \lambda^{1+1/m} g(x)$$

$$+ \sum_{k=m}^{\infty} \frac{\alpha(k)}{k!} \left(\frac{m!}{(m-1)\alpha(m)} \right)^{k/m} \lambda^{k/m} \left(g(x)^k - kg(x) \right) + o(\lambda),$$

$$= \lambda x + \frac{\lambda}{m-1} (g(x)^m - mg(x)) + o(\lambda), \quad x \in [0,1].$$

In other words, when m is the smallest integer such that α is non-zero, we find that in

the limit as $\lambda \searrow 0$,

$$mg(x) - g(x)^m - x(m-1) = 0.$$

Applying the inversion theorem once more, we obtain the series

$$g(x) = \sum_{j=1}^{\infty} \frac{x^j}{j!} \lim_{z \to 0} \left[\frac{d^{j-1}}{dz^{j-1}} \left(\frac{z}{f(z)} \right)^j \right], \tag{2.30}$$

where $f(x) = m/(m-1)z - 1/(m-1)z^m$. We compute

$$
\begin{aligned}
\left(\frac{z}{f(z)} \right)^j &= \left(\frac{m-1}{m} \right)^j \left(\frac{1}{1 - z^{m-1}/m} \right)^j \\
&= \left(\frac{m-1}{m} \right)^j \left(\sum_{n=0}^{\infty} \left(\frac{1}{m} \right)^n z^{n(m-1)} \right)^j \\
&= \left(\frac{m-1}{m} \right)^j \sum_{n=0}^{\infty} \left(\frac{1}{m} \right)^n \binom{j-1+n}{n} z^{n(m-1)},
\end{aligned}
$$

using the product formula for series. To get the summands for (2.30) we need to look at the $(k-1)^{th}$ derivative of the above series, evaluated at $z = 0$. Hence we have that

$$
\frac{d^{j-1}}{dz^{j-1}} \left(\frac{z}{f(z)} \right)^j \bigg|_{z=0} =
\begin{cases}
\left(\frac{m-1}{m} \right)^j \left(\frac{1}{m} \right)^{\frac{j-1}{m-1}} \binom{m\left(\frac{j-1}{m-1} \right)}{\frac{j-1}{m-1}} (j-1)! & \text{if } m-1 \text{ divides } j-1, \\
& \text{or } j = 1 \\
0 & \text{otherwise.}
\end{cases}
$$

Plugging this into (2.30) and comparing coefficients of x^j, we have

$$
\lim_{\lambda \searrow 0} p_j =
\begin{cases}
\frac{1}{j} \left(\frac{m-1}{m} \right)^j \left(\frac{1}{m} \right)^{\frac{j-1}{m-1}} \binom{m\left(\frac{j-1}{m-1} \right)}{\frac{j-1}{m-1}} & \text{if } m-1 \text{ divides } j-1, \text{ or } j = 1 \\
0 & \text{otherwise.}
\end{cases}
$$

Hence, we can conclude that, for large values of j, under these conditions,

$$\lim_{\lambda \searrow 0} p_j \sim j^{-3/2},$$

regardless of the values of $\alpha(k)$, $k \geq 2$.

$\square$

Remark 2.17. *This result can perhaps be a little surprising, so we'll provide some intuition. As the universal exponent is 3/2 this implies the mean cluster size diverges to infinity and the number of clusters falls below order n. We'll take the example from before where $\alpha(k) = \delta_{k,3} + 2\delta_{k,4}$. The coalescence rates of 3 clusters is scaled by n^{-2} and those of 4 clusters is scaled by n^{-3}. When there are order n number of clusters this means the total rates for both are matched at order n. However, once the number*

of clusters decreases to order n^β, where $\beta < 1$, this becomes unbalanced in favour of coalescence involving 3 clusters by a factor of $n^{1-\beta}$. Hence, coalescence events involving 3 clusters is infinitely more frequent than coalescence events involving 4 clusters, and thus it is only the results of the former that we see in the infinite limit.

CHAPTER 3

THE FAST FRAGMENTATION-COALESCENCE PROCESS

In the previous chapter, when defining the dynamics of our process on a system with n particles, we were scaling our k-coalescence rates by n^{1-k} to ensure that the total coalescence rate was of the same order as the total fragmentation rate (in the $\lambda_n \equiv \lambda$ case). This allowed us to see both dynamics in the infinite limit. However, when the fragmentation rate converged to zero with n, and therefore the total coalescence rate outweighed the total fragmentation rate, we did not see the fragmentation dynamics in the infinite limit.

Is this still the case in the realm of exchangeable fragmentation-coalescence processes? Such processes require compatibility (where the behaviour of the first n integers of a process defined on $n + 1$ integers is distributed like the process defined on n integers) and hence we cannot scale the coalescence rates by system size, n. If we take similar coalescence and fragmentation mechanisms from Chapter 2, but without any scaling the coalescence rate with system size, does this mean that we don't see the fragmentation dynamics in the infinite limit?

As an example, if we coalesce pairs of blocks at rate c (a scaled Kingman coalescent) and fragment blocks as before at rate λ, then a naïve intuition would suggest that coalescence dominates fragmentation rate, because the total coalescence rate is of order n^2 and the total fragmentation rate is of order n. In particular, a naïve prediction would be that, regardless of the values of c and λ, the process would come down from infinity, just as Kingman's coalescent would. However, perhaps surprisingly, this intuition will turn out to be false and there is in fact a phase transition in the quantity $2\lambda/c$, where on one side the process comes down from infinity and on the other it stays infinite.

We introduced exchangeable fragmentation-coalescence (EFC) processes in Chapter 1, which were first explored by Berestycki [11]. Such processes are, in a rough sense, constructed by allowing a homogeneous fragmentation (Bertoin [16]) and an exchangeable coalescent (Schweinsberg [67], Möhle & Sagitov [59]) to run simultaneously. The

questions that often arise from these processes are about how many blocks the process has, and the sizes of the blocks. This will be our focus too.

The chapter is laid out as follows: Section 3.1 introduces the fast fragmentation-coalescence process, and the main results we have proved about this process. Section 3.2 is devoted to proving the main result about this process. Section 3.3 proves some further results about the block counting process and the excursion process behind it.

3.1   Introduction and main results

Chapter 1 goes into more detail as to the history and construction of these processes, we just give a brief reminder of the main points. EFC processes take place on the space of partitions of $\mathbb{N}$, although they can equivalently be looked at on the space of decreasing sequences with unit mass, $\mathcal{S}^{\downarrow}$. The notion of coalescence and fragmentation will act on the blocks that make up the partition. We have two operators, **Coag** and **Frag**, which define the actions of a coalescence and fragmentation event on a partition below.

$$\mathbf{Coag} : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$$
$$(\pi, \pi') \mapsto \pi''$$

where

$$\pi_i'' := \bigcup_{j \in \pi_i'} \pi_j,$$

and

$$\mathbf{Frag} : \mathcal{P} \times \mathcal{P} \times \mathbb{N} \to \mathcal{P}$$
$$(\pi, \pi', k) \mapsto \tilde{\pi}$$

where the blocks of $\tilde{\pi}$ consist of the blocks π_i, if $i \neq k$, and $\pi_k \cap \pi_j'$ for $j \in \mathbb{N}$.

The dynamics of this process are defined by two independent Poisson point processes, PPP_C and PPP_F, which govern the coalescence and fragmentation as follows. For each of these we need an exchangeable measure, C and F, on $\mathcal{P}$, which by Proposition 3 in [11] can be decomposed as follows

$$C = c_k \sum_{i<j} \delta_{\epsilon_{i,j}} + \mu_{\nu_{\mathrm{Coag}}}$$
$$F = c_e \sum_{n \in \mathbb{N}} \delta_{\epsilon_n} + \mu_{\nu_{\mathrm{Disl}}}$$

where c_k, c_e are constants which govern the speed of Kingman's coalescent and erosion respectively, $\nu_{\mathrm{Coag}}, \nu_{\mathrm{Disl}}$ are exchangeable measures on $\mathcal{S}^{\downarrow}$, and μ_{ν} is Kingman's

paintbox measure generated by ν. If $(t, \pi(t))$ is an atom of PPP_C, then

$$\Pi(t) = \mathbf{Coag}(\Pi(t-), \pi(t)),$$

and if $(t, \pi(t), k(t))$ is an atom of PPP_F, then

$$\Pi(t) = \mathbf{Frag}(\Pi(t-), \pi(t), k(t)).$$

The partition contains a lot of information, which can make it difficult to understand the true behaviour of these processes. Hence, we tend to look at processes which condense this information down to amounts which are much easier to understand. We look at two such processes. The first is the block counting process N, which is the number of non-empty blocks of Π. The second is the process $X = (|\Pi_1|, |\Pi_2|, \ldots)^{\downarrow}$, which measures the asymptotic frequency of these blocks, which is defined below for the ith block of Π

$$|\Pi_i(t)| = \lim_{n \to \infty} \frac{\{1, \ldots, n\} \cap \Pi_i(t)}{n}.$$

By Theorem 7 in [11], we know that these asymptotic frequencies exist for all blocks at all times t almost surely and also that X is a Feller process on the space $\mathcal{S}^{\downarrow}$. As a reminder the definition of a Feller process is stated below.

Definition 3.1 (Feller Process). *Let S be a locally compact, separable metric space. A continuous time Markov process $(Y_t)_{t \geq 0}$ on S is a Feller process if, for all continuous functions $f : S \to \mathbb{R}$, its associated semigroup $(P_t^Y)_{t \geq 0}$ satisfies*

(i) $P_t^Y f$ is a continuous function for all $t \geq 0$,

(ii) $P_t^Y f(y) \to f(y)$, as $t \downarrow 0$, for all $y \in S$

3.1.1 The 'fast' fragmentation-coalescence process

We will start with an informal outline of and the motivation behind what we call the 'fast' fragmentation-coalescence process. Recall Kingman's coalescent from Chapter 1, which is the exchangeable coalescent process where every pair of blocks coalesces at rate $c > 0$. It was shown that this coalescent comes down from infinity, that is,

$$\inf\{t > 0 : N_t < \infty\} = 0 \quad \mathbb{P}_\pi - \text{a.s.}$$

so the first hitting time of a state with finitely many blocks is 0 almost surely.

It was shown by Berestycki [11] that this was still the case for an EFC process with coalescent mechanism given by Kingman's coalescent and certain fragmentation mechanisms. These fragmentation mechanisms allow each block to fragment at a finite rate into finitely many smaller blocks, with an additional condition that the dislocation

measure, ν_{Disl}, must satisfy

$$\sum_k \log(k)\nu_{\mathrm{Disl}}\left(\left\{x \in \mathcal{S}^{\downarrow} : \sum_{i=1}^{k+1} x_i = 1\right\}\right) < \infty.$$

This allowed the block counting process N_t to use results for logistic branching processes from Lambert [50].

This leads us to an interesting question: does there exist a finite rate fragmentation mechanism which prevents Kingman's coalescent from coming down from infinity? This is the motivation behind the fast fragmentation-coalescence process.

Hence, we borrow the fragmentation mechanism from Chapter 2, where every block fragments into its constituent singletons at rate λ. This is exchangeable, as we will see in the formal definition, and is the most extreme finite rate fragmentation mechanism. Therefore, if this EFC process still comes down from infinity with this fragmentation mechanism, then it would be fairly safe to assume all finite rate fragmentation mechanisms still allow Kingman's coalescent to come down from infinity.

To answer this question we must look at the Markov chain $N := (N_t : t \geq 0)$ on $\mathbb{N} \cup \{\infty\}$, which represents the number of blocks in the fast fragmentation-coalescence process. Its transitions are specified by the Q-matrix having entries given by the aforesaid Kingman and fragmentation dynamic, so that

$$Q_{i,j} = \begin{cases} c\binom{i}{2} & \text{if } j = i-1, \\ \lambda i & \text{if } j = \infty. \end{cases}$$

This is due to Kingman's Theorem (Theorem 1.19) which states that all exchangeable random partitions can be constructed using a paintbox. Hence, all exchangeable random partitions have blocks which either have positive asymptotic frequency or are singletons, and if there are singletons then the set of all singletons has positive asymptotic frequency (i.e. there are infinitely many of them). Hence, if an exchangeable random partition has finitely many blocks, then all blocks must be infinite in size. Therefore if one block fragments, then the result is infinitely many singletons forming and thus N jumps to infinity.

Informally, we are interested in understanding whether the fragmentation mechanism will be felt when N is large, or would the Kingman coalescent part simply dominate the process in those states, hence meaning that the process comes down from infinity. In this setting this corresponds to whether the state $\{\infty\}$ is absorbing or recurrent for N. That is to say, we want to know whether it is possible to construct a recurrent extension of the process N beyond its first hitting time of $\{\infty\}$, when issued from a point in $\mathbb{N}$. The idea that the process 'comes down from infinity' is then clearly captured in the notion that N instantaneously visits $\mathbb{N}$ after entering the state $\{\infty\}$.

Formal definition

To formally define the fast fragmentation-coalescence process we must specify $c_k, c_e, \nu_{\text{Disl}}$ and ν_{Coag}. These two constants and two exchangeable measures will then specify the measures which determine the Poisson point processes that govern the dynamics of Π.

As our coalescent part shall be Kingman's coalescent, we will take $c_k = c > 0$ and $\nu_{\text{Coag}} = 0$. For our fragmentation part, if we recall $\mathbf{0} = (\{1\}, \{2\}, \ldots)$, then we take $F = \lambda \delta_{\mathbf{0}}$, for $\lambda > 0$. That is to say, by our definition of the **Frag** operator, each block is fragmented entirely into its constituent singletons. This is a valid exchangeable measure as we can take $\nu_{\text{Disl}} = \lambda \delta_{\underline{0}}$, where $\underline{0}$ is the mass partition made of an infinite sequence of zeros. Then with no erosion present, i.e. $c_e = 0$, the paintbox of ν_{Disl} gives F as required.

3.1.2   Notation

A reminder and introduction to some of the notation used throughout this chapter:

- $\mathcal{P}$: the space of all partitions on $\mathbb{N}$.

- $\mathcal{P}_n$: the space of all partitions on $\{1, \ldots n\}$, where $n \in \mathbb{N}$.

- $\mathcal{P}_B$: the space of all partitions on $B \subset \mathbb{N}$.

- $\Pi = (\Pi(t), t \geq 0)$: a right-continuous $\mathcal{P}$-valued Markov process. It will generally represent the fast fragmentation-coalescence process defined above.

- $\Pi^{(n)} = (\Pi^{(n)}(t), t \geq 0)$: is the restriction of Π to $\{1, \ldots, n\}$.

- $N = (N(t), t \geq 0)$: the process with value the number of non-empty blocks of Π, $N(t) = \#\Pi(t)$.

- $M = (M(t), t \geq 0)$: is the process defined as $M(t) := 1/N(t)$ when $N(t) < \infty$ and $M(t) = 0$ if $N(t) = \infty$.

- $\mathbb{P}_\pi(\cdot) := \mathbb{P}(\cdot \mid \Pi(0) = \pi)$.

- $\mathbb{P}_n(\cdot) := \mathbb{P}(\cdot \mid N(0) = n)$.

- $\mathbb{E}_\pi[\cdot] := \mathbb{E}[\cdot \mid \Pi(0) = \pi]$,

- $\mathbb{E}_n[\cdot] := \mathbb{E}[\cdot \mid N_0 = n]$,

- $\tau_k := \inf\{t \geq 0 : N(t) = k\}$, $k \in \mathbb{N} \cup \{\infty\}$.

- $\tau_k^{(n)} := \inf\{t \geq 0 : \#\Pi^{(n)}(t) = k\}$, $k \leq n$.

- ρ: stationary distribution of Π (existence proven by Theorem 8 in [11]),

- ρ_M: stationary distribution of M (existence to be proven),

- $\mathcal{E}$: the path space of M,

- ϵ: an excursion path of M,

- ζ: the excursion length of ϵ,

- $\mathbb{Q}$: the associated excursion measure for M.

3.1.3 Main results

The main result we have for this process is that there is a phase transition in coming down from infinity for Π. It turns out to be more convenient to study the reciprocal process $M := 1/N$. The case that there is a recurrent extension of M from 0 corresponds to the ability of the fast fragmentation-coalescence process to come down from infinity. Moreover, if $M = 0$ is an absorbing state then the fast fragmentation-coalescence process stays infinite. It transpires that $\theta := 2\lambda/c$ is the quantity that governs this behaviour.

Theorem 3.2 (Phase transition)**.**

(i) *If $\theta \in (0, 1)$, then $M := (M(t) : t \geq 0)$ is a recurrent Feller process on $\{1/n : n \in \mathbb{N}\} \cup \{0\}$ such that 0 is instantaneously regular (that is to say 0 is a not a holding point) and not sticky (that is to say $\int_0^\infty \mathbf{1}_{\{M(s)=0\}}\mathrm{d}s = 0$ almost surely).*

(ii) *If $\theta \geq 1$, then 0 is an absorbing state for M.*

The remainder of our focus in this area is on the subcritical case, where Π comes down from infinity. The above theorem alludes to the existence of an excursion theory for the process M away from 0. In addition, we know from Theorem 8 in [11] that there exists a stationary probability measure ρ that Π converges to in distribution to, as $t \to \infty$. Although, we do not have an explicit construction of ρ, we can find the stationary distribution of M, ρ_M, and therefore that of the number of blocks of Π under stationarity.

Theorem 3.3 (Stationary distribution)**.** *If $\theta \in (0, 1)$, then M has stationary distribution given by*

$$\rho_M(1/k) = \frac{(1-\theta)}{\Gamma(\theta)}\frac{\Gamma(k-1+\theta)}{\Gamma(k+1)}, \quad k \in \mathbb{N}.$$

A simple corollary of this theorem is that the proportion of time spent with all integers in one block tends to $1 - \theta$ almost surely. Another property that can be captured in the recurrent case is that, in the appropriate sense, the rate of coming down from

infinity matches that of Kingman's Coalescent. To this end, let us denote by $\mathbb{P}_{1/n}$, for $n \in \mathbb{N} \cup \{+\infty\}$, the probabilities of M. Furthermore, define the excursion space, $\mathcal{E}$,

$$\mathcal{E} := \{\epsilon : \mathbb{R}^+ \to \mathbb{R}, \text{ where } \epsilon(0) = 0 \text{ and } \epsilon^{-1}(\mathbb{R}\backslash\{0\}) = (0, \zeta) \text{ for some } \zeta > 0\},$$

with associated excursion measure $\mathbb{Q}$ on $\mathcal{E}$ (defined by Itô's Theorem). Further background is supplied in Section 3.3.

Theorem 3.4 (Speed of coming down from infinity)**.** *Suppose that $\theta \in (0, 1)$.*

(i) Let $e_{1/k}^{(\infty)}$ be the expected first hitting time of $1/k$ by M under $\mathbb{P}_0$. Then

$$e_{1/k}^{(\infty)} = \frac{2}{c(1 - \theta)k}.$$

(ii) Let ϵ denote an excursion of M, then

$$\lim_{t \downarrow 0} \frac{t}{\epsilon(t)} = \frac{2}{c}, \quad \mathbb{Q}\text{-}a.e.$$

The latter of these results is illustrated in Figure 3-1.
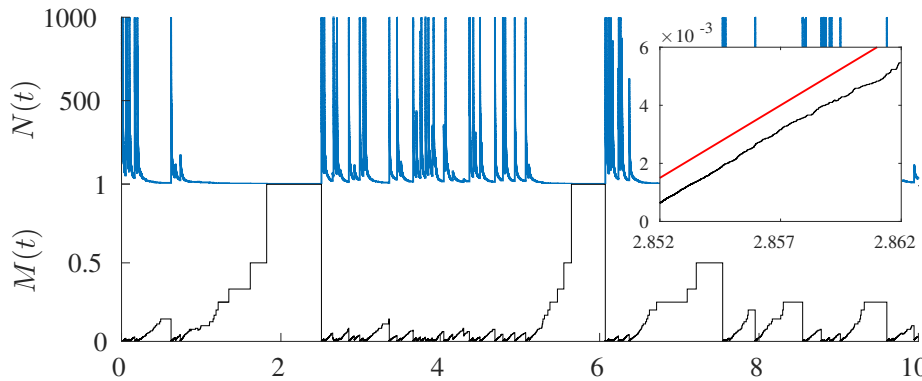
Figure 3-1: *A computer simulation of the trajectories of N (blue) and M (black) restricted to $n = 10^6$ integers, with $c = 1$ and $\lambda = 0.2$. The inset shows detail of typical behaviour near M = 0; the red line here illustrates the 'speed' predicted by Theorem 3.4.*

3.2 Phase transition in coming down from infinity

This section is devoted to the proof of Theorem 3.2, which shows the existence of a phase transition in θ for Π coming down from infinity. We start by proving four technical lemmas that, when combined, will show the behaviour of Π in the subcritical case, $\theta \in (0, 1)$, and therefore prove part (i) of the theorem. Let $T = \inf\{t > 0 : N(t) < \infty\}$, i.e. the time needed to observe a finite state. The next Lemma shows that when $\theta < 1$,

we have $T = 0$ a.s., which establishes that the process instantaneously comes down from infinity.

Lemma 3.5. *If $\theta \in (0, 1)$ and $\pi \in \mathcal{P}$, then $\mathrm{P}_\pi(T = 0) = 1$.*

Proof. To prove this lemma we will uniformly bound the expected hitting time of the state with one block when starting from a state with n blocks. This will therefore prove that T is almost surely finite. To complete the proof we will show that if T is almost surely finite, then it must almost surely be zero. We do this by showing that the expected value of T is bounded above by the expected time for the process to reach a state with k blocks, for all $k \in \mathbb{N}$, and showing that this converges to zero as k tends to infinity.

We first look at the probability of hitting the state with 1 block before any fragmentation event occurring when the process starts with n blocks. Suppose we label this probability $p_{n,1}$. Recall that $\theta := 2\lambda/c$. From the definition of the model, holding times are exponential random variables in states with finitely many blocks, say k, and the total rate of coalescence is $c\binom{k}{2}$ with the total rate of fragmentation being λk. Hence we may conclude that

$$p_{n,1} = \prod_{k=2}^{n} \frac{c\binom{k}{2}}{\lambda k + c\binom{k}{2}} = \prod_{k=2}^{n} \frac{k-1}{k-1+\theta} = \frac{\Gamma(n)\Gamma(1+\theta)}{\Gamma(n+\theta)} \sim \Gamma(1+\theta) n^{-\theta}, \qquad (3.1)$$

as n tends to infinity. We now find an expression for the expected time until the first fragmentation event. We do this by splitting this expected time over the number of blocks the process has when it fragments. Write $\mathbb{P}_n$ to denote probabilities when the process starts with n blocks. Recall first that $\tau_\infty := \inf\{t \geq 0 : N(t) = \infty\}$, hence

$$\mathbb{E}_n[\tau_\infty] = \sum_{k=1}^{n} \mathbb{E}_n\left[\tau_\infty | N(\tau_\infty-) = k\right] \mathbb{P}_n(N(\tau_\infty-) = k) \qquad (3.2)$$

First we find the probability that the first fragmentation event occurs when the process is in a state with k blocks.

$$\begin{aligned}
\mathbb{P}_n(N(\tau_\infty-) = k) &= \frac{\lambda k}{\lambda k + c\binom{k}{2}} \prod_{j=k+1}^{n} \frac{c\binom{j}{2}}{\lambda j + c\binom{j}{2}} \\
&= \frac{\theta}{k-1+\theta} \prod_{j=k+1}^{n} \frac{j-1}{j-1+\theta} \\
&= \frac{\theta}{k-1+\theta} \frac{\Gamma(k+\theta)\Gamma(n)}{\Gamma(n+\theta)\Gamma(k)} \\
&= \frac{\theta\Gamma(n)\Gamma(k-1+\theta)}{\Gamma(n+\theta)\Gamma(k)}. \qquad (3.3)
\end{aligned}$$

Next we bound the other term in the sum in (3.2). Note further that N can only

decrease in steps of size one and hence can't skip any integer when decreasing (we refer to this as having skip-free downward paths). In addition, the holding time in a state with j blocks is exponentially distributed with rate $c\binom{j}{2} + \lambda j$. Hence, conditioned on the first fragmentation event occurring when there are k blocks, τ_∞ is a sum of these exponential random variables. Therefore

$$
\begin{aligned}
\mathbb{E}_n[\tau_\infty | N(\tau_\infty -) = k] &= \sum_{j=k}^{n} \frac{2}{2\lambda j + cj(j-1)} \\
&\leq \frac{2}{ck(k-1+\theta)} + \sum_{j=k+1}^{n} \frac{2}{cj(j-1)} \\
&\leq \frac{2}{ck(k-1+\theta)} + \frac{2}{ck}.
\end{aligned}
\tag{3.4}
$$

We can now combine (3.3) and (3.4) to bound (3.2),

$$
\begin{aligned}
\mathbb{E}_n[\tau_\infty] &= \sum_{k=1}^{n} \mathbb{E}\left[\tau_\infty | N(\tau_\infty - = k)\right] \mathbb{P}_n(N(\tau_\infty -) = k) \\
&\leq \sum_{k=1}^{n} \left(\frac{2}{ck(k-1+\theta)} + \frac{2}{ck} \right) \frac{\theta \Gamma(n) \Gamma(k-1+\theta)}{\Gamma(n+\theta)\Gamma(k)} \\
&= \frac{2\theta \Gamma(n)}{c\Gamma(n+\theta)} \sum_{k=1}^{n} \frac{\Gamma(k-1+\theta)}{\Gamma(k+1)} \left(1 + \frac{1}{k-1+\theta} \right) \\
&\leq \frac{2(1+\theta)\Gamma(n)}{c\Gamma(n+\theta)} \sum_{k=1}^{n} \frac{\Gamma(k-1+\theta)}{\Gamma(k+1)}.
\end{aligned}
\tag{3.5}
$$

as $1/(k-1+\theta)$ is decreasing in k.

We can now use this to bound $\mathbb{E}_n[\tau_1^{(n)}]$, and therefore $\mathbb{E}_\pi[\tau_1^{(n)}]$. There will be a number of fragmentation events before the first hitting time of the state with 1 block. These fragmentation events will cause the number of blocks to increase by some number depending on the size of the block fragmented. As N is a skip-free process when decreasing, we have $\mathbb{E}_k[\tau_1^{(n)}] \leq \mathbb{E}_n[\tau_1^{(n)}]$ for all $k \leq n$. Therefore, if every fragmentation event moved the process back to the state with n blocks again, instead of some number between 2 and n, the expected hitting time of the state with 1 block will be larger.

Hence, consider a slightly different Markov process on $\mathcal{P}_n$, say $\widehat{\Pi}^{(n)}$, where the rates of coalescence and fragmentation are the same as $\Pi^{(n)}$ (the process Π restricted to $\{1, \ldots, n\}$), but when fragmentation occurs, we return to the state $\mathbf{0}|_{[n]}$. We can consider, for $\#\widehat{\Pi}^{(n)}$, the number of times the system attempts to descend to 1 from the initial state n, where a failure corresponds to a fragmentation event occurring before it reaches state 1. It is a geometric random variable with success rate $p_{n,1}$ and thus the expected value of this random variable (expected number of attempts until success) will be $p_{n,1}^{-1}$. Equation (3.5) tells us the expected amount of time each failure will take,

we have

$$\mathbb{E}_n[\inf\{t > 0 : \#\widehat{\Pi}^{(n)}(t) = 1\}] = p_{n,1}^{-1}\mathbb{E}_n[\tau_\infty].$$

Using the same argument as for Π, it is straightfoward to see that,

$$\mathbb{E}_k[\inf\{t > 0 : \#\widehat{\Pi}^{(n)}(t) = 1\}] \le \mathbb{E}_n[\inf\{t > 0 : \#\widehat{\Pi}^{(n)}(t) = 1\}],$$

for $1 \le k \le n$. The transition rates of $\#\widehat{\Pi}^{(n)}$ from k to $k-1$ are identical to $\#\Pi^{(n)}$, but now at rate λk, $\#\widehat{\Pi}^{(n)}$ transitions to state n, whereas $\#\Pi^{(n)}$ transitions at the same rate but to some state between $k+1$ and n depending on the size of the block fragmented. Hence by the monotonicity property of both $\#\Pi^{(n)}$ and $\#\widehat{\Pi}^{(n)}$, and the fact that both are skip-free when decreasing, we see that

$$e_1^{(n)} := \mathbb{E}_n[\tau_1^{(n)}] \le \mathbb{E}_n[\inf\{t > 0 : \#\widehat{\Pi}^{(n)}(t) = 1\}] = p_{n,1}^{-1}\mathbb{E}_n[\tau_\infty].$$

Hence, by (3.1) and (3.5), we have

$$
\begin{aligned}
e_1^{(n)} &\le \frac{2(1+\theta)}{c}\left(\frac{\Gamma(n+\theta)}{\Gamma(n)\Gamma(1+\theta)}\right)\frac{\Gamma(n)}{\Gamma(n+\theta)}\sum_{k=1}^n \frac{\Gamma(k-1+\theta)}{\Gamma(k+1)} \\
&\le \frac{2(1+\theta)}{c\Gamma(1+\theta)}\sum_{k=1}^n \frac{\Gamma(k-1+\theta)}{\Gamma(k+1)} \\
&\le D\sum_{k=1}^n k^{\theta-2} \\
&\le D\sum_{k=1}^\infty k^{\theta-2},
\end{aligned}
$$

where D is a constant that does not depend on n or on the initial state π. The above is finite if and only if $\theta \in (0,1)$, uniformly for all n. It follows that of $\theta < 1$, then $\sup_n e_1^{(n)} < \infty$.

Now let $\tau_1^{(n)} = \inf\{t : \Pi^{(n)}(t) = [n]\}$ be the first time that the integers $1, \ldots, n$ are all in the same block. Then $\tau_1^{(1)} \le \tau_1^{(2)} \le \cdots$, and so $\tau_1^{(n)}$ increases to some limit $\tau_1^{(\infty)}$. As $\sup_n \mathbb{E}_n[\tau_1^{(n)}] = \sup_n e_1^{(n)} < \infty$, it follows from the Monotone Convergence Theorem that $\mathbb{E}_\pi[\tau_1^{(\infty)}] < \infty$ and so $\tau_1^{(\infty)} < \infty$ a.s. Hence, $N(\tau_1^{(\infty)}) = 1$, and so $\tau_1^{(\infty)} = \tau_1$. It follows that τ_1 and thus T are $\mathbb{P}_\pi$-almost surely finite.

To prove that $T = 0$ almost surely, we bound the expected time for N to hit k from initial state n uniformly in n and prove that this converges to 0 as $k \to \infty$. The proof is very similar to the argument above. To this end, let $p_{n,k}$ be the probability of N hitting the state with k blocks before a fragmentation event occurs. A similar

calculation to (3.1) gives

$$p_{n,k} = \prod_{j=k+1}^{n} \frac{c\binom{j}{2}}{\lambda j + c\binom{j}{2}} = \frac{\Gamma(k+\theta)\Gamma(n)}{\Gamma(n+\theta)\Gamma(k)}. \tag{3.6}$$

Then, using the same argument as for $e_1^{(n)}$, the expected time to hit a state with k blocks, written $e_k^{(n)}$, satisfies

$$
\begin{aligned}
e_k^{(n)} &\leq p_{n,k}^{-1} \mathbb{E}_n[\tau_\infty] \\
&\leq \left(\frac{\Gamma(n+\theta)\Gamma(k)}{\Gamma(k+\theta)\Gamma(n)} \right) \frac{2(1+\theta)\Gamma(n)}{c\Gamma(n+\theta)} \sum_{j=1}^{n} \frac{\Gamma(j-1+\theta)}{\Gamma(j+1)} \\
&\leq \frac{2(1+\theta)\Gamma(k)}{c\Gamma(k+\theta)} \sum_{j=1}^{\infty} \frac{\Gamma(j-1+\theta)}{\Gamma(j+1)},
\end{aligned}
$$

and so

$$
\begin{aligned}
\sup_n e_k^{(n)} &\leq \frac{2(1+\theta)\Gamma(k)}{c\Gamma(k+\theta)} \sum_{j=1}^{\infty} \frac{\Gamma(j-1+\theta)}{\Gamma(j+1)} \\
&\leq D_2 k^{-\theta} \sum_{j=1}^{\infty} j^{\theta-2} \to 0, \tag{3.7}
\end{aligned}
$$

for some constant D_2, as $k \to \infty$, because the series is finite for $\theta \in (0,1)$ and $\Gamma(k+a)/\Gamma(k+b) \sim k^{a-b}$ as $k \to \infty$. Let $\tau_k^{(n)}$ denote the first time the process Π_n has k blocks. Then $\tau_k^{(k)} \leq \tau_k^{(k+1)} \leq \cdots$, so $\tau_k^{(n)}$ increases to some limit $\tau_k^{(\infty)}$. As $\sup_n \mathbb{E}_n[\tau_k^{(n)}] = \sup_n e_k^{(n)} < \infty$, it follows from the Monotone Convergence Theorem that $\mathbb{E}_\pi[\tau_k^{(\infty)}] < \infty$ and so $\tau_k^{(\infty)} < \infty$ a.s. Hence, $N(\tau_k^{(\infty)}) = k$, and so $\tau_k^{(\infty)} = \tau_k$. Thus, as T is the first hitting time of *any* finite state

$$0 \leq \mathbb{E}_\pi[T] \leq \mathbb{E}_\pi[\tau_k], \quad \text{for all } k \in \mathbb{N},$$
$$\to 0,$$

as $k \to \infty$, by (3.7). It follows that $T = 0$ $\mathbb{P}_\pi$-almost surely, as required. $\qquad\square$

The next result shows that, when the process starts from the partition of the positive integers into singletons, although the process immediately comes down from infinity, there are also fragmentation events at arbitrarily small times which cause the number of blocks to become infinite.

Lemma 3.6. *Let $S = \inf\{t > 0 : N(t) = \infty\}$. If $\theta \in (0,1)$ and π is a partition with infinitely many blocks, then $\mathrm{P}_\pi(S = 0) = 1$.*

Proof. Let F_k be the number of fragmentations that occur before the first time the

number of blocks reaches k, so that $p_{n,k} = \mathbb{P}_n(F_k = 0)$ is the probability that N drops from n to k without a fragmentation occurring. When a fragmentation occurs, the process N must first return to a state with n blocks before it can reach a state with k blocks. Hence, we can conclude that, for $n \geq k$, the random variable F_k is stochastically dominates a geometric random variable with success probability $p_{n,k}$. Therefore, appealing to (3.6), for all $j, k \in \mathbb{N}$, we have

$$\mathbb{P}_\pi(F_k > j) \geq \left(1 - \frac{\Gamma(k+\theta)\Gamma(n)}{\Gamma(n+\theta)\Gamma(k)}\right)^j \to 1, \quad \text{as } n \to \infty.$$

We can therefore conclude that for all $k \in \mathbb{N}$, the number of fragmentation events that occur before the process reaches any state with k blocks is infinite almost surely. The result follows. $\qquad\square$

The next lemma proves that the process N (resp. M) is not sticky at $\{\infty\}$ (resp. $\{0\}$).

Lemma 3.7. *If $\theta \in (0,1)$ and $\pi \in \mathcal{P}$, then $\int_0^\infty \mathbf{1}_{\{N(t)=\infty\}}\,\mathrm{d}t = 0$, $\mathbb{P}_\pi$-almost surely.*

Proof. To prove this result we will decompose $\mathbb{E}_\pi[\tau_1]$ into a sum over the expected amount of time spent with $N(t) = k$ before this hitting time for $k \in \{2, \ldots\} \cup \{\infty\}$. Then we will show that actually summing these over only the finite k gives $\mathbb{E}_\pi[\tau_1]$, so the contribution of the time spent with infinitely many blocks is zero.

Let $N^{(n)}(t)$ be the number of blocks in the partition $\Pi^{(n)}(t)$. For $k, n \in \mathbb{N} \cup \{+\infty\}$ with $2 \leq k \leq n$, let

$$g_k^{(n)} = \mathbb{E}_\pi\left[\int_0^{\tau_1^{(n)}} \mathbf{1}_{\{N^{(n)}(t)=k\}}\,\mathrm{d}t\right]$$

be the expected amount of time for which the process $\Pi^{(n)}$ has k blocks, before the time $\tau_1^{(n)}$. To calculate $g_k^{(\infty)}$, note that each time the process N visits the state k, it has probability $p_{k,1}$ of reaching the state 1 before fragmentation, and if there is a fragmentation the process must return to k before reaching the state 1. This is because if N is finite, then every block is infinite in size by Kingman's Theorem. Therefore, assuming there are initially at least k blocks, the process makes $p_{k,1}^{-1}$ visits to k on average before time τ_1, and thus

$$g_k^{(\infty)} = \frac{1}{p_{k,1}} \cdot \frac{1}{\lambda k + c\binom{k}{2}}.$$

When the process $N^{(n)}$ visits k, there is still probability $p_{k,1}$ that the process reaches the state with one block before fragmentation, but the process could have fewer than k blocks after fragmentation as the block is finite in size. Hence it could still have

another chance to reach the state 1 before returning to k. It follows that

$$g_k^{(n)} \leq g_k^{(\infty)}$$

for $2 \leq k \leq n$. Thus,

$$\mathbb{E}_\pi[\tau_1] = \sum_{k=2}^\infty g_k^{(\infty)} + g_\infty^{(\infty)} \geq \sum_{k=2}^n g_k^{(n)} + g_\infty^{(\infty)} = \mathbb{E}_\pi[\tau_1^{(n)}] + g_\infty^{(\infty)}.$$

Because the times $\tau_1^{(n)}$ increase to τ_1, which has finite mean as shown in the proof of Lemma 3.5, it follows by letting $n \to \infty$ and using the Monotone Convergence Theorem that $g_\infty^{(\infty)} = 0$. That is, with probability one, the set of times that N spends in the state ∞ before time τ_1 has Lebesgue measure zero. This is sufficient to establish the result. $\qquad\square$

To prove part (i) of Theorem 3.2, it remains to show that M and N are strong Markov processes. It is known from results of Berestycki [11] that the partition-valued EFFC process Π is a Feller process. However, while the processes M and N clearly evolve in a Markovian way when there are only finitely many blocks, one could be concerned about whether the Markov property holds when there are infinitely many blocks, especially in view of the unusual behaviour described in Lemmas 3.5 and 3.6. In particular, there is the question of whether knowing that the partition has infinitely many blocks provides sufficient information about the partition to determine how the number of blocks evolves in the future. The lemma below settles this question.

Lemma 3.8. *If $\theta \in (0, 1)$, then $(M(t), t \geq 0)$ and $(N(t), t \geq 0)$ are Feller processes.*

Proof. We prove this result by showing first that N is a Markov process and then that it is a Feller process. The main issue is how N behaves in the state $\{\infty\}$, hence we will look at a modified process where the times when $N(t) > k$ are cut out. Then we show that the transition probabilities of N are a limit, as $k \to \infty$, of the transition probabilities of this modified process, which will show that N is a Markov process. This modified process will then be used to show that N is a Feller process.

For $k \in \mathbb{N}$ and $t \geq 0$, let $S_k(t) = \inf\{u : \int_0^u \mathbf{1}_{\{N(s) \leq k\}}\, ds > t\}$. Then let

$$\hat{N}_k(t) = N(S_k(t)), \qquad t \geq 0.$$

Note that the process $\hat{N}_k$ is the same as the original process N, except that the periods during which the partition has more than k blocks are cut out. After every fragmentation event, the process $\hat{N}_k$ jumps to k. Therefore, $(\hat{N}_k(t), t \geq 0)$ is a continuous-time Markov chain with state space $\{1, \ldots, k\}$ and transition rates $\hat{Q}_{j,j-1} = c\binom{j}{2}$ for $2 \leq j \leq k$ and $\hat{Q}_{j,k} = \lambda j$ for $1 \leq j \leq k - 1$. Let $p_t^k(i, j) = P(\hat{N}_k(s + t) = j | \hat{N}_k(s) = i)$,

$t \geq 0$, $i, j \in \{1, \ldots, k\}$, denote the transition probabilities associated with this chain.

Because $\int_0^\infty \mathbf{1}_{\{N(t)=\infty\}} \, dt = 0$ a.s. by Lemma 3.7, it follows that for all $t \geq 0$, we have $S_k(t) \downarrow t$ a.s. as $k \to \infty$. Since $(N(t), t \geq 0)$ is right-continuous by Proposition 1 in [49], it follows that $\hat{N}_k(t) \to N(t)$ a.s. as $k \to \infty$. Therefore, for all times $t_1 < \cdots < t_m$ and positive integers $j_1, \ldots, j_m$, we have

$$\lim_{k\to\infty} \mathbb{P}_\pi(\hat{N}_k(t_1) = j_1, \ldots, \hat{N}_k(t_m) = j_m) = \mathbb{P}_\pi(N(t_1) = j_1, \ldots, N(t_m) = j_m)$$

by the Dominated Convergence Theorem. By applying this result when $m = 1$ and the initial partition has i blocks, we obtain for all $i \in \mathbb{N}$, $j \in \mathbb{N}$, and $t > 0$, the existence of the limit

$$p_t(i, j) := \lim_{k\to\infty} p_t^k(i, j).$$

Likewise, by considering an initial condition in which the partition has infinitely many blocks, we obtain for all $j \in \mathbb{N}$ and $t > 0$ the existence of the limit

$$p_t(\infty, j) := \lim_{k\to\infty} p_t^k(k, j).$$

Also, because $\int_0^\infty \mathbf{1}_{\{N(t)=\infty\}} \, dt = 0$ a.s. by Lemma 3.7, it is not hard to see that $\mathbb{P}_\pi(N(t) = \infty) = 0$ for all $t > 0$ and $\pi \in \mathcal{P}$. Therefore, for all $t > 0$, we let $p_t(i, \infty) := 0$ for all $i \in \mathbb{N} \cup \{+\infty\}$. It then follows that, if π has i blocks, then for $j_1, \ldots, j_m \in \mathbb{N} \cup \{+\infty\}$,

$$\mathrm{P}_\pi(N(t_1) = j_1, \ldots, N(t_m) = j_m) = p_{t_1}(i, j_i) p_{t_2 - t_1}(j_1, j_2) \ldots p_{t_m - t_{m-1}}(j_{m-1}, j_m).$$

Thus, $(N(t), t \geq 0)$ is a continuous-time Markov process with transition probabilities p_t.

It remains to check that N, and therefore M, is Feller. Let $f : \mathbb{N} \cup \{+\infty\} \to (0, \infty)$ be a continuous function, which in this setting means that $\lim_{n\to\infty} f(n) = f(\infty)$. Note that the function f must be bounded. Using $\mathbb{P}_n$ to denote the law of N started from n, we need to show that

1. For all $n \in \mathbb{N} \cup \{+\infty\}$, we have $\lim_{t\to 0} \mathbb{E}_n[f(N(t))] = f(n)$.

2. For all $t > 0$, the function $n \mapsto \mathbb{E}_n[f(N(t))]$ is continuous.

The first of these claims follows immediately from the right continuity of $(N(t), t \geq 0)$, see Proposition 1 of [49], the boundedness of f, and the Dominated Convergence Theorem. To prove the second claim, we need to show that $\lim_{n\to\infty} \mathbb{E}_n[f(N(t))] = \mathbb{E}_\infty[f(N(t))]$. It suffices to show that for all $j \in \mathbb{N}$ and $t > 0$, we have

$$\lim_{n\to\infty} p_t(n, j) = p_t(\infty, j).$$

For $n \leq k < \infty$, let $\zeta_{n,k} = \inf\{t : \hat{N}_k(t) = n\}$. Observe that $S_k(t + \zeta_{n,k}) \downarrow t + \tau_n$ as $k \to \infty$. Also, if the initial partition has infinitely many blocks, then $\tau_n \downarrow 0$ as $n \to \infty$ by Lemma 3.5. Therefore, using the right continuity on $(N(t), t \geq 0)$ in the first two lines and the strong Markov property of $(\hat{N}_k(t), t \geq 0)$ in the third line, we get

$$
\begin{aligned}
p_t(\infty, j) &= \lim_{n \to \infty} \mathbb{P}_{\mathbf{0}}(N(t + \tau_n) = j) \\
&= \lim_{n \to \infty} \lim_{k \to \infty} \mathbb{P}_{\mathbf{0}}(\hat{N}_k(t + \zeta_{n,k}) = j) \\
&= \lim_{n \to \infty} \lim_{k \to \infty} p_t^k(n, j) \\
&= \lim_{n \to \infty} p_t(n, j),
\end{aligned}
$$

which completes the proof. $\qquad\square$

Now we can combine the previous four lemmas to prove the first part of Theorem 3.2. The second part of the theorem, which deals with the super-critical case $\theta \geq 1$, requires a different method, which involves showing that excursions from $\{\infty\}$ for N can't exist almost surely.

Proof of Theorem 3.2. (i) The process M is a Feller process, and thus strong Markov, by Lemma 3.8. That 0 is a regular point of M follows from Lemma 3.6, and that 0 is not a holding point follows from Lemma 3.5. That 0 is non-sticky is a consequence of Lemma 3.7. The proof of part (i) of Theorem 3.2 is now complete.

(ii) To prove part (ii) of the Theorem, we consider the excursions from infinity of the process N and show that no such excursions can exist. Let u_k be the expected waiting time in a state with k blocks, specifically

$$
u_k = \frac{2}{2\lambda k + ck(k-1)}.
$$

Fix $t > 0$, and let $s_k(t)$ be the expected time spent in a state with k blocks during excursions that started before time t. Also, let $E_k(t)$ be the expected number of excursions started before time t that reach a state with k blocks. Then

$$
s_k(t) = E_k(t) \cdot u_k,
$$

and in particular note that because once an excursion has reached a state with k blocks, the probability that it reaches the state with just one block is $p_{k,1}$, hence

$$
E_k(t) = p_{k,1} E_1(t)
$$

where

$$p_{k,1} = \frac{\Gamma(1+\theta)\Gamma(k)}{\Gamma(k+\theta)}$$

from (3.1) and therefore

$$s_1(t) = E_1(t) \cdot u_1$$
$$= \frac{1}{\lambda} E_k(t) \cdot p_{k,1},$$

Hence, for all $k \in \mathbb{N}$ we have

$$\frac{s_k(t)}{s_1(t)} = \frac{\lambda u_k}{p_{k,1}},$$

as long as $s_1(t) > 0$. Thus, assuming for a contradiction, that this is the case we have that

$$\sum_{k=1}^{\infty} \frac{s_k(t)}{s_1(t)} = \frac{2\lambda}{c} \sum_{k=1}^{\infty} \frac{1}{p_{k,1}} \frac{1}{k(k-1+\theta)}$$
$$= \theta \sum_{k=1}^{\infty} \frac{\Gamma(k+\theta)}{\Gamma(k)\Gamma(1+\theta)k(k-1+\theta)},$$

which is $+\infty$ if, and only if, $\lambda/c \geq 1/2$, as then $\theta \geq 1$. However, if $s_1(t) > 0$, then the expected total time spent on excursions that start before time t must be finite as the expected length of any one excursion must be finite. Otherwise, it would be the case that $E_1(t) = 0$ for all $t \geq 0$, as $\mathbb{E}_\infty[\tau_1]$ would have to be infinite for an excursion to have expected infinite length, because $\mathbb{E}_1[\tau_\infty] = 1/\lambda < \infty$. Therefore $s_1(t) = 0$, which would be a contradiction. Hence, we must conclude that $s_1(t) = 0$ and therefore $s_k(t) = 0$ for all $k \in \mathbb{N}$. That is Π stays infinite almost surely. $\qquad\square$

3.3 Excursions from 0 in the subcritical case

For the entirety of this section, we will assume that $\theta \in (0, 1)$. Therefore, Theorem 3.2 states that M is a Feller (and hence strong Markov) process, such that zero is a regular and instantaneous point. Standard theory now allows us to invoke the existence of a local time at zero for M, denoted by $L = (L_t : t \geq 0)$. Therefore, we can exploit the powerful machinery of excursion theory to prove further important results, for example, finding the stationary distribution of M (and therefore N) and the speed of coming down from infinity in these cases.

We will therefore introduce some of the background and the main results of interest to us from the field, our main sources being Bertoin [15], Dellacherie & Meyer [26] and Itô [42]. We consider the particular case where the sample space is $\mathbb{R}$ and let X be a strong Markov process on the probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P})$. Let $\mathbb{P}_x$ be the law of X

started from $x \in \mathbb{R}$. We provide a brief construction of the local time of X at x, $L_x(t)$. A far more detailed construction can be found in Chapter 4 of [15].

First, note that for some fixed $c > 0$, the probability that there is at least one excursion of length greater than c is one, because x is an instantaneous point and paths are right-continuous. Let l be the length of an excursion, then clearly

$$\mathbb{P}_x(l > c | l > a) > 0, \quad \text{for all } a \in (0, c).$$

Hence, we can define a function $\gamma : (0, \infty] \to (0, \infty)$ as

$$\gamma(a) = \begin{cases} 1/\mathbb{P}_x(l > c | l > a) & \text{if } a < c \\ \mathbb{P}_x(l > a | l > c) & \text{if } a \geq c \end{cases}$$

In addition, define $Z_a(t)$ to be the number of excursions of at least length a, started before time t. Hence, Theorem 4 of [15] gives the construction of the local time.

Theorem 3.9. *The following assertions hold a.s.:*

1. *For all $t \geq 0$, $Z_a(t)/\gamma(a)$ converges as a tends to $0+$, the limit is denoted by $L_x(t)$.*

2. *The mapping $t \to L_x(t)$ is incresing and continuous, it is called the local time of X at x.*

A very important process is the inverse local time, L_x^{-1}, of X at x defined as

$$L_x^{-1}(t) = \inf\{s \geq 0 : L_x(s) > t\}, \quad t \geq 0.$$

This process captures the excursion intervals of X away from x, which are all of the form $(L_x^{-1}(t-), L_x^{-1}(t))$ for all $t \geq 0$ such that $L_x^{-1}(t) > L_x^{-1}(t-)$. Now we can introduce the excursion process itself. It takes values on the space of all paths of X started from x, which we will denote $\mathcal{E}_x$, where

$$\mathcal{E}_x := \{\epsilon : \mathbb{R}^+ \to \mathbb{R}, \text{ where } \epsilon(0) = x \text{ and } \epsilon^{-1}(\mathbb{R}\backslash\{x\}) = (0, \zeta), \text{ for some } \zeta > 0\}.$$

The excursion process, $Y = (Y(t), t \geq 0)$, takes values in $\mathcal{E}$ such that $Y(t)$ is the path taken by X on the interval $(L_x^{-1}(t-), L_x^{-1}(t))$.

Theorem 3.10 (Itô [42])**.** *Y is a Poisson point process on $\mathcal{E}$ with characteristic measure $\mathbb{Q}$, where $\mathbb{Q}$ is defined such that*

$$\mathbb{Q}(\cdot \, | \zeta > s),$$

is the law of X condtioned to have it's first excursion with lifetime greater than s.

Remark 3.11. $\mathbb{Q}$ *is often referred to as the excursion measure. It is often not a finite measure, though it is σ-finite. As local times are unique only up to a constant, so are these measures.*

We can now note some results of importance to us about local times and excursion theory.

Theorem 3.12. *There exists a constant $d \geq 0$ such that for all $t \geq 0$,*

$$\int_0^t \mathbb{1}_{\{X_s = x\}} ds = dL_x(t). \tag{3.8}$$

If $d = 0$, then x is called a non-sticky point of X. Now we can specify the law of the inverse local time.

Theorem 3.13. *The inverse local time is a subordinator with*

$$\mathbb{E}\left[e^{-qL_x^{-1}(t)}\right] = \exp\left\{-t\Phi(q)\right\},$$

where

$$\Phi(q) = dq + q \int_0^\infty e^{-qr} \mathbb{Q}(\zeta > r) dr,$$

where d is the same constant from (3.8).

We can use the excursion process to find important properties about the original Markov process X. For example, Chapter XIX.46 of Dellacherie and Meyer [26] shows when you can find the stationary distribution of X, ρ_X.

Theorem 3.14. *Suppose $\mathbb{Q}(\zeta) < \infty$. Then, for all $y \in \mathbb{R}$,*

$$\rho_X(X = y) = \frac{\mathbb{Q}\left(\int_0^\zeta \mathbb{1}_{\{\epsilon(t) = y\}} dt\right)}{\mathbb{Q}(\zeta)}.$$

Moving back to the fast fragmentation-coalescence process, in the subcritical case 0 is instantaneous and regular for M, so it follows that L^{-1} has infinite activity (as in, the path of L^{-1} contains infinitely many, arbitrarily small positive jumps for any finite time horizon). The periods of time where the process is in a state with finitely many blocks correspond to the excursions away from zero for M. Moreover, the fact that the state 0 is not sticky for M (by Lemma 3.7) implies that L^{-1} is pure jump with no drift component (by Theorems 3.8 and 3.13). We will use this local time, and associated excursion theory, to prove several results about the fast fragmentation-coalescence process in the subcritical case

3.3.1   Stationary distribution

We know, from Theorem 8 [11], that the fast fragmentation-coalescence has a stationary distribution, ρ. There are very few non-trivial EFC processes where ρ is explicitly known. One such example is given by Bertoin [19], involving the Poisson-Dirichlet distribution. Unfortunately, it is not explicitly known in this case either (I discuss the possibility of finding this stationary distribution in Chapter 4). However, we can find a partial result for ρ; that is, for $k \in \mathbb{N}$, the value of

$$\rho(\{\pi \in \mathcal{P} : \#\pi = k\}) = \rho_M(1/k),$$

as a function of θ and k.

Proof of Theorem 3.3. Classical results from excursion theory, cf. Chapter XIX.46 of Dellacherie and Meyer [26], give us a way to construct the stationary distribution, ρ_M, of M using the excursion measure. To this end, let us introduce the canonical space of excursions, $\mathcal{E}$, that is càdlàg measurable mappings $\epsilon : (0, \zeta] \to \{1/n : n \in \mathbb{N}\} \cup \{0\}$, where $\zeta = \inf\{t > 0 : \epsilon(t) = 0\}$ and $\lim_{t \downarrow 0} \epsilon(t) = 0$, with associated excursion measure (not a probability measure) $\mathbb{Q}$.

First note, that ρ_M has no atom at zero as it is not a sticky point, equivalently, the inverse local time has no linear component. Again, referring to Chapter XIX.46 of Dellacherie and Meyer [26], we have that, when $\mathbb{Q}(\zeta) < \infty$, for $k \in \mathbb{N}$,

$$\rho_M(1/k) = \frac{\mathbb{Q}\left(\int_0^\zeta \mathbf{1}_{\{\epsilon(t) = 1/k\}} \mathrm{d}t\right)}{\mathbb{Q}(\zeta)}.$$

It's difficult to compute this quantity directly starting from zero, however, we can appeal to a technique that uses the strong Markov property for excursions; see Section VI.48 of [64]. Let $\sigma_{1/n} = \inf\{t > 0 : \epsilon(t) = 1/n\}$, then we see that, for $n > k$,

$$\mathbb{Q}\left(\int_0^\zeta \mathbf{1}_{\{\epsilon(t) = 1/k\}} \mathrm{d}t\right) = \mathbb{Q}(\sigma_{1/n} < \zeta)\mathbb{E}_n\left[\int_0^{\tau_\infty} \mathbf{1}_{\{N(t) = k\}} \mathrm{d}t\right],$$

where $\mathbb{E}_n$ is expectation of the process given we start in any state with n blocks and τ_∞ is the time of the first fragmentation event and therefore matches ζ. In any excursion, ϵ, Π visits a state with k blocks only once at most. This is because once we are in a state with fewer than k blocks there must be a fragmentation event before Π can be in a state with k blocks again, which means it will be a different excursion. Hence, the above expectation can break down into the probability of reaching a state with k blocks given you start in one with n, times the expected amount of time spent in a

state k blocks before leaving, thus, appealing to (3.6) we see that

$$
\mathbb{Q}\left(\int_0^\zeta \mathbf{1}_{\{\epsilon(t)=1/k\}}\mathrm{d}t\right) = \mathbb{Q}(\sigma_{1/n} < \zeta)p_{n,k}\frac{2}{ck(k-1)+2\lambda k}
$$

$$
= \mathbb{Q}(\sigma_{1/n} < \zeta)\frac{\Gamma(n)}{\Gamma(n+\theta)}\frac{2}{c}\frac{\Gamma(k-1+\theta)}{\Gamma(k+1)}. \tag{3.9}
$$

Therefore, as the left-hand side is positive and finite, we may conclude that there exists a constant C such that

$$
\mathbb{Q}(\sigma_{1/n} < \zeta)\frac{\Gamma(n)}{\Gamma(n+\theta)} = C. \tag{3.10}
$$

As excursion measures are only defined up to a multiplicative constant we can take $C = 1$ without loss of generality. In addition, from (3.9), we have that

$$
\rho_M(1/k) \propto \frac{2}{c}\frac{\Gamma(k-1+\theta)}{\Gamma(k+1)}.
$$

Note that the right hand side of this equation is $O(k^{-(2-\theta)})$ and so, as $\theta \in (0,1)$, we can normalise this into a probability measure using the identity from [61]

$$
\sum_{j=0}^{n-1}\frac{\Gamma(j-\beta)}{\Gamma(j+1)} = -\frac{\Gamma(n-\beta)}{\beta\Gamma(n)}, \quad \text{for } 0 < \beta < 1, \tag{3.11}
$$

hence the normalisation constant Z^{-1} satisfies

$$
Z^{-1} = \frac{2}{c}\sum_{k=1}^{\infty}\frac{\Gamma(k-1+\theta)}{\Gamma(k+1)} = \frac{2}{c}\lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{\Gamma(k-1+\theta)}{\Gamma(k+1)},
$$

$$
= \frac{2}{c}\lim_{n\to\infty}\left(\frac{\Gamma(\theta)}{1-\theta} - \frac{\Gamma(n-1+\theta)}{(1-\theta)\Gamma(n)}\right),
$$

$$
= \frac{\Gamma(1+\theta)}{\lambda(1-\theta)}
$$

as $\theta \in (0,1)$, which gives the desired result. $\qquad\square$

3.3.2 Speed of coming down from infinity

There are several results for the Kingman coalescent with regards to the speed with which it comes down from infinity. First, as the hitting times of the set of partitions with k blocks is a sum of exponential random variables of rate $c\binom{n}{2}$ for $n = k+1,\ldots,$, the expected hitting time of that set is $2/(ck)$, where c is the rate of the coalescent. Second, that the small-time behaviour of $N(t)$ is given as follows

$$
\lim_{t\searrow 0} tN(t) = \frac{2}{c}
$$

almost surely. That is the speed of coming down from infinity is $2/(ct)$. We will show three results in this section. That the speed of coming down from infinity is unaffected by the fragmentation (as long as $\theta \in (0,1)$), and that this speed is constant across all excursions from 0 and not just as t tends to zero. However, we show that the expected first hitting time of the set of partitions with k blocks is affected by the fragmentation rate. This is summarised more formally in Theorem 3.4.

Proof of Theorem 3.4. (i) Basic Markov chain theory tells us that the stationary distribution probabilities in each state are equal to the inverse of the mean return time from that state. That is for a general Markov chain with stationary distribution π

$$\pi_i = \frac{1}{q_i \mathbb{E}_i[T_i]}$$

where T_i is the first return time of i, and q_i is the jump rate of the chain out of state i. Hence, for M,

$$\frac{1}{\rho_M(1/k)} = \left(c\binom{k}{2} + \lambda k \right) \mathbb{E}_{1/k}[\text{Time for } M \text{ to return to } 1/k]$$
$$= \left(c\binom{k}{2} + \lambda k \right) \left(\mathbb{E}_{1/k}[\text{Time to hit } 0] + e_{1/k}^{(\infty)} \right),$$

as to return to a state with k blocks you must first fragment (that is hit 0, as all fragmentation events send M to 0), then come down from infinity and reach a state with k blocks once more. Rearranging this shows that

$$e_{1/k}^{(\infty)} = \frac{1}{\rho_M(1/k)(c\binom{k}{2} + \lambda k)} - \mathbb{E}_{1/k}[\text{Time to first fragmentation event}].$$

Using Theorem 3.3, we see that

$$e_{1/k}^{(\infty)} = \frac{\Gamma(\theta)\Gamma(k+1)}{(1-\theta)\Gamma(k-1+\theta)(c\binom{k}{2} + \lambda k)} - \sum_{j=1}^{k} p_{k,j} \frac{\lambda j}{c\binom{j}{2} + \lambda j} \sum_{i=j}^{k} \frac{1}{c\binom{i}{2} + \lambda i},$$

as the expected time to the first fragmentation event can be split over how many blocks

you have just before you fragment. Therefore, using (3.6),

$$
\begin{aligned}
e_{1/k}^{(\infty)} &= \frac{2\Gamma(\theta)\Gamma(k)}{c(1-\theta)\Gamma(k-1+\theta)(k-1+\theta)} \\
&\qquad - \sum_{j=1}^{k} \frac{\Gamma(j+\theta)\Gamma(k)}{\Gamma(k+\theta)\Gamma(j)} \frac{\theta}{j-1+\theta} \sum_{i=j}^{k} \frac{2}{ci(i-1)+2\lambda i} \\
&= \frac{2\Gamma(\theta)\Gamma(k)}{c(1-\theta)\Gamma(k+\theta)} - \theta \sum_{j=1}^{k} \frac{\Gamma(j-1+\theta)\Gamma(k)}{\Gamma(k+\theta)\Gamma(j)} \sum_{i=j}^{k} \frac{2}{ci(i-1)+2\lambda i} \\
&= \frac{2\Gamma(\theta)\Gamma(k)}{c(1-\theta)\Gamma(k+\theta)} - \frac{2\theta\Gamma(k)}{c\Gamma(k+\theta)} \sum_{i=1}^{k} \frac{1}{i(i-1+\theta)} \sum_{j=1}^{i} \frac{\Gamma(j-1+\theta)}{\Gamma(j)},
\end{aligned}
$$

which, by (3.11), gives us that

$$
\begin{aligned}
e_{1/k}^{(\infty)} &= \frac{2\Gamma(\theta)\Gamma(k)}{c(1-\theta)\Gamma(k+\theta)} - \frac{2\Gamma(k)}{c\Gamma(k+\theta)} \frac{\Gamma(\theta)\Gamma(k+2)-(k+1)\Gamma(k+\theta)}{(1-\theta)\Gamma(k+2)} \\
&= \frac{2}{c(1-\theta)k},
\end{aligned}
$$

as required.

(ii) We prove this using a similar method to Theorem 3.3. By appealing to the same technique that uses the strong Markov property of excursions, let $k \in \mathbb{N}$ and denote by $\mathbb{Q}_k$ the measure $\mathbb{Q}$ conditioned on $\{\zeta > \sigma_{1/k}\}$, where ζ is the excursion length. Under $\mathbb{Q}_k$, an excursion from 0 of M looks like a scaled Kingman coalescent, but with slightly accelerated rates, until it reaches a state with k blocks. Hence, we can use Aldous' construction of Kingman's coalescent [2]. Define random variables φ_j, for $j \geq 1$, as

$$
\varphi_j = \sum_{i=j+1}^{\infty} \xi_i,
$$

where ξ_i are independent exponential, with rate $c\binom{i}{2} + \lambda i$. Under the aforementioned conditioning, φ_j is the hitting time of a state with j blocks when $j \geq k$. Let U_j be IID uniform random variables on (0,1), $j \geq 1$. Then for all j, draw a vertical line of length φ_j at point U_j on the unit interval. At time t, where $\varphi_j < t < \varphi_{j-1}$, $j \geq k$, look at the subintervals of [0,1] with endpoints $\{0, 1, U_1, \ldots, U_{j-1}\}$. The lengths of the subintervals have the same distribution as the asymptotic frequencies of the blocks of Π conditional on $\zeta > \sigma_{1/k}$, when Π has j blocks.

Then, under the measure $\mathbb{Q}_k$, for large j we have

$$
\mathbb{E}[\varphi_j] = \sum_{i=j+1}^{\infty} \frac{1}{c\binom{i}{2}+\lambda i} \sim \frac{2}{cj}, \quad \text{and} \quad \mathrm{Var}[\varphi_j] = \sum_{i=j+1}^{\infty} \left(\frac{1}{c\binom{i}{2}+\lambda i}\right)^2 \sim \frac{4}{3c^2 j^3},
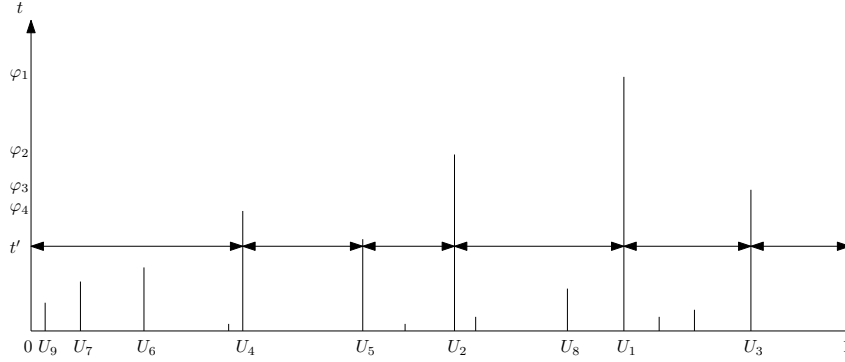$$

Figure 3-2: *A figure showing an example of this construction. Time moves up the graph, and the arrows show the width of the subintervals at time t' which correspond to the asympmtotic frequencies of the blocks in this realisation of Kingman's coalescent at time t'.*

which is the exact same asymptotic behaviour as for a Kingman coalescent of rate c. Hence we may conclude, using Aldous' method (by considering a time-reversal of the process, and using the CLT for independent sums) as used for the Kingman coalescent case, that

$$\lim_{t\downarrow 0} \frac{t}{\epsilon(t)} = \lim_{t\downarrow 0} tN(t) = \frac{2}{c},$$

$\mathbb{Q}_k$-almost surely. As this is independent of k, and the sets

$$\mathcal{E}^{(k)} := \{\epsilon \in \mathcal{E} : \epsilon(\zeta-) \geq 1/k\} \to \mathcal{E}$$

as $k \to \infty$, we may conclude this occurs $\mathbb{Q}$-almost everywhere. □

3.3.3   Properties of the excursion process

In addition to the above results, we have some natural questions to ask about the excursions of M from 0 and the structure of the excursion measure $\mathbb{Q}$. The main two we focus on are the excursion height and the excursion length under $\mathbb{Q}$. The latter of these is saved as part of the discussion in the final chapter. We can, however, find an explicit expression for the excursion height of M under $\mathbb{Q}$.

Proposition 3.15. *The distribution of the excursion height of M is given by*

$$\mathbb{Q}(M(\zeta-) = 1/k) = \theta \frac{\Gamma(k-1+\theta)}{\Gamma(k)}.$$

Proof. Let $n > k$. For the excursion height to reach $1/k$, it must first reach $1/n$ as M is skip-free upwards. Hence

$$\mathbb{Q}(M(\zeta-) = 1/k) = \mathbb{Q}(\sigma_{1/n} < \zeta)\mathbb{Q}_n(M(\zeta-) = 1/k).$$

The final quantity on the right-hand side is the probability that a fragmentation event occurs when you have k blocks when you start with n blocks. This was calculated earlier as the quantity $r_k^{(n)}$, so we use (3.3) to find that

$$\mathbb{Q}(M(\zeta-) = 1/k) = \mathbb{Q}(\sigma_{1/n} < \zeta)\frac{\theta\Gamma(n)\Gamma(k-1+\theta)}{\Gamma(n+\theta)\Gamma(k)} = \theta\frac{\Gamma(k-1+\theta)}{\Gamma(k)},$$

where the last equality comes from the proof of Theorem 3.3, namely (3.10). □

3.4 Further results

To gain information about the asymptotic frequencies of the blocks, we look at a sized-biased pick from those blocks. The natural way to do this is to study the block containing 1, which is always Π_1 as the blocks are ordered by least element. This is also known as the 'tagged' fragment. For example, in homogeneous fragmentations the process

$$\xi = -\log(|\Pi_1(t)|)$$

is a subordinator (see Theorem 3 [16]) with drift coefficient c_e, killing rate

$$k = c_e + \int_{S^\downarrow}\left(1 - \sum_{j=1}^\infty x_j\right)\nu_{\mathrm{Disl}}(d\mathbf{x}),$$

and Lévy measure

$$L(dx) = e^{-x}\sum_{j=1}^\infty \nu_{\mathrm{Disl}}(-\log x_j \in dx), \quad x \in (0, \infty).$$

Clearly, because in our EFC process $|\Pi_1(t)|$ is not monotonic, it's unlikely that we're going to get such nice results as these. However, we can find some moment results; we do this by using a link between the kth moments of $|\Pi_1(t)|$ and $\Pi_1^{(k+1)}(t)$ in the process restricted to $[k+1]$. The latter of which is

Proposition 3.16. *Let $|\Pi_1(t)|$ be the asymptotic frequency of the block containing 1 at time t. Then, for all $\lambda > 0$,*

(i)
$$\mathbb{E}_{\mathbf{0}}[|\Pi_1(t)|] = \frac{c}{c+\lambda}\left(1 - e^{-(c+\lambda)t}\right),$$

(ii)
$$\mathbb{E}_{\mathbf{0}}[|\Pi_1(t)|^2] = \frac{c}{c+\lambda}\frac{3c}{3c+\lambda}\left(1 - \frac{3c+\lambda}{2c}e^{-(c+\lambda)t} + \frac{c+\lambda}{2c}e^{-(3c+\lambda)t}\right).$$

Our proof makes use of the Poissonian construction of EFC processes.

Proof. We have from [16], that

$$\mathbb{E}_{\mathbf{0}}[|\Pi_1(t)|^k] = \mathbb{P}_{\mathbf{0}}(\Pi_1^{(k+1)}(t) = [k+1]),$$

as by definition of the asymptotic frequency of a block, we have

$$|\Pi_1(t)| = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \mathbb{1}_{\{j \in \Pi_1(t)\}},$$

and by exchangeability

$$\mathbb{P}_{\mathbf{0}}(j_1, \ldots, j_{k+1} \in \Pi_1(t)) = \mathbb{P}_{\mathbf{0}}(\Pi_1^{(k+1)}(t) = [k+1]), \quad \text{for all distinct } j_1, \ldots, j_{k+1} \geq 2,$$

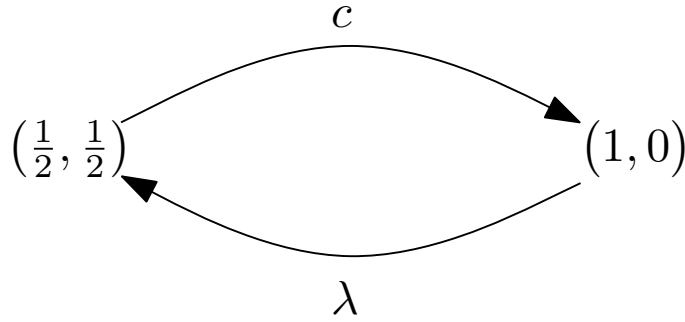combining the above two properties with dominated convergence gives us that claimed result.

Figure 3-3: *A figure displaying the transitions of $X^{(2)}$.*

Hence, for part (i) we study the process $\Pi^{(2)}$, so restrict our view to the first two integers. $\Pi^{(2)}$ is a two-state strong Markov jump process. If $\Pi^{(2)}(t) = \{\{1\}, \{2\}\}$, then at rate c it transitions to the state $\{\{1, 2\}\}$. If $\Pi^{(2)}(t) = \{\{1, 2\}\}$, then at rate λ it transitions to the state $\{\{1\}, \{2\}\}$. To simplify the process for restrictions of Π to higher numbers of integers, look at the equivalent process $X^{(2)}$ on $\mathcal{S}^{\downarrow}$ which has two states $(1, 0)$ and $(1/2, 1/2)$ representing the asymptotic frequencies of the blocks (the sequences are truncated to two places).

Therefore, we see that

$$\mathbb{E}_{\mathbf{0}}[|\Pi_1(t)|] = \mathbb{P}\left(X^{(2)}(t) = (1, 0) \,\Big|\, X^{(2)}(0) = \left(\frac{1}{2}, \frac{1}{2}\right)\right)$$
$$= \frac{c}{c + \lambda}\left(1 - e^{-(c+\lambda)t}\right),$$

by standard results from continuous time Markov chains, as required.

For part (ii), we don't study the process $\Pi^{(3)}$, but rather the process $X^{(3)}$ as it has

fewer states. Figure 3-4 shows the transitions of $X^{(3)}$. The $3c$ is due to there being three pairs coalescing at the same rate when in the state $(1/3, 1/3, 1/3)$ and the reason why fragmentation only happens at rate λ regardless of the number of blocks in the state is because only at most one block is not a singleton in $\Pi^{(3)}$.

$$\left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) \xrightleftharpoons[\lambda]{3c} \left(\tfrac{2}{3}, \tfrac{1}{3}, 0\right) \xrightarrow{c} \left(1, 0, 0\right)$$

$$\lambda$$
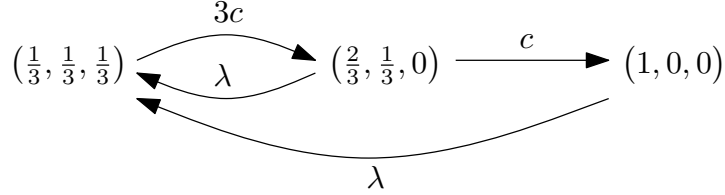
Figure 3-4: *A figure displaying the transitions of $X^{(3)}$.*

Therefore, using the same standard results from continuous time Markov chains, we see that

$$\mathbb{E}_{\mathbf{0}}[|\Pi_1(t)|^2] = \mathbb{P}\left(X^{(3)}(t) = (1,0,0)\,\Big|\, X^{(3)}(0) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right)$$

$$= \frac{c}{c+\lambda}\frac{3c}{3c+\lambda}\left(1 - \frac{3c+\lambda}{2c}e^{-(c+\lambda)t} + \frac{c+\lambda}{2c}e^{-(3c+\lambda)t}\right),$$

as required.

$\square$

Remark 3.17. *This result holds for all $\theta \in (0, \infty)$. Also, other moments can be calculated in a similar way, unfortunately the event $\{\Pi_1^{(k+1)}(t) = [k+1]\}$ gets very complicated very quickly.*

I will conclude by discussing some of the open questions that have arisen from the work done in previous chapters.

4.1 Finite fragmentation-coalescence processes

Theorem 2.1 gave sufficient conditions under which the finite fragmentation-coalescence process converged to its mean-field estimate. Our two main open questions relate to how strong the conditions on α need to be, and whether the fragmentation mechanism can be generalised, much like how the coalescence mechanism has been.

4.1.1 Is the condition on α tight?

The condition on α required for Theorem 2.1, that

$$\alpha(k) \leq C \exp(\gamma k \log \log k), \quad \text{for all } k \in \mathbb{N}, \tag{4.1}$$

for some constants $C > 0$ and $\gamma < 1$, arises in the proof of Lemma 2.4 when bounding an error term. Specifically using a recent bound on Bell's numbers [9]. Are there examples of α that violate the conditions but the fragmentation-coalescence process still converges in the thermodynamic limit?

The weakest condition that is required on α a priori is that

$$\sum_{k=2}^{\infty} \frac{\alpha(k)}{(k-2)!} < \infty, \tag{4.2}$$

otherwise multiple quantities are divergent and none of the questions we want to ask make sense. Where, between the condition in Theorem 2.1 (4.1) and the one above

(4.2), does the true condition lie?

An initial look at the question would suggest it is closer to the latter, but this will need careful manipulation of the complicated $\beta_n(x,t)$ term defined in (2.7), in particular the terms with higher powers of $1/n$.

4.1.2 Different fragmentation mechanisms

Can the proof of Theorem 2.1 be adapted to allow for different fragmentation mechanisms? In particular, the case of binary fragmentation which we discussed in Chapter 1, which is of particular interest.

The complication that arises from this change is that the fragmentation mechanism we looked at gives the very nice $-2\lambda Y_n(t)$ term in the derivative of $Y_n(t)$ and it is unlikely that binary fragmentation will give such nice terms. It is more likely to give series involving G like the coalescent mechanism does. For example, for a $CF^{(n)}$ process with fragmentation kernel $F(i,j)$, then in the derivative of $\mathbb{E}[G_n(x,t)]$ the fragmentation would give terms like

$$\frac{\partial}{\partial t}\mathbb{E}[G_n(x,t)] = \frac{1}{2}\sum_{j=2}^{n}\sum_{i=1}^{j-1}F(i,j-i)\left(x^i + x^{j-i} - x^j\right)w_{n,j}(t) + \dots$$

Even in the case $F \equiv 1$, this gives complex terms which could cause the existing method to break down, as it is tricky to even write this in terms of G_n as opposed to $w_{n,j}$. Hence, we perhaps need a new generating function, or even need to consider working with the vector $w_n(t)$ itself.

4.1.3 Different coagulation kernels

Much was made in the introductory chapter about the different coagulation kernels, K, which governed the rate at which two blocks coalesced based on their sizes. We looked at the case of the unit kernel, where $K \equiv 1$. What can we say about the process if we replace that kernel with a different one? The multiplicative case, where $K(i,j) = ij$, has been covered by Ráth and Tóth [63] though not in the case of multiple blocks merging, and there are many more of interest.

4.2 The fast fragmentation-coalescence process

Theorem 3.2 showed the existence of a phase transition in $\theta = 2\lambda/c$ when the fast fragmentation-coalescence process comes down from infinity. In addition, in the case $\theta < 1$ it showed the existence of a local time for the process $M_t = 1/N_t$ at zero which was exploited to find the stationary distribution of M in Theorem 3.3 and further results

in Theorem 3.4. Our main open questions are in relation to finding the stationary distribution of Π when $\theta \in (0, 1)$, and finding out more about the process when $\theta \geq 1$.

4.2.1 The subcritical case

Here I will briefly outline the open questions I have in the case where $\theta \in (0, 1)$. I will also state and prove a couple of partial results, which would hopefully be useful in calculating the stationary distribution of the process Π in this case.

Stationary distribution

We have the stationary distribution of M and therefore a partial result for the stationary distribution of Π. Can this be extended to find ρ itself? This boils down to finding the distribution of the asymptotic frequencies of blocks under stationarity, given there are k of them. Combining this with ρ_M and exchangeability will allow us to fully express ρ.

We can consider a different excursion process, this time involving Π instead of M. These are excursions away from the state $\mathbf{0} = \{\{1\}, \{2\}, \ldots\}$ and returning to that state. These are rather different from the ones involving M, as we note that $\mathbf{0}$ is an irregular point so the return time is positive almost surely. This is because the only way to return to state $\mathbf{0}$ is to reach the state with only 1 block, $\mathbf{1}$, and then wait for it to fragment. Hence, the length of these excursions is an exponential amount of time of rate $\Gamma(1 + \theta)$ on the local time scale for M as we need to wait for an excursion of M that reaches height 1.

Let $\mathcal{E}'$ be the path space of Π in $\mathcal{P}$, ϵ' a path, ζ' the excursion length and $\mathbb{Q}'$ the associated excursion measure. A natural check to perform is that calculating the stationary measure ρ of Π on sets of the form $\{\pi \in \mathcal{P} : \#\pi = k\}$ with this excursion process agrees with that of ρ_M.

Proposition 4.1. *For all $k \in \mathbb{N}$,*

$$\rho(\{\pi \in \mathcal{P} : \#\pi = k\}) = \rho_M(1/k).$$

Proof. Standard excursion theory from [26] tells us that

$$\rho(\{\pi \in \mathcal{P} : \#\pi = k\}) = \frac{\mathbb{Q}'\left(\int_0^{\zeta'} \mathbb{1}_{\{\#\Pi(s)=k\}}ds\right)}{\mathbb{Q}'(\zeta')}.$$

The denominator is for normalising only, so we will just check the numerator. We already know that the excursion will end once we reach the state with all integers in one block and then fragment it, and because N is skip-free when decreasing, Π must pass through a state with k blocks in order to reach the state with 1 block. The

probability of going from k blocks to 1 block before a fragmentation is given below

$$p_{k,1} = \prod_{i=2}^{k} \frac{c\binom{k}{2}}{c\binom{k}{2} + \lambda k} = \frac{\Gamma(1+\theta)\Gamma(k)}{\Gamma(k+\theta)}.$$

Therefore, the number of times a state with k blocks is visited in an excursion of Π is a Geometric random variable with success probability $p_{k,1}$. When in a state with k blocks you spend an exponential amount of time in such a state with rate $ck(k-1+\theta)/2$. This time is independent of the number of times a state with k blocks is visited. Hence

$$\mathbb{Q}'\left(\int_0^{\zeta'} \mathbb{1}_{\{\#\Pi(s)=k\}} ds\right) = \frac{2}{ck(k-1+\theta)} \frac{1}{p_{k,1}} = \frac{2}{ck(k-1+\theta)} \frac{\Gamma(k+\theta)}{\Gamma(k)\Gamma(1+\theta)}$$

$$= \frac{2}{c} \frac{\Gamma(k-1+\theta)}{\Gamma(1+\theta)\Gamma(k+1)}. \tag{4.3}$$

This is the same ratio of gamma functions as for ρ_M, hence ρ and ρ_M agree, as required.

$\square$

Corollary 4.2.

$$\mathbb{Q}'(\zeta') = \frac{1}{\lambda(1-\theta)}$$

Proof. $\mathbb{Q}'(\zeta')$ normalises (4.3) to a probability distribution. Therefore, we require

$$\mathbb{Q}'(\zeta')^{-1} \sum_{k=1}^{\infty} \frac{2}{c} \frac{\Gamma(k-1+\theta)}{\Gamma(1+\theta)\Gamma(k+1)} = 1.$$

Hence, simple computation shows us that

$$\mathbb{Q}'(\zeta') = \frac{2}{c\theta(1-\theta)} = \frac{1}{\lambda(1-\theta)},$$

as required.

$\square$

Remark 4.3. *This makes heuristic sense: if θ is close to 1, then we would expect to take a long time to reach the state with 1 block due to the fragmentation preventing the chain from reaching there. If λ is small, then once we've reached the state with 1 block we will then wait longer for it to fragment and thus for Π to return to $\mathbf{0}$.*

It is hoped that this excursion process will shed some light on the stationary distribution of Π.

Excursion process

In Theorem 3.13 in Section 1.5 we mentioned the result concerning the inverse local time and the fact that it is a subordinator with explicitly known Laplace exponent.

We wish to find this Laplace exponent in the case of the inverse local time of M for the fast fragmentation-coalescence process.

As M is non-sticky at zero, we know that the drift coefficient d is zero. Hence to find Φ it remains to find the jump part of the Laplace exponent. Standard excursion theory tells us that

$$q \int_0^\infty e^{-qr} \mathbb{Q}(\zeta > r) dr = \mathbb{Q}\left(1 - e^{-q\zeta}\right),$$

then we can calculate this quantity by splitting over the height of the process at $\zeta-$

$$\mathbb{Q}\left(1 - e^{-q\zeta}\right) = \sum_{k=1}^\infty \mathbb{Q}(M(\zeta-) = 1/k)\mathbb{Q}(1 - e^{-q\zeta}| M(\zeta-) = 1/k). \tag{4.4}$$

We have by Proposition 3.15 that

$$\mathbb{Q}(M(\zeta-) = 1/k) = \theta \frac{\Gamma(k - 1 + \theta)}{\Gamma(k)}. \tag{4.5}$$

Hence, we just need to calculate the other part. First, note that conditioning on $M(\zeta-) = 1/k$ makes $\mathbb{Q}$ a finite measure. Second, that conditioning on the height being $1/k$ makes $\zeta = \sum_{j=k}^\infty \mathbb{e}_{c\binom{j}{2}+\lambda j}$ and independent of the height. Hence

$$\mathbb{Q}(1 - e^{-q\zeta}| M(\zeta-) = 1/k) = 1 - \mathbb{Q}\left(\exp\left(-q \sum_{j=k}^\infty \mathbb{e}_{c\binom{j}{2}+\lambda j}\right)\right),$$

$$= 1 - \prod_{j=k}^\infty \frac{c\binom{j}{2} + \lambda j}{c\binom{j}{2} + \lambda j + q}. \tag{4.6}$$

Therefore, we can put (4.4), (4.5) and (4.6) together to get an expression for the Laplace exponent

$$\Phi(q) = \theta \sum_{k=1}^\infty \frac{\Gamma(k - 1 + \theta)}{\Gamma(k)}\left(1 - \prod_{j=k}^\infty \frac{c\binom{j}{2} + \lambda j}{c\binom{j}{2} + \lambda j + q}\right).$$

This, unfortunately, is not particularly enlightening. Hence, an open problem here is to find a simplification of the above series. We have that

$$\prod_{j=k}^\infty \frac{c\binom{j}{2} + \lambda j}{c\binom{j}{2} + \lambda j + q} = \frac{\Gamma\left(k - \frac{1-\theta}{2} + \frac{1}{2}\sqrt{(1+\theta)^2 - \frac{8q}{c}}\right)\Gamma\left(k - \frac{1-\theta}{2} - \frac{1}{2}\sqrt{(1+\theta)^2 - \frac{8q}{c}}\right)}{\Gamma(k)\Gamma(k - 1 + \theta)},$$

but still this does not seem to help too much.

4.2.2 The critical and supercritical cases

Theorem 3.2 shows that in the critical case $\theta = 1$ we have that the process stays infinite, which is the same as the supercritical case. Are there any properties of Π where the critical and supercritical cases have different behaviour? Possibilities to consider include scaling limits for $N^{(n)}(t)$ (examples of work in this field come from Möhle [58] among others), or the size of the block containing the integer 1. These are interesting questions for both the critical and supercritical case regardless of whether there is any difference between the two.

Another question to ask involves Theorem 9 in [11]. Berestycki gives sufficient conditions such that the stationary distribution of Π, ρ, gives mass only to partitions with finitely many blocks. The conditions are that

(i) $c_e = 0$ and $\nu_{\text{Disl}}(\mathcal{S}^\downarrow) < \infty$,

(ii) $\nu_{\text{Disl}}(\{x \in \mathcal{S}^\downarrow : x_k > 0 \,\forall k \in \mathbb{N}\}) = 0$,

(iii) $\sum_k \log(k)\nu_{\text{Disl}}\left(\left\{x \in \mathcal{S}^\downarrow : \sum_{i=1}^{k+1} x_i = 1\right\}\right) < \infty$,

(iv) $\nu_{\text{Coag}}(\mathcal{S}^\downarrow) = 0$ and $c_\kappa > 0$.

The fast fragmentation-coalescence process meets all of these conditions as it's coalescent part is Kingman only, and $\nu_{\text{Disl}} = \lambda\delta_{\underline{0}}$. However, in the proof of this theorem it was stated that these conditions were identical to a set identified in Proposition 15 of [11], which replaces condition (ii) with

(v) $\nu_{\text{Disl}}(\mathcal{S}^\downarrow \backslash \Delta_f) = 0$,

where $\Delta_f = \{x \in \mathcal{S}^\downarrow : \exists n \in \mathbb{N} \text{ s.t. } \sum_{i=1}^n x_i = 1\}$. That is ν_{Disl} only gives mass to sequences which have finitely many nonzero elements and these elements add to one. However, these two conditions are different as $\mathcal{S}^\downarrow \backslash \Delta_f$ contains any sequences that have a dust part, i.e. that sum to less than one, for example $\underline{0}$, whereas $\{x \in \mathcal{S}^\downarrow : x_k > 0 \forall k \in \mathbb{N}\}$ does not. So this change in conditions ignores all sequences that have a dust part. Therefore, this question about whether the stationary distribution only gives mass to partitions with finitely many blocks is still open for the fast fragmentation-coalescence process in the critical and supercritical cases. The subcritical case is dealt with by Theorem 3.3. Theorem 10 in [11] states that ρ does not charge partitions with dust, regardless of the value of θ, so it remains to show that it does not charge partitions with an infinite number of blocks, all with positive asymptotic frequency.

4.2.3 Generalisation of the fragmentation

One immediate question is whether it was necessary in Theorem 3.2, Theorem 3.3 and Theorem 3.4 that the blocks fragmented entirely into singletons. One would expect that

so long as every fragmentation event results in infinitely many blocks, the structure of the resulting blocks has no effect on N. That is the motivation behind the following conjecture, which is the equivalent of Theorem 3.2. First recall the definition of an important set from [11].

$$\Delta_f := \left\{ x \in \mathcal{S}^{\downarrow} : \exists i \in \mathbb{N} \text{ s.t. } \sum_{j=1}^{i} x_j = 1 \right\}$$

which are all the elements in $\mathcal{S}^{\downarrow}$ which when used to create the paintbox measure result in partitions with finitely many blocks almost surely. Hence, when used as the fragmentation mechanism, result in finitely many more blocks forming.

Let $\tilde{\Pi}$ be the EFC process defined by taking $\nu_{\text{Coag}} = c_e = 0$, $\nu_{\text{Disl}}(\Delta_f) = 0$, $\nu_{\text{Disl}}(\mathcal{S}^{\downarrow} \backslash \Delta_f) = \lambda$ and $c_\kappa = c$. This is Kingman's coalescent at rate c, with a fragmentation mechanism of finite rate which forms infinitely many blocks. Furthermore, let $\theta := 2\lambda/c$.

Conjecture 4.4.   *(i) If $\theta \in (0,1)$, then $M := 1/N$ is a recurrent Feller process on $\{1/n : n \in \mathbb{N}\} \cup \{0\}$ such that 0 is instantaneously regular and not sticky.*

(ii) If $\theta \geq 1$, then 0 is an absorbing state for M.

4.2.4   Generalisation of the coalescence

Another natural extension to the work already done in this chapter is to generalise the process further, and see if there are similar results. The first obvious choice is to replace the Kingman coalescent part with the more general Λ-coalescent introduced in Chapter 1. As already discussed, Schweinsberg [68] gave necessary and sufficient criteria on Λ such that a Λ-coalescent comes down from infinity. So, the question is, once we add the same fragmentation mechanism already introduced, which of these processes still come down from infinity? In addition, would a version of the stronger result from Theorem 3.2 be valid; allowing for the existence of a local time at zero for M?

As stated earlier, Kingman's coalescent is the 'quickest' simple coalescent to come down from infinity. If there is a phase transition in c, λ in whether the process comes down from infinity or not, then it may suggest that any Λ-coalescent with no Kingman part will not be able to do so. Let Π be the EFC process with coalescent part defined by Λ and fragmentation mechanism as in the previous sections with $\lambda \in (0, \infty)$. The notion above is formalised in the following conjecture.

Conjecture 4.5. *Let Λ be a measure on $[0,1]$ such that $\Lambda([0,1]) = 1$, $\Lambda(\{1\}) = 0$, and $\Lambda = c\delta_0 + (1-c)\Lambda_1$ where $c \in [0,1]$, $\Lambda_1([0,1]) = 1$ and $\Lambda_1(\{0\}) = 0$. Define $\theta := 2\lambda/c$ as before, if $c > 0$.*

(i) If $c > 0$, then the result is the same as in Theorem 3.2. That is

(a) If $\theta \in (0,1)$, then $M := (M(t) : t \geq 0)$ is a recurrent Feller process on $\{1/n : n \in \mathbb{N}\} \cup \{0\}$ such that 0 is instantaneously regular (that is to say 0 is a not a holding point) and not sticky (that is to say $\int_0^\infty \mathbf{1}_{\{M(s)=0\}} \mathrm{d}s = 0$ almost surely).

(b) If $\theta \geq 1$, then 0 is an absorbing state for M.

(ii) If $c = 0$, then 0 is an absorbing state for M for all $\lambda \in (0, \infty)$.

The main complications that arise come from the fact that M (and therefore N) is no longer a skip-free process if $c < 1$. Hence, most of the calculations in Lemma 3.5 are no longer valid, and more importantly, are replaced with calculations that are far more complex.

A slightly different question for Λ-coalescents that come down from infinity is is there some measure of how 'strong' fragmentation has to be to prevent the EFC process where they are combined from coming down from infinity? Answering such a question might give additional intuition as to why the phase transition in Theorem 3.2 exists.

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