Optimal Stopping Problems for the Maximum Process

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SUMMARY

A cornerstone in the theory of optimal stopping for the maximum process is a result known as Peskir's maximality principle. It has proved to be a powerful tool to solve optimal stopping problems involving the maximum process under the assumption that the driving process X is a time-homogeneous diffusion. In this thesis we adapt Peskir's maximality principle to allow for X a spectrally negative Lévy processes, thereby providing a general method to approach optimal stopping problems for the maximum process driven by spectrally negative Lévy processes. We showcase this by explicitly solving three optimal stopping problems and the capped versions thereof. Here capped version means a modification of the original optimal stopping problem in the sense that the payoff is bounded from above by some constant. Moreover, we discuss applications of the aforementioned optimal stopping problems in option pricing in financial markets whose price process is driven by an exponential spectrally negative Lévy process. Finally, to further highlight the applicability of our general method, we present the solution to the problem of predicting the time at which a positive self-similar Markov process with one-sided jumps attains its maximum or minimum.

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CONTENTS

CHAPTER 1

INTRODUCTION

This thesis is concerned with optimal stopping problems for the maximum process driven by spectrally negative Lévy processes or positive self-similar Markov processes with one-sided jumps. Generally, the area of optimal stopping considers problems of the following form. One observes a random evolution whose future cannot be predicted and the aim is to stop it in a certain "optimal" way. For instance, imagine an investor holding a financial product which, when sold, pays a certain monetary amount depending on the performance of a stock. It is then their goal to choose the optimal moment to sell it in order to maximise the expected profit. This type of problem and similar ones are usually found in the area of mathematical finance [13, 33, 41]. Another example arises in mathematical statistics and goes by the name of quickest detection problem [33]. A system with potentially hazardous outcome (for instance seismic waves which indicate when an earthquake is about to occur) is observed and a decision when to send out an alarm has to be made. In this case the task is to minimise the expectation of a function of the decision error and/or observation time. Other areas of application include financial engineering or stochastic analysis [33].

The mathematical theory of optimal stopping is vast and has been developed by many people. Some of the major contributions/ideas to form the theory can be found in [12, 28, 31, 32, 33, 42, 43, 44] to name but a few. For a recent and excellent account of the general theory of optimal stopping and for applications in the aforementioned areas we recommend [33]. The latter (see at the end of Section 2) also contains a more detailed historic account of the different contributions over the past seventy years.

1.1 Outline of thesis and main results

This thesis consists of four self-contained chapters (excluding the introduction) and as a result there is some overlap between the different chapters. The second chapter has been accepted for publication in the *Annals of Applied Probability* as [29] and the third one, which is joint work with A. E. Kyprianou, has been accepted for publication in *Acta Applicandae Mathematicae* as [23]. The fourth chapter has been submitted and the fifth, which is joint work with A. E. Kyprianou and E. Baurdoux, presents some recent work.

As indicated above, this thesis consists of four chapters each of which deals with a different optimal stopping problem. Although they seem to be different, they are all connected in the sense that the same approach (except for Chapter 5) was used to solve them. It is a "guess and verify" approach, that is, one guesses a candidate solution and then verifies that it is indeed a solution. Typically, the verification part is quite long and involves a careful analysis of an ordinary differential equation as well as some tools from stochastic calculus. The tools from stochastic analysis are readily available, whereas the ordinary differential equations we will encounter have to be treated separately from case to case. A common feature, however, is that they all involve so-called scale functions, a special family of functions associated with spectrally negative Lévy processes [6, 20, 21]. This might seem unpleasant at a first glance as most scale functions are not known explicitly. Nevertheless, in recent years, enough of their analytical properties have been established (see [20] for an excellent summary) so that we can actually analyse ordinary differential equations involving them with the help of some phase plane analysis. Moreover, on the positive side, the use of scale functions allows us to formulate most of our results in a very neat and compact way.

The main contribution of this thesis lies in the "guess" part which is based on a good understanding of the problem at an intuitive level as well as results from the general theory of optimal stopping [33]. More precisely, we will provide a general method which allows us to derive candidate solutions for a certain class of optimal stopping problems. It is essentially an adaptation of Peskir's famous maximality principle [31] to our setting; see Subsection 1.2.1.

Let us spend some time describing the content of each of the chapters in more detail. To this end, let $X = \{X_t : t \geq 0\}$ be a spectrally negative Lévy process adapted to a filtration $\mathbb{F}$; that is to say, a one-dimensional process which has stationary and independent increments, and càdlàg paths with only negative discontinuities, but which does not have monotone paths. Associate with X the maximum process $\overline{X} = \{\overline{X}_t : t \geq 0\}$, where $\overline{X}_t := \sup_{0\leq u\leq t} X_u$, $t \geq 0$. Denote by $\mathbb{E}_{x,s}$ the expectation given that the two-dimensional strong Markov process $(X, \overline{X})$ starts at $x \leq s$. Furthermore, introduce the constant $\epsilon \in \mathbb{R} \cup \{\infty\}$ which will be referred to as "cap". The special role it plays throughout this thesis should become clear in due course. Finally, let $q > 0$ be a discount factor and $\mathcal{M}$ the set of (possibly infinite) $\mathbb{F}$-stopping times. We are now in a position to describe the content of each chapter.

Chapter 2

In this chapter we solve the optimal stopping problem

$$V(x,s) = \sup_{\tau\in\mathcal{M}} \mathbb{E}_{x,s}[e^{-q\tau+\overline{X}_\tau\wedge\epsilon}]. \tag{1.1}$$

This problem was introduced and studied in [38, 39] under the assumption that $\epsilon=\infty$ and X is a linear Brownian motion. Their results were then extended to allow for X a spectrally negative Lévy process in [2]. Due to its connection to pricing Russian options, problem (1.1) is sometimes referred to as "Russian" optimal stopping problem [33, 41]. Here, we generalise the aforementioned results by additionally introducing a cap $\epsilon\in\mathbb{R}\cup\{\infty\}$ which bounds the payoff from above, and hence the name "cap". At least when $\epsilon=\infty$, one possible technique to solve (1.1) is a reduction to a one-dimensional problem for the process $\overline{X}-X$ via an exponential change of measure; see [2, 39]. This is not possible when $\epsilon\in\mathbb{R}$ and therefore (1.1) has to be treated as a genuine two-dimensional optimal stopping problem for the pair $(X,\overline{X})$.

Moreover, we are interested in a "barrier version" of (1.1), that is, (1.1) but with $\mathcal{M}$ replaced by the set of all stopping times $\tau\in\mathcal{M}$ such that $\tau\leq\tau_{\tilde{\epsilon}}^-$, where $\tau_{\tilde{\epsilon}}^- := \inf\{t\geq 0 : X_t\leq\tilde{\epsilon}\}$ for some $\tilde{\epsilon}$. This means that the decision to stop has to be made before X drops below level $\tilde{\epsilon}$ – in some sense this captures the idea of a "lower" cap. This problem was proposed and solved in [40] again assuming that $\epsilon=\infty$ and X is a linear Brownian motion. We extend this to allow for an "upper" cap $\epsilon\in\mathbb{R}$ and X a spectrally negative Lévy process. Our main contribution here is an excursion theoretic calculation to obtain the solution in closed form.

Chapter 3

The focus of this chapter is on the optimal stopping problem

$$V(x,s) = \sup_{\tau\in\mathcal{M}} \mathbb{E}_{x,s}[e^{-q\tau}(e^{\overline{X}_\tau\wedge\epsilon}-K)^+], \tag{1.2}$$

where $K>0$ and $\epsilon>\log(K)$. For $\epsilon=\infty$ and X a linear Brownian motion, the problem was solved in [17, 30] and for $\epsilon=\infty$ and X a jump-diffusion it was solved in [14]. Our contribution here is an extension of these results to allow for X a spectrally negative Lévy process X and a cap ϵ. Furthermore, (1.2) constitutes a specific example of an optimal stopping problem for the maximum process where the analogue of Peskir's maximality principle is verified in a spectrally negative Lévy setting.

Chapter 4

In this chapter we study the optimal stopping problem

$$V(x,s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s}[e^{-q\tau}(e^{\overline{X}_\tau \wedge \epsilon} - Ke^{X_\tau})^+], \tag{1.3}$$

where $K > 0$. Again, provided $\epsilon = \infty$, it has been considered in [5] when X is a linear Brownian motion and in [14] when X is a jump-diffusion. We extend these results to allow for X a spectrally negative Lévy process as well as a cap ϵ. Note that in contrast to (1.1) and (1.2), the payoff in (1.3) does not only depend on the maximum process $\overline{X}$, but also on X itself. Similarly to (1.1), one may reduce (1.3) to a one-dimensional problem for the process $\overline{X} - X$ via an exponential change of measure provided that $\epsilon = \infty$. If $\epsilon \in \mathbb{R}$ this is not possible and one has to treat it as a two-dimensional problem for the pair $(X, \overline{X})$.

At this point it is worth mentioning that all the stopping problems (1.1)–(1.3) have applications in mathematical finance in the area of pricing American type options in financial markets where the underlying price process is an exponential spectrally negative Lévy process. In particular, the cap ϵ can be interpreted as a means of moderating the payoff of such an option. We will not go into details here, this connection is discussed at the beginning of Chapters 2–4.

Finally, let us summarise the content of the last chapter which treats an optimal stopping problem for the class of positive self-similar Markov processes with one-sided jumps.

Chapter 5

Imagine a transient diffusion process X in $(0,\infty)$ such that $X_t \to \infty$ as $t \to \infty$ and denote by $\hat{\theta}$ the time at which X attains its pathwise global infimum. Can we stop "as close as possible" to $\hat{\theta}$, that is, can we find a stopping time τ that minimises $\mathbb{E}[|\hat{\theta} - \tau| - \hat{\theta}]$ amongst all X-stopping times? This problem, which belongs to the class of prediction problems within optimal stopping, was recently solved in [15], and in the special case when X is a d-dimensional Bessel process for $d > 2$, the optimal stopping time is given by

$$\tau^* = \inf\{t \geq 0 : X_t \geq \lambda^* \underline{X}_t\}, \tag{1.4}$$

where $\lambda^* > 0$ is a solution of some polynomial and $\underline{X}_t := \inf_{0 \leq u \leq t} X_u$, $t \geq 0$. The family of d-dimensional Bessel processes for $d > 2$ also belongs to the class of positive self-similar Markov processes, and hence (1.4) can (up to a time-change) be expressed as the first upcrossing time above a certain level of the Lamperti representation ξ (a Lévy process) of X reflected at its infimum. This suggests that the prediction problem can be solved for the class of positive self-similar Markov processes drifting to infinity and that its solution can be reduced to a one-dimensional optimal stopping problem for a reflected Lévy process via the Lamperti transformation. The aim of this chapter is to show that this is indeed possible. Moreover, we will formulate and solve the analogue of the prediction problem above for positive self-similar Markov processes which continuously approach zero or jump onto zero. All of this is done under the assumption that the positive self-similar Markov process only has one-sided jumps, but we discuss at the end how one might get rid of this assumption.

The remainder of this introductory chapter is devoted to explaining the common technique used to solve (1.1)–(1.3). When doing so, we try to be as general as we can, since we believe that the computations presented below might be useful in the future to solve optimal stopping problems that are similar to the ones considered in this thesis. In addition, we will clarify the special role of the cap ϵ. At this point we should also say that at the time of writing the paper that constitutes Chapter 2, most of the connections explained in Subsections 1.2.1 and 1.2.2 were not known to us. One may therefore see Subsections 1.2.1 and 1.2.2 as a complement to Chapters 2–4 as well as a summary of the method applied in Chapters 2–4. We recommend to read the remainder of this chapter after Chapters 2–4.

1.2 The guessing method

The goal of this section is to present an adaptation of Peskir's maximality principle [31] to our setting. Consider the optimal stopping problem

$$V(x,s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s}[e^{-q\tau} f(\overline{X}_\tau) - \int_0^\tau e^{-qt} c(X_t, \overline{X}_t)\, dt]. \tag{1.5}$$

Here $f : \mathbb{R} \to (0,\infty)$ is a continuously differentiable and strictly increasing function such that $\lim_{s\to\infty} f(s) = \infty$ and $\lim_{s\to-\infty} f(s) = 0$ (the "payoff" function) and $c : \mathbb{R}^2 \to [0,\infty)$ is a continuous function (the "cost" function). Moreover, we will temporarily abuse the notation and use $\mathcal{M}$ for the set of all $\mathbb{F}$-stopping times such that $\mathbb{E}_{x,s}[\int_0^\tau e^{-qt} c(X_t, \overline{X}_t)\, dt] < \infty$. Originally the maximality principle [31] was established for (1.5) under the assumption that X is a time-homogeneous diffusion, $f(s) = s$ and $c(x,s) = c(x)$. The key observation in this thesis is that the steps that led to the maximality principle can be carried over to our setting when X is a spectrally negative Lévy process. The reason why this is possible becomes apparent when looking at [31] more closely. The two crucial facts used in [31] were continuity of the paths of the maximum process $\overline{X}$ and the solvability of the two-sided exit problem in terms of scale functions (for diffusions); cf. Chapter VII in [37]. Now for a spectrally negative Lévy process it is still true that the process $\overline{X}$ is continuous and the two-sided exit problem is also solvable in terms of scale functions (for spectrally negative Lévy processes); cf. [6, 20, 21].

1.2.1 An adaptation of Peskir's maximality principle

The aim is to derive a candidate solution for (1.5) by adapting the method described in [31] to our setting. We will make use of the general theory of optimal stopping and the notion of scale functions for spectrally negative Lévy processes. For background reading on the former we refer to [33], for the latter we suggest [6, 20, 21].

We begin by heuristically motivating a class of stopping times in which we will look for the optimal stopping time. To this end, note that the process $(X, \overline{X})$ can only move upwards by climbing up the diagonal in the (x, s)-plane; see Figure 1.1. The dynamics of $(X, \overline{X})$ are such that $\overline{X}$ remains constant at times when X is undertaking an excursion below $\overline{X}$. During such periods the discounting in the payoff as well as the penalisation by the cost function (if $c(x, s) > 0$) is detrimental. One should therefore not allow X to drop too far below $\overline{X}$ in value as otherwise the time it will take X to recover to the value of its previous maximum will prove to be costly in terms of the gain. More specifically, given a current value s of $\overline{X}$, there should be a point $g(s) > 0$ such that if the process $(X, \overline{X})$ reaches or jumps over the point $(s - g(s), s)$ we should stop instantly; see Figure 1.1. In more mathematical terms, we expect an optimal stopping time of the form

$$\tau_g := \inf\{t \geq 0 : \overline{X}_t - X_t \geq g(\overline{X}_t)\} \tag{1.6}$$

for some function $g : \mathbb{R} \to (0, \infty)$. This qualitative guess can be turned into a quan-

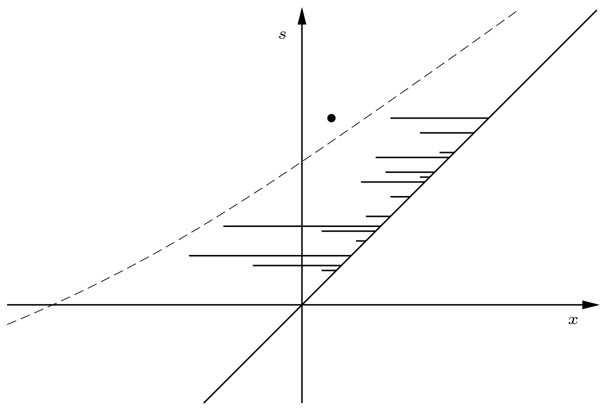

Fig. 1.1 *An illustration of a potential optimal stopping boundary $s \mapsto s - g(s)$ (dashed line). The horizontal lines (and the dot) are meant to schematically indicate the trace of an excursion of X away from the running maximum. The candidate optimal strategy τ_g then consists of continuing if the height of the excursion away from the running maximum s does not exceed $g(s)$; otherwise we stop.*

titative guess. To this end, assume that X is of unbounded variation. We will deal with the bounded variation case later; see page 8. From the general theory of optimal stopping (cf. Section 13 in [33]) we informally expect the value function

$$V_g(x,s) = \mathbb{E}_{x,s}\big[e^{-q\tau_g} f(\overline{X}_{\tau_g}) - \int_0^{\tau_g} e^{-qt} c(X_t, \overline{X}_t)\, dt\big] \tag{1.7}$$

to satisfy the system

$$\begin{aligned} \Gamma V_g(x,s) = qV_g(x,s) + c(x,s) \quad & \text{for } s - g(s) < x < s \text{ with } s \text{ fixed},\\ \tfrac{\partial V_g}{\partial s}(x,s)\big|_{x=s-} = 0 \quad & \text{(normal reflection)},\\ V_g(x,s)|_{x=(s-g(s))+} = f(s) \quad & \text{(instantaneous stopping)}, \end{aligned} \tag{1.8}$$

where Γ is the infinitesimal generator of the process X under $\mathbb{P}_0$. Moreover, the principle of smooth fit [28, 33] suggests that this system should be complemented by

$$\frac{\partial V_g}{\partial x}(x,s)\big|_{x=(s-g(s))+} = 0 \quad \text{(smooth fit)}. \tag{1.9}$$

Note that, although the smooth fit condition is not necessarily part of the general theory, it is imposed since by the "rule of thumb" outlined in Section 7 in [1] it should hold in this setting because of path regularity. Applying the strong Markov property at $\tau_s^+ := \inf\{t > 0 : X_t > s\}$ and using that $(X_{\tau_s^+}, \overline{X}_{\tau_s^+}) = (s,s)$ due to the spectral negativity of X yields

$$\begin{aligned} V_g(x,s) &= f(s)\mathbb{E}_{x,s}\big[e^{-q\tau^-_{s-g(s)}} 1_{\{\tau^-_{s-g(s)} < \tau_s^+\}}\big] - \mathbb{E}_{x,s}\big[\int_0^{\tau_g} e^{-qt} c(X_t, \overline{X}_t)\, dt 1_{\{\tau^-_{s-g(s)} < \tau_s^+\}}\big] \\ &\quad + \mathbb{E}_{x,s}[e^{-q\tau_s^+} 1_{\{\tau^-_{s-g(s)} > \tau_s^+\}}]\mathbb{E}_{s,s}[e^{-q\tau_g} f(\overline{X}_{\tau_g})] \\ &\quad - \mathbb{E}_{x,s}\big[\int_0^{\tau_g} e^{-qt} c(X_t, \overline{X}_t)\, dt 1_{\{\tau^-_{s-g(s)} > \tau_s^+\}}\big] \\ &= f(s)\mathbb{E}_{x,s}\big[e^{-q\tau^-_{s-g(s)}} 1_{\{\tau^-_{s-g_\epsilon(s)} < \tau_s^+\}}\big] + \mathbb{E}_{x,s}[e^{-q\tau_s^+} 1_{\{\tau^-_{s-g(s)} > \tau_s^+\}}]V_g(s,s) \\ &\quad - \mathbb{E}_x\big[\int_0^{\tau^-_{s-g(s)} \wedge \tau_s^+} e^{-qt} c(X_t, s)\, dt\big]. \end{aligned}$$

Denoting by $W^{(q)}$ and $Z^{(q)}$ the q-scale functions associated with X (cf. [6, 20, 21]), it is possible to rewrite the previous equation. Specifically, using (iii) of Theorem 8.1 and Theorem 8.7 in [21] gives

$$\begin{aligned} V_g(x,s) &= f(s)\Big(Z^{(q)}(x-s+g(s)) - W^{(q)}(x-s+g(s))\frac{Z^{(q)}(g(s))}{W^{(q)}(g(s))}\Big) \\ &\quad + \frac{W^{(q)}(x-s+g(s))}{W^{(q)}(g(s))} V_g(s,s) \end{aligned}$$

$$-\int_0^{g(s)} c(y+s-g(s),s)u^{(q)}(x-s+g(s),y)\,dy,$$

where $u^{(q)}(\cdot,\cdot)$ is the q-resolvent density of X upon leaving $[0,g(s)]$ so that

$$u^{(q)}(x-s+g(s),y)=\frac{W^{(q)}(x-s+g(s))W^{(q)}(g(s)-y)}{W^{(q)}(g(s))}-W^{(q)}(x-s+g(s)-y).$$

Now using the principle of smooth fit (1.9) and the fact that $W^{(q)}(0+)=0$ (cf. Lemma 3.1 in [20]) gives

$$0=\lim_{x\downarrow s-g(s)}\frac{\partial V_g}{\partial x}(x,s)=\lim_{x\downarrow s-g(s)}\frac{W^{(q)\prime}(x-s+g(s))}{W^{(q)}(g(s))}I(s), \tag{1.10}$$

where

$$I(s)=V_g(s,s)-f(s)Z^{(q)}(g(s))-\int_0^{g(s)} c(y+s-g(s),s)W^{(q)}(g(s)-y)\,dy.$$

It is known from Lemma 3.2 in [20] that the first factor on the right-hand side of (1.10) tends to a strictly positive value or infinity which shows that

$$V_g(s,s)=f(s)Z^{(q)}(g(s))+\int_0^{g(s)} c(y+s-g(s),s)W^{(q)}(g(s)-y)\,dy.$$

This would mean that $s-g(s)<x<s$ we have

$$V_g(x,s)=f(s)Z^{(q)}(x-s+g(s))+\int_{s-g(s)}^{x} c(y,s)W^{(q)}(x-y)\,dy. \tag{1.11}$$

Note that in order to obtain the previous equality we have used that $W^{(q)}(z)=0$ for $z<0$. Having derived the form of a candidate optimal value function for (1.5), we still need to do the same for g. Using the normal reflection condition in (1.8) shows that our candidate function g should satisfy the first order non-linear differential equation

$$g'(s)=1-\frac{f'(s)Z^{(q)}(g(s))+\int_0^{g(s)} c_2(s-y,s)W^{(q)}(y)\,dy}{\big(f(s)q+c(s-g(s),s)\big)W^{(q)}(g(s))}, \tag{1.12}$$

where the subscript two in $c_2(\cdot,\cdot)$ means the derivative with respect to the second argument.

If X is of bounded variation, we informally expect from the general theory that V_g satisfies the first two equations of (1.8). Additionally, the principle of continuous fit [1, 32] suggests that the system should be complemented by

$$V_g(x,s)|_{x=(s-g(s))+}=f(s)\quad\text{(continuous fit)}.$$

So far we have derived a candidate solution of (1.5) up to choosing a solution of (1.12). Note that equation (1.12) comes with no initial or boundary condition and hence might have many solutions. Thus, a-priori it is not clear which solution to choose. In order to resolve this issue, one may first perform a phase plane analysis of (1.12) to obtain an overview of the different solutions of (1.12) and then apply Peskir's famous maximality principle [31] which tells us which solution is the "right" one to choose. Note that Peskir's result was established in a different setting, however, an inspection of the arguments in [31] reveals that, at least formally, an analogue of the maximality principle should hold here too. We say that a solution $s \mapsto g^*(s)$ of (1.12) is the optimal stopping boundary for (1.5) if the stopping time τ_g associated with it [see (1.6)] is optimal for (1.5).

Minimality principle: The optimal stopping boundary $s \mapsto g^*(s)$ for (1.5) is the minimal solution of (1.12) satisfying $g^*(s) > 0$ for all $s \in \mathbb{R}$.

Remark 1.1. Note that the functions g here would correspond to $s - g(s)$ in [31]. This is the reason why we obtain a minimality principle rather than a maximality principle as in [31].

Remark 1.2. In Chapter 3 we explicitly verify that the minimality principle holds in a specific example, and, although we do not prove it, it is clear that it also holds for the optimal stopping problems considered in Chapters 2 and 4.

Remark 1.3. In Chapters 2 and 4 we do not carry out a phase plane analysis as the ordinary differential equation turns out to be autonomous and we are able to construct the desired solutions explicitly. However, in Chapter 3 the ordinary differential equation is not autonomous anymore and the phase plane analysis is an essential tool on the way to the solution of the optimal stopping problem.

Remark 1.4. From an analytical point of view the procedure above is nothing else than a probabilistic method to derive a candidate solution of the free-boundary problem (1.8). It seems reasonable to ask why one does not try to solve (1.8) directly. In some cases this is possible and, for instance, done in [17, 30, 38]. The reason why this works in the latter is the fact that they consider problems of the form (1.5) under the assumption that X is a linear Brownian motion and $c \equiv 0$. In this case Γ is a well known second-order differential operator and it is possible to make an ansatz for the general solution of the first equation in (1.8). However, when X is a spectrally negative Lévy process Γ becomes a nonlocal integro-differential operator and it is difficult to make an ansatz, especially if additionally $c(x,s) > 0$. In some sense the probabilistic approach above avoids pre-knowledge of the general solution of the first equation of (1.8).

Summing up, we have derived a candidate value function of the form (1.11) and a candidate optimal stopping time of the form (1.6), where g should be the minimal solution of (1.12) that never hits zero. Now if one wants to solve (1.1) explicitly (at least in some specific cases) it is important to construct the minimal solution of (1.12). This is possible via a limiting procedure and leads us to the special role of the cap ϵ which is discussed in the next subsection.

1.2.2 The special role of the cap ϵ

In this subsection we investigate the role of the cap ϵ and the capped version of (1.5), that is,

$$V_\epsilon(x,s) = \sup_{\tau\in\mathcal{M}} \mathbb{E}_{x,s}[e^{-q\tau} f(\overline{X}_\tau \wedge \epsilon) - \int_0^\tau e^{-qt} c(X_t, \overline{X}_t)\, dt], \tag{1.13}$$

where $\epsilon \in \mathbb{R}\cup\{\infty\}$. This is nothing else than (1.5) with $f(s)$ replaced by $f(s\wedge\epsilon)$ and for $\epsilon=\infty$ they coincide. At a first glance, guessing the solution for (1.13) seems more difficult due to the additional cap ϵ, but a moment of thought reveals that this is not true. The only difference to (1.5) is that once the process $\overline{X}$ has reached level ϵ, it is necessary to stop immediately because of the exponential discounting ($q>0$) and the penalisation due the cost function (if $c(x,s)>0$). However, the rest of the argument in Subsection 1.6 still goes through and hence one expects a candidate optimal value function $V_{g_\epsilon}(x,s)$ of the form (1.11) and the candidate optimal stopping time τ_{g_ϵ} of the form (1.6), where $g_\epsilon : (-\infty,\epsilon) \to (0,\infty)$ is solution to (1.12) with $\lim_{s\uparrow\epsilon} g_\epsilon(s) = 0$ and $g_\epsilon(s) = 0$ for $s>\epsilon$. This last requirement, which reflects the fact that once the process $\overline{X}$ has reached level ϵ it is necessary to stop, is a boundary condition which tells us which solution of (1.5) to choose. In this case the minimality principle is not necessary.

It seems that (1.5) can be obtained by letting $\epsilon\uparrow\infty$ in (1.13). This raises the following question: Can we obtain the solution of (1.5) from the solutions of (1.13) by some kind of limiting procedure? The answer is affirmative and in order to explain this, assume temporarily that we have solved (1.13) for every $\epsilon\in\mathbb{R}$. These solutions are denoted (as in the previous paragraph) by V_{g_ϵ} and τ_{g_ϵ}. Now, informally, one would expect that the solution of (1.5) is given by $V_{g_\infty}(x,s) := \lim_{\epsilon\uparrow\infty} V_{g_\epsilon}(x,s)$ for $x\le s$ and τ_{g_∞} of the form (1.6) with $g_\infty(s) := \lim_{s\uparrow\epsilon} g_\epsilon(s)$ for $s\in\mathbb{R}$. But if g_∞ is indeed the solution of (1.5), does it coincide with the minimal solution mentioned in the minimality principle above? We show in Chapter 3 that the limiting procedure can be made rigorous and that the resulting solution g_∞ is indeed the minimal one as in the minimality principle. Moreover, although we do not prove it, these observations should generally be true for (1.5) and (1.13). Hence, the capped problems (1.13) can be interpreted as "building blocks" for the uncapped problem (1.5).

This idea of obtaining the minimal solution of (1.12) by approximating it with solutions g_ϵ that hit zero was already implicitly contained in part (II) of the proof of Theorem 3.1 in [31], but the difference is that in [31] the sequence of solutions of (1.12) was chosen differently (they used an initial condition instead of a boundary condition).

In Peskir's language, our solutions g_ϵ correspond in [31] to the so-called "bad-good" solutions, "bad" because they are not the optimal boundary for the uncapped problem (1.5), "good" as they can be used to approximate the optimal boundary of the latter. The advantage of choosing the solution of (1.12) according to a boundary condition is that it gives a probabilistic interpretation of the "bad-good" solutions, namely that they correspond to an optimal stopping boundary, not for the uncapped problem (1.5), but the capped version of it. Finally, it is worth noting that this was already observed in [10] in a slightly different context; see the remark just after Proposition 3.1 in [10].

1.2.3 Limitations of the method

A natural question is to ask how important it was to work with a spectrally negative Lévy process. Replacing the spectrally negative Lévy processes with a general Lévy process should, in principle, not change the solutions qualitatively. For instance, it still seems reasonable that the optimal stopping time is of the form (1.6). However, when it comes down to computing things more explicitly it seems unclear how to proceed. Naively, one could assume that the optimal stopping time is of the form (1.6) and define V_g as in (1.7) and try to replicate the argument in Subsection 1.2.1. Unfortunately, this does not go very far and stops with the expression

$$\begin{aligned} V_g(x,s) &= f(s)\mathbb{E}_{x,s}\big[e^{-q\tau^-_{s-g(s)}}1_{\{\tau^-_{s-g_\epsilon(s)}<\tau^+_s\}}\big] + \mathbb{E}_{x,s}[e^{-q\tau^+_s}1_{\{\tau^-_{s-g(s)}>\tau^+_s\}}V_g(X_{\tau^+_s},\overline{X}_{\tau^+_s})] \\ &\quad -\mathbb{E}_x[\int_0^{\tau^-_{s-g(s)}\wedge\tau^+_s} e^{-qt}c(X_t,s)\,dt]. \end{aligned}$$

Unless one can now express the quantities on the right-hand side more explicitly for a general Lévy process it seems not possible to continue with the method in Subsection 1.2.1. Of course, this should by no means imply that it cannot be done using a different approach.

CHAPTER 2

RUSSIAN OPTION

This paper concerns optimal stopping problems driven by the running maximum of a spectrally negative Lévy process X. More precisely, we are interested in modifications of the Shepp–Shiryaev optimal stopping problem [2, 38, 39]. First, we consider a capped version of the Shepp–Shiryaev optimal stopping problem and provide the solution explicitly in terms of scale functions. In particular, the optimal stopping boundary is characterised by an ordinary differential equation involving scale functions and changes according to the path variation of X. Secondly, in the spirit of [40], we consider a modification of the capped version of the Shepp–Shiryaev optimal stopping problem in the sense that the decision to stop has to be made before the process X falls below a given level.

2.1 Introduction

Let $X = \{X_t : t \geq 0\}$ be a spectrally negative Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$ satisfying the natural conditions; cf. [7], Section 1.3, page 39. For $x \in \mathbb{R}$, denote by $\mathbb{P}_x$ the probability measure under which X starts at x and for simplicity write $\mathbb{P}_0 = \mathbb{P}$. We associate with X the maximum process $\overline{X} = \{\overline{X}_t : t \geq 0\}$ given by $\overline{X}_t := s \vee \sup_{0 \leq u \leq t} X_u$ for $t \geq 0, s \geq x$. The law under which $(X, \overline{X})$ starts at (x, s) is denoted by $\mathbb{P}_{x,s}$.

In this paper we are mainly interested in the following optimal stopping problem:

$$V_\epsilon^*(x, s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s}\big[e^{-q\tau + \overline{X}_\tau \wedge \epsilon}\big], \tag{2.1}$$

where $\epsilon \in \mathbb{R}, q > 0, (x, s) \in E := \{(x_1, s_1) \in \mathbb{R}^2 \,|\, x_1 \leq s_1\}$, and $\mathcal{M}$ is the set of all finite $\mathbb{F}$-stopping times. Since the constant ϵ bounds the process $\overline{X}$ from above, we refer to it as the upper cap. Due to the fact that the pair $(X, \overline{X})$ is a strong Markov process, (2.1) has also a Markovian structure and hence the general theory of optimal stopping [33] suggests that the optimal stopping time is the first entry time of the process $(X, \overline{X})$ into some subset of E. Indeed, it turns out that under some assumptions on q and $\psi(1)$, where ψ is the Laplace exponent of X (see ($*$), page 16, for a formal definition), the solution of (2.1) is given by

$$\tau_\epsilon^* = \inf\{t \geq 0 : \overline{X}_t - X_t \geq g_\epsilon(\overline{X}_t)\}$$

for some function g_ϵ which is characterised as a solution to a certain ordinary differential equation involving scale functions. The function $s \mapsto s - g_\epsilon(s)$ is sometimes referred to as the optimal stopping boundary. We will show that the shape of the optimal boundary has different characteristics according to the path variation of X. The solution of problem (2.1) is closely related to the solution of the Shepp–Shiryaev optimal stopping problem

$$V^*(x,s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s}\big[e^{-q\tau + \overline{X}_\tau}\big], \tag{2.2}$$

which was first studied by Shepp and Shiryaev [38, 39] for the case when X is a linear Brownian motion and later by Avram, Kyprianou and Pistorius [2] for the case when X is a spectrally negative Lévy process. Shepp and Shiryaev [38] introduced the problem as a means to pricing Russian options. In the latter context the solution of (2.2) can be viewed as the fair price of such an option. If we introduce a cap ϵ, an analogous interpretation of the solution of (2.1) applies, but for a Russian option whose payoff was moderated by capping it at a certain level (a fuller description is given in Section 2.2).

Our method for solving (2.1) consists of a verification technique, that is, we heuristically derive a candidate solution and then verify that it is indeed a solution. In particular, we will make use of the principle of smooth or continuous fit [1, 28, 32, 33] in a similar way to [31, 38].

It is also natural to ask for a modification of (2.1) with a lower cap. Whilst this is already included in the starting point of the maximum process $\overline{X}$, there is a stopping problem that captures this idea of lower cap in the sense that the decision to exercise has to be made before X drops below a certain level. Specifically, consider

$$V^*_{\epsilon_1,\epsilon_2}(x,s) = \sup_{\tau \in \mathcal{M}_{\epsilon_1}} \mathbb{E}_{x,s}\big[e^{-q\tau + \overline{X}_\tau \wedge \epsilon_2}\big], \tag{2.3}$$

where $\epsilon_1, \epsilon_2 \in \mathbb{R}$ such that $\epsilon_1 < \epsilon_2$, $q > 0, \mathcal{M}_{\epsilon_1} := \{\tau \in \mathcal{M} \,|\, \tau \leq T_{\epsilon_1}\}$ and T_{ϵ_1} is given by $T_{\epsilon_1} := \inf\{t \geq 0 : X_t \leq \epsilon_1\}$. In the special case of no cap ($\epsilon_2 = \infty$), this problem was considered by Shepp, Shiryaev and Sulem [40] for the case where X is a linear Brownian motion. Inspired by their result we expect the optimal stopping time to be of the form $T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}$, where $\tau^*_{\epsilon_2}$ is the optimal stopping time in (2.1). Our main contribution here is that, with the help of excursion theory (cf. [6, 21]), we find a closed form expression for the value function associated with the strategy $T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}$, thereby allowing us to verify that it is indeed an optimal strategy.

This paper is organised as follows. In Section 2.2 we provide some motivation for studying (2.1) and (2.3). Then we introduce some more notation and collect some auxiliary results in Section 2.3. Our main results are presented in Section 2.4, followed by their proofs in Sections 2.5 and 2.6. Finally, some numerical examples are given in Section 2.7.

2.2 Application to pricing capped Russian options

The aim of this section is to give some motivation for studying (2.1) and (2.3).

Consider a financial market consisting of a riskless bond and a risky asset. The value of the bond $B = \{B_t : t \geq 0\}$ evolves deterministically such that

$$B_t = B_0 e^{rt}, \quad B_0 > 0, r \geq 0, t \geq 0. \tag{2.4}$$

The price of the risky asset is modelled as the exponential spectrally negative Lévy process

$$S_t = S_0 e^{X_t}, \quad S_0 > 0, t \geq 0. \tag{2.5}$$

In order to guarantee that our model is free of arbitrage we will assume that $\psi(1) = r$. If $X_t = \mu t + \sigma W_t$, where $W = \{W_t : t \geq 0\}$ is a standard Brownian motion, we get the standard Black–Scholes model for the price of the asset. Extensive empirical research has shown that this (Gaussian) model is not capable of capturing certain features (such as skewness and heavy tails) which are commonly encountered in financial data, for example, returns on stocks. To accommodate for these problems, an idea, going back to [27], is to replace the Brownian motion as the model for the log-price by a general Lévy process X; cf. [9]. Here we will restrict ourselves to the model where X is given by a spectrally negative Lévy process. This restriction is mainly motivated by analytical tractability. It is worth mentioning, however, that Carr and Wu [8] as well as Madan and Schoutens [25] have offered empirical evidence to support the case of a model in which the risky asset is driven by a spectrally negative Lévy process for appropriate market scenarios.

A capped Russian option is an option which gives the holder the right to exercise at any almost surely finite stopping time τ yielding payouts

$$e^{-\alpha\tau}\left(M_0 \vee \sup_{0 \leq u \leq \tau} S_u \wedge C\right), \quad C > M_0 \geq S_0, \alpha > 0.$$

The constant M_0 can be viewed as representing the "starting" maximum of the stock price (say, over some previous period $(-t_0, 0]$). The constant C can be interpreted as cap and moderates the payoff of the option. The value $C = \infty$ is also allowed and corresponds to no moderation at all. In this case we just get the normal Russian option. Finally, when $C = \infty$ it is necessary to choose α strictly positive to guarantee that it is optimal to stop in finite time and that the value is finite; cf. Proposition 2.1.

Standard theory of pricing American-type options [41] directs one to solving the optimal stopping problem

$$V_r(M_0, S_0, C) = B_0 \sup_{\tau} \mathbb{E}\left[B_\tau^{-1} e^{-\alpha\tau}\left(M_0 \vee \sup_{0 \le u \le \tau} S_u \wedge C\right)\right] \tag{2.6}$$

where the supremum is taken over all $[0, \infty)$-valued $\mathbb{F}$-stopping times. In other words, we want to find a stopping time which optimises the expected discounted claim. The right-hand side of (2.6) may be rewritten as

$$V_r(M_0, S_0, C) = V_\epsilon^*(x, s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s}\big[e^{-q\tau + \overline{X}_\tau \wedge \epsilon}\big],$$

where $q = r + \alpha, x = \log(S_0), s = \log(M_0)$ and $\epsilon = \log(C)$.

In (2.6) one might only allow stopping times that are smaller or equal than the first time the risky asset S drops below a certain barrier. From a financial point of view this corresponds to a default time after which all economic activity stops; cf. [40]. Including this additional feature leads in an analogous way to the above optimal stopping problem (2.3).

2.3 Notation and auxiliary results

The purpose of this section is to introduce some notation and collect some known results about spectrally negative Lévy processes. Moreover, we state the solution of the Shepp–Shiryaev optimal stopping problem (2.2) which will play an important role throughout this paper.

2.3.1 Spectrally negative Lévy processes

It is well known that a spectrally negative Lévy process X is characterised by its Lévy triplet (γ, σ, Π), where $\sigma \ge 0, \gamma \in \mathbb{R}$ and Π is a measure on $(-\infty, 0)$ satisfying the condition $\int_{(-\infty,0)}(1 \wedge x^2)\,\Pi(dx) < \infty$. By the Lévy–Itô decomposition, X may be represented in the form

$$X_t = \sigma B_t - \gamma t + X_t^{(1)} + X_t^{(2)}, \tag{2.7}$$

where $\{B_t : t \ge 0\}$ is a standard Brownian motion, $\{X_t^{(1)} : t \ge 0\}$ is a compound Poisson process with discontinuities of magnitude bigger than or equal to one and $\{X_t^{(2)} : t \ge 0\}$

is a square integrable martingale with discontinuities of magnitude strictly smaller than one and the three processes are mutually independent. In particular, if X is of bounded variation, the decomposition reduces to

$$X_t = \mathtt{d}t - \eta_t \tag{2.8}$$

where $\mathtt{d} > 0$, and $\{\eta_t : t \geq 0\}$ is a driftless subordinator. Further, the spectral negativity of X ensures existence of the Laplace exponent ψ of X, that is, $\mathbb{E}[e^{\theta X_1}] = e^{\psi(\theta)}$ for $\theta \geq 0$, which is known to take the form

$$\psi(\theta) = -\gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,0)} \left(e^{\theta x} - 1 - \theta x 1_{\{x>-1\}}\right) \Pi(dx). \tag{$*$}$$

Its right-inverse is defined by

$$\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}$$

for $q \geq 0$.

For any spectrally negative Lévy process having $X_0 = 0$ we introduce the family of martingales

$$\exp(cX_t - \psi(c)t),$$

defined for any $c \in \mathbb{R}$ for which $\psi(c) = \log \mathbb{E}[\exp(cX_1)] < \infty$, and further the corresponding family of measures $\{\mathbb{P}^c\}$ with Radon–Nikodym derivatives

$$\left.\frac{d\mathbb{P}^c}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp(cX_t - \psi(c)t). \tag{2.9}$$

For all such c the measure $\mathbb{P}^c_x$ will denote the translation of $\mathbb{P}^c$ under which $X_0 = x$. In particular, under $\mathbb{P}^c_x$ the process X is still a spectrally negative Lévy process; cf. Theorem 3.9 in [21].

2.3.2 Scale functions

A special family of functions associated with spectrally negative Lévy processes is that of scale functions (cf. [21]) which are defined as follows. For $q \geq 0$, the q-scale function $W^{(q)} : \mathbb{R} \longrightarrow [0, \infty)$ is the unique function whose restriction to $(0, \infty)$ is continuous and has Laplace transform

$$\int_0^\infty e^{-\theta x} W^{(q)}(x)\, dx = \frac{1}{\psi(\theta) - q}, \qquad \theta > \Phi(q),$$

and is defined to be identically zero for $x \leq 0$. Equally important is the scale function $Z^{(q)} : \mathbb{R} \longrightarrow [1, \infty)$ defined by

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(z)\, dz.$$

The passage times of X below and above $k \in \mathbb{R}$ are denoted by

$$\tau_k^- = \inf\{t > 0 : X_t \leq k\} \quad \text{and} \quad \tau_k^+ = \inf\{t > 0 : X_t \geq k\}.$$

We will make use of the following four identities. For $q \geq 0$ and $x \in (a, b)$ it holds that

$$\mathbb{E}_x\big[e^{-q\tau_b^+} 1_{\{\tau_b^+ < \tau_a^-\}}\big] = \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)}, \tag{2.10}$$

$$\mathbb{E}_x\big[e^{-q\tau_a^-} 1_{\{\tau_b^+ > \tau_a^-\}}\big] = Z^{(q)}(x-a) - W^{(q)}(x-a)\frac{Z^{(q)}(b-a)}{W^{(q)}(b-a)}, \tag{2.11}$$

for $q > 0$ and $x \in \mathbb{R}$ it holds that

$$\mathbb{E}_x\big[e^{-q\tau_0^-} 1_{\{\tau_0^- < \infty\}}\big] = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x), \tag{2.12}$$

and finally for $q > 0$ we have

$$\lim_{x \to \infty} \frac{Z^{(q)}(x)}{W^{(q)}(x)} = \frac{q}{\Phi(q)}. \tag{2.13}$$

Identities (2.10)–(2.12) can be found in Theorem 8.1 of [21] and identity (2.13) is Lemma 3.3 of [20]. For each $c \geq 0$ we denote by $W_c^{(q)}$ the q-scale function with respect to the measure $\mathbb{P}^c$. A useful formula (cf. Lemma 8.4 of [21]) linking scale functions under different measures is given by

$$W^{(q)}(x) = e^{\Phi(q)x} W_{\Phi(q)}(x) \tag{2.14}$$

for $q \geq 0$ and $x \geq 0$.

We conclude this subsection by stating some known regularity properties of scale functions; cf. Lemma 2.4, Corollary 2.5, Theorem 3.10, Lemma 3.1 and Lemma 3.2 of [20].

Smoothness: For all $q \geq 0$,

$$W^{(q)}|_{(0,\infty)} \in \begin{cases} C^1(0,\infty), & \text{if } X \text{ is of bounded variation and } \Pi \text{ has no atoms,} \\ C^1(0,\infty), & \text{if } X \text{ is of unbounded variation and } \sigma = 0, \\ C^2(0,\infty), & \sigma > 0. \end{cases}$$

Continuity at the origin: For all $q \geq 0$,

$$W^{(q)}(0+) = \begin{cases} \mathtt{d}^{-1}, & \text{if } X \text{ is of bounded variation,} \\ 0, & \text{if } X \text{ is of unbounded variation.} \end{cases} \tag{2.15}$$

Right-derivative at the origin: For all $q \geq 0$,

$$W_+^{(q)\prime}(0+) = \begin{cases} \frac{q+\Pi(-\infty,0)}{\mathtt{d}^2}, & \text{if } \sigma = 0 \text{ and } \Pi(-\infty,0) < \infty, \\ \frac{2}{\sigma^2}, & \text{if } \sigma > 0 \text{ or } \Pi(-\infty,0) = \infty, \end{cases} \tag{2.16}$$

where we understand the second case to be $+\infty$ when $\sigma = 0$.

For technical reasons, we require for the rest of the paper that $W^{(q)}$ is in $C^1(0,\infty)$ [and hence $Z^{(q)} \in C^2(0,\infty)$]. This is ensured by henceforth assuming that Π is atomless whenever X has paths of bounded variation.

2.3.3 Solution to the Shepp–Shiryaev optimal stopping problem

In order to state the solution of the Shepp–Shiryaev optimal stopping problem, we introduce the function $f : [0,\infty) \to \mathbb{R}$ which is defined as

$$f(z) = Z^{(q)}(z) - qW^{(q)}(z).$$

It can be shown (cf. page 6 of [3]) that, when $q > \psi(1)$, the function f is strictly decreasing to $-\infty$ and hence within this regime

$$k^* := \inf\{z \geq 0 : Z^{(q)}(z) \leq qW^{(q)}(z)\} \in [0,\infty).$$

In particular, when $q > \psi(1)$, then $k^* = 0$ if and only if $W^{(q)}(0+) \geq q^{-1}$. Also, note that the requirement $W^{(q)}(0+) \geq q^{-1}$ implies $q \geq \mathtt{d} > \psi(1)$. We now give a reformulation of a part of Theorem 1 in [3].

Proposition 2.1.

(a) Suppose that $q > \psi(1)$ and $W^{(q)}(0+) < q^{-1}$. Then the solution of (2.2) is given by

$$V^*(x,s) = e^s Z^{(q)}(x - s + k^*)$$

with optimal strategy

$$\tau^* := \inf\{t \geq 0 \,:\, \overline{X}_t - X_t \geq k^*\}.$$

(b) If $W^{(q)}(0+) \geq q^{-1}$ (and hence $q > \psi(1)$), then the solution of (2.2) is given by $V^*(x,s) = e^s$ and optimal strategy $\tau^* = 0$.

(c) If $q \le \psi(1)$, then $V^*(x,s) = \infty$.

The result in part (b) of Proposition 2.1 is not surprising. If $W^{(q)}(0+) \ge q^{-1}$, then X is necessarily of bounded variation with $\mathtt{d} \le q$ which implies that the process $e^{-qt+\overline{X}_t}$, $t \ge 0$, is pathwise decreasing. As a result we have for $\tau \in \mathcal{M}$ the inequality $\mathbb{E}_{x,s}\big[e^{-q\tau+\overline{X}_\tau}\big] \le e^s$ and hence (b) follows. An analogous argument shows that $V^*_\epsilon(x,s) = e^{s\wedge\epsilon}$ for $(x,s) \in E$ with optimal strategy $\tau^*_\epsilon = 0$ and $V^*_{\epsilon_1,\epsilon_2}(x,s) = e^{s\wedge\epsilon_2}$ for $(x,s) \in E$ with optimal strategy $\tau^*_{\epsilon_1,\epsilon_2} = 0$. Therefore, we will not consider the regime $W^{(q)}(0+) \ge q^{-1}$ in what follows. Note, however, that the parameter regime $q \le \psi(1)$ will not be degenerate for (2.1) and (2.3) due to the upper cap which prevents the value function from exploding.

2.4 Main results

2.4.1 Maximum process with upper cap

The first result ensures existence of a function g_ϵ which, as will follow in due course, describes the optimal stopping boundary in (2.1).

Lemma 2.2. *Let $\epsilon \in \mathbb{R}$ be given.*

a) If $q > \psi(1)$ and $W^{(q)}(0+) < q^{-1}$, then $k^* \in (0,\infty)$.

b) If $q \le \psi(1)$, then $k^* = \infty$.

c) Under the assumptions in (a) *or* (b) *stated above, there exists a unique solution $g_\epsilon : (-\infty,\epsilon) \to (0,k^*)$ of the ordinary differential equation*

$$g'_\epsilon(s) = 1 - \frac{Z^{(q)}(g_\epsilon(s))}{qW^{(q)}(g_\epsilon(s))} \quad \text{on } (-\infty,\epsilon) \tag{2.17}$$

satisfying $\lim_{s\uparrow\epsilon} g_\epsilon(s) = 0$ and $\lim_{s\downarrow-\infty} g_\epsilon(s) = k^*$.

Next, extend g_ϵ to the whole real line by setting $g_\epsilon(s) = 0$ for $s \ge \epsilon$. We now present the solution of (2.1).

Theorem 2.3. *Let $\epsilon \in \mathbb{R}$ be given and suppose that $q > \psi(1)$ and $W^{(q)}(0+) < q^{-1}$ or $q \le \psi(1)$. Then the solution of (2.1) is given by*

$$V^*_\epsilon(x,s) = e^{s\wedge\epsilon} Z^{(q)}(x - s + g_\epsilon(s))$$

with corresponding optimal strategy

$$\tau^*_\epsilon := \inf\{t \ge 0 : \overline{X}_t - X_t \ge g_\epsilon(\overline{X}_t)\},$$

where g_ϵ is given in Lemma 2.2.

Define the continuation region

$$C^*_\epsilon = C^* := \{(x,s) \in E \,|\, s < \epsilon, s - g_\epsilon(s) < x \leq s\}$$

and the stopping region $D^*_\epsilon = D^* := E \setminus C^*$. The shape of the boundary separating them, that is, the optimal stopping boundary, is of particular interest. Theorem 2.3 together with (2.15) and (2.17) shows that

$$\lim_{s\uparrow\epsilon} g'_\epsilon(s) = \begin{cases} -\infty, & \text{if } X \text{ is of unbounded variation,} \\ 1 - \mathtt{d}/q, & \text{if } X \text{ is of bounded variation.} \end{cases}$$

Also, using (2.13) we see that

$$\lim_{s\to-\infty} g'_\epsilon(s) = \begin{cases} 0, & \text{if } q > \psi(1) \text{ and } W^{(q)}(0+) < q^{-1}, \\ 1 - \Phi(q)^{-1}, & \text{if } q \leq \psi(1). \end{cases}$$

This (qualitative) behaviour of g_ϵ and the resulting shape of the continuation and stopping region are illustrated in Figure 2.1. Note in particular that the shape of g_ϵ at ϵ (and consequently the optimal boundary) changes according to the path variation of X. The horizontal and vertical lines in Figure 2.1 are meant to schematically indicate

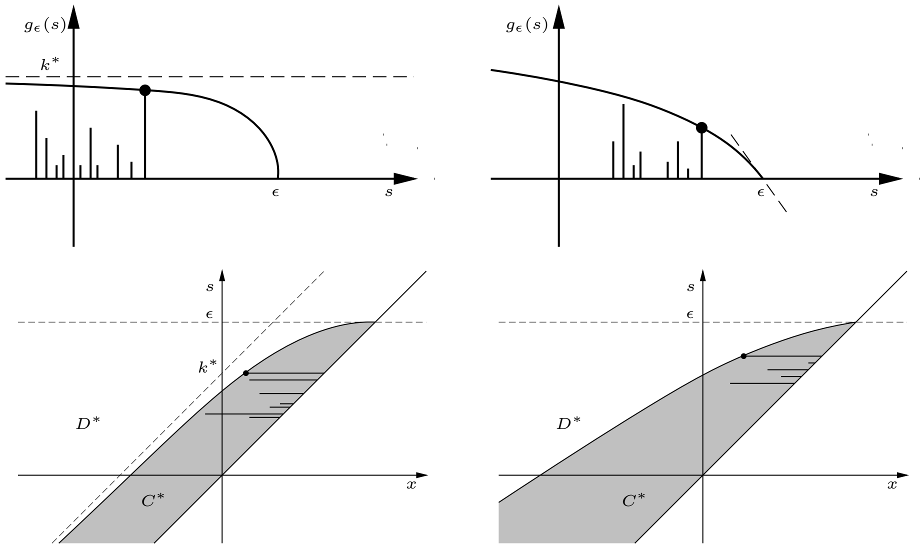

Fig. 2.1 *For the two pictures on the left it is assumed that* $q > \psi(1)$ *and* $W^{(q)}(0+) = 0$*, whereas on the right it is assumed that* $q \leq \psi(1)$*.*

the trace of the excursions of X away from the running maximum. We thus see that the optimal strategy consists of continuing if the height of the excursion away from the running supremum s does not exceed $g_\epsilon(s)$; otherwise we stop.

2.4.2 Maximum process with upper and lower cap

Inspired by the result in [40], we expect the strategy $T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}$ to be optimal, where $\tau^*_{\epsilon_2}$ is given in Theorem 2.3 and $T_{\epsilon_1} = \inf\{t \geq 0 : X_t \leq \epsilon_1\}$. This means that the optimal boundary is expected to be a vertical line at ϵ_1 combined with the curve described by g_{ϵ_2} characterised in Lemma 2.2. Before we can proceed, we need to introduce an auxiliary quantity, namely the point on the s-axis where the vertical line at ϵ_1 and the optimal boundary corresponding to g_{ϵ_2} intersect; see Figure 2.2. If $q > \psi(1)$ and $W^{(q)}(0+) < q^{-1}$ or $q \leq \psi(1)$ define the map $a_{\epsilon_2} : (-\infty, \epsilon_2) \to (0, k^*)$ by $a_{\epsilon_2}(s) := s - g_{\epsilon_2}(s)$. It follows by definition of g_{ϵ_2} that a_{ϵ_2} is continuous, strictly increasing and satisfies $\lim_{s\uparrow\epsilon_2} a_{\epsilon_2}(s) = \epsilon_2$ and $\lim_{s\downarrow-\infty} a_{\epsilon_2}(s) = -\infty$. Therefore the intermediate value theorem guarantees existence of a unique $A_{\epsilon_1,\epsilon_2} = A \in (-\infty, \epsilon_2)$ such that $A - g_{\epsilon_2}(A) = \epsilon_1$. Our candidate optimal strategy $T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}$ splits the set E into the continuation regions

$$
\begin{aligned}
C^*_{I,\epsilon_1,\epsilon_2} = C^*_I &:= \{(x,s) \in E : \epsilon_1 < x \leq s, \epsilon_1 < s < A\}, \\
C^*_{II,\epsilon_1,\epsilon_2} = C^*_{II} &:= \{(x,s) \in E : s - g_{\epsilon_2}(s) < x \leq s, A \leq s < \epsilon_2\}
\end{aligned}
$$

and the stopping region $E \backslash (C^*_I \cup C^*_{II})$. Additionally, define $E_{\epsilon_1} := \{(x,s) \in E : x > \epsilon_1\}$.

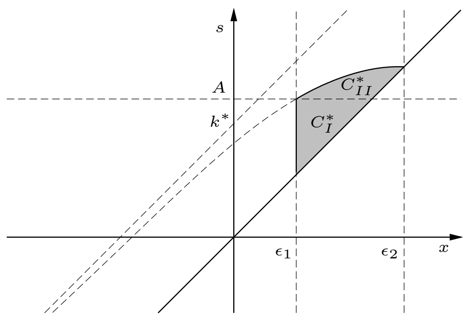

Fig. 2.2 *A qualitative picture of the continuation and stopping region under the assumption that $q > \psi(1)$ and $W^{(q)}(0+) = 0$; cf. Theorem 2.5.*

Clearly, if $(x,s) \in E \setminus E_{\epsilon_1}$, then the only stopping time in $\mathcal{M}_{\epsilon_1}$ is $\tau = 0$ and hence the optimal value function is given by $e^{s\wedge\epsilon_2}$. Furthermore, when $(x,s) \in E$ such that $s \geq A$ and $x > \epsilon_1$, we have $\tau^*_{\epsilon_2} \leq T_{\epsilon_1}$, so that the optimality of $\tau^*_{\epsilon_2}$ in (2.1) implies $V^*_{\epsilon_1,\epsilon_2}(x,s) = V^*_{\epsilon_2}(x,s)$. Consequently, the interesting case is really $(x,s) \in C^*_I$. The key to verifying that $T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}$ is optimal, is to find the value function associated with it.

Lemma 2.4. *Let $\epsilon_1 < \epsilon_2$ be given, and suppose that $q > \psi(1)$ and $W^{(q)}(0+) < q^{-1}$ or*

$q \leq \psi(1)$. *Define*

$$V_{\epsilon_1,\epsilon_2}(x,s) := \begin{cases} V^*_{\epsilon_2}(x,s), & (x,s) \in C^*_{II}, \\ U_{\epsilon_1,\epsilon_2}(x,s), & (x,s) \in C^*_{I}, \\ e^{s\wedge\epsilon_2}, & \textit{otherwise}, \end{cases}$$

where $V^*_{\epsilon_2}$ *is given in Theorem 2.3,*

$$U_{\epsilon_1,\epsilon_2}(x,s) := e^s Z^{(q)}(x-\epsilon_1) + e^{\epsilon_1} W^{(q)}(x-\epsilon_1) \int_{s-\epsilon_1}^{g_{\epsilon_2}(A)} e^t \frac{Z^{(q)}(t)}{W^{(q)}(t)}\, dt$$

and $A \in (-\infty, \epsilon_2)$ *is the unique constant such that* $A - g_{\epsilon_2}(A) = \epsilon_1$. *We then have, for* $(x,s) \in E$,

$$\mathbb{E}_{x,s}\left[e^{-q(T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}) + \overline{X}_{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}}\wedge\epsilon_2}\right] = V_{\epsilon_1,\epsilon_2}(x,s).$$

Our main contribution here is the expression for $U_{\epsilon_1,\epsilon_2}$, thereby allowing us to verify that the strategy $T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}$ is still optimal. In fact, this is the content of the next result.

Theorem 2.5. *Let* $\epsilon_1 < \epsilon_2$ *be given and suppose that* $q > \psi(1)$ *and* $W^{(q)}(0+) < q^{-1}$ *or* $q \leq \psi(1)$. *Then the solution to* (2.3) *is given by* $V^*_{\epsilon_1,\epsilon_2} = V_{\epsilon_1,\epsilon_2}$ *with corresponding optimal strategy* $\tau^*_{\epsilon_1,\epsilon_2} = T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}$, *where* $\tau^*_{\epsilon_2}$ *is given in Theorem 2.3.*

It is also possible to obtain the solution of (2.3) with lower cap only. To this end, define when $q > \psi(1)$ and $W^{(q)}(0+) < q^{-1}$ the constant function $g_\infty(s) := k^*$ and $A_{\epsilon_1,\infty} := \epsilon_1 + k^*$.

Corollary 2.6. *Let* $\epsilon_1 \in \mathbb{R}$ *and suppose that* $\epsilon_2 = \infty$, *that is, there is no upper cap.*

(a) Assume that $q > \psi(1)$ *and that* $W^{(q)}(0+) < q^{-1}$. *Then the solution to* (2.3) *is given by*

$$V^*_{\epsilon_1,\infty}(x,s) = \begin{cases} V^*(x,s), & (x,s) \in C^*_{II,\epsilon_1,\infty}, \\ U_{\epsilon_1,\infty}(x,s), & (x,s) \in C^*_{I,\epsilon_1,\infty}, \\ e^s, & \textit{otherwise}, \end{cases} \tag{2.18}$$

where V^* *is given in Proposition 2.1 and*

$$U_{\epsilon_1,\infty}(x,s) = e^s Z^{(q)}(x-\epsilon_1) + e^{\epsilon_1} W^{(q)}(x-\epsilon_1) \int_{s-\epsilon_1}^{k^*} e^t \frac{Z^{(q)}(t)}{W^{(q)}(t)}\, dt.$$

The corresponding optimal strategy is given by $\tau^*_{\epsilon_1,\infty} = T_{\epsilon_1} \wedge \tau^*$, *where* τ^* *is given in Proposition 2.1.*

(b) If $q \leq \psi(1)$, *then* $V^*_{\epsilon_1,\infty}(x,s) = \infty$ *for* $(x,s) \in E_{\epsilon_1}$ *and* $V^*_{\epsilon_1,\infty}(x,s) = e^s$ *otherwise.*

Remark 2.7. In Theorem 2.3 there is no lower cap, and hence it seems natural to obtain Theorem 2.3 as a corollary to Theorem 2.5. This would be possible if one merged the proofs of Theorem 2.3 and Theorem 2.5 appropriately. However, a merged proof would still contain the main arguments of both the proof of Theorem 2.3 and the proof of Theorem 2.5 (note that the proof of Theorem 2.5 makes use of Theorem 2.3). Therefore, and also for presentation purposes, we choose to present them separately.

Finally, if $X_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$, $t \geq 0$, where $\mu \in \mathbb{R}, \sigma > 0$ and W_t, $t \geq 0$, is a standard Brownian motion, then Corollary 2.6 is nothing else than Theorem 3.1 in [40]. However, this is not immediately clear and requires a simple but lengthy computation which is provided in Section 2.7.

2.5 Guess and verify via principle of smooth or continuous fit

Let us consider the solution to (2.1) from an intuitive point of view. We shall restrict ourselves to the case where $q > \psi(1)$ and $W^{(q)}(0+) < q^{-1}$. It follows from what was said at the beginning of Subsection 2.3.3 that $k^* \in (0, \infty)$.

It is clear that if $(x, s) \in E$ such that $x \geq \epsilon$, then it is optimal to stop immediately since one cannot obtain a higher payoff than ϵ, and waiting is penalised by exponential discounting. If x is much smaller than ϵ, then the cap ϵ should not have too much influence, and one expects that the optimal value function V_ϵ^* and the corresponding optimal strategy τ_ϵ^* look similar to the optimal value function V^* and optimal strategy τ^* of problem (2.2). On the other hand, if x is close to the cap, then the process X should be stopped "before" it is a distance k^* away from its running maximum. This can be explained as follows: The constant k^* in the solution to problem (2.2) quantifies the acceptable "waiting time" for a possibly much higher running supremum at a later point in time. But if we impose a cap, there is no hope for a much higher supremum and therefore "waiting the acceptable time" for problem (2.2) does not pay off in the situation with cap. With exponential discounting we would therefore expect to exercise earlier. In other words, we expect an optimal strategy of the form

$$\tau_{g_\epsilon} = \inf\{t \geq 0 \, : \, \overline{X}_t - X_t \geq g_\epsilon(\overline{X})\}$$

for some function g_ϵ satisfying $\lim_{s \to -\infty} g_\epsilon(s) = k^*$ and $\lim_{s \to \epsilon} g_\epsilon(s) = 0$.

This qualitative guess can be turned into a quantitative guess by an adaptation of the argument in Section 3 of [31] to our setting. To this end, assume that X is of unbounded variation ($W^{(q)}(0+) = 0$). We will deal with the bounded variation case later. From the general theory of optimal stopping (cf. [33], Section 13) we informally expect the value function

$$V_{g_\epsilon}(x, s) = \mathbb{E}_{x,s}\big[e^{-q\tau_{g_\epsilon} + \overline{X}_{\tau_{g_\epsilon}}}\big]$$

to satisfy the system

$$\begin{aligned} \Gamma V_{g_\epsilon}(x,s) = qV_{g_\epsilon}(x,s) \quad & \text{for } s-g_\epsilon(s)<x<s \text{ with } s \text{ fixed},\\ \tfrac{\partial V_{g_\epsilon}}{\partial s}(x,s)\big|_{x=s-} = 0 \quad & \text{(normal reflection)}, \\ V_{g_\epsilon}(x,s)|_{x=(s-g_\epsilon(s))+} = e^s \quad & \text{(instantaneous stopping)}, \end{aligned} \tag{2.19}$$

where Γ is the infinitesimal generator of the process X under $\mathbb{P}_0$. Moreover, the principle of smooth fit [28, 33] suggests that this system should be complemented by

$$\frac{\partial V_{g_\epsilon}}{\partial x}(x,s)\big|_{x=(s-g_\epsilon(s))+} = 0 \quad \text{(smooth fit)}. \tag{2.20}$$

Note that, although the smooth fit condition is not necessarily part of the general theory, it is imposed since by the "rule of thumb" outlined in Section 7 in [1] it should hold in this setting because of path regularity. This belief will be vindicated when we show that system (2.19) with (2.20) leads to the solution of problem (2.1). Applying the strong Markov property at τ_s^+ and using (2.10) and (2.11) shows that

$$\begin{aligned} V_{g_\epsilon}(x,s) &= e^s\mathbb{E}_{x,s}\big[e^{-q\tau^-_{s-g_\epsilon(s)}}1_{\{\tau^-_{s-g_\epsilon(s)}<\tau_s^+\}}\big] \\ &\quad +\mathbb{E}_{x,s}\big[e^{-q\tau_s^+}1_{\{\tau^-_{s-g_\epsilon(s)}>\tau_s^+\}}\big]\mathbb{E}_{s,s}\big[e^{-q\tau_{g_\epsilon}+\overline{X}_{\tau_{g_\epsilon}}}\big] \\ &= e^s\left(Z^{(q)}(x-s+g_\epsilon(s)) - W^{(q)}(x-s+g_\epsilon(s))\frac{Z^{(q)}(g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))}\right) \\ &\quad +\frac{W^{(q)}(x-s+g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))}V_{g_\epsilon}(s,s). \end{aligned}$$

Furthermore, the smooth fit condition implies

$$0 = \lim_{x\downarrow s-g_\epsilon(s)}\frac{\partial V_{g_\epsilon}}{\partial x}(x,s) = \lim_{x\downarrow s-g_\epsilon(s)}\frac{W^{(q)\prime}(x-s+g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))}\big(V_{g_\epsilon}(s,s)-e^sZ^{(q)}(g_\epsilon(s))\big).$$

By (2.16) the first factor tends to a strictly positive value or infinity which shows that $V_{g_\epsilon}(s,s)=e^sZ^{(q)}(g_\epsilon(s))$. This means that for $(x,s)\in E$ such that $s-g_\epsilon(s)<x<s$ we have

$$V_{g_\epsilon}(x,s) = e^sZ^{(q)}(x-s+g_\epsilon(s)). \tag{2.21}$$

Having derived the form of a candidate optimal value function V_{g_ϵ}, we still need to do the same for g_ϵ. Using the normal reflection condition in (2.19) shows that our candidate function g_ϵ should satisfy the ordinary differential equation

$$Z^{(q)}(g_\epsilon(s)) + qW^{(q)}(g_\epsilon(s))(g_\epsilon'(s)-1) = 0.$$

If X is of bounded variation [$W^{(q)}(0+)\in(0,q^{-1})$], we informally expect from the general theory that V_{g_ϵ} satisfies the first two equations of (2.19). Additionally, the principle of continuous fit [1, 32] suggests that the system should be complemented by

$$V_{g_\epsilon}(x,s)|_{x=(s-g_\epsilon(s))+} = e^s \quad \text{(continuous fit)}.$$

A very similar argument as above produces the same candidate value function and the same ordinary differential equation for g_ϵ.

2.6 Proofs of main results

Proof of Lemma 2.2. The idea is to define a suitable bijection H from $(0,k^*)$ to $(-\infty,\epsilon)$ whose inverse satisfies the differential equation and the boundary conditions.

First consider the case $q > \psi(1)$ and $W^{(q)}(0+) < q^{-1}$. It follows from the discussion at the beginning of Subsection 2.3.3 that $k^* \in (0,\infty)$ and that the function

$$s \mapsto h(s) := 1 - \frac{Z^{(q)}(s)}{qW^{(q)}(s)}$$

is negative on $(0,k^*)$. Moreover, $\lim_{s\downarrow 0} h(s) \in [-\infty,0)$ and $\lim_{s\uparrow k^*} h(s) = 0$. These properties imply that the function $H : (0,k^*) \to (-\infty,\epsilon)$ defined by

$$H(s) := \epsilon + \int_0^s \left(1 - \frac{Z^{(q)}(\eta)}{qW^{(q)}(\eta)}\right)^{-1} d\eta = \epsilon + \int_0^s \frac{qW^{(q)}(\eta)}{qW^{(q)}(\eta) - Z^{(q)}(\eta)}\, d\eta \tag{2.22}$$

is strictly decreasing. If we can also show that the integral tends to $-\infty$ as s approaches k^*, we could deduce that H is a bijection from $(0,k^*)$ to $(-\infty,\epsilon)$. Indeed, appealing to l'Hôpital's rule and using (2.14) we obtain

$$\begin{aligned}
\lim_{z\uparrow k^*} \frac{qW^{(q)}(z) - Z^{(q)}(z)}{k^* - z} &= \lim_{z\uparrow k^*} qW^{(q)}(z) - qW^{(q)\prime}(z) \\
&= \lim_{z\uparrow k^*} qe^{\Phi(q)z}\big((1-\Phi(q))W_{\Phi(q)}(z) - W'_{\Phi(q)}(z)\big) \\
&= qe^{\Phi(q)k^*}\big((1-\Phi(q))W_{\Phi(q)}(k^*) - W'_{\Phi(q)}(k^*)\big).
\end{aligned}$$

Denote the term on the right-hand side by c, and note that $c < 0$ due to the fact that $W_{\Phi(q)}$ is strictly positive and increasing on $(0,\infty)$ and since $\Phi(q) > 1$ for $q > \psi(1)$. Hence there exists a $\delta > 0$ and $0 < z_0 < k^*$ such that $c - \delta < \frac{qW^{(q)}(z)-Z^{(q)}(z)}{k^*-z}$ for all $z_0 < z < k^*$. Thus

$$\frac{1}{qW^{(q)}(z) - Z^{(q)}(z)} < \frac{1}{(c-\delta)(k^*-z)} < 0 \quad \text{for } z_0 < z < k^*.$$

This shows that

$$\lim_{s\uparrow k^*} H(s) \le \epsilon + \lim_{s\uparrow k^*} \int_{z_0}^{s} \frac{qW^{(q)}(\eta)}{(c-\delta)(k^*-\eta)}\, d\eta = -\infty.$$

The discussion above permits us to define $g_\epsilon := H^{-1} \in C^1((-\infty,\epsilon);(0,k^*))$. In particular, differentiating g_ϵ gives

$$g_\epsilon'(s) = \frac{1}{H'(g_\epsilon(s))} = 1 - \frac{Z^{(q)}(g_\epsilon(s))}{qW^{(q)}(g_\epsilon(s))}$$

for $s \in (-\infty,\epsilon)$, and g_ϵ satisfies $\lim_{s\to-\infty} g_\epsilon(s) = k^*$ and $\lim_{s\uparrow\epsilon} g_\epsilon(s) = 0$ by construction.

As for the case $q \le \psi(1)$, note that by (2.12) we have

$$Z^{(q)}(x) - qW^{(q)}(x) \ge Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x) > 0 \tag{2.23}$$

for $x \ge 0$ which shows that $k^* = \infty$. Moreover, (2.23) together with (2.13) implies that the map $s \mapsto h(s)$ is negative on $(0,\infty)$, satisfies $\lim_{s\uparrow\infty} h(s) = 1 - \Phi(q)^{-1} \le 0$ and

$$\lim_{s\downarrow 0} h(s) = \begin{cases} -\infty, & \text{if } X \text{ is of unbounded variation,} \\ 1 - \frac{\mathtt{d}}{q}, & \text{if } X \text{ is of bounded variation.} \end{cases}$$

Since $q \le \psi(1)$ implies that $q < \mathtt{d}$ whenever X is of bounded variation, we conclude that $\lim_{s\downarrow 0} h(s) \in [-\infty, 0)$. Defining $H : (0,\infty) \to (-\infty,\epsilon)$ as in (2.22), one deduces similarly as above that H is a continuously differentiable bijection whose inverse satisfies the requirements.

We finish the proof by addressing the question of uniqueness. To this end, assume that there is another solution $\tilde{g}$. In particular, $\tilde{g}'(s) = h(\tilde{g}(s))$ for $s \in (s_1,\epsilon) \subset (-\infty,\epsilon)$ and hence

$$s_1 = \epsilon - \int_{(s_1,\epsilon)} d\eta = \epsilon + \int_{(s_1,\epsilon)} \frac{|\tilde{g}'(s)|}{h(\tilde{g}(s))}\, ds = \epsilon + \int_0^{\tilde{g}(s_1)} \frac{1}{h(s)}\, ds = H(\tilde{g}(s_1))$$

which implies that $\tilde{g} = H^{-1} = g_\epsilon$. □

Proof of Theorem 2.3. Define the function

$$V_\epsilon(x,s) := e^{s\wedge\epsilon} Z^{(q)}(x - s + g_\epsilon(s))$$

for $(x,s) \in E$ and let $\tau_{g_\epsilon} := \inf\{t \ge 0 : \overline{X}_t - X_t \ge g_\epsilon(\overline{X}_t)\}$, where g_ϵ is as in Lemma 2.2. Because of the infinite horizon and Markovian claim structure of problem (2.1) it is enough to check the following conditions:

(i) $V_\epsilon(x,s) \geq e^{s\wedge\epsilon}$ for all $(x,s) \in E$;

(ii) the process $e^{-qt}V_\epsilon(X_t,\overline{X}_t)$, $t \geq 0$, is a right-continuous $\mathbb{P}_{x,s}$-supermartingale for $(x,s) \in E$;

(iii) $V_\epsilon(x,s) = \mathbb{E}_{x,s}\big[e^{-q\tau_{g_\epsilon}+\overline{X}_{\tau_{g_\epsilon}}\wedge\epsilon}\big]$ for all $(x,s) \in E$.

To see why these are sufficient conditions, note that (i) and (ii) together with Fatou's lemma in the second inequality and Doob's stopping theorem in the third inequality show that for $\tau \in \mathcal{M}$,

$$\begin{aligned}\mathbb{E}_{x,s}\big[e^{-q\tau+\overline{X}_\tau\wedge\epsilon}\big] &\leq \mathbb{E}_{x,s}\big[e^{-q\tau}V_\epsilon(X_\tau,\overline{X}_\tau)\big]\\ &\leq \liminf_{t\to\infty}\mathbb{E}_{x,s}\big[e^{-q(t\wedge\tau)}V_\epsilon(X_{t\wedge\tau},\overline{X}_{t\wedge\tau})\big]\\ &\leq V_\epsilon(x,s),\end{aligned}$$

which in view of (iii) implies $V_\epsilon^* = V_\epsilon$ and $\tau_\epsilon^* = \tau_{g_\epsilon}$.

The remainder of this proof is devoted to checking conditions (i)–(iii). Condition (i) is clearly satisfied since $Z^{(q)}$ is bigger or equal to one by definition.

Supermartingale property (ii)**:** Given the inequality

$$\mathbb{E}_{x,s}\big[e^{-qt}V_\epsilon(X_t,\overline{X}_t)\big] \leq V_\epsilon(x,s), \quad (x,s) \in E, \tag{2.24}$$

the supermartingale property is a consequence of the Markov property of the process $(X,\overline{X})$. Indeed, for $u \leq t$ we have

$$\begin{aligned}\mathbb{E}_{x,s}\big[e^{-qt}V_\epsilon(X_t,\overline{X}_t)\big|\mathcal{F}_u\big] &= e^{-qu}\mathbb{E}_{X_u,\overline{X}_u}\big[e^{-q(t-u)}V_\epsilon(X_{t-u},\overline{X}_{t-u})\big]\\ &\leq e^{-qu}V_\epsilon(X_u,\overline{X}_u).\end{aligned}$$

We now prove (2.24), first under the assumption that $W^{(q)}(0+) = 0$, that is, X is of unbounded variation. Let Γ be the infinitesimal generator of X and formally define the function $\Gamma Z^{(q)} : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ by

$$\begin{aligned}\Gamma Z^{(q)}(x) &:= -\gamma Z^{(q)\prime}(x) + \frac{\sigma^2}{2}Z^{(q)\prime\prime}(x)\\ &\quad + \int_{(-\infty,0)}\big(Z^{(q)}(x+y) - Z^{(q)}(x) - yZ^{(q)\prime}(x)1_{\{y\geq -1\}}\big)\,\Pi(dy).\end{aligned}$$

For $x < 0$ the quantity $\Gamma Z^{(q)}(x)$ is well defined and $\Gamma Z^{(q)}(x) = 0$. However, for $x > 0$ one needs to check whether the integral part of $\Gamma Z^{(q)}(x)$ is well defined. This is done in Lemma 2.9 (see Section 2.8) which shows that this is indeed the case. Moreover, as shown in Section 3.2 of [34], it holds that

$$\Gamma Z^{(q)}(x) = qZ^{(q)}(x), \quad x \in (0,\infty).$$

Now fix $(x,s) \in E$ and define the semimartingale $Y_t := X_t - \overline{X}_t + g_\epsilon(\overline{X}_t)$, $t \geq 0$. Applying an appropriate version of the Itô–Meyer formula (cf. Theorem 71, Chapter IV of [36]) to $Z^{(q)}(Y_t)$ yields $\mathbb{P}_{x,s}$-a.s.

$$\begin{aligned} Z^{(q)}(Y_t) &= Z^{(q)}(x-s+g_\epsilon(s)) + m_t + \int_0^t \Gamma Z^{(q)}(Y_u)\,du \\ &\quad + \int_0^t Z^{(q)\prime}(Y_u)(g_\epsilon'(\overline{X}_u)-1)\,d\overline{X}_u, \end{aligned} \tag{2.25}$$

where

$$\begin{aligned} m_t &= \int_{0+}^t \sigma Z^{(q)\prime}(Y_{u-})dB_u + \int_{0+}^t Z^{(q)\prime}(Y_{u-})dX_u^{(2)} \\ &\quad + \sum_{0<u\leq t}\big(\Delta Z^{(q)}(Y_u) - \Delta X_u Z^{(q)\prime}(Y_{u-})1_{\{\Delta X_u \geq -1\}}\big) \\ &\quad - \int_0^t \int_{(-\infty,0)} \big(Z^{(q)}(Y_{u-}+y) - Z^{(q)}(Y_{u-}) - yZ^{(q)\prime}(Y_{u-})1_{\{y\geq -1\}}\big)\,\Pi(dy)du \end{aligned}$$

and $\Delta X_u = X_u - X_{u-}$, $\Delta Z^{(q)}(Y_u) = Z^{(q)}(Y_u) - Z^{(q)}(Y_{u-})$. The fact that $\Gamma Z^{(q)}$ is not defined at zero is not a problem as the time Y spends at zero has Lebesgue measure zero anyway. By the boundedness of $Z^{(q)\prime}$ on $(-\infty, g_\epsilon(s)]$ the first two stochastic integrals in the expression for m_t are zero-mean martingales and by the compensation formula (cf. Corollary 4.6 of [21]) the third and fourth term constitute a zero-mean martingale. Next, recall that $V_\epsilon(x,s) = e^{s\wedge\epsilon}Z^{(q)}(x-s+g_\epsilon(s))$ and use stochastic integration by parts for semimartingales (cf. Corollary 2 of Theorem 22, Chapter II of [36]) to deduce that

$$\begin{aligned} e^{-qt}V_\epsilon(X_t,\overline{X}_t) &= V_\epsilon(x,s) + M_t + \int_0^t e^{-qu+\overline{X}_u\wedge\epsilon}(\Gamma Z^{(q)}(Y_u) - qZ^{(q)}(Y_u))\,du \\ &\quad + \int_0^t e^{-qu+\overline{X}_u\wedge\epsilon}\big(Z^{(q)}(Y_u)1_{\{\overline{X}_u\leq\epsilon\}} + Z^{(q)\prime}(Y_u)(g_\epsilon'(\overline{X}_u)-1)\big)\,d\overline{X}_u \end{aligned} \tag{2.26}$$

where $M_t = \int_{0+}^t e^{-qu+\overline{X}_u\wedge\epsilon}dm_u$ is a zero-mean martingale. The first integral is non-positive since $\Gamma Z^{(q)}(y) - qZ^{(q)}(y) \leq 0$ for all $y \in \mathbb{R}\setminus\{0\}$. The last integral vanishes since the process $\overline{X}_u$ only increments when $\overline{X}_u = X_u$ and by definition of g_ϵ. Thus, taking expectations on both sides yields

$$\mathbb{E}_{x,s}\Big[e^{-qt}V_\epsilon(X_t,\overline{X}_t)\Big] \leq V_\epsilon(x,s).$$

If $W^{(q)}(0+) \in (0, q^{-1})$ (X has bounded variation), then the Itô–Meyer formula is nothing more than an appropriate version of the change of variable formula for Stieltjes integrals and the rest of the proof follows the same line of reasoning as above. The only change worth mentioning is that the generator of X takes a different form. Specifically, one has to work with

$$\Gamma Z^{(q)}(x) = \mathtt{d}Z^{(q)\prime}(x) + \int_{(-\infty,0)} \big(Z^{(q)}(x+y) - Z^{(q)}(x)\big)\Pi(dy)$$

which satisfies all the required properties by Lemma 2.9 (see Section 2.8) and Section 3.2 in [34].

This completes the proof of the supermartingale property.

Verification of condition (iii)**:** The assertion is clear for $(x,s) \in D^*$. Hence, suppose that $(x,s) \in C^*$. The assertion now follows from the proof of the supermartingale property (ii). More precisely, replacing t by $t \wedge \tau_{g_\epsilon}$ in (2.26) and recalling that we have $(\Gamma - q)Z^{(q)}(y) = 0$ for $y > 0$ shows that

$$\mathbb{E}_{x,s}\big[e^{-q(t\wedge\tau_{g_\epsilon})}V_\epsilon(X_{t\wedge\tau_{g_\epsilon}}, \overline{X}_{t\wedge\tau_{g_\epsilon}})\big] = V_\epsilon(x,s).$$

Using that $\tau_{g_\epsilon} < \infty$ a.s. and dominated convergence, one obtains the desired equality. $\square$

Proof of Lemma 2.4. For $(x,s) \in C_I^*$, write

$$\begin{aligned}\mathbb{E}_{x,s}\big[e^{-q(T_{\epsilon_1}\wedge\tau^*_{\epsilon_2})+\overline{X}_{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}\wedge\epsilon_2}}\big] &= \mathbb{E}_{x,s}\Big[e^{-q(T_{\epsilon_1}\wedge\tau^*_{\epsilon_2})+\overline{X}_{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}}}1_{\{T_{\epsilon_1}>\tau_A^+\}}\Big] \\ &\quad+\mathbb{E}_{x,s}\Big[e^{-q(T_{\epsilon_1}\wedge\tau^*_{\epsilon_2})+\overline{X}_{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}}}1_{\{T_{\epsilon_1}<\tau_A^+\}}\Big]\end{aligned}$$

and denote the first expectation on the right by I_1 and the second expectation by I_2. An application of the strong Markov property at τ_A^+ and the definition of $V^*_{\epsilon_2}$ (see Theorem 2.3) give

$$I_1 = \mathbb{E}_{x,s}\Big[e^{-q\tau_A^+}1_{\{T_{\epsilon_1}>\tau_A^+\}}\Big]\mathbb{E}_{A,A}\Big[e^{-q\tau^*_{\epsilon_2}+\overline{X}_{\tau^*_{\epsilon_2}}}\Big] = \frac{W^{(q)}(x-\epsilon_1)}{W^{(q)}(A-\epsilon_1)}e^A Z^{(q)}(g_{\epsilon_2}(A)).$$

Recalling that $s < A$ and using the strong Markov property at τ_s^+ yields

$$\begin{aligned} I_2 &= e^s\mathbb{E}_{x,s}\Big[e^{-qT_{\epsilon_1}}1_{\{T_{\epsilon_1}<\tau_s^+\}}\Big] \\ &\quad+\mathbb{E}_{x,s}\Big[e^{-q\tau_s^+}1_{\{T_{\epsilon_1}>\tau_s^+\}}\Big]\mathbb{E}_{s,s}\Big[e^{-qT_{\epsilon_1}+\overline{X}_{T_{\epsilon_1}}}1_{\{T_{\epsilon_1}<\tau_A^+\}}\Big] \\ &= e^s\Big(Z^{(q)}(x-\epsilon_1) - W^{(q)}(x-\epsilon_1)\frac{Z^{(q)}(s-\epsilon_1)}{W^{(q)}(s-\epsilon_1)}\Big)\end{aligned}$$

$$
\begin{aligned}
&\quad+\frac{W^{(q)}(x-\epsilon_1)}{W^{(q)}(s-\epsilon_1)}\mathbb{E}_{s,s}\Big[e^{-qT_{\epsilon_1}+\overline{X}_{T_{\epsilon_1}}}1_{\{T_{\epsilon_1}<\tau_A^+\}}\Big]\\
&= e^s\Big(Z^{(q)}(x-\epsilon_1)-W^{(q)}(x-\epsilon_1)\frac{Z^{(q)}(s-\epsilon_1)}{W^{(q)}(s-\epsilon_1)}\Big)\\
&\quad+\frac{W^{(q)}(x-\epsilon_1)}{W^{(q)}(s-\epsilon_1)}e^s\mathbb{E}_{0,0}\Big[e^{-q\tau^-_{\epsilon_1-s}+\overline{X}_{\tau^-_{\epsilon_1-s}}}1_{\{\tau^-_{\epsilon_1-s}<\tau^+_{A-s}\}}\Big]. \qquad (2.27)
\end{aligned}
$$

Next, we compute the expectation on the right-hand side of (2.27) by excursion theory. To be more precise, we are going to make use of the compensation formula of excursion theory, and hence we shall spend a moment setting up some necessary notation. In doing so, we closely follow pages 221–223 in [2] and refer the reader to Chapters 6 and 7 in [6] for background reading. The process $L_t := \overline{X}_t$ serves as local time at 0 for the Markov process $\overline{X}-X$ under $\mathbb{P}_{0,0}$. Write $L^{-1} := \{L_t^{-1} : t \geq 0\}$ for the right-continuous inverse of L. The Poisson point process of excursions indexed by local time shall be denoted by $\{(t,\varepsilon_t) : t \geq 0\}$, where

$$
\varepsilon_t = \{\varepsilon_t(s) := X_{L_t^{-1}} - X_{L^{-1}_{t-}+s} : 0 < s < L_t^{-1} - L_{t-}^{-1}\}
$$

whenever $L_t^{-1} - L_{t-}^{-1} > 0$. Accordingly, we refer to a generic excursion as $\varepsilon(\cdot)$ (or just ε for short as appropriate) belonging to the space $\mathcal{E}$ of canonical excursions. The intensity measure of the process $\{(t,\varepsilon_t) : t \geq 0\}$ is given by $dt \times dn$, where n is a measure on the space of excursions (the excursion measure). A functional of the canonical excursion that will be of interest is $\overline{\varepsilon} = \sup_{s<\zeta}\varepsilon(s)$, where $\zeta(\varepsilon)=\zeta$ is the length of an excursion. A useful formula for this functional that we shall make use of is the following [cf. [21], equation (8.26)]:

$$
n(\overline{\varepsilon} > x) = \frac{W'(x)}{W(x)} \qquad (2.28)
$$

provided that x is not a discontinuity point in the derivative of W [which is only a concern when X is of bounded variation, but we have assumed that in this case Π is atomless and hence W is continuously differentiable on $(0,\infty)$]. Another functional that we will also use is $\rho_a := \inf\{s > 0 : \varepsilon(s) > a\}$, the first passage time above a of the canonical excursion ε. We now proceed with the promised calculation involving excursion theory. Specifically, an application of the compensation formula in the second equality and using Fubini's theorem in the third equality gives

$$
\begin{aligned}
&\mathbb{E}\Big[e^{-q\tau^-_{\epsilon_1-s}+L_{\tau^-_{\epsilon_1-s}}}1_{\{\tau^-_{\epsilon_1-s}<\tau^+_{A-s}\}}\Big]\\
&=\mathbb{E}\Bigg[\sum_{0<t<\infty}e^{-qL^{-1}_{t-}+t}1_{\left\{\substack{\overline{\varepsilon}_u\leq u-\epsilon_1+s\,\forall\,u<t\\ t<A-s}\right\}}1_{\{\overline{\varepsilon}_t>t-\epsilon_1+s\}}e^{-q\rho_{t-\epsilon_1+s}(\varepsilon_t)}\Bigg]\\
&=\mathbb{E}\Bigg[\int_0^{A-s}dt\,e^{-qL_t^{-1}+t}1_{\{\overline{\varepsilon}_u\leq u-\epsilon_1+s\,\forall\,u<t\}}\int_{\mathcal{E}}1_{\{\overline{\varepsilon}>t-\epsilon_1+s\}}e^{-q\rho_{t-\epsilon_1+s}(\varepsilon)}n(d\varepsilon)\Bigg]
\end{aligned}
$$

$$= \int_0^{A-s} e^{t-\Phi(q)t}\mathbb{E}\Big[e^{-qL_t^{-1}+\Phi(q)t}1_{\{\overline{\varepsilon}_u \le u-\epsilon_1+s\,\forall\, u<t\}}\Big]\hat{f}(t-\epsilon_1+s)\,dt,$$

where in the first equality the time index runs over local times and the sum is the usual shorthand for integration with respect to the Poisson counting measure of excursions, and $\hat{f}(u) = \frac{Z^{(q)}(u)W^{(q)\prime}(u)}{W^{(q)}(u)} - qW^{(q)}(u)$ is an expression taken from Theorem 1 in [2]. Next, note that L_t^{-1} is a stopping time and hence a change of measure according to (2.9) shows that the expectation inside the integral can be written as

$$\mathbb{P}^{\Phi(q)}\big[\overline{\varepsilon}_u \le u-\epsilon_1+s \text{ for all } u<t\big].$$

Using the properties of the Poisson point process of excursions (indexed by local time) and with the help of (2.28) and (2.14) we may deduce

$$\begin{aligned}
\mathbb{P}^{\Phi(q)}\big[\overline{\varepsilon}_u \le u-\epsilon_1+s \text{ for all } u<t\big] &= \exp\left(-\int_0^t n_{\Phi(q)}(\overline{\varepsilon} > u-\epsilon_1+s)\,du\right)\\
&= e^{\Phi(q)t}\frac{W^{(q)}(s-\epsilon_1)}{W^{(q)}(t-\epsilon_1+s)},
\end{aligned}$$

where $n_{\Phi(q)}$ denotes the excursion measure associated with X under $\mathbb{P}^{\Phi(q)}$. By a change of variables and the fact that $A-\epsilon_1 = g_{\epsilon_2}(A)$ we further obtain

$$\begin{aligned}
\mathbb{E}_{0,0}\Big[e^{-q\tau^-_{\epsilon_1-s}+L_{\tau^-_{\epsilon_1-s}}}1_{\{\tau^-_{\epsilon_1-s}<\tau^+_{A-s}\}}\Big] &= W^{(q)}(s-\epsilon_1)e^{\epsilon_1-s}\int_{s-\epsilon_1}^{g_{\epsilon_2}(A)} e^t\frac{\hat{f}(t)}{W^{(q)}(t)}\,dt\\
&= -W^{(q)}(s-\epsilon_1)e^{\epsilon_1-s}\int_{s-\epsilon_1}^{g_{\epsilon_2}(A)} e^t\left(\frac{Z^{(q)}}{W^{(q)}}\right)'(t)dt.
\end{aligned}$$

Integrating by parts on the right-hand side, plugging the resulting expression into (2.27) and finally adding I_1 and I_2 gives the result. □

Proof of Theorem 2.5. Recall that $T_{\epsilon_1} = \inf\{t \ge 0 : X_t \le \epsilon_1\}$ and from Lemma 2.4 that, for $(x,s) \in E$,

$$V_{\epsilon_1,\epsilon_2}(x,s) = \mathbb{E}_{x,s}\big[e^{-q(T_{\epsilon_1}\wedge\tau^*_{\epsilon_2})+\overline{X}_{T_{\epsilon_1}\wedge\tau^*_{\epsilon_2}}\wedge\epsilon_2}\big]. \tag{2.29}$$

Similarly to the proof of Theorem 2.3, it is now enough to prove that:

(i) $V_{\epsilon_1,\epsilon_2}(x,s) \ge e^{s\wedge\epsilon_2}$ for all $(x,s) \in E_{\epsilon_1}$;

(ii) $e^{-q(t\wedge T_{\epsilon_1})}V_{\epsilon_1,\epsilon_2}(X_{t\wedge T_{\epsilon_1}}, \overline{X}_{t\wedge T_{\epsilon_1}})$, $t \ge 0$, is a right-continuous $\mathbb{P}_{x,s}$-supermartingale for all $(x,s) \in E_{\epsilon_1}$.

Condition (i) is clearly satisfied, so we devote the remainder of this proof to checking condition (ii).

Supermartingale property (ii)**:** Throughout this proof, let

$$\tilde{E} := \{(x,s) \in E : x > \epsilon_1 \text{ and } s \geq A\}.$$

Let $Y_t := e^{-qt} V_{\epsilon_1,\epsilon_2}(X_t, \overline{X}_t)$ for $t \geq 0$. Analogously to the proof of Theorem 2.3, it suffices to show that for $(x,s) \in E_{\epsilon_1}$ we have the inequality

$$\mathbb{E}_{x,s}\big[Y_{t \wedge T_{\epsilon_1}}\big] \leq V_{\epsilon_1,\epsilon_2}(x,s). \tag{2.30}$$

For $(x,s) \in \tilde{E}$ inequality (2.30) can be extracted from the proof of Theorem 2.3 where it is shown that the process $e^{-qt} V^*_{\epsilon_2}(X_t, \overline{X}_t)$, $t \geq 0$, is a $\mathbb{P}_{x,s}$-supermartinagle for all $(x,s) \in E$. In particular, the process Y_t, $t \geq 0$, is a $\mathbb{P}_{x,s}$-supermartingale for $(x,s) \in \tilde{E}$. The supermartingale property is preserved when stopping at T_{ϵ_1} and therefore we obtain, for $(x,s) \in \tilde{E}$,

$$\mathbb{E}_{x,s}\big[Y_{t \wedge T_{\epsilon_1}}\big] \leq V_{\epsilon_1,\epsilon_2}(x,s). \tag{2.31}$$

Thus, it remains to establish (2.30) for $(x,s) \in C^*_I$. To this end, we first prove that the process $Y_{t \wedge T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}}$, $t \geq 0$, is a $\mathbb{P}_{x,s}$-martingale. The strong Markov property gives

$$\mathbb{E}_{x,s}\big[Y_{T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}} \big| \mathcal{F}_t\big] = Y_{T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}} 1_{\{T_{\epsilon_1} \wedge \tau^*_{\epsilon_2} \leq t\}} + e^{-qt} \mathbb{E}_{X_t, \overline{X}_t}\big[Y_{T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}}\big] 1_{\{T_{\epsilon_1} \wedge \tau^*_{\epsilon_2} > t\}}. \tag{2.32}$$

By definition of $V_{\epsilon_1,\epsilon_2}$ we see that

$$Y_{T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}} = \begin{cases} \exp\big(-qT_{\epsilon_1} + \overline{X}_{T_{\epsilon_1}}\big), & \text{on } \{T_{\epsilon_1} \leq \tau^*_{\epsilon_2}\}, \\ \exp(-q\tau^*_{\epsilon_2} + \overline{X}_{\tau^*_{\epsilon_2}}), & \text{on } \{T_{\epsilon_1} > \tau^*_{\epsilon_2}\}, \end{cases}$$

which shows that the second term on the right-hand side of (2.32) equals

$$\begin{aligned}
&e^{-qt} \mathbb{E}_{X_t, \overline{X}_t}\Big[e^{-q(T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}) + \overline{X}_{T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}}}\Big](1_{\{t \leq \tau^+_A\}} + 1_{\{t > \tau^+_A\}}) 1_{\{T_{\epsilon_1} \wedge \tau^*_{\epsilon_2} > t\}} \\
&= \Big(e^{-qt} U_{\epsilon_1,\epsilon_2}(X_t, \overline{X}_t) 1_{\{t \leq \tau^+_A\}} + e^{-qt} V^*_{\epsilon_2}(X_t, \overline{X}_t) 1_{\{t > \tau^+_A\}}\Big) 1_{\{T_{\epsilon_1} \wedge \tau^*_{\epsilon_2} > t\}} \\
&= e^{-qt} V_{\epsilon_1,\epsilon_2}(X_t, \overline{X}_t) 1_{\{T_{\epsilon_1} \wedge \tau^*_{\epsilon_2} > t\}} \\
&= Y_t 1_{\{T_{\epsilon_1} \wedge \tau^*_{\epsilon_2} > t\}}.
\end{aligned}$$

Thus, $\mathbb{E}_{x,s}\big[Y_{T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}} \big| \mathcal{F}_t\big] = Y_{t \wedge T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}}$ which implies that $Y_{t \wedge T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}}$, $t \geq 0$, is a martingale. Again using the strong Markov property we further obtain for $(x,s) \in C^*_I$,

$$\begin{aligned}
\mathbb{E}_{x,s}\big[Y_{t \wedge T_{\epsilon_1}} \big| \mathcal{F}_{\tau^*_{\epsilon_2}}\big] &= Y_{t \wedge T_{\epsilon_1}} 1_{\{t \wedge T_{\epsilon_1} \leq \tau^*_{\epsilon_2}\}} \\
&\quad + e^{-q\tau^*_{\epsilon_2}} \mathbb{E}_{X_{\tau^*_{\epsilon_2}}, \overline{X}_{\tau^*_{\epsilon_2}}}\big[Y_{(t-u) \wedge T_{\epsilon_1}}\big]\Big|_{u = \tau^*_{\epsilon_2}} 1_{\{t \wedge T_{\epsilon_1} > \tau^*_{\epsilon_2}\}} \\
&\leq Y_{t \wedge T_{\epsilon_1}} 1_{\{t \wedge T_{\epsilon_1} \leq \tau^*_{\epsilon_2}\}} + e^{-q\tau^*_{\epsilon_2}} V_{\epsilon_1,\epsilon_2}(X_{\tau^*_{\epsilon_2}}, \overline{X}_{\tau^*_{\epsilon_2}}) 1_{\{t \wedge T_{\epsilon_1} > \tau^*_{\epsilon_2}\}} \\
&= Y_{t \wedge T_{\epsilon_1} \wedge \tau^*_{\epsilon_2}},
\end{aligned}$$

where the inequality follows from (2.31) and the fact that $(X_{\tau^*_{\epsilon_2}} \overline{X}_{\tau^*_{\epsilon_2}}) \in \tilde{E}$ on the set $\{t \wedge T_{\epsilon_1} > \tau^*_{\epsilon_2}\}$. Thus, $\mathbb{E}_{x,s}\big[Y_{t\wedge T_{\epsilon_1}}\big] \leq U_{\epsilon_1,\epsilon_2}(x,s) = V_{\epsilon_1,\epsilon_2}(x,s)$ for $(x,s) \in C^*_I$. □

Proof of Corollary 2.6. Part (a) follows from the proof of Theorem 2.5 by replacing g_ϵ with $g_\infty(s) = k^*$ and A by $\epsilon_1 + k^*$. For part (b), let $\epsilon_1 \in \mathbb{R}$ be given and recall that due to the assumption $q \leq \psi(1)$ we have $\lim_{s\downarrow -\infty} g_{\epsilon_1}(s) = \infty$. For an arbitrary $\delta > \epsilon_1$, the uniqueness in Lemma 2.2 implies that

$$g_\delta(s) = g_{\epsilon_1}(s - \delta + \epsilon_1), \quad s \in (-\infty, \delta).$$

It follows that $\lim_{\delta\uparrow\infty} g_\delta(s) = \infty$ for $s \in \mathbb{R}$ and that $\lim_{\delta\uparrow\infty} g_\delta(A_\delta) = \infty$. Hence, for $(x,s) \in E_{\epsilon_1}$, we have

$$V^*_{\epsilon_1,\infty}(x,s) := \sup_{\tau\in\mathcal{M}_{\epsilon_1}} \mathbb{E}_{x,s}\big[e^{-q(T_{\epsilon_1}\wedge\tau)+\overline{X}_{T_{\epsilon_1}\wedge\tau}}\big] \geq \lim_{\delta\uparrow\infty} V^*_{\epsilon_1,\delta}(x,s) = \infty.$$

On the other hand, if $(x,s) \in E \setminus E_{\epsilon_1}$, then clearly $V^*_{\epsilon_1,\infty}(x,s) = e^s$. This completes the proof. □

2.7 Examples

The solutions of (2.1) and (2.3) are given semi-explicitly in terms of scale functions and a specific solution g_ϵ and g_{ϵ_2} respectively of the ordinary differential equation (2.17). The aim of this section is to look at some examples where the solutions of (2.1) and (2.3) can be computed more explicitly. For simplicity, we will assume from now on that every spectrally negative Lévy process X considered below is such that $q > \psi(1)$ and $W^{(q)}(0+) < q^{-1}$. Also assume to begin with that there is an upper cap ϵ only.

A first step towards more explicit solutions of (2.1) is looking at processes X where explicit expressions for $W^{(q)}$ and $Z^{(q)}$ are available. In recent years various authors have found several processes whose scale functions are explicitly known (Example 1.3, Chapter 4 and Section 5.5 in [20], for instance). Here, however, we would additionally like to find g_ϵ explicitly. To the best of our knowledge, we do not know of any examples where this is possible. One might instead try to solve (2.17) numerically, but this is not straightforward as there is no initial point to start a numerical scheme from and, moreover, the possibility of g_ϵ having infinite gradient at ϵ might lead to inaccuracies in the numerical scheme. Therefore, we follow a different route which avoids these difficulties. Instead of looking at g_ϵ, we rather focus on its inverse

$$H(s) = \epsilon + \int_0^s \left(1 - \frac{Z^{(q)}(\eta)}{qW^{(q)}(\eta)}\right)^{-1} d\eta, \quad s \in (0,k^*), \tag{2.33}$$

where $k^* \in (0,\infty)$ is the unique root of $Z^{(q)}(z) - qW^{(q)}(z) = 0$. In fact, passing to the inverse is a standard trick in this setting and, for instance, used in [33]. It turns out that in some cases (including the Black–Scholes model) H can be computed explicitly. Since H is the inverse of g_ϵ, plotting $(H(y), y)$ for $y \in (0,k^*)$, yields visualisations of $s \mapsto g_\epsilon(s)$ for $s \in (-\infty,\epsilon)$; see Figures 2.3–2.5. Similarly, plotting $(H(y)-y, H(y))$ for $y \in (0,k^*)$, produces graphical representations of the optimal stopping boundary in the (x,s)-plane; see Figures 2.3–2.5. Unfortunately, it is often the case that we cannot compute the integral in (2.33) explicitly in which case one might use numerical integration in Matlab to obtain an approximation of the integral. The procedure just described is carried out below for different examples of X.

2.7.1 Brownian motion with drift and compound Poisson jumps

Consider the process

$$X_t = \sigma W_t + \mu t - \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0,$$

where $\sigma > 0$, $\mu \in \mathbb{R}$, W_t, $t \geq 0$, is a standard Brownian motion, N_t, $t \geq 0$, is a Poisson process with intensity $a > 0$ and ξ_i, $i \in \mathbb{N}$, are i.i.d. random variables which are exponentially distributed with parameter $\rho > 0$. The processes W and N as well as the sequence ξ_i are assumed to be mutually independent. The Laplace exponent of X is given by

$$\psi(\theta) = \frac{\sigma^2}{2}\theta^2 + \mu\theta - \frac{a\theta}{\rho+\theta}, \quad \theta \geq 0.$$

It is known (cf. Example 1.3 in [20] and Subsection 8.2 of [2]) that

$$W^{(q)}(x) = \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} + \frac{e^{-\zeta_1 x}}{\psi'(-\zeta_1)} + \frac{e^{-\zeta_2 x}}{\psi'(-\zeta_2)}, \quad x \geq 0, \tag{2.34}$$

where $-\zeta_2 < -\rho < -\zeta_1 < 0 < \Phi(q)$ are the three real solutions of the equation $\psi(\theta) = q$, and that, for $x \geq 0$,

$$Z^{(q)}(x) = D_1 e^{\Phi(q)x} + D_2 e^{-\zeta_1 x} + D_3 e^{-\zeta_2 x}, \tag{2.35}$$

where $D_1 = \frac{q}{\Phi(q)\psi'(\Phi(q))}$, $D_2 = \frac{q}{-\zeta_1\psi'(-\zeta_1)}$ and $D_3 = \frac{q}{-\zeta_2\psi'(-\zeta_2)}$.

As a first example consider $\sigma = 0$. In this case $\psi(\theta) = q$ reduces to a quadratic equation, and one can calculate explicitly

$$\begin{aligned}
\zeta_1 &= \frac{1}{2\mu}\Big(\sqrt{(a+q-\mu\rho)^2 + 4\mu q\rho} - (a+q-\mu\rho)\Big), \\
\Phi(q) &= \frac{1}{2\mu}\Big(\sqrt{(a+q-\mu\rho)^2 + 4\mu q\rho} + (a+q-\mu\rho)\Big).
\end{aligned}$$

Moreover, it follows that

$$k^* = \frac{1}{\zeta_1 + \phi(q)} \log\left(\frac{\Phi(q)\psi'(\Phi(q))(\zeta_1+1)}{\zeta_1\psi'(-\zeta_1)(1-\Phi(q))}\right).$$

Using elementary algebra and integration one finds, for $s \in (0, k^*)$,

$$\begin{aligned} H(s) &= \epsilon + \int_0^s \left(\frac{D_1\Phi(q)e^{(\Phi(q)+\zeta_1)x}}{D_1(\Phi(q)-1)e^{(\Phi(q)+\zeta_1)x} - D_2(\zeta_1+1)}\right) dx \\ &\quad - \int_0^s \frac{D_2\zeta_1 e^{-(\zeta_1+\Phi(q))x}}{D_1(\Phi(q)-1) - D_2(\zeta_1+1)e^{-(\zeta_1+\Phi(q))x}} dx \\ &= \epsilon + \int_0^s \left(\frac{\Phi(q)e^{Ax}}{Be^{Ax} - CD} - \frac{\zeta_1 e^{-Ax}}{C^{-1}B - De^{-Ax}}\right) dx \\ &= \epsilon + \frac{\Phi(q)}{AB}\log\left|\frac{Be^{As} - CD}{B - CD}\right| - \frac{\zeta_1}{AD}\log\left|\frac{B - CDe^{-As}}{B - CD}\right|, \end{aligned}$$

where $A := \zeta_1 + \Phi(q), B := \Phi(q) - 1, C := \frac{\Phi(q)\psi'(\Phi(q))}{-\zeta_1\psi'(-\zeta_1)}$ and $D := \zeta_1 + 1$. An example for a certain choice of parameters is given in Figure 2.3.

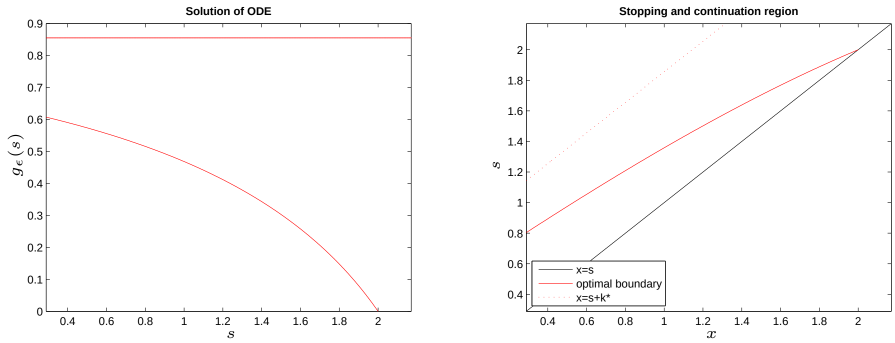

Fig. 2.3 *An illustration of* $s \mapsto g_\epsilon(s)$ *and the corresponding optimal boundary for* $q = 1.6$, $\epsilon = 2$, $\sigma = 0$, $\mu = 3$, $a = 3$ *and* $\rho = 0.1$.

Next, assume $\sigma > 0$ and $\rho = \infty$; that is, X is a linear Brownian motion. In particular, this includes the Black–Scholes model. Again, as explained in Example 1.3 of [20], the equation $\psi(\theta) = q$ reduces to a quadratic equation and $\zeta_1 = \delta - \gamma$ and $\Phi(q) = \delta + \gamma$, where

$$\gamma := -\frac{\mu}{\sigma^2} \quad \text{and} \quad \delta := \frac{1}{\sigma^2}\sqrt{\mu^2 + 2q\sigma^2}.$$

Furthermore, (2.34) and (2.35) may be rewritten on $x \geq 0$ as

$$W^{(q)}(x) = \frac{2}{\sigma^2\delta}e^{\gamma x}\sinh(\delta x) \quad \text{and} \quad Z^{(q)}(x) = e^{\gamma x}\cosh(\delta x) - \frac{\gamma}{\delta}e^{\gamma x}\sinh(\delta x), \tag{2.36}$$

and one can compute

$$k^* = \frac{1}{\Phi(q) + \zeta_1} \log\Big(\frac{1 + \zeta_1^{-1}}{1 - \Phi(q)^{-1}}\Big). \tag{2.37}$$

Using elementary algebra in the first and formula 2.447.1 of [16] in the second equality one obtains, for $s \in (0, k^*)$,

$$\begin{aligned} H(s) &= \epsilon + \frac{2q}{\sigma^2 \delta} \int_0^{s\delta} \frac{\sinh(x)}{(2q/\sigma^2 + \gamma)\cosh(x) - \delta \sinh(x)}\, dx \\ &= \epsilon + \frac{2q}{\sigma^2 \delta\big(F^2 - \delta^2\big)} \left(F\delta s - \delta \log \left| \frac{\sinh\big(\tanh^{-1}(-\delta F^{-1})\big)}{\sinh\big(\delta s + \tanh^{-1}(-\delta F^{-1})\big)} \right| \right), \end{aligned}$$

where $F := 2q/\sigma^2 + \gamma$. An example for a certain parameter choice is provided in Figure 2.4.

In the next example we combine the first example with the second one. More precisely, suppose that $\sigma >$ and $\rho \in (0, \infty)$, that is, a linear Brownian motion with exponential jumps. In this case we are unable to compute k^* and H explicitly. We therefore find k^* numerically and use numerical integration to obtain an approximation of k^* and H respectively; see Figure 2.4.

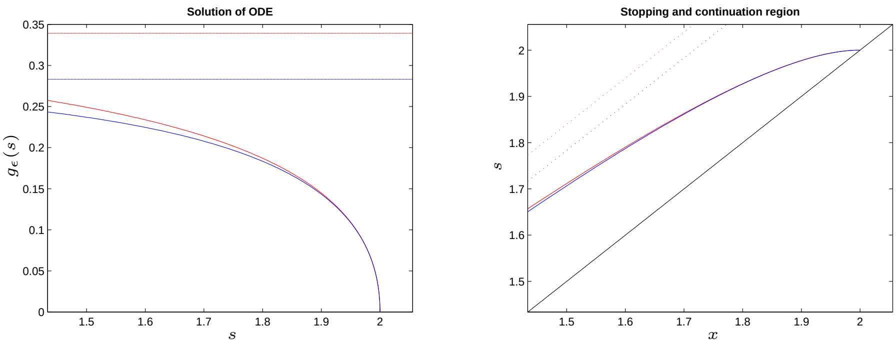

Fig. 2.4 *Left: A visualisation of $s \mapsto g_\epsilon(s)$ for when $q = 3$, $\epsilon = 2$, $\sigma = 1$ and $\mu = 2$ (red) and $q = 3$, $\epsilon = 2$, $\sigma = 1$, $\mu = 2$, $a = 3$ and $\rho = 0.1$ (blue). Right: An illustration of the corresponding optimal stopping boundaries.*

2.7.2 Stable jumps

Suppose that X is an α-stable process, $\alpha \in (1, 2]$, with Laplace exponent $\psi(\theta) = \theta^\alpha$, $\theta \geq 0$. It is known (cf. Example 4.17 of [20] and Subsection 8.3 of [2]) that, for $x \geq 0$,

$$W^{(q)}(x) = x^{\alpha-1} E_{\alpha,\alpha}(qx^\alpha) \quad \text{and} \quad Z^{(q)}(x) = E_{\alpha,1}(qx^\alpha),$$

where $E_{\alpha,\beta}$ is the two-parameter Mittag–Leffler function which is defined for $\alpha, \beta > 0$ as

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)}.$$

Again, using numerical integration and a Matlab function that computes the Mittag–Leffler function (cf. [35]) one may approximate k^* and H respectively; see Figure 2.5. Additionally, we have computed the value function for a choice of parameters (Figure 2.6).

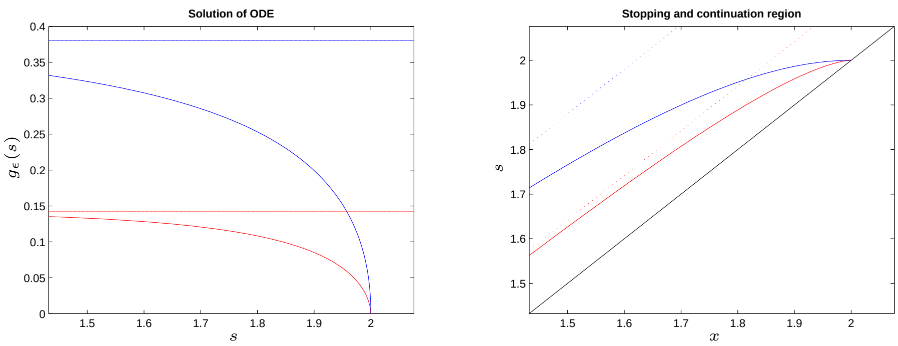

Fig. 2.5 *Left: A visualisation of $s \mapsto g_\epsilon(s)$ when $q = 2$ and $\epsilon = 2$ and X is either a linear Brownian motion (blue curve, $\sigma = \sqrt{2}$, $\mu = 0$) or an α-stable process (red curve, $\alpha = 1.6$).*

If one considers a lower cap ϵ_1 and an upper cap ϵ_2, then the only thing that changes for the optimal boundary is that one has to include an additional vertical line at the value of the lower cap ϵ_1. However, introducing a lower cap will make a difference, that is, the value functions $V^*_{\epsilon_2}(x,s)$ and $V^*_{\epsilon_1,\epsilon_2}(x,s)$ will be different for $(x,s) \in C^*_{I,\epsilon_1,\epsilon_2}$; see Theorems 2.3 and 2.5. Exploiting the fact that H is the inverse of g_{ϵ_2} in a similar way as above, one may also obtain numerical approximations of the value functions $V^*_{\epsilon_2}(x,s)$ and $V^*_{\epsilon_1,\epsilon_2}(x,s)$; see Figure 2.6.

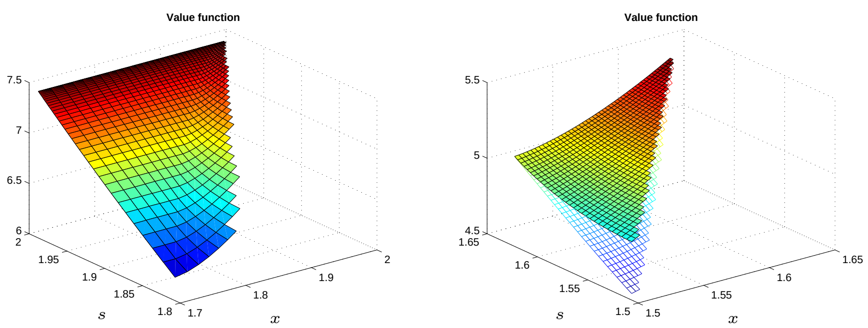

Fig. 2.6 *Left: A visualisation of $V^*_\epsilon(x,s)$ when X is α-stable with parameter choice $q = 3$, $\epsilon = 2$ and $\alpha = 1.6$. Right: An illustration of the difference between $V^*_{\epsilon_2}(x,s)$ (darker surface) and $V^*_{\epsilon_1,\epsilon_2}(x,s)$ (lighter surface) on $C^*_{I,\epsilon_1,\epsilon_2}$ for the same X and same parameters as on the left except that $\epsilon_1 = 1.5$ and $\epsilon_2 = \epsilon$. In this case $A \approx 1.63$, where A is formally defined in Subsection 2.4.2.*

2.7.3 Maximum process with lower cap only

Assume the same setting as in the second example above, that is, $X_t = \sigma W_t + \mu t$, $t \geq 0$. The scale functions and k^* are given by (2.36) and (2.37) respectively. If we suppose that there is a lower cap $\epsilon_1 \in \mathbb{R}$ and no upper cap ($\epsilon_2 = \infty$), then Corollary 2.6 can be rewritten more explicitly as follows.

Lemma 2.8. *The V^* and $U_{\epsilon_1,\infty}$ part of the optimal value function $V^*_{\epsilon_1,\infty}$ are given by*

$$V^*(x,s) = \frac{1}{\Phi(q)+\zeta_1}\left(\Phi(q)\left(\frac{e^x}{e^{s-k^*}}\right)^{-\zeta_1} + \zeta_1\left(\frac{e^x}{e^{s-k^*}}\right)^{\Phi(q)}\right)$$

and

$$\begin{aligned} U_{\epsilon_1,\infty}(x,s) &= \left(\frac{e^x}{e^{\epsilon_1}}\right)^{-\zeta_1}\left[-\frac{e^{\epsilon_1}}{\beta}\left(\int_{\beta(s-\epsilon_1)}^{\beta k^*}\frac{e^{u(1+y)}}{e^u-1}\,du - e^{k^*\Phi(q)}\right)\right] \\ &\quad + \left(\frac{e^x}{e^{\epsilon_1}}\right)^{\Phi(q)}\left[\frac{e^{\epsilon_1}}{\beta}\left(\int_{\beta(s-\epsilon_1)}^{\beta k^*}\frac{e^{uy}}{e^u-1}\,du - e^{-k^*\zeta_1}\right)\right]. \end{aligned}$$

where $\beta = \Phi(q) + \zeta_1 = 2\delta$ and $y = \beta^{-1}$.

The proof of this result is a lengthy computation provided in Subsection 2.8.2. Finally, if we set $\epsilon_1 = \epsilon$, $\mu = r - \sigma^2/2$ for some $r \geq 0$ and $q = \lambda + r$ for some $\lambda > 0$ we recover Theorem 3.1 of [40].

2.8 Appendix

2.8.1 Complementary Results on the Infinitesimal Generator of X

In this section we provide some results concerning the infinitesimal generator of X when applied to the scale function $Z^{(q)}$.

First assume that X is of unbounded variation, and define an operator $(\Gamma, \mathcal{D}(\Gamma))$ as follows. $\mathcal{D}(\Gamma)$ stands for the family of functions $f \in C^2(0,\infty)$ such that the integral

$$\int_{(-\infty,0)} \left(f(x+y) - f(x) - yf'(x)1_{\{y\geq -1\}}\right)\Pi(dy)$$

is absolutely convergent for all $x > 0$. For any $f \in \mathcal{D}(\Gamma)$, we define the function $\Gamma f : (0,\infty) \to \mathbb{R}$ by

$$\Gamma f(x) = -\gamma f'(x) + \frac{\sigma^2}{2}f''(x) + \int_{(-\infty,0)} \left(f(x+y) - f(x) - yf'(x)1_{\{y\geq -1\}}\right)\Pi(dy).$$

Similarly, if X is of bounded variation, then $\mathcal{D}(\Gamma)$ stands for the family of $f \in C^1(0,\infty)$

such that the integral

$$\int_{(-\infty,0)} \big(f(x+y)-f(x)\big)\,\Pi(dy)$$

is absolutely convergent for all $x > 0$, and for $f \in \mathcal{D}(\Gamma)$, we define the function $\Gamma f : (0,\infty) \to \mathbb{R}$ by

$$\Gamma f(x) = \mathtt{d} f'(x) + \int_{(-\infty,0)} \big(f(x+y)-f(x)\big)\Pi(dy).$$

In the sequel it should always be clear from the context in which of the two cases we are and therefore there should be no ambiguity when writing $\mathcal{D}(\Gamma)$ and Γ.

Lemma 2.9. *We have that $Z^{(q)} \in \mathcal{D}(\Gamma)$ and the function $x \mapsto \Gamma Z^{(q)}(x)$ is continuous on $(0,\infty)$.*

Proof. We prove the unbounded and bounded variation case separately.

Unbounded variation: To show that $Z^{(q)} \in \mathcal{D}(\Gamma)$ it is enough to check that the integral part of $\Gamma Z^{(q)}$ is absolutely convergent since $Z^{(q)} \in C^2(0,\infty)$. Fix $x > 0$ and write the integral part of $\Gamma Z^{(q)}$ as

$$\begin{aligned}&\int_{(-\infty,-\delta)} \big|Z^{(q)}(x+y) - Z^{(q)}(x) - yZ^{(q)\prime}(x)1_{\{y\geq -1\}}\big|\Pi(dy)\\ &+\int_{(-\delta,0)} \big|Z^{(q)}(x+y) - Z^{(q)}(x) - yZ^{(q)\prime}(x)1_{\{y\geq -1\}}\big|\Pi(dy)\end{aligned}$$

where the value $\delta = \delta(x) \in (0,1)$ is chosen such that $x - \delta > 0$. For $y \in (-\infty,-\delta)$ the monotonicity of $Z^{(q)}$ implies

$$\big|Z^{(q)}(x+y) - Z^{(q)}(x) - yZ^{(q)\prime}(x)1_{\{y\geq -1\}}\big| \leq 2Z^{(q)}(x) + Z^{(q)\prime}(x) \tag{2.38}$$

and for $y \in (-\delta, 0)$, using the mean value theorem, we have

$$\begin{aligned}&|Z^{(q)}(x+y) - Z^{(q)}(x) - yZ^{(q)\prime}(x)|\\ &= q|y||W^{(q)}(\xi(y)) - W^{(q)}(x)| \quad \text{where } \xi(y) \in (x+y,x)\\ &= q|y|\bigg|\int_{\xi(y)}^{x} W^{(q)\prime}(z)\,dz\bigg|\\ &\leq qy^2 \sup_{z\in[x-\delta,x]} W^{(q)\prime}(z).\end{aligned} \tag{2.39}$$

Using these two estimates and defining $C(\delta) = \int_{(-\delta,0)} y^2\Pi(dy) < \infty$, we see that

$$\begin{aligned}&\int_{(-\infty,0)} \big|Z^{(q)}(x+y) - Z^{(q)}(x) - yZ^{(q)\prime}(x)1_{\{y\geq -1\}}\big|\Pi(dy)\\ &\leq \big(2Z^{(q)}(x) + Z^{(q)\prime}(x)\big)\Pi(-\infty,-\delta) + qC(\delta) \sup_{z\in[x-\delta,x]} W^{(q)\prime}(z) < \infty.\end{aligned}$$

For continuity, let $x > 0$ and choose $\delta = \delta(x) \in (0,1)$ such that $x - 2\delta > 0$ as well as a sequence $(x_n)_{n\in\mathbb{N}}$ converging to x. Moreover, let $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $|x_n - x| < \delta$. In particular, it holds that $x_n - \delta > 0$ for $n \geq n_0$ and hence, using the estimates in (2.38) and (2.39), we have for all $n \geq n_0$

$$\begin{aligned}
&|Z^{(q)}(x_n + y) - Z^{(q)}(x_n) - yZ^{(q)\prime}(x_n)1_{\{y\geq -1\}}| \\
&\leq qy^2 \sup_{z\in[x_n-\delta,x_n]} W^{(q)\prime}(z)1_{\{y\geq-\delta\}} + \big(2Z^{(q)}(x_n) + Z^{(q)\prime}(x_n)\big)1_{\{y<-\delta\}} \\
&\leq qy^2 \sup_{z\in[x-2\delta,x+\delta]} W^{(q)\prime}(z)1_{\{y\geq-\delta\}} + \big(2Z^{(q)}(x+\delta) + Z^{(q)\prime}(x+\delta)\big)1_{\{y<-\delta\}}.
\end{aligned}$$

Since the last term is Π-integrable, the continuity assertion follows by dominated convergence and the fact that $Z^{(q)} \in C^2(0,\infty)$.

Bounded variation: To show that $Z^{(q)} \in \mathcal{D}(\Gamma)$ it is enough to show that the integral part of $\Gamma Z^{(q)}$ is absolutely convergent since $Z^{(q)} \in C^1(0,\infty)$. Using the monotonicity and the definition of $Z^{(q)}$, it is easy to see that for fixed $x > 0$,

$$\begin{aligned}
&\int_{(-\infty,0)} |Z^{(q)}(x+y) - Z^{(q)}(x)|\,\Pi(dy) \\
&\leq 2Z^{(q)}(x)\Pi(-\infty,-1) + qW^{(q)}(x)\int_{(-1,0)} |y|\,\Pi(dy) < \infty.
\end{aligned}$$

The continuity assertion follows in a straightforward manner from dominated convergence and the fact that $Z^{(q)} \in C^1(0,\infty)$. □

2.8.2 A lengthy computation

Proof of Lemma 2.8. The first part is a short calculation using the definition of γ, δ, ζ_1, $\Phi(q)$ and that $\cosh(z) = \frac{e^z+e^{-z}}{2}$ and $\sinh(z) = \frac{e^z-e^{-z}}{2}$. As for the second part, recall that, for $(x,s) \in C_I^* \cup D_I^*$,

$$U_{\epsilon_1,\infty}(x,s) = e^s Z^{(q)}(x-\epsilon_1) + e^{\epsilon_1}W^{(q)}(x-\epsilon_1)\int_{s-\epsilon_1}^{k^*} e^t \frac{Z^{(q)}(t)}{W^{(q)}(t)}\,dt.$$

It is easy to see that

$$e^t\frac{Z^{(q)}(t)}{W^{(q)}(t)} = e^t\frac{\delta\sigma^2}{2}\left(\frac{1}{1-e^{-2\delta t}} + \frac{1}{e^{2\delta t}-1}\right) - e^t\frac{\gamma\sigma^2}{2}$$

which, after a change of variables, gives

$$\begin{aligned}
\int_{s-\epsilon_1}^{k^*} e^t\frac{Z^{(q)}(t)}{W^{(q)}(t)}\,dt \;&=\; \frac{\sigma^2}{4}\left(\int_{\beta(s-\epsilon_1)}^{\beta k^*} \frac{e^{u(1+y)}}{e^u-1}\,du + \int_{\beta(s-\epsilon_1)}^{\beta k^*} \frac{e^{uy}}{e^u-1}\,du\right) \\
&\quad + \frac{\gamma\sigma^2}{2}(e^{s-\epsilon_1} - e^{k^*}),
\end{aligned}$$

where $\beta = \Phi(q) + \zeta_1 = 2\delta$ and $y = \beta^{-1}$. Denote the first integral on the right-hand side I_1 and the second integral I_2. After some algebra one obtains

$$\begin{aligned} U_{\epsilon_1,\infty}(x,s) &= \frac{e^s}{2}\big(e^{\Phi(q)(x-\epsilon_1)} + e^{-\zeta_1(x-\epsilon_1)}\big) - \frac{e^{\epsilon_1+k^*}\gamma}{\beta}\big(e^{\Phi(q)(x-\epsilon_1)} - e^{-\zeta(x-\epsilon_1)}\big) \\ &\quad -\frac{e^{\epsilon_1}}{2\beta}e^{-\zeta_1(x-\epsilon_1)}I_1 + \frac{e^{\epsilon_1}}{2\beta}e^{\Phi(q)(x-\epsilon_1)}I_2 \\ &\quad +\frac{e^{\epsilon_1}}{2\beta}e^{\Phi(q)(x-\epsilon_1)}I_1 - \frac{e^{\epsilon_1}}{2\beta}e^{-\zeta_1(x-\epsilon_1)}I_2. \end{aligned} \tag{2.40}$$

Next, note that the last two terms on the right-hand side of (2.40) can be rewritten as

$$\begin{aligned} &\frac{e^{\epsilon_1}}{2\beta}\big(e^{\Phi(q)(x-\epsilon_1)} + e^{-\zeta_1(x-\epsilon_1)}\big)(I_1 - I_2) - \frac{e^{\epsilon_1}}{2\beta}e^{-\zeta_1(x-\epsilon_1)}I_1 + \frac{e^{\epsilon_1}}{2\beta}e^{\Phi(q)(x-\epsilon_1)}I_2 \\ &= \frac{e^{\epsilon_1}}{2}\big(e^{\Phi(q)(x-\epsilon_1)} + e^{-\zeta_1(x-\epsilon_1)}\big)\big(e^{k^*} - e^{s-\epsilon_1}\big) - \frac{e^{\epsilon_1}}{2\beta}e^{-\zeta_1(x-\epsilon_1)}I_1 + \frac{e^{\epsilon_1}}{2\beta}e^{\Phi(q)(x-\epsilon_1)}I_2 \end{aligned}$$

where the equality follows from evaluating $I_1 - I_2$. Plugging this into (2.40) and simplifying yields

$$\begin{aligned} U_{\epsilon_1,\infty}(x,s) &= -e^{-\zeta_1(x-\epsilon_1)}e^{\epsilon_1}\beta^{-1}I_1 + e^{\Phi(q)(x-\epsilon_1)}e^{\epsilon_1}\beta^{-1}I_2 + e^{\epsilon_1+\Phi(q)(x-\epsilon_1)}e^{k^*}\beta^{-1}\zeta_1 \\ &\quad +e^{\epsilon_1-\zeta_1(x-\epsilon_1)}e^{k^*}\beta^{-1}\Phi(q). \end{aligned}$$

Rearranging the terms completes the proof. □

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CHAPTER 3

AMERICAN LOOKBACK OPTION

This paper concerns an optimal stopping problem driven by the running maximum of a spectrally negative Lévy process X. More precisely, we are interested in capped versions of the American lookback optimal stopping problem [14, 17, 30], which has its origins in mathematical finance, and provide semi-explicit solutions in terms of scale functions. The optimal stopping boundary is characterised by an ordinary first-order differential equation involving scale functions and, in particular, changes according to the path variation of X. Furthermore, we will link these capped problems to Peskir's maximality principle [31].

3.1 Introduction

Let $X = \{X_t : t \geq 0\}$ be a spectrally negative Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$ satisfying the natural conditions; cf. [7], Section 1.3, page 39. For $x \in \mathbb{R}$, denote by $\mathbb{P}_x$ the probability measure under which X starts at x and for simplicity write $\mathbb{P}_0 = \mathbb{P}$. We associate with X the maximum process $\overline{X} = \{\overline{X}_t : t \geq 0\}$ where $\overline{X}_t := s \vee \sup_{0 \leq u \leq t} X_u$ for $t \geq 0, x \leq s$. The law under which $(X, \overline{X})$ starts at (x, s) is denoted by $\mathbb{P}_{x,s}$.

We are interested in the following optimal stopping problem:

$$V_\epsilon^*(x,s) := \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s}\big[e^{-q\tau}(e^{\overline{X}_\tau \wedge \epsilon} - K)^+\big], \tag{3.1}$$

where $q \geq 0, K \geq 0, \epsilon \in (\log(K), \infty]$, $(x,s) \in E := \{(x_1, s_1) \in \mathbb{R}^2 \,|\, x_1 \leq s_1\}$, and $\mathcal{M}$ is the set of all $\mathbb{F}$-stopping times (not necessarily finite). In particular, on $\{\tau = \infty\}$ we set $e^{-q\tau}(e^{\overline{X}_\tau \wedge \epsilon} - K)^+ := \limsup_{t \to \infty} e^{-qt}(e^{\overline{X}_t \wedge \epsilon} - K)^+$. This problem is, at least in the case $\epsilon = \infty$, classically associated with mathematical finance. It arises in the context of pricing American lookback options [14, 17, 30] and its solution may be viewed as the fair price for such an option. If $\epsilon \in (\log(K), \infty)$, an analogous interpretation applies for an American lookback option whose payoff is moderated by capping it at a certain level (a fuller description is given in Section 3.2).

When $K = 0$ and $\epsilon = \infty$, (3.1) is known as the Shepp–Shiryaev optimal stopping problem which was first studied by Shepp and Shiryaev [38, 39] for the case when X is a linear Brownian motion and later by Avram, Kyprianou and Pistorius [2] for the case when X is a spectrally negative Lévy process. If $K = 0$ and $\epsilon \in \mathbb{R}$ then the problem is a capped version of the Shepp–Shiryaev optimal stopping problem and was considered by Ott [29]. Therefore, our main focus in this paper will be the case $K > 0$ which we henceforth assume.

Our objective is to solve (3.1) for $\epsilon = (\log(K), \infty)$ by a "guess and verify" technique and to use this to obtain the solution to (3.1) when $\epsilon = \infty$ via a limiting procedure. Our work extends and complements results by Conze and Viswanathan [11], Guo and Shepp [17], Pedersen [30] and Gapeev [14] all of which solve (3.1) for $\epsilon = \infty$ and X a linear Brownian motion or a jump-diffusion.

As we shall see, the general theory of optimal stopping [33, 42] and the principle of smooth or continuous fit [1, 28, 32, 33] (and the results in [14, 17, 29, 30]) strongly suggest that under some assumptions on q and $\psi(1)$, where ψ is the Laplace exponent of X, the optimal strategy for (3.1) is of the form

$$\tau_\epsilon^* = \inf\{t \geq 0 : \overline{X}_t - X_t \geq g_\epsilon(\overline{X}_t) \text{ and } \overline{X}_t > \log(K)\} \tag{3.2}$$

for some strictly positive solution g_ϵ of the differential equation

$$g_\epsilon'(s) = 1 - \frac{e^s Z^{(q)}(g_\epsilon(s))}{(e^s - K)qW^{(q)}(g_\epsilon(s))} \quad \text{on } (\log(K), \epsilon), \tag{3.3}$$

where $W^{(q)}$ and $Z^{(q)}$ are the so-called q-scale functions associated with X; see Section 3.3. In particular, we will find that the optimal stopping boundary $s \mapsto s - g_\epsilon(s)$ changes shape according to the path variation of X. This has already been observed in [29] in the case of the capped version of the Shepp–Shiryaev optimal stopping problem. It will also turn out that our solutions exhibit a pattern suggested by Peskir's maximality principle [31]. In fact, we will be able to give a reformulation of our main results in terms of Peskir's maximality principle.

We conclude this section with an overview of the paper. In Section 3.2 we give an application of our results in the context or pricing capped American lookback options. Section 3.3 is an auxiliary section introducing some necessary notation, followed by Section 3.4 which gives an overview of the different parameter regimes considered. Sections 3.5 and 3.7 deal with the "guess" part of our "guess and verify" technique and our main results, which correspond to the "verify" part, are presented in Section 3.6.

The proofs of our main results can then be found in Section 3.9. Finally, Section 3.8 provides an explicit example where X is a linear Brownian motion.

3.2 Application to pricing capped American lookback options

The aim of this section is to give some motivation for studying (3.1).

Consider a financial market consisting of a riskless bond and a risky asset. The value of the bond $B = \{B_t : t \geq 0\}$ evolves deterministically such that

$$B_t = B_0 e^{rt}, \quad B_0 > 0, r \geq 0, t \geq 0. \tag{3.4}$$

The price of the risky asset is modelled as the exponential spectrally negative Lévy process

$$S_t = S_0 e^{X_t}, \quad S_0 > 0, t \geq 0. \tag{3.5}$$

In order to guarantee that our model is free of arbitrage we will assume that $\psi(1) = r$. If $X_t = \mu t + \sigma W_t$, where $W = \{W_t : t \geq 0\}$ is a standard Brownian motion, we get the standard Black–Scholes model for the price of the asset. Extensive empirical research has shown that this (Gaussian) model is not capable of capturing certain features (such as skewness and heavy tails) which are commonly encountered in financial data, for example, returns on stocks. To accommodate for these problems, an idea, going back to [27], is to replace the Brownian motion as the model for the log-price by a general Lévy process X; cf. [9]. Here we will restrict ourselves to the model where X is given by a spectrally negative Lévy process. This restriction is mainly motivated by analytical tractability. It is worth mentioning, however, that Carr and Wu [8] as well as Madan and Schoutens [25] have offered empirical evidence to support the case of a model in which the risky asset is driven by a spectrally negative Lévy process for appropriate market scenarios.

A capped American lookback option is an option which gives the holder the right to exercise at any stopping time τ yielding payouts

$$L_\tau := e^{-\alpha\tau}\left[\left(M_0 \vee \sup_{0\leq u\leq\tau} S_u \wedge C\right) - K\right]^+, \quad C > M_0 \geq S_0, \alpha \geq 0.$$

The constant M_0 can be viewed as representing the "starting" maximum of the stock price (say, over some previous period $(-t_0, 0]$). The constant C can be interpreted as cap and moderates the payoff of the option. The value $C = \infty$ is also allowed and corresponds to no moderation at all. In this case we just get a normal American lookback option. Finally, when $C = \infty$ it is necessary to choose α strictly positive to guarantee that it is optimal to stop in finite time and that the value is finite; cf. Section 3.6.

Standard theory of pricing American-type options [41] directs one to solving the optimal stopping problem

$$V_r(M_0, S_0, C) := B_0 \sup_\tau \mathbb{E}\big[B_\tau^{-1} L_\tau\big] \tag{3.6}$$

where the supremum is taken over all $\mathbb{F}$-stopping times. In other words, we want to find a stopping time which optimises the expected discounted claim. The right-hand side of (3.6) may be rewritten as

$$\sup_\tau \mathbb{E}_{x,s}\big[e^{-q\tau}(e^{\overline{X}_\tau \wedge \epsilon} - K)^+\big],$$

where $q = r + \alpha, x = \log(S_0), s = \log(M_0)$ and $\epsilon = \log(C)$. Hence, we recognise (3.1) which is the problem of interest in this article.

3.3 Preliminaries

It is well known that a spectrally negative Lévy process X is characterised by its Lévy triplet (γ, σ, Π), where $\sigma \geq 0, \gamma \in \mathbb{R}$ and Π is a measure on $(-\infty, 0)$ satisfying the condition $\int_{(-\infty,0)}(1 \wedge x^2)\,\Pi(dx) < \infty$. By the Lévy–Itô decomposition, the latter may be represented in the form

$$X_t = \sigma B_t - \gamma t + X_t^{(1)} + X_t^{(2)}, \tag{3.7}$$

where $\{B_t : t \geq 0\}$ is a standard Brownian motion, $\{X_t^{(1)} : t \geq 0\}$ is a compound Poisson process with discontinuities of magnitude bigger than or equal to one and $\{X_t^{(2)} : t \geq 0\}$ is a square integrable martingale with discontinuities of magnitude strictly smaller than one and the three processes are mutually independent. In particular, if X is of bounded variation, the decomposition reduces to

$$X_t = \mathtt{d}t - \eta_t \tag{3.8}$$

where $\mathtt{d} > 0$, and $\{\eta_t : t \geq 0\}$ is a driftless subordinator. Further let

$$\psi(\theta) := \mathbb{E}\big[e^{\theta X_1}\big], \qquad \theta \geq 0,$$

be the Laplace exponent of X which is known to take the form

$$\psi(\theta) = -\gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,0)} \big(e^{\theta x} - 1 - \theta x 1_{\{x > -1\}}\big)\,\Pi(dx).$$

Moreover, ψ is strictly convex and infinitely differentiable and its derivative at zero characterises the asymptotic behaviour of X. Specifically, X drifts to $\pm\infty$ or oscillates according to whether $\pm\psi'(0+) > 0$ or, respectively, $\psi'(0+) = 0$. The right-inverse of ψ is defined by

$$\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}$$

for $q \geq 0$.

For any spectrally negative Lévy process having $X_0 = 0$ we introduce the family of martingales

$$\exp(cX_t - \psi(c)t), \tag{3.9}$$

defined for any $c \in \mathbb{R}$ for which $\psi(c) = \log \mathbb{E}[\exp(cX_1)] < \infty$, and further the corresponding family of measures $\{\mathbb{P}^c\}$ with Radon–Nikodym derivatives

$$\left.\frac{d\mathbb{P}^c}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp(cX_t - \psi(c)t). \tag{3.10}$$

For all such c the measure $\mathbb{P}^c_x$ will denote the translation of $\mathbb{P}^c$ under which $X_0 = x$. In particular, under $\mathbb{P}^c_x$ the process X is still a spectrally negative Lévy process; cf. Theorem 3.9 in [21].

A special family of functions associated with spectrally negative Lévy processes is that of scale functions (cf. [21]) which are defined as follows. For $q \geq 0$, the q-scale function $W^{(q)} : \mathbb{R} \longrightarrow [0, \infty)$ is the unique function whose restriction to $(0, \infty)$ is continuous and has Laplace transform

$$\int_0^\infty e^{-\theta x} W^{(q)}(x)\, dx = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q),$$

and is defined to be identically zero for $x \leq 0$. Equally important is the scale function $Z^{(q)} : \mathbb{R} \longrightarrow [1, \infty)$ defined by

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(z)\, dz.$$

The passage times of X below and above $k \in \mathbb{R}$ are denoted by

$$\tau_k^- = \inf\{t > 0 : X_t \leq k\} \quad \text{and} \quad \tau_k^+ = \inf\{t > 0 : X_t \geq k\}.$$

We will make use of the following two identities; cf. [2]. For $q \geq 0$ and $x \in (a, b)$ it holds that

$$\mathbb{E}_x\big[e^{-q\tau_b^+} I_{\{\tau_b^+ < \tau_a^-\}}\big] = \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)}, \tag{3.11}$$

$$\mathbb{E}_x\big[e^{-q\tau_a^-} I_{\{\tau_b^+ > \tau_a^-\}}\big] = Z^{(q)}(x-a) - W^{(q)}(x-a)\frac{Z^{(q)}(b-a)}{W^{(q)}(b-a)}. \tag{3.12}$$

For each $c \geq 0$ we denote by $W_c^{(q)}$ the q-scale function with respect to the measure $\mathbb{P}^c$. A useful formula (cf. Lemma 8.4 of [21]) linking scale functions under different measures is given by

$$W^{(q)}(x) = e^{\Phi(q)x} W_{\Phi(q)}(x) \tag{3.13}$$

for $q \geq 0$ and $x \geq 0$.

We conclude this section by stating some known regularity properties of scale functions; cf. [20].

Smoothness: For all $q \geq 0$,

$$W^{(q)}|_{(0,\infty)} \in \begin{cases} C^1(0,\infty), & \text{if } X \text{ is of bounded variation and } \Pi \text{ has no atoms,} \\ C^1(0,\infty), & \text{if } X \text{ is of unbounded variation and } \sigma = 0, \\ C^2(0,\infty), & \sigma > 0. \end{cases}$$

Continuity at the origin: For all $q \geq 0$,

$$W^{(q)}(0+) = \begin{cases} \mathtt{d}^{-1}, & \text{if } X \text{ is of bounded variation,} \\ 0, & \text{if } X \text{ is of unbounded variation.} \end{cases} \tag{3.14}$$

Right-derivative at the origin: For all $q \geq 0$,

$$W_+^{(q)\prime}(0+) = \begin{cases} \frac{q+\Pi(-\infty,0)}{\mathtt{d}^2}, & \text{if } \sigma = 0 \text{ and } \Pi(-\infty,0) < \infty, \\ \frac{2}{\sigma^2}, & \text{if } \sigma > 0 \text{ or } \Pi(-\infty,0) = \infty, \end{cases} \tag{3.15}$$

where we understand the second case to be $+\infty$ when $\sigma = 0$.

For technical reasons, we require for the rest of the paper that $W^{(q)}$ is in $C^1(0,\infty)$ [and hence $Z^{(q)} \in C^2(0,\infty)$]. This is ensured by henceforth assuming that Π is atomless whenever X is of bounded variation.

3.4 The different parameter regimes

Our analysis distinguishes between the following parameter regimes.

Main cases:

- $q > 0$ and $\epsilon \in (\log(K), \infty)$;
- $q > 0 \vee \psi(1)$ and $\epsilon = \infty$.

Special cases:

- $q = 0$ and $\epsilon \in (\log(K), \infty)$;
- $q = 0$ and $\epsilon = \infty$;

- $0 < q \leq \psi(1)$ and $\epsilon = \infty$.

3.5 Candidate solution for the main cases

The aim of this section is to derive a candidate solution to (3.1) for the main cases via the principle of smooth or continuous fit [1, 28, 32, 33].

We begin by heuristically motivating a class of stopping times in which we will look for the optimal stopping time under the assumption that $q > 0$ and $\epsilon \in (\log(K), \infty)$. Because $e^{-qt}(e^{\overline{X}_{t\wedge\epsilon}} - K)^+ = 0$ as long as $(X, \overline{X})$ is in the set

$$C_{II}^* := \{(x, s) \in E : s \leq \log(K)\},$$

it is intuitively clear that it is never optimal to stop the process $(X, \overline{X})$ in C_{II}^*. Moreover, as the process $(X, \overline{X})$ can only move upwards by climbing up the diagonal in the (x, s)-plane (Figure 3.1), it can only leave C_{II}^* through the point $(\log(K), \log(K))$. Therefore, one should not exercise until the process $(X, \overline{X})$ has hit $(\log(K), \log(K))$. It is possible that this never happens as X might escape to $-\infty$ before reaching level $\log(K)$. On the other hand, if the process $(X, \overline{X})$ is in $\{(x, s) \in E : s \geq \epsilon\}$, it should

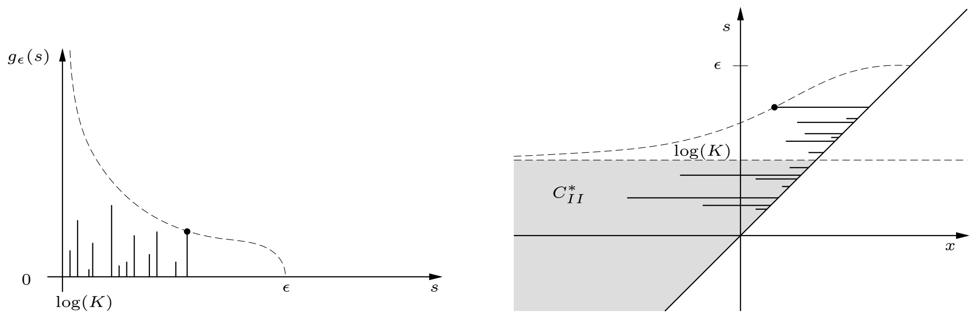

Fig. 3.1 *An illustration of a possible function g_ϵ and the corresponding stopping boundary $s \mapsto s - g_\epsilon(s)$. The vertical and horizontal lines are meant to schematically indicate the trace of an excursion of X away from the running maximum. The candidate optimal strategy τ_{g_ϵ} then consists of continuing if the height of the excursion away from the running maximum s does not exceed $g_\epsilon(s)$; otherwise we stop.*

be stopped immediately due to the discounting as the spatial part of the payout is deterministic and fixed at $e^\epsilon - K$ in value. The remaining case is when $(X, \overline{X})$ is in $\{(x, s) \in E : \log(K) < s < \epsilon\}$ in which case we can argue in the same way as described in [31], Section 3, page 6: The dynamics of the process $(X, \overline{X})$ are such that $\overline{X}$ remains constant at times when X is undertaking an excursion below $\overline{X}$. During such periods the discounting in the payoff is detrimental. One should therefore not allow X to drop too far below $\overline{X}$ in value as otherwise the time it will take X to recover to the value of its previous maximum will prove to be costly in terms of the gain on account of exponential discounting. More specifically, given a current value s, $s \in (\log(K), \epsilon)$, of

$\overline{X}$, there should be a point $g_\epsilon(s) > 0$ such that if the process X reaches or jumps below the value $s - g_\epsilon(s)$ we should stop instantly; see Figure 3.1. In more mathematical terms, we expect an optimal stopping time of the form

$$\tau_{g_\epsilon} := \inf\{t \geq 0 : \overline{X}_t - X_t \geq g_\epsilon(\overline{X}_t) \text{ and } \overline{X}_t > \log(K)\} \tag{3.16}$$

for some function $g_\epsilon : (\log(K), \epsilon) \to (0, \infty)$ such that $\lim_{s \uparrow \epsilon} g_\epsilon(s) = 0$ and $g_\epsilon(s) = 0$ for $s > \epsilon$. This is illustrated in Figure 3.1. For $(x, s) \in E$, we define the value function associated with τ_{g_ϵ} by

$$V_{g_\epsilon}(x, s) := \mathbb{E}_{x,s}\big[e^{-q\tau_{g_\epsilon}}(e^{\overline{X}_{\tau_{g_\epsilon}} \wedge \epsilon} - K)^+\big]. \tag{3.17}$$

Now suppose for the moment that we have chosen a function g_ϵ. The strong Markov property and Theorem 3.12 of [21] then imply that, for $(x, s) \in C^*_{II}$,

$$\begin{aligned} V_{g_\epsilon}(x, s) &= e^{-\Phi(q)(\log(K)-x)}\mathbb{E}_{\log(K),\log(K)}\big[e^{-q\tau_{g_\epsilon}}(e^{\overline{X}_{\tau_{g_\epsilon}} \wedge \epsilon} - K)\big] \\ &= e^{-\Phi(q)(\log(K)-x)} \lim_{s \downarrow \log(K)} V_{g_\epsilon}(s, s). \end{aligned}$$

This means that V_{g_ϵ} is determined on C^*_{II} as soon as V_{g_ϵ} is known on

$$E_1 := \{(x, s) \in E : s > \log(K)\}.$$

This leaves us with two key questions:

- How should one choose g_ϵ?
- Given g_ϵ, what does $V_{g_\epsilon}(x, s)$ look like for $(x, s) \in E_1$?

These questions can be answered heuristically in the spirit of the method applied in Section 3 of [31], but adapted to the case when X is a spectrally negative Lévy processes (rather than a diffusion). More precisely, as we shall see in more detail in Section 3.7, the general theory of optimal stopping [33, 42] together with the principle of smooth or continuous fit [1, 28, 32, 33] suggest that g_ϵ should be a solution to the ordinary differential equation

$$g'_\epsilon(s) = 1 - \frac{e^s Z^{(q)}(g_\epsilon(s))}{(e^s - K)qW^{(q)}(g_\epsilon(s))} \quad \text{on } (\log(K), \epsilon), \tag{3.18}$$

and that

$$V_{g_\epsilon}(x, s) = (e^{s \wedge \epsilon} - K)Z^{(q)}(x - s + g_\epsilon(s)), \qquad (x, s) \in E_1.$$

Note that there might be many solutions to (3.18) without an initial/boundary condition. However, we are specifically looking for the solution satisfying $\lim_{s \uparrow \epsilon} g_\epsilon(s) = 0$. Summing up, we have suggested/found a candidate stopping time τ_{g_ϵ} and candidate value function V_{g_ϵ}.

As for the case $q > 0 \vee \psi(1)$ and $\epsilon = \infty$, one might let ϵ tend to infinity which informally yields a candidate stopping time of the form (3.16) with g_ϵ replaced with g_∞, where g_∞ should satisfy (3.18), but on $(\log(K), \infty)$ instead of $(\log(K), \epsilon)$. The corresponding value function V_{g_∞} is then expected to be of the form

$$V_{g_\infty}(x,s) = (e^s - K)Z^{(q)}(x - s + g_\infty(s)), \qquad (x,s) \in E_1.$$

If we are to identify g_∞ as a solution to (3.18), we need an initial/boundary condition which in this case can be found as follows. For $s \gg K$ the payoff in (3.1) resembles the payoff of the Shepp–Shiryaev optimal stopping problem [2, 21, 29, 38] and hence we expect $s \mapsto s - g_\infty(s)$ to look similar to the optimal boundary of the Shepp–Shiryaev optimal stopping problem for $s \gg K$. Therefore, we expect that $\lim_{s\uparrow\infty} g_\infty(s) = k^*$, where $k^* > 0$ is the unique root of the equation $Z^{(q)}(s) - qW^{(q)}(s) = 0$; cf. [2, 29].

These heuristic arguments are made rigorous in the next section.

3.6 Main results

3.6.1 The different solutions of the ODE

In this subsection we investigate, for $q > 0$, the solutions of the ordinary differential equation

$$g'(s) = 1 - \frac{e^s Z^{(q)}(g(s))}{(e^s - K)qW^{(q)}(g(s))} \tag{3.19}$$

whose graph lies in

$$U := \{(s,H) \in \mathbb{R}^2 : s > \log(K), H > 0\}.$$

As already hinted in the previous section, these solutions will play an important role. But before we analyse (3.19), recall that the requirement $W^{(q)}(0+) < q^{-1}$ is the same as asking that either X is of unbounded variation or X is of bounded variation with $\mathtt{d} > q$. Similarly, the condition $W^{(q)}(0+) \geq q^{-1}$ means that X is of bounded variation with $0 < \mathtt{d} \leq q$. Also note that $W^{(q)}(0+) \geq q^{-1}$ implies $q \geq \mathtt{d} > \psi(1)$.

The existence of solutions to (3.19) and their behaviour under the different parameter regimes is summarised in the next result.

Lemma 3.1. *Assume that $q > 0$. For $\epsilon \in (\log(K), \infty)$, we have the following.*

(a) If $q > \psi(1)$ and $W^{(q)}(0+) < q^{-1}$, then there exists a unique solution $g_\epsilon : (\log(K), \epsilon) \to (0, \infty)$ to (3.19) such that $\lim_{s\uparrow\epsilon} g_\epsilon(s) = 0$.

(b) If $W^{(q)}(0+) \geq q^{-1}$ [and hence $q > \psi(1)$], then there exists a unique solution

$g_\epsilon : (\log(K), \epsilon \wedge \beta) \to (0,\infty)$ *to* (3.19) *such that* $\lim_{s \uparrow \epsilon \wedge \beta} g_\epsilon(s) = 0$. *Here, the constant* β *is given by* $\beta := \log\big(K(1 - \mathtt{d}/q)^{-1}\big) \in (0, \infty]$.

(c) *If* $q \le \psi(1)$, *then there exists a unique solution* $g_\epsilon : (\log(K), \epsilon) \to (0, \infty)$ *to* (3.19) *such that* $\lim_{s \uparrow \epsilon} g_\epsilon(s) = 0$.

For $\epsilon = \infty$, *we have in particular:*

(d) *If* $q > \psi(1)$ *and* $W^{(q)}(0+) < q^{-1}$, *then there exists a unique solution* $g_\infty : (\log(K), \infty) \to (0, \infty)$ *to* (3.19) *such that* $\lim_{s \uparrow \infty} g_\infty(s) = k^*$, *where the constant* $k^* \in (0, \infty)$ *is the unique root of* $Z^{(q)}(s) - qW^{(q)}(s) = 0$.

(e) *If* $W^{(q)}(0+) \ge q^{-1}$ [*and hence* $q > \psi(1)$], *then there exists a unique solution* $g_\infty : (\log(K), \beta) \to (0, \infty)$ *to* (3.19) *such that* $\lim_{s \uparrow \beta} g_\infty(s) = 0$. *The constant* β *is as in* (b).

Moreover, all the solutions mentioned in (a)–(e) *tend to* $+\infty$ *as* $s \downarrow \log(K)$. *Also note that if* $\beta \le \epsilon$ *then the solutions in* (b) *and* (e) *coincide. Finally, the qualitative behaviour of the solutions of* (3.19) *is displayed in Figures 3.2–3.4.*

We will henceforth use the following convention: If a solution to (3.19) is not defined for all $s \in (\log(K), \infty)$, we extend it to $(\log(K), \infty)$ by setting it equal to zero wherever it is not defined (typically $s \ge \epsilon$).

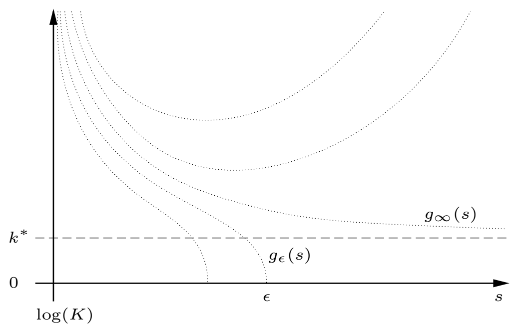

Fig. 3.2 *A schematic illustration of the solutions of* (3.19) *when* $q > \psi(1)$ *and* $W^{(q)}(0+) = 0$. *If* $q > \psi(1)$ *and* $W^{(q)}(0+) \in (0, q^{-1})$, *then the solutions look the same except that they hit zero with finite gradient (since* $W^{(q)}(0+) > 0$*).*

3.6.2 Verification of the case $q > 0$ and $\epsilon \in (\log(K), \infty)$

We are now in a position to state our first main result.

Theorem 3.2. *Suppose that* $q > 0$ *and* $\epsilon \in (\log(K), \infty)$. *Then the solution to* (3.1) *is given by*

$$V_\epsilon^*(x,s) = \begin{cases} (e^{s \wedge \epsilon} - K) Z^{(q)}(x - s + g_\epsilon(s)), & (x,s) \in E_1, \\ e^{-\Phi(q)(\log(K) - x)} A_\epsilon, & (x,s) \in C_{II}^*, \end{cases} \tag{3.20}$$

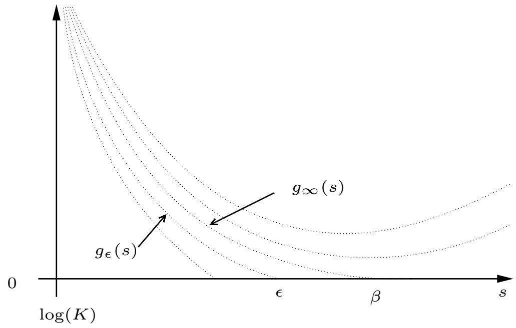

Fig. 3.3 *A schematic illustration of the solutions of* (3.19) *when* $W^{(q)}(0+) \geq q^{-1}$ *and* $\epsilon < \beta$.

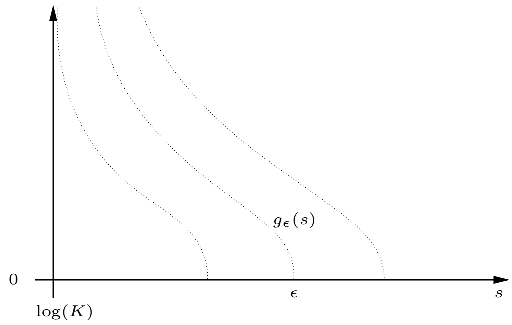

Fig. 3.4 *A schematic illustration of the solutions of* (3.19) *when* $q \leq \psi(1)$ *and* $W^{(q)}(0+) = 0$. *If* $q \leq \psi(1)$ *and* $W^{(q)}(0+) \in (0, q^{-1})$, *then the solutions look the same except that they hit zero with finite gradient (since* $W^{(q)}(0+) > 0$*).*

with value $A_\epsilon \in (0, \infty)$ *given by*

$$A_\epsilon := \mathbb{E}_{\log(K),\log(K)}\left[e^{-q\tau_\epsilon^*}(e^{\overline{X}_{\tau_\epsilon^*\wedge\epsilon}} - K)\right] = \lim_{s\downarrow\log(K)} (e^s - K)Z^{(q)}(g_\epsilon(s)),$$

and optimal stopping time

$$\tau_\epsilon^* = \inf\{t \geq 0 \, : \, \overline{X}_t - X_t \geq g_\epsilon(\overline{X}_t) \text{ and } \overline{X}_t > \log(K)\}, \tag{3.21}$$

where g_ϵ *is given in Lemma 3.1. Moreover,*

$$\mathbb{P}_{x,s}[\tau_\epsilon^* < \infty] = \begin{cases} 1, & \text{if } \psi'(0+) \geq 0, \\ e^{-\Phi(q)(\log(K)-x)}, & \text{if } \psi'(0+) < 0. \end{cases}$$

Remark 3.3. With the help of excursion theory, it is possible to obtain an alternative representation for $V_\epsilon^*(s,s)$ for $\log(K) \leq s < \epsilon \wedge \beta$; see Subsection 3.10.2 for the relevant computations. Specifically, under the same assumptions as in Theorem 3.2, we have

$$V_\epsilon^*(s,s) \;=\; \int_s^{\epsilon\wedge\beta} (e^t - K)\hat{f}(g_\epsilon(t)) \exp\left(- \int_s^t \frac{W^{(q)\prime}(g_\epsilon(u))}{W^{(q)}(g_\epsilon(u))}\, du \right) dt \tag{3.22}$$

$$+(e^{\epsilon\wedge\beta}-K)\exp\Big(-\int_s^{\epsilon\wedge\beta}\frac{W^{(q)\prime}(g_\epsilon(u))}{W^{(q)}(g_\epsilon(u))}\,du\Big)$$

where $\hat{f}(u)=\frac{Z^{(q)}(u)W^{(q)\prime}(u)}{W^{(q)}(u)}-qW^{(q)}(u)$ and we set $\beta=\infty$ unless $W^{(q)}(0+)\geq q^{-1}$, in which case we take $\beta=\log\big(K(1-\mathtt{d}/q)^{-1}\big)$ as before. In particular, we can identify the value A_ϵ as the above expression, setting $s=\log(K)$.

Remark 3.4. Assume that $(x,s)\in E$ such that $\log(K)<s<\epsilon\wedge\beta$ and set $\beta=\infty$ unless $W^{(q)}(0+)\geq q^{-1}$. The excursion theoretic calculation that led to (3.22) contains an additional result, namely that $\mathbb{P}_{x,s}[\tau_\epsilon^*=\tau_{\epsilon\wedge\beta}^+]\in(0,1)$. To see this, note that it follows from the computation in Subsection 3.10.2 that

$$\mathbb{E}_{x,s}[e^{-q\tau_\epsilon^*}1_{\{\tau_\epsilon^*=\tau_{\epsilon\wedge\beta}^+\}}]=\exp\Big(-\int_s^{\epsilon\wedge\beta}\frac{W^{(q)\prime}(g_\epsilon(u))}{W^{(q)}(g_\epsilon(u))}\,du\Big).$$

Hence, the claim follows provided the integral on the right-hand side is strictly positive and finite. Indeed, changing variables according to $v=g_\epsilon(u)$ and using the explicit form of g_ϵ' gives

$$\int_s^{\epsilon\wedge\beta}\frac{W^{(q)\prime}(g_\epsilon(u))}{W^{(q)}(g_\epsilon(u))}\,du=\int_0^{g_\epsilon(s)}\frac{W^{(q)\prime}(v)}{y(v)}\,dv,$$

where $y(v):=\frac{e^{g_\epsilon^{-1}(v)}}{q(e^{g_\epsilon^{-1}(v)}-K)}Z^{(q)}(v)-W^{(q)}(v)$ and g_ϵ^{-1} is the inverse of g_ϵ. Using (3.14) one may then deduce that $y(v)$ is bounded on $(0,g_\epsilon(s)]$ by a constant, say $C>0$, and that

$$\int_0^{g_\epsilon(s)}\frac{W^{(q)\prime}(v)}{y(v)}\,dv\leq C^{-1}\int_0^{g_\epsilon(s)}W^{(q)\prime}(v)\,dv=C^{-1}(W^{(q)}(g_\epsilon(s))-W^{(q)}(0)).$$

This proves the claim. A similar phenomenon in a different context has been observed in [22].

Let us now discuss some consequences of Theorem 3.2. Firstly, it shows that if $\psi'(0+)\geq 0$ the stopping problem has an optimal solution in the smaller class of $[0,\infty)$-valued $\mathbb{F}$-stopping times. On the other hand, if there is a possibility that the process X drifts to $-\infty$ before reaching $\log(K)$, which occurs exactly when $\psi'(0+)<0$, then the probability that τ_ϵ^* is infinite is strictly positive and τ_ϵ^* is only optimal in the class of $[0,\infty]$-valued $\mathbb{F}$-stopping times.

Secondly, when $W^{(q)}(0+)\geq q^{-1}$ or, equivalently, X is of bounded variation with $q\geq\mathtt{d}$, the result shows that $g_\epsilon(s)$ hits the origin at $\epsilon\wedge\beta$, where $\beta=\log\big(K(1-\mathtt{d}/q)^{-1}\big)$; see Figure 3.5. Intuitively speaking, if $\beta<\epsilon$, the discounting is so strong that it is best to stop even before reaching level ϵ. On the other hand, if $\beta\geq\epsilon$, it would be better to wait longer, but as there is a cap we are forced to stop as soon as we have reached it.

As already observed in [29], it is also the case in our setting that, if $W^{(q)}(0+)<q^{-1}$, the slope of g_ϵ at ϵ [and hence the shape of the optimal boundary $s\mapsto s-g_\epsilon(s)$] changes

according to the path variation of X. Specifically, it holds that

$$\lim_{s\uparrow\epsilon} g'_\epsilon(s) = \begin{cases} -\infty, & \text{if } X \text{ is of unbounded variation,} \\ 1 - \frac{e^\epsilon \mathtt{d}}{(e^\epsilon - K)q}, & \text{if } X \text{ is of bounded variation.} \end{cases}$$

Next, introduce the sets

$$\begin{aligned} C_I^* = C_{I,\epsilon}^* &:= \{(x,s) \in E \,:\, s > \log(K), x > s - g_\epsilon(s)\}, \\ D^* = D_\epsilon^* &:= \{(x,s) \in E \,:\, s > \log(K), x \leq s - g_\epsilon(s)\}. \end{aligned} \tag{3.23}$$

Two examples of g_ϵ and the corresponding continuation region $C_I^* \cup C_{II}^*$ and stopping region D^* are pictorially displayed in Figure 3.5.

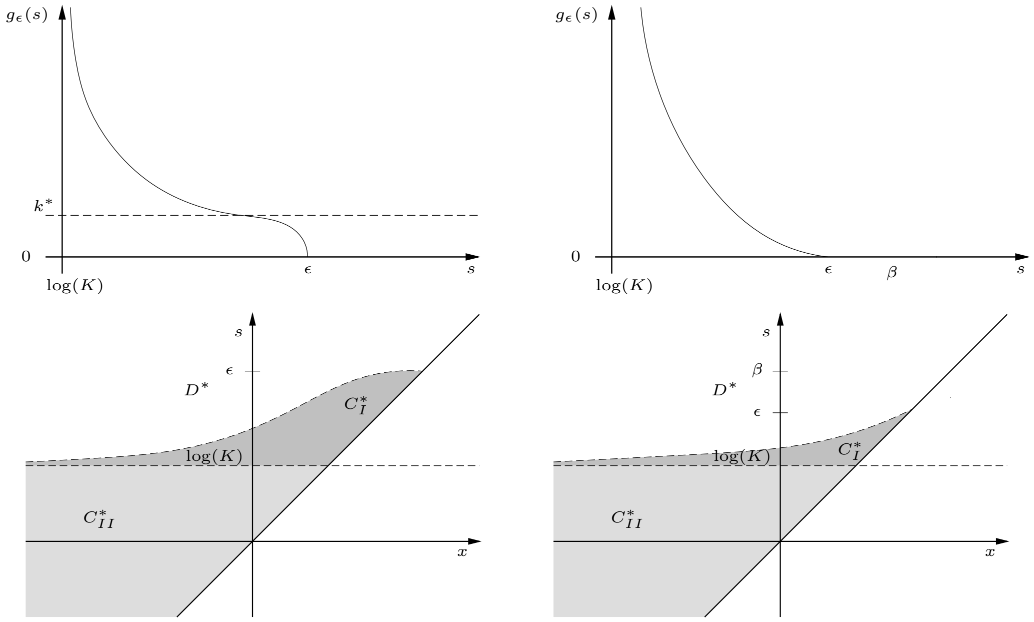

Fig. 3.5 *For the two pictures on the left-hand side it is assumed that $q > 0$ and $W^{(q)}(0+) = 0$, whereas on the right-hand side it is assumed that $q > 0$, $W^{(q)}(0+) \geq q^{-1}$ and $\epsilon < \beta$.*

3.6.3 Verification of the case $q > 0 \vee \psi(1)$ and $\epsilon = \infty$

The analogous result to Theorem 3.2 reads as follows.

Theorem 3.5. *Suppose that $q > 0 \vee \psi(1)$ and $\epsilon = \infty$. Then the solution to (3.1) is given by*

$$V_\infty^*(x,s) = \begin{cases} (e^s - K)Z^{(q)}(x - s + g_\infty(s)), & (x,s) \in E_1, \\ e^{-\Phi(q)(\log(K)-x)} A_\infty, & (x,s) \in C_{II}^*, \end{cases} \tag{3.24}$$

with value $A_\infty \in (0,\infty)$ *given by*

$$A_\infty := \mathbb{E}_{\log(K),\log(K)}\big[e^{-q\tau^*_\infty}(e^{\overline{X}_{\tau^*_\infty}} - K)\big] = \lim_{s\downarrow\log(K)} (e^s - K)Z^{(q)}(g_\infty(s)),$$

and optimal stopping time

$$\tau^*_\infty = \inf\{t \geq 0 \,:\, \overline{X}_t - X_t \geq g_\infty(\overline{X}_t) \text{ and } \overline{X}_t > \log(K)\}, \tag{3.25}$$

where g_∞ *is given in Lemma 3.1. Moreover,*

$$\mathbb{P}_{x,s}[\tau^*_\infty < \infty] = \begin{cases} 1, & \text{if } \psi'(0+) \geq 0,\\ e^{-\Phi(q)(\log(K)-x)}, & \text{if } \psi'(0+) < 0. \end{cases}$$

Remark 3.6. As in Remark 3.3, $V^*_\infty(s,s)$ can be identified as the integral in (3.22) with $\epsilon = \infty$ for $\log(K) \leq s < \beta$ in the case $W^{(q)}(0+) \geq q^{-1}$. Otherwise it is identified as

$$V^*_\infty(s,s) = \int_s^\infty (e^t - K)\hat{f}(g_\infty(t)) \exp\left(-\int_s^t \frac{W^{(q)\prime}(g_\infty(u))}{W^{(q)}(g_\infty(u))}\, du\right) dt,$$

where $\hat{f}(u) = \frac{Z^{(q)}(u)W^{(q)\prime}(u)}{W^{(q)}(u)} - qW^{(q)}(u)$ as before; see again the computations in Subsection 3.10.2. In particular, one obtains an alternative expression for A_∞.

Similarly to Theorem 3.2 one sees again that if $\psi'(0+) \geq 0$ there is an optimal stopping time in the class of all $[0,\infty)$-valued $\mathbb{F}$-stopping times. Furthermore, let $C^*_I = C^*_{I,\infty}$ and $D^* = D^*_\infty$ denote the same sets as in (3.23), but with g_∞ instead of g_ϵ. The (qualitative) behaviour of g_∞ and the resulting shape of the continuation region $C^*_I \cup C^*_{II}$ and stopping region D^* are illustrated in Figure 3.6.

3.6.4 The special cases

In this subsection we deal with the cases that have not been considered yet, that is, the special cases; see Section 3.4.

Lemma 3.7. *Suppose that* $q = 0$ *and* $\epsilon \in (\log(K),\infty)$*.*

(a) When $\psi'(0+) < 0$ *and* $\Phi(0) \neq 1$*, then the solution to* (3.1) *is given by*

$$V^*_\epsilon(x,s) = \begin{cases} e^\epsilon - K, & s \geq \epsilon,\\ e^s - K + \frac{e^{x\Phi(0)}}{\Phi(0)-1}\big(e^{s(1-\Phi(0))} - e^{\epsilon(1-\Phi(0))}\big), & \log(K) \leq s < \epsilon,\\ e^{-\Phi(0)(\log(K)-x)}A_\epsilon, & s < \log(K), \end{cases}$$

where $A_\epsilon := \frac{K^{\Phi(0)}(K^{1-\Phi(0)} - e^{\epsilon(1-\Phi(0))})}{\Phi(0)-1}$*, and* $\tau^*_\epsilon = \tau^+_\epsilon$*. If* $\Phi(0) = 1$*, then the middle term on the right-hand side in the expression for* $V^*_\epsilon(x,s)$ *has to be replaced by*

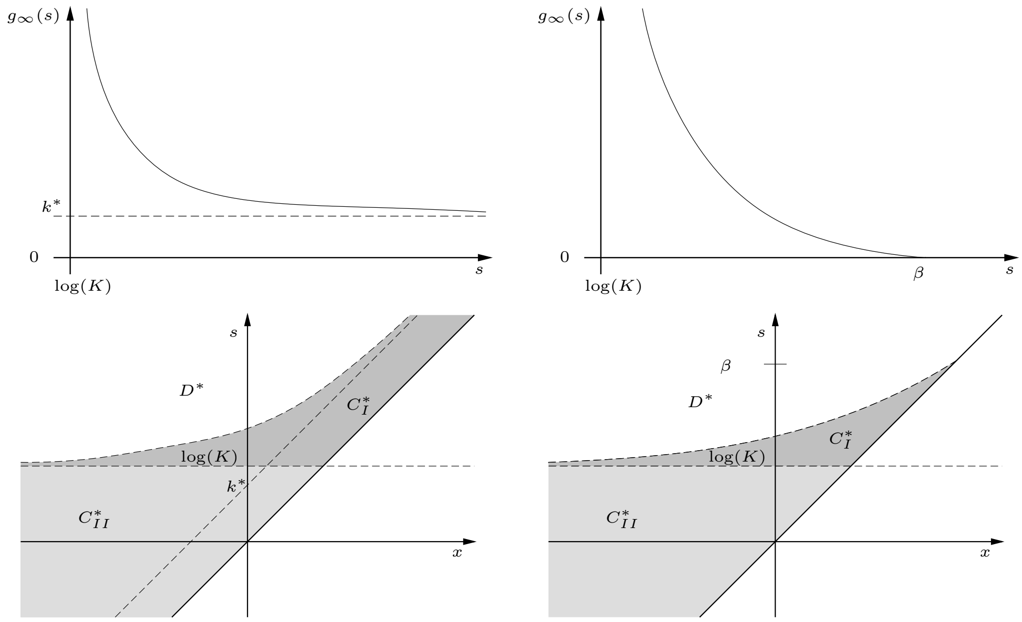

Fig. 3.6 *For the two pictures on the left it is assumed that* $q > 0 \vee \psi(1)$ *and* $W^{(q)}(0+) < q^{-1}$*, whereas on the right it is assumed that* $q > 0 \vee \psi(1)$ *and* $W^{(q)}(0+) \geq q^{-1}$.

$e^s - K + e^x(\epsilon - s)$ *and* A_ϵ *by* $K(\epsilon - \log(K))$.

(b) *When* $\psi'(0+) \geq 0$*, then solution to* (3.1) *is given by* $V_\epsilon^* \equiv e^\epsilon - K$ *and* $\tau_\epsilon^* = \tau_\epsilon^+$.

Note that although the optimal stopping time is the same in both parts of Lemma 3.7, in (a) it attains the value infinity with positive probability, whereas in (b) this happens with probability zero. Hence, in (b) there is actually an optimal stopping time in the class of $[0, \infty)$-valued $\mathbb{F}$-stopping times.

Lemma 3.8. *Suppose that* $\epsilon = \infty$.

(a) *Assume that* $q = 0$*. If* $\psi'(0+) < 0$ *and* $\Phi(0) > 1$*, we have*

$$V_\infty^*(x,s) = \begin{cases} e^s - K + \frac{e^{x\Phi(0)+s(1-\Phi(0))}}{\Phi(0)-1}, & s \geq \log(K), \\ e^{-\Phi(0)(\log(K)-x)}\frac{K}{\Phi(0)-1}, & s < \log(K), \end{cases} \tag{3.26}$$

and the optimal stopping time is given by $\tau_\infty^* = \infty$*. On the other hand, if either* $\psi'(0+) < 0$ *and* $\Phi(0) \leq 1$ *or* $\psi'(0+) \geq 0$*, then* $V_\infty^*(x,s) \equiv \infty$ *and* $\tau_\infty^* = \infty$.

(b) *When* $0 < q \leq \psi(1)$*, we have* $V_\infty^*(x,s) \equiv \infty$.

The second part in the Lemma 3.8 is intuitively clear. If $0 < q \leq \psi(1)$, then the average upwards motion of X (and hence $\overline{X}$) is stronger than the discounting. On the other hand, $\psi'(0+) < 0$ means that X will drift to $-\infty$ and thus X will eventually attain its maximum (in the pathwise sense). Of course, we do not know when this happens, but since there is no discounting we do not mind waiting forever. The other cases in Lemma 3.8 have a similar interpretation.

3.6.5 The maximality principle

The maximality principle was understood as a powerful tool to solve a class of stopping problems for the maximum process associated with a one-dimensional time-homogeneous diffusion [31]. Although we work with a different class of processes, our main results [Lemma 3.1, Theorem 3.2, Theorem 3.5 and Lemma 3.8(b)] can be reformulated through the maximality principle.

Lemma 3.9. *Suppose that $q > 0$ and $\epsilon \in (\log(K), \infty)$. Define the set*

$$\mathcal{S} := \big\{ g|_{(\log(K),\epsilon)} \,\big|\, g \textit{ is a solution to } (3.19) \textit{ defined at least on } (\log(K), \epsilon) \big\}.$$

Let g_ϵ^* be the minimal solution in $\mathcal{S}$. Then the solution to (3.1) *is given by* (3.20) *and* (3.21) *with g_ϵ replaced by g_ϵ^*.*

In the case that there is a cap, it cannot happen that the value function becomes infinite. This changes when there is no cap.

Lemma 3.10. *Let $q > 0$ and $\epsilon = \infty$.*

1. *Let g_∞^* denote the minimal solution to* (3.19) *which does not hit zero (whenever such a solution exists). Then the solution to* (3.1) *is given by* (3.24) *and* (3.25) *with g_∞ replaced by g_∞^*.*

2. *If every solution to* (3.19) *hits zero, then the value function in* (3.1) *is given by $V_\infty^*(x, s) \equiv \infty$.*

Remark 3.11.

1. We select the minimal solution rather than the maximal one as in [31], since our functions $g_\epsilon(s)$ are the analogue of $s - g_\epsilon(s)$ in [31].

2. The "right" boundary conditions which were used to select g_ϵ and g_∞ from the class of solutions of (3.19) (see Section 3.5) are not used in the formulation of Lemmas 3.9 and 3.10. In fact, by choosing the minimal solution, it follows as a consequence that g_ϵ^* and g_∞^* have exactly the "right" boundary conditions. Put differently, the "minimality principle" is a means of selecting the "good" solution from the class of all solutions of (3.19). This is a reformulation of [31] in our specific setting.

3. A similar observation is contained in [10], but in a slightly different setting.

4. If $\epsilon = \infty$, the solutions to (3.19) that hit zero correspond to the so-called "bad-good" solutions in [31]; "bad" since they do not give the optimal boundary, "good" as they can be used to approximate the optimal boundary.

3.7 Guess via principle of smooth or continuous fit

Our proofs are essentially based on a "guess and verify" technique. Here we provide the missing details from Section 3.5 on how to "guess" a candidate solution. The following presentation is an adaptation of the argument of Section 3 of [31] to our setting.

Assume that $q > 0$ and $\epsilon \in (\log(K), \epsilon)$. Let $g_\epsilon : (\log(K), \epsilon) \to (0, \infty)$ be continuously differentiable and define the stopping time τ_{g_ϵ} as in (3.16) and let V_{g_ϵ} be as in (3.17). For simplicity assume from now on that X is of unbounded variation (if X is of bounded variation a similar argument based on the principle of continuous fit applies, see [1, 32, 33]). From the general theory of optimal stopping, [33, 42], we would expect that V_{g_ϵ} satisfies for $(x, s) \in E$ such that $\log(K) < s < \epsilon$ the system

$$\begin{aligned} \Gamma V_{g_\epsilon}(x,s) &= qV_{g_\epsilon}(x,s) && \text{for } s - g_\epsilon(s) < x < s \text{ with } s \text{ fixed,} \\ \tfrac{\partial V_{g_\epsilon}}{\partial s}(x,s)\big|_{x=s-} &= 0 && \text{(normal reflection),} \\ V_{g_\epsilon}(x,s)|_{x=(s-g_\epsilon(s))+} &= e^s - K && \text{(instantaneous stopping),} \end{aligned} \tag{3.27}$$

where Γ is the infinitesimal generator of the process X under $\mathbb{P}$. For functions $h \in C_0^\infty(\mathbb{R})$ and $z \in \mathbb{R}$, it is given by

$$\begin{aligned} \Gamma h(z) &= -\gamma h'(z) + \frac{\sigma^2}{2} h''(z) \\ &\quad + \int_{(-\infty,0)} \big(h(z+y) - h(z) - yh'(z)1_{\{y \geq -1\}}\big)\,\Pi(dy). \end{aligned} \tag{3.28}$$

Here $C_0^\infty(\mathbb{R})$ denotes the class of infinitely differentiable functions h on $\mathbb{R}$ such that h and its derivatives vanish at infinity. In addition, the principle of smooth fit (cf. [28, 33]) suggests that the system above should be complemented by

$$\frac{\partial V_{g_\epsilon}}{\partial x}(x,s)\big|_{x=(s-g_\epsilon(s))+} = 0 \quad \text{(smooth fit).} \tag{3.29}$$

Note that the smooth fit condition is not necessarily part of the general theory, it is imposed since by the "rule of thumb" outlined in Section 7 in [1] one suspects it should hold in this setting because of path regularity. This belief will be vindicated when we show that system (3.27) and (3.29) leads to the desired solution. Applying the strong Markov property at τ_s^+ and using (3.11) and (3.12) shows that

$$V_{g_\epsilon}(x,s) = (e^s - K)\bigg(Z^{(q)}(x - s + g_\epsilon(s)) - W^{(q)}(x - s + g_\epsilon(s))\frac{Z^{(q)}(g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))}\bigg)$$

$$+\frac{W^{(q)}(x-s+g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))}V_{g_\epsilon}(s,s).$$

Furthermore, the smooth fit condition (3.29) implies

$$\begin{aligned} 0 &= \lim_{x\downarrow s-g_\epsilon(s)} \frac{\partial V_{g_\epsilon}}{\partial x}(x,s) \\ &= \lim_{x\downarrow s-g_\epsilon(s)} \frac{W^{(q)\prime}(x-s+g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))}\big(V_{g_\epsilon}(s,s)-(e^s-K)Z^{(q)}(g_\epsilon(s))\big). \end{aligned}$$

By (3.15) the first factor tends to a strictly positive value or infinity which shows that $V_{g_\epsilon}(s,s) = (e^s-K)Z^{(q)}(g_\epsilon(s))$. This would mean that for all $(x,s)\in E$ such that $\log(K)<s<\epsilon$ we have

$$V_{g_\epsilon}(x,s) = (e^s-K)Z^{(q)}(x-s+g_\epsilon(s)). \tag{3.30}$$

Finally, using the normal reflection condition shows that our candidate function g_ϵ should satisfy the first-order differential equation

$$g_\epsilon'(s) = 1-\frac{e^sZ^{(q)}(g_\epsilon(s))}{(e^s-K)qW^{(q)}(g_\epsilon(s))} \quad \text{on } (\log(K),\epsilon). \tag{3.31}$$

3.8 Example

Suppose that $X_t = (\mu-\frac{1}{2}\sigma^2)t+\sigma W_t$, $t\geq 0$, where $\mu\in\mathbb{R},\sigma>0$ and W_t, $t\geq 0$, is a standard Brownian motion. It is well known that in this case the scale functions are given by

$$W^{(q)}(x) = \frac{2}{\sigma^2\delta}e^{\gamma x}\sinh(\delta x) \quad \text{and} \quad Z^{(q)}(x) = e^{\gamma x}\cosh(\delta x)-\frac{\gamma}{\delta}e^{\gamma x}\sinh(\delta x),$$

on $x\geq 0$, where $\delta(q)=\delta=\sqrt{(\frac{\mu}{\sigma^2}-\frac{1}{2})^2+\frac{2q}{\sigma^2}}$ and $\gamma=\frac{1}{2}-\frac{\mu}{\sigma^2}$. Additionally, let $\gamma_1 := \gamma-\delta$ and $\gamma_2 := \gamma+\delta = \Phi(q)$ both of which are the roots of the quadratic equation $\frac{\sigma^2}{2}\theta^2+(\mu-\frac{\sigma^2}{2})\theta-q=0$ and satisfy $\gamma_2>0>\gamma_1$. Using the specific form of $Z^{(q)}$ and $W^{(q)}$ it is straightforward to obtain the following result.

Lemma 3.12. *Let $\epsilon=\infty$ and assume that $q>\psi(1)$ or, equivalently, $q>\mu$. Then the solution to* (3.1) *is given by*

$$V_\infty^*(x,s) = \begin{cases} e^s-K, & (x,s)\in D^*, \\ \frac{e^s-K}{\gamma_2-\gamma_1}\big(\gamma_2 e^{\gamma_1(x-s+g_\infty(s))}-\gamma_1 e^{\gamma_2(x-s+g_\infty(s))}\big), & (x,s)\in C_I^*, \\ e^{-\gamma_2(\log(K)-x)}\frac{\gamma_1}{\gamma_1-\gamma_2}A_\infty, & (x,s)\in C_{II}^*, \end{cases}$$

where $A_\infty = \lim_{s\downarrow\log(K)}(e^s-K)e^{\gamma_2 g_\infty(s)}$. The corresponding optimal strategy is given *by* $\tau_\infty^* := \inf\{t \geq 0 : \overline{X}_t - X_t \geq g_\infty(\overline{X}_t)$ *and* $\overline{X}_t > \log(K)\}$, *where* g_∞ *is the unique strictly positive solution to the differential equation*

$$g'_\infty(s) = 1 - \frac{e^s}{e^s - K}\left(\frac{\gamma_2^{-1}e^{\gamma_2 g_\infty(s)} - \gamma_1^{-1}e^{\gamma_1 g_\infty(s)}}{e^{\gamma_2 g_\infty(s)} - e^{\gamma_1 g_\infty(s)}}\right) \quad on\ (\log(K), \infty)$$

such that $\lim_{s\uparrow\infty} g_\infty(s) = k^*$, *where the constant* $k^* \in (0, \infty)$ *is given by*

$$k^* = \frac{1}{\gamma_2 - \gamma_1}\log\left(\frac{1-\gamma_1^{-1}}{1-\gamma_2^{-1}}\right).$$

Lemma 3.12 is nothing else than Theorem 2.5 of [30] or Theorem 1 of [17] which shows that our results are consistent with the existing literature.

3.9 Proof of main results

Proof of Lemma 3.1. Recall that $q > 0$. We distinguish three cases:

- $q > \psi(1)$ and $W^{(q)}(0+) < q^{-1}$;
- $W^{(q)}(0+) \geq q^{-1}$ (and hence $q > \psi(1)$, see beginning of Subsection 3.6.1);
- $\psi(1) \geq q$.

The case $q > \psi(1)$ **and** $W^{(q)}(0+) < q^{-1}$**:** The assumptions imply that the function $H \mapsto Z^{(q)}(H) - qW^{(q)}(H)$ is strictly decreasing on $(0, \infty)$ and has a unique root $k^* \in (0, \infty)$; cf. Proposition 2.1 of [29]. In particular, $\frac{Z^{(q)}(H)}{qW^{(q)}(H)} > 1$ for $H < k^*$, $\frac{Z^{(q)}(H)}{qW^{(q)}(H)} < 1$ for $H > k^*$ and $\frac{Z^{(q)}(k^*)}{qW^{(q)}(k^*)} = 1$. It is also known that the mapping $H \mapsto \frac{Z^{(q)}(H)}{qW^{(q)}(H)}$ is strictly decreasing on $(0, \infty)$ (cf. first remark in Section 3 of [34]) and that $\lim_{H\to\infty} \frac{Z^{(q)}(H)}{qW^{(q)}(H)} = \Phi(q)^{-1}$; cf. Lemma 1 of [2]. We will make use of these properties below.

The ordinary differential equation (3.19) has, at least locally, a unique solution for every starting point $(s_0, H_0) \in U$ by the Picard–Lindelöf theorem (cf. Theorem 1.1 in [18]), on account of local Lipschitz continuity of the field. It is well known that these unique local solutions can be extended to their maximal interval of existence; cf. Theorem 3.1 of [18]. Hence, whenever we speak of a solution to (3.19) from now on, we implicitly mean the unique maximal one. In order to analyse (3.19), we sketch its direction field based on various qualitative features of the ordinary differential equation. The 0-isocline, that is, the points (s, H) in U satisfying $1 - \frac{e^s Z^{(q)}(H)}{(e^s - K)qW^{(q)}(H)} = 0$, is given by the graph of

$$f(H) = \log\left(K\left(1 - \frac{Z^{(q)}(H)}{qW^{(q)}(H)}\right)^{-1}\right), \quad H \in (k^*, \infty). \tag{3.32}$$

Using the analytical properties of the map $H \mapsto Z^{(q)}(H)/(qW^{(q)}(H))$ given at the beginning of the paragraph above, one deduces that f is strictly decreasing on (k^*, ∞) and that $\eta := \lim_{H\uparrow\infty} f(H) = \log(K(1-\Phi(q)^{-1})^{-1})$ and $\lim_{H\downarrow k^*} f(H) = \infty$. Moreover, the inverse of f, which exists due to the strict monotonicity of f, will be denoted by f^{-1}. Using the 0-isocline and what was said in the paragraph above, we obtain qualitatively the direction field shown in Figure 3.7.

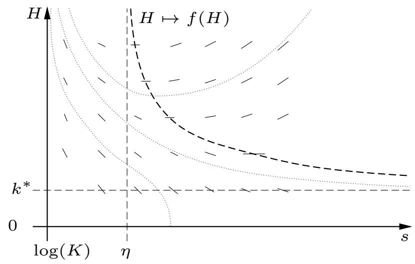

Fig. 3.7 *A qualitative picture of the direction field when $q > \psi(1)$ and $W^{(q)}(0+) = 0$. The case when $W^{(q)}(0+) \in (0, q^{-1})$ is similar except that the solutions (finer line) hit zero with finite slope instead of infinite slope (since $W^{(q)}(0+) > 0$).*

We continue by investigating two types of solutions. Let $s_0 > \log(K)$ and let $g(s)$ be the solution such that $g(s_0) = k^*$ which is defined on the maximal interval of existence, say I_g, of g. From the specific form of the direction field and the fact that solutions tend to the boundary of U (cf. Theorem 3.1 of [18]), we infer that $I_g = (\log(K), \tilde{s})$ for some $\tilde{s} > s_0$, $\lim_{s\uparrow\tilde{s}} g(s) = 0$ and $\lim_{s\downarrow\log(K)} g(s) = \infty$. In other words, the solutions of (3.19) which intersect the horizontal line $H = k^*$ come from infinity and eventually hit zero [with infinite gradient if $W^{(q)}(0+) = 0$ and with finite gradient if $W^{(q)}(0+) \in (0, q^{-1})$]. Next, suppose that $s_0 > \eta$ and let $g(s)$ be the solution such that $g(s_0) = f^{-1}(s_0)$. Similarly to above, we conclude that $I_g = (\log(K), \infty)$, $\lim_{s\uparrow\infty} g(s) = \infty$ and $\lim_{s\downarrow\log(K)} g(s) = \infty$. Put differently, every solution that intersects the 0-isocline comes from infinity and tends to infinity.

Let $\mathcal{S}^-$ be the set of solutions of (3.19) whose range contains the value k^* and $\mathcal{S}^+$ the set of solutions of (3.19) whose graph $s \mapsto g(s)$ intersects the 0-isocline; see Figure 3.7. Both these sets are nonempty as explained in the previous paragraph. For fixed $s^* > \eta$ define

$$
\begin{aligned}
H_-^* &:= \sup\{H \in (0,\infty) \,|\, \text{there exists } g \in \mathcal{S}^- \text{ such that } g(s^*) = H\}, \\
H_+^* &:= \inf\{H \in (0,\infty) \,|\, \text{there exists } g \in \mathcal{S}^+ \text{ such that } g(s^*) = H\}.
\end{aligned}
$$

It follows that $k^* \leq H_-^* \leq H_+^* \leq f^{-1}(s^*)$ and we claim that $H_-^* = H_+^*$. Suppose this was false and choose H_1, H_2 such that $H_-^* < H_1 < H_2 < H_+^*$. Denote by g_1 the solution to (3.19) such that $g_1(s^*) = H_1$ and by g_2 the solutions of (3.19) such that $g(s^*) = H_2$. Both these solutions must lie between the 0-isocline and the horizontal line $H = k^*$. In particular, it holds that $I_{g_1} = I_{g_2} = (\log(K), \infty)$ and

$$\lim_{s\to\infty} g_1(s) = \lim_{s\to\infty} g_2(s) = k^*. \tag{3.33}$$

Furthermore, set $F(s,H) := 1 - \frac{e^s Z^{(q)}(H)}{(e^s - K)qW^{(q)}(H)}$ for $(s,H) \in U$ and observe that, from earlier remarks, for fixed s, it is an increasing function in H. Using this and the fact that $g_1(s) < g_2(s)$ for all $s > \log(K)$ we may write [using the equivalent integral formulation of (3.19)]

$$g_2(s) - g_1(s) = H_2 - H_1 + \int_{s^*}^{s} F(u, g_2(u)) - F(u, g_1(u))\, du \geq H_2 - H_1 > 0$$

for $s > \log(K)$. This contradicts (3.33) and hence $H_-^* = H_+^*$. Denote by g_∞ be the solution to (3.19) such that $g_\infty(s^*) = H_-^*$. By construction, g_∞ lies above all the solutions in $\mathcal{S}^-$ and below all the solutions in $\mathcal{S}^+$. In particular, $I_{g_\infty} = (\log(K), \infty)$ and $\lim_{s\to\infty} g_\infty(s) = k^*$.

So far we have found that there are (at least) three types of solutions of (3.19) and, in fact, there are no more; that is, any solution to (3.19) either lies in $\mathcal{S}^- \cup \mathcal{S}^+$ or coincides with g_∞. To see this, note that the graph of g_∞ splits U into two disjoint sets. If $(s,H) \in U$ lies above the graph of g_∞, then the specific form of the field implies that the solution, g say, through (s,H) must intersect the vertical line $s = s^*$ and $g(s^*) > H_+^*$; thus $g \in \mathcal{S}^+$. Similarly, one may deduce that the solution through a point lying below the graph of g_∞ must intersect the horizontal line $H = k^*$ and therefore lies in $\mathcal{S}^-$.

Finally, we claim that given $\epsilon > \log(K)$, there exists a unique solution g_ϵ of (3.19) such that $I_{g_\epsilon} = (\log(K), \epsilon)$ and $\lim_{s\uparrow\epsilon} g_\epsilon(s) = 0$. Indeed, define the sets

$$\begin{aligned} s_\epsilon^+ &:= \sup\{s \in (\log(K), \infty) \,|\, \exists\, g \in \mathcal{S}^- \text{ s.t. } I_g \subsetneq (\log(K), \epsilon) \text{ and } g(s) = k^*\}, \\ s_\epsilon^- &:= \inf\{s \in (\log(K), \infty) \,|\, \exists\, g \in \mathcal{S}^- \text{ s.t. } (\log(K), \epsilon) \subsetneq I_g \text{ and } g(s) = k^*\}. \end{aligned}$$

One can then show by a similar argument as above that $s_\epsilon^- = s_\epsilon^+$. The solution through s_+^*, denoted g_ϵ, is then the desired one.

This whole discussion is summarised pictorially in Figure 3.2.

The case $W^{(q)}(0+) \geq q^{-1}$: Similarly to the first case, one sees that under the current assumptions it is still true that f is strictly decreasing on $(0,\infty)$ and

$$\eta := \lim_{H\uparrow\infty} f(H) = \log(K(1 - \Phi(q)^{-1})^{-1}).$$

Moreover, recalling that $W^{(q)}(0+) = \mathtt{d}^{-1}$, one deduces that $\lim_{H\downarrow 0} f(H) = \beta$, where

$$\beta := \log(K(1-\mathtt{d}/q)^{-1}) \in (0,\infty].$$

Analogously to the first case, one may use this information to qualitatively draw the direction field which is shown in Figure 3.8.

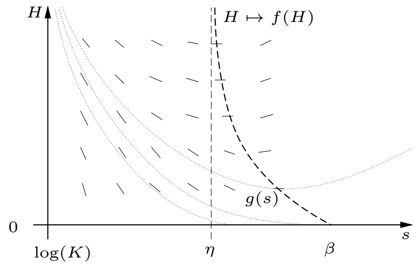

Fig. 3.8 *A qualitative picture of the direction field when* $W^{(q)}(0+) \geq q^{-1}$. *The constants* η *and* β *are given by* $\eta = \log(K(1-1/\Phi(q))^{-1})$ *and* $\beta = \log(K(1-\mathtt{d}/q)^{-1})$.

As in the first case, one may show that there are again three types of solutions; the ones that intersect the 0-isocline $[H \mapsto f(H)]$ and never hit zero, the ones that hit zero before β and the one which lies in between the other two types. One may also show that for a given $\epsilon \in (\log(K),\infty)$ there exists a unique solution g_ϵ such that $I_{g_\epsilon} = (\log(K), \epsilon\wedge\beta)$ and $\lim_{s\to\epsilon\wedge\beta} g_\epsilon(s) = 0$. This is pictorially displayed in Figure 3.3.

The case $\psi(1) \geq q$: Under this assumption it holds that $\Phi(q) \leq 1$ which together with equation (8.9) of [21] implies that

$$Z^{(q)}(H) - qW^{(q)}(H) \geq Z^{(q)}(H) - \frac{q}{\Phi(q)}W^{(q)}(H) > 0$$

for $H > 0$. This in turn means that $Z^{(q)}(H)/qW^{(q)}(H) > 1$ for $H > 0$. One may again draw the direction field and argue along the same line as above to deduce that all solutions of (3.19) are strictly decreasing, escape to infinity and hit zero [with infinite gradient if $W^{(q)}(0+) = 0$ and with finite gradient if $W^{(q)}(0+) \in (0, q^{-1})$]. Again, an argument as in the first case shows that for a given $\epsilon > \log(K)$ there exists a unique solution g_ϵ such that $I_{g_\epsilon} = (\log(K), \epsilon)$ and $\lim_{s\to\epsilon} g_\epsilon(s) = 0$. This was already pictorially displayed in Figure 3.4. $\square$

Proof of Theorem 3.2. The proof consists of five steps (i)–(v) which will imply the result. Before we go through these steps, recall that

$$\limsup_{t\to\infty} e^{-qt}(e^{\overline{X}_{t\wedge\epsilon}} - K) = 0 \quad \mathbb{P}_{x,s}\text{-a.s.} \tag{3.34}$$

for $(x,s) \in E$ and let τ_ϵ^* be given as in (3.21). Moreover, define the function

$$V_\epsilon(x,s) := (e^{s\wedge\epsilon} - K)Z^{(q)}(x - s + g_\epsilon(s))$$

for $(x,s) \in E_1 = \{(x,s) \in E : s > \log(K)\}$. We claim that

(i) $\mathbb{E}_{x,s}[e^{-qt}V_\epsilon(X_t, \overline{X}_t)] \leq V_\epsilon(x,s)$ for $(x,s) \in E_1$;

(ii) $V_\epsilon(x,s) = \mathbb{E}_{x,s}\big[e^{-q\tau_\epsilon^*}(e^{\overline{X}_{\tau_\epsilon^*}\wedge\epsilon} - K)\big]$ for $(x,s) \in E_1$.

Verification of (i)**:** We first prove (i) under the assumption that X is of unbounded variation, that is, $W^{(q)}(0+) = 0$. To this end, let Γ be the infinitesimal generator of X defined in (3.28). Although the function $Z^{(q)}$ is only in $C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ and it is a-priori not clear whether Γ applied to $Z^{(q)}$ is well defined, one may, at least formally, define $\Gamma Z^{(q)} : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by

$$\begin{aligned}\Gamma Z^{(q)}(x) \quad := \quad & -\gamma Z^{(q)\prime}(x) + \frac{\sigma^2}{2}Z^{(q)\prime\prime}(x) \\ & + \int_{(-\infty,0)} \big(Z^{(q)}(x+y) - Z^{(q)}(x) - yZ^{(q)\prime}(x)1_{\{y\geq-1\}}\big)\,\Pi(dy).\end{aligned}$$

For $x < 0$ the quantity $\Gamma Z^{(q)}(x)$ is well defined and $\Gamma Z^{(q)}(x) = 0$. On the other hand, for $x > 0$ one needs to check whether the integral part in $\Gamma Z^{(q)}(x)$ is well defined. This is done in Lemma A.1 of [29], which shows that this is indeed the case. Moreover, as shown in Section 3.2 of [34], it holds that

$$\Gamma Z^{(q)}(x) = qZ^{(q)}(x), \quad x \in (0,\infty).$$

Now fix $(x,s) \in E_1$ and define the semimartingale $Y_t := X_t - \overline{X}_t + g_\epsilon(\overline{X}_t)$, $t \geq 0$. Applying an appropriate version of the Itô–Meyer formula (cf. Theorem 71, Chapter VI of [36]) to $Z^{(q)}(Y_t)$ yields $\mathbb{P}_{x,s}$-a.s.

$$\begin{aligned}Z^{(q)}(Y_t) \quad = \quad & Z^{(q)}(x - s + g_\epsilon(s)) + m_t + \int_0^t \Gamma Z^{(q)}(Y_u)\,du \\ & + \int_0^t Z^{(q)\prime}(Y_u)(g_\epsilon'(\overline{X}_u) - 1)\,d\overline{X}_u,\end{aligned}$$

where

$$\begin{aligned}m_t \quad = \quad & \int_{0+}^t \sigma Z^{(q)\prime}(Y_{u-})dB_u + \int_{0+}^t Z^{(q)\prime}(Y_{u-})dX_u^{(2)} \\ & + \sum_{0<u\leq t} \Delta Z^{(q)}(Y_u) - \Delta X_u Z^{(q)\prime}(Y_{u-})1_{\{\Delta X_u\geq-1\}} \\ & - \int_0^t \int_{(-\infty,0)} Z^{(q)}(Y_{u-}+y) - Z^{(q)}(Y_{u-}) - yZ^{(q)\prime}(Y_{u-})1_{\{y\geq-1\}}\,\Pi(dy)du\end{aligned}$$

and $\Delta X_u = X_u - X_{u-}$, $\Delta Z^{(q)}(Y_u) = Z^{(q)}(Y_u) - Z^{(q)}(Y_{u-})$. The fact that $\Gamma Z^{(q)}$ is not defined at zero is not a problem as the time Y spends at zero has zero Lebesgue measure anyway. By the boundedness of $Z^{(q)\prime}$ on $(-\infty, g_\epsilon(s)]$ the first two stochastic integrals in the expression for m_t are zero-mean martingales and by the compensation formula (cf. Corollary 4.6 of [21]) the third and fourth term constitute a zero-mean martingale. Next, use stochastic integration by parts for semimartingales (cf. Corollary 2 of Theorem 22, Chapter II of [36]) to deduce that $\mathbb{P}_{x,s}$-a.s.

$$\begin{aligned} e^{-qt}V_\epsilon(X_t, \overline{X}_t) &= V_\epsilon(x,s) + M_t + \int_0^t e^{-qu}(e^{\overline{X}_u \wedge \epsilon} - K)(\Gamma - q)Z^{(q)}(Y_u)\,du \\ &\quad + \int_0^t e^{-qu}(e^{\overline{X}_u \wedge \epsilon} - K)Z^{(q)\prime}(Y_u)(g_\epsilon'(\overline{X}_u) - 1)\,d\overline{X}_u \qquad (3.35) \\ &\quad + \int_0^t e^{-qu + \overline{X}_u} Z^{(q)}(Y_u) 1_{\{\overline{X}_u \leq \epsilon\}}\,d\overline{X}_u \end{aligned}$$

where $M_t = \int_{0+}^t e^{-qu}(e^{\overline{X}_u \wedge \epsilon} - K)\,dm_u$ is a zero-mean martingale. The first integral is nonpositive since $(\Gamma - q)Z^{(q)}(y) \leq 0$ for all $y \in \mathbb{R} \setminus \{0\}$. The last two integrals vanish since the process $\overline{X}_u$ only increments when $\overline{X}_u = X_u$ and by definition of g_ϵ. Thus, taking expectations on both sides of (3.35) gives (i) if X is of unbounded variation.

If $W^{(q)}(0+) \in (0, q^{-1})$ or $W^{(q)}(0+) \geq q^{-1}$ (X has bounded variation), then the Itô–Meyer formula is nothing more than an appropriate version of the change of variable formula for Stieltjes integrals and one may obtain (i) in the same way as above. The only change worth mentioning is that the generator of X takes a different form. Specifically, for $h \in C_0^\infty(\mathbb{R})$ and $z \in \mathbb{R}$ it is given by

$$\Gamma h(z) = \mathtt{d}h'(z) + \int_{(-\infty,0)} \big(h(z+y) - h(z)\big)\Pi(dy).$$

As above, we want to apply Γ to $Z^{(q)}$ which is only in $C^1(\mathbb{R} \setminus \{0\})$. However, at least formally, we may define $\Gamma Z^{(q)} : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by

$$\Gamma Z^{(q)}(x) = \mathtt{d}Z^{(q)\prime}(x) + \int_{(-\infty,0)} \big(Z^{(q)}(x+y) - Z^{(q)}(x)\big)\Pi(dy).$$

This expression is well defined and $\Gamma Z^{(q)}$ satisfies all the properties required in the proof by Lemma A.1 of [29]. This completes the proof of (i).

Verification of (ii)**:** Recalling that $(\Gamma - q)Z^{(q)}(y) = 0$ for $y > 0$, we see from (3.35) that $\mathbb{E}_{x,s}\big[e^{-q(t \wedge \tau_\epsilon^*)} V(X_{t \wedge \tau_\epsilon^*}, \overline{X}_{t \wedge \tau_\epsilon^*})\big] = V_\epsilon(x,s)$ and hence (ii) follows by dominated convergence.

Next, recall that $A_\epsilon := \mathbb{E}_{\log(K),\log(K)}[e^{-q\tau_\epsilon^*}(e^{\overline{X}_{\tau_\epsilon^*}\wedge\epsilon} - K)]$ and note that

$$A_\epsilon = \lim_{s\downarrow\log(K)} \mathbb{E}_{s,s}[e^{-q\tau_\epsilon^*}(e^{\overline{X}_{\tau_\epsilon^*}\wedge\epsilon} - K)] = \lim_{s\downarrow\log(K)} (e^s - K)Z^{(q)}(g_\epsilon(s)),$$

where in the second equality we have used (ii) on page 64. Now extend the definition of the function V_ϵ to

$$V_\epsilon(x,s) = \begin{cases} (e^{s\wedge\epsilon} - K)Z^{(q)}(x - s + g_\epsilon(s)), & (x,s) \in E_1, \\ e^{-\Phi(q)(\log(K)-x)}A_\epsilon, & (x,s) \in C^*_{II}. \end{cases} \tag{3.36}$$

We claim that:

(iii) $V_\epsilon(x,s) \geq (e^{s\wedge\epsilon} - K)^+$ for $(x,s) \in E$;

(iv) $\mathbb{E}_{x,s}[e^{-qt}V_\epsilon(X_t, \overline{X}_t)] \leq V_\epsilon(x,s)$ for $(x,s) \in E$;

(v) $V_\epsilon(x,s) = \mathbb{E}_{x,s}\big[e^{-q\tau_\epsilon^*}(e^{\overline{X}_{\tau_\epsilon^*}\wedge\epsilon} - K)\big]$ for $(x,s) \in E$.

Condition (iii) is clear from the definition of $Z^{(q)}$ and V_ϵ.

Verification of condition (iv)**:** In view of (i), it is enough to show (iv) for $(x,s) \in C^*_{II}$. In order to prove this, set $Y_t = e^{-qt}V_\epsilon(X_t, \overline{X}_t)$, $t \geq 0$, and observe that

$$\mathbb{E}_{\log(K),\log(K)}[Y_t] = \lim_{s\downarrow\log(K)} \mathbb{E}_{s,s}[Y_t] \leq \lim_{s\downarrow\log(K)} V_\epsilon(s,s),$$

where in the inequality we have used (i). Combining this with the strong Markov property, we obtain on $\{\tau^+_{\log(K)} < \infty\}$ for $(x,s) \in C^*_{II}$,

$$\begin{aligned} \mathbb{E}_{x,s}\Big[Y_t\Big|\mathcal{F}_{\tau^+_{\log(K)}}\Big] &= Y_t 1_{\{t\leq\tau^+_{\log(K)}\}} + e^{-q\tau^+_{\log(K)}}\mathbb{E}_{\log(K),\log(K)}[Y_{t-u}]\big|_{u=\tau^+_{\log(K)}} 1_{\{t>\tau^+_{\log(K)}\}} \\ &\leq Y_t 1_{\{t\leq\tau^+_{\log(K)}\}} + Y_{\tau^+_{\log(K)}} 1_{\{t>\tau^+_{\log(K)}\}} \\ &= Y_{t\wedge\tau^+_{\log(K)}}. \end{aligned}$$

Hence, taking expectations on both sides and using (3.34) shows that, for $(x,s) \in C^*_{II}$, we have $\mathbb{E}_{x,s}[Y_t] \leq \mathbb{E}_{x,s}\big[Y_{t\wedge\tau^+_{\log(K)}}\big]$. Since $Y_{t\wedge\tau^+_{\log(K)}}$ is a $\mathbb{P}_{x,s}$-martingale for $(x,s) \in C^*_{II}$ [see (3.9)] the inequality in (iv) follows.

Verification of condition (v)**:** By the strong Markov property, Theorem 3.12 of [21] and the definition of A_ϵ and V_ϵ we have

$$\mathbb{E}_{x,s}\big[e^{-q\tau_\epsilon^*}(e^{\overline{X}_{\tau_\epsilon^*}\wedge\epsilon} - K)^+\big] = e^{-\Phi(q)(\log(K)-x)}A_\epsilon = V_\epsilon(x,s)$$

for $(x,s) \in C^*_{II}$. This together with (iii) gives assertion (v).

We are now in a position to prove Theorem 3.2. Inequality (iv) and the Markov property of $(X,\overline{X})$ imply that the process $e^{-qt}V_\epsilon(X_t,\overline{X}_t)$, $t\geq 0$, is a $\mathbb{P}_{x,s}$-supermartingale for $(x,s)\in E$. Using (3.34), (iii), Fatou's lemma in the second inequality and the supermartingale property of $e^{-qt}V_\epsilon(X_t,\overline{X}_t)$, $t\geq 0$, and Doob's optional stopping theorem in the third inequality shows that for $\tau\in\mathcal{M}$,

$$\begin{aligned}
\mathbb{E}_{x,s}\big[e^{-q\tau}(e^{\overline{X}_\tau\wedge\epsilon}-K)\big] &= \mathbb{E}_{x,s}\big[e^{-q\tau}(e^{\overline{X}_\tau\wedge\epsilon}-K)1_{\{\tau<\infty\}}\big]\\
&\leq \mathbb{E}_{x,s}\big[e^{-q\tau}V_\epsilon(X_\tau,\overline{X}_\tau)1_{\{\tau<\infty\}}\big]\\
&\leq \liminf_{t\to\infty}\mathbb{E}_{x,s}\big[e^{-q(t\wedge\tau)}V_\epsilon(X_{t\wedge\tau},\overline{X}_{t\wedge\tau})\big]\\
&\leq V_\epsilon(x,s).
\end{aligned}$$

This together with (v) shows that $V_\epsilon^*=V_\epsilon$ and that τ_ϵ^* is optimal. □

Proof of Theorem 3.5. Recall that under the current assumptions Lemma 3.14 in Section 3.10 implies that

$$\limsup_{t\to\infty} e^{-qt}(e^{\overline{X}_t}-K)^+ = 0 \quad \mathbb{P}_{x,s}\text{-a.s.} \tag{3.37}$$

$$\mathbb{E}_{x,s}\Big[\sup_{0\leq t<\infty} e^{-qt+\overline{X}_t}\Big]<\infty \tag{3.38}$$

for $(x,s)\in E$, from which it follows that

$$\sup_{\tau\in\mathcal{M}}\mathbb{E}_{x,s}\big[e^{-q\tau}(e^{\overline{X}_\tau}-K)^+\big]<\infty$$

for $(x,s)\in E$. Also, for $\epsilon\in(\log(K),\infty)$, let V_ϵ^*,A_ϵ, τ_ϵ^* and g_ϵ be as in Theorem 3.2 and g_∞,τ_∞^* as stated in Theorem 3.5. An inspection of the proof of Lemma 3.1 and Theorem 3.2 of [18] show that $g_\infty(s)=\lim_{\epsilon\uparrow\infty}g_\epsilon(s)$ for $s>\log(K)$ which in turn implies that $\lim_{\epsilon\uparrow\infty}\tau_\epsilon^*=\tau_\infty^*$ $\mathbb{P}_{x,s}$-a.s. for all $(x,s)\in E$. Furthermore, recall that $A_\infty:=\mathbb{E}_{\log(K),\log(K)}[e^{-q\tau_\infty^*}(e^{\overline{X}_{\tau_\infty^*}}-K)]$ and define

$$V_\infty(x,s):=\begin{cases}(e^s-K)Z^{(q)}(x-s+g_\infty(s)), & (x,s)\in E_1,\\ e^{-\Phi(q)(\log(K)-x)}A_\infty, & (x,s)\in C_{II}^*.\end{cases}$$

Now, using (3.37), (3.38) and dominated convergence, we see that

$$\lim_{\epsilon\to\infty}A_\epsilon=\lim_{\epsilon\to\infty}\mathbb{E}_{\log(K),\log(K)}\big[e^{-q\tau_\epsilon^*}\big(e^{\overline{X}_{\tau_\epsilon^*}\wedge\epsilon}-K\big)\big]=A_\infty$$

and

$$A_\infty = \lim_{s\downarrow\log(K)}\mathbb{E}_{s,s}\big[e^{-q\tau_\infty^*}\big(e^{\overline{X}_{\tau_\infty^*}}-K\big)\big]$$

$$
\begin{aligned}
&= \lim_{s\downarrow\log(K)} \lim_{\epsilon\to\infty} \mathbb{E}_{s,s}\big[e^{-q\tau_\epsilon^*}\big(e^{\overline{X}_{\tau_\epsilon^*}} - K\big)\big] \\
&= \lim_{s\downarrow\log(K)} (e^s - K)Z^{(q)}(g_\infty(s)).
\end{aligned}
$$

It follows in particular that $V_\infty(x,s) = \lim_{\epsilon\uparrow\infty} V_\epsilon^*(x,s)$ for $(x,s) \in E$. Next, we claim that:

(i) $V_\infty(x,s) \geq (e^s - K)^+$ for $(x,s) \in E$;

(ii) $\mathbb{E}_{x,s}[e^{-qt}V_\infty(X_t, \overline{X}_t)] \leq V_\infty(x,s)$ for $(x,s) \in E$;

(iii) $V_\infty(x,s) = \mathbb{E}_{x,s}\big[e^{-q\tau_\infty^*}(e^{\overline{X}_{\tau_\infty^*}} - K)\big]$ for $(x,s) \in E$.

Condition (i) is clear from the definition of $Z^{(q)}$ and V_∞. To prove (ii), use Fatou's lemma and (i) of the proof of Theorem 3.2 to show that

$$
\begin{aligned}
\mathbb{E}_{x,s}[e^{-qt}V_\infty(X_t, \overline{X}_t)] &\leq \liminf_{\epsilon\to\infty} \mathbb{E}_{x,s}[e^{-qt}V_\epsilon^*(X_t, \overline{X}_t)] \\
&\leq \liminf_{\epsilon\to\infty} V_\epsilon^*(x,s) \\
&= V_\infty(x,s)
\end{aligned}
$$

for $(x,s) \in E$. As for (iii), using (3.37), (3.38) and dominated convergence we deduce that

$$
\begin{aligned}
V_\infty(x,s) &= \lim_{\epsilon\to\infty} V_\epsilon^*(x,s) \\
&= \lim_{\epsilon\to\infty} \mathbb{E}_{x,s}\big[e^{-q\tau_\epsilon^*}(e^{\overline{X}_{\tau_\epsilon^*}\wedge\epsilon} - K)\big] \\
&= \mathbb{E}_{x,s}\big[e^{-q\tau_\infty^*}(e^{\overline{X}_{\tau_\infty^*}} - K)\big]
\end{aligned}
$$

for $(x,s) \in E$. The proof of the theorem is now completed by using (i)–(iii) in the same way as in the proof of Theorem 3.2 to show that $V_\infty^* = V_\infty$ and that τ_∞^* is optimal. □

Remark 3.13. Instead of proving Theorem 3.5 via a limiting procedure, it would be possible to prove it analogously to Theorem 3.2 by going through the Itô–Meyer formula. We chose to present the proof above as it emphasises that the capped version of (3.1) [$\epsilon \in (\log(K), \infty)$], is a building block for the uncapped version of (3.1) ($\epsilon = \infty$) rather than an isolated problem in itself.

Proof of Lemma 3.7. First assume that $\psi'(0+) < 0$ and fix $(x,s) \in E$ such that $\log(K) \leq s \leq \epsilon$. Since the supremum process $\overline{X}$ is increasing and there is no discounting, it follows that

$$
V_\infty^*(x,s) = \mathbb{E}_{x,s}\Big[e^{\overline{X}_{\tau_\epsilon^+}}\Big] - K = \mathbb{E}_{x,s}[e^{\overline{X}_\infty\wedge\epsilon}] - K = e^x\mathbb{E}_{0,s-x}[e^{\overline{X}_\infty\wedge(\epsilon-x)}] - K.
$$

The fact that $\psi'(0+) < 0$ implies that $\sup_{0\leq u<\infty} X_u$ is exponentially distributed with parameter $\Phi(0) > 0$ under $\mathbb{P}_0$; see equation (8.4) in [21]. Thus, if $\Phi(0) \neq 1$, one calculates

$$V_\infty^*(x,s) = e^s + \frac{e^{x\Phi(0)}}{\Phi(0)-1}\big(e^{s(1-\Phi(0))} - e^{\epsilon(1-\Phi(0))}\big) - K.$$

Similarly, if $\Phi(0) = 1$, we have $V_\epsilon^*(x,s) = e^s - K + e^x(\epsilon - s)$.

On the other hand, if $(x,s) \in E$ such that $s < \log(K)$ then an application of the strong Markov property at $\tau^+_{\log(K)}$ and Theorem 3.12 of [21] gives

$$\begin{aligned} V_\infty^*(x,s) &= \mathbb{E}_{x,s}\Big[\Big(e^{\overline{X}_{\tau_\epsilon^+}} - K\Big)^+\Big] \\ &= e^{-\Phi(0)(\log(K)-x)}\mathbb{E}_{\log(K),\log(K)}\Big[e^{\overline{X}_{\tau_\epsilon^+}} - K\Big]. \end{aligned}$$

The last expression on the right-hand side is known from the computations above and hence the first part of the proof follows.

As for the second part, it is known that $\psi'(0+) \geq 0$ implies that $\mathbb{P}_{x,s}[\tau_\epsilon^+ < \infty] = 1$ for $(x,s) \in E$ and since there is no discounting the claim follows. $\square$

Proof of Lemma 3.8. The first part follows by taking limits in Lemma 3.7, since by monotone convergence we have

$$V_\infty^*(x,s) = \mathbb{E}_{x,s}\big[(e^{\overline{X}_\infty} - K)^+\big] = \lim_{\epsilon\uparrow\infty}\mathbb{E}_{x,s}\big[(e^{\overline{X}_{\tau_\epsilon^+\wedge\epsilon}} - K)^+\big] = \lim_{\epsilon\uparrow\infty} V_\epsilon^*(x,s).$$

As for the second part, note that $V_\infty^*(x,s) \geq \lim_{\epsilon\uparrow\infty} V_\epsilon^*(x,s)$ and hence it is enough to show that the limit equals infinity. To this end, observe that under the current assumptions we have $\lim_{\epsilon\uparrow\infty} g_\epsilon(s) = \infty$ for $s > \log(K)$; see Lemma 3.1(c). This in conjunction with the fact that $\lim_{z\to\infty} Z^{(q)}(z) = \infty$ shows that, for $(x,s) \in E$ such that $s > \log(K)$,

$$\lim_{\epsilon\to\infty} V_\epsilon^*(x,s) = \lim_{\epsilon\to\infty}(e^{s\wedge\epsilon} - K)Z^{(q)}(x - s + g_\epsilon(s)) = \infty.$$

On the other hand, if $(x,s) \in E$ such that $s \leq \log(K)$, the claim follows provided that $\lim_{\epsilon\to\infty} A_\epsilon = \infty$. Indeed, using the strong Markov property and Theorem 3.12 of [21] one may deduce that

$$A_\epsilon \geq \mathbb{E}_{\log(K),\log(K)}\big[e^{-q\tau_s^+}1_{\{\tau_s^+<\tau_\epsilon^*\}}\big]V_\epsilon^*(s,s).$$

The second factor on the right-hand side increases to $+\infty$ as $\epsilon \uparrow \infty$ by the first part of the proof and thus the proof is complete. $\square$

3.10 Appendix

3.10.1 An auxiliary result

Lemma 3.14. *If $q > \psi(1)$ we have for $(x,s) \in E$ that*

$$\mathbb{E}_{x,s}\Big[\sup_{0 \le t < \infty} e^{-qt+\overline{X}_t}\Big] < \infty.$$

In particular, $\limsup_{t\to\infty} e^{-qt+\overline{X}_t} = 0$ $\mathbb{P}_{x,s}$*-a.s. for* $(x,s) \in E$.

Proof of Lemma 3.14. We want to show that

$$\int_0^\infty \mathbb{P}_{x,s}\Big[\sup_{0 \le t < \infty} e^{-qt+\overline{X}_t} > y\Big]\,dy < \infty. \tag{3.39}$$

First note that it is enough to consider the above integral over the interval (e^s, ∞), since for $y < e^s$ the probability inside the integral is equal to one. Next, for $y > e^s$ define $\gamma = \log(y) - x > 0$ and write

$$\begin{aligned} \mathbb{P}_{x,s}\Big[\sup_{0 \le t < \infty} e^{-qt+\overline{X}_t} > y\Big] &= \mathbb{P}\Big[\sup_{0 \le t < \infty}\Big(\Big(\sup_{0 \le u \le t} X_u \vee (s-x)\Big) - \gamma - qt\Big) > 0\Big] \\ &\le \mathbb{P}[X_t - qt > \gamma \text{ for some } t]. \end{aligned}$$

The term on the right-hand side is the probability that the spectrally negative Lévy process $\tilde{X}_t := X_t - qt$, $t \ge 0$, with Laplace exponent $\psi_{\tilde{X}}(\theta) = \psi(\theta) - q\theta$, $\theta \ge 0$, reaches level γ. Thus,

$$\mathbb{P}_{x,s}\Big[\sup_{0 \le t < \infty} e^{-qt+\overline{X}_t} > y\Big] \le e^{-\Phi_{\tilde{X}}(0)\gamma} = e^{\Phi_{\tilde{X}}(0)x} y^{-\Phi_{\tilde{X}}(0)},$$

where $\Phi_{\tilde{X}}$ is the right-inverse of $\psi_{\tilde{X}}$. Hence, the integral (3.39) converges provided $\Phi_{\tilde{X}}(0) > 1$. This is indeed satisfied because $\psi_{\tilde{X}}$ is convex and $\psi_{\tilde{X}}(1) = \psi(1) - q < 0$ by assumption.

As for the second assertion, let $\delta > 0$ such that $q - \delta > \psi(1)$. By the first part we may now, for $(x,s) \in E$, infer that $\sup_{0 \le t < \infty} e^{-(q-\delta)t+\overline{X}_t} < \infty$ $\mathbb{P}_{x,s}$-a.s. and hence

$$\limsup_{t\to\infty} e^{-qt+\overline{X}_t} = \limsup_{t\to\infty} e^{-\delta t} e^{-(q-\delta)t+\overline{X}_t} = 0. \tag{3.40}$$

This completes the proof. ∎

3.10.2 An excursion theoretic calculation

Our aim is to compute the value $\mathbb{E}_{s,s}\big[e^{-q\tau_\epsilon^*}\big(e^{\overline{X}_{\tau_\epsilon^*}\wedge\epsilon} - K\big)\big]$ for $s \in [\log(K), \epsilon)$ with the help of excursion theory; see Remark 3.3. We shall spend a moment setting up some necessary notation. In doing so, we closely follow pages 221–223 in [2] and refer the reader to Chapters 6 and 7 in [6] for background reading. The process $L_t := \overline{X}_t$ serves as local time at 0 for the Markov process $\overline{X} - X$ under $\mathbb{P}_{0,0}$. Write $L^{-1} := \{L_t^{-1} : t \geq 0\}$ for the right-continuous inverse of L. The Poisson point process of excursions indexed by local time shall be denoted by $\{(t, \varepsilon_t) : t \geq 0\}$, where

$$\varepsilon_t = \{\varepsilon_t(s) := X_{L_t^{-1}} - X_{L_{t-}^{-1}+s} : 0 < s < L_t^{-1} - L_{t-}^{-1}\}$$

whenever $L_t^{-1} - L_{t-}^{-1} > 0$. Accordingly, we refer to a generic excursion as $\varepsilon(\cdot)$ (or just ε for short as appropriate) belonging to the space $\mathcal{E}$ of canonical excursions. The intensity measure of the process $\{(t, \varepsilon_t) : t \geq 0\}$ is given by $dt \times dn$, where n is a measure on the space of excursions (the excursion measure). A functional of the canonical excursion that will be of interest is $\overline{\varepsilon} = \sup_{s<\zeta} \varepsilon(s)$, where $\zeta(\varepsilon) = \zeta$ is the length of an excursion. A useful formula for this functional that we shall make use of is the following [cf. [21], equation (8.26)]:

$$n(\overline{\varepsilon} > x) = \frac{W'(x)}{W(x)} \tag{3.41}$$

provided that x is not a discontinuity point in the derivative of W [which is only a concern when X is of bounded variation, but we have assumed that in this case Π is atomless and hence W is continuously differentiable on $(0, \infty)$]. Another functional that we will also use is $\rho_a := \inf\{s > 0 : \varepsilon(s) > a\}$, the first passage time above a of the canonical excursion ε.

We now proceed with the promised calculation involving excursion theory. First, assume that $\log(K) < \epsilon < \infty$ and $\beta = \infty$. Note that for $\log(K) \leq s < \epsilon$,

$$\begin{aligned} \mathbb{E}_{s,s}\big[e^{-q\tau_\epsilon^*}\big(e^{\overline{X}_{\tau_\epsilon^*}\wedge\epsilon} - K\big)\big] &= \mathbb{E}_{s,s}\big[e^{-q\tau_\epsilon^*}\big(e^{\overline{X}_{\tau_\epsilon^*}\wedge\epsilon} - K\big)1_{\{\tau_\epsilon^*<\tau_\epsilon^+\}}\big] \\ &\quad + \mathbb{E}_{s,s}\big[e^{-q\tau_\epsilon^*}\big(e^{\overline{X}_{\tau_\epsilon^*}\wedge\epsilon} - K\big)1_{\{\tau_\epsilon^*=\tau_\epsilon^+\}}\big]. \end{aligned} \tag{3.42}$$

We compute the two terms on the right-hand side separately. An application of the compensation formula in the second equality and using Fubini's theorem in the third equality gives for $\log(K) \leq s < \epsilon$,

$$\begin{aligned} &\mathbb{E}_{s,s}\big[e^{-q\tau_\epsilon^*}\big(e^{\overline{X}_{\tau_\epsilon^*}\wedge\epsilon} - K\big)1_{\{\tau_\epsilon^*<\tau_\epsilon^+\}}\big] \\ &= \mathbb{E}\left[\sum_{0<t<\epsilon-s} e^{-qL_{t-}^{-1}}(e^{t+s} - K)1_{\{\overline{\varepsilon}_u \leq g_\epsilon(u+s)\,\forall\, u<t\}}1_{\{\overline{\varepsilon}_t > g_\epsilon(t+s)\}}e^{-q\rho_{g_\epsilon(s+t)}(\varepsilon_t)}\right] \\ &= \mathbb{E}\left[\int_0^{\epsilon-s} dt\, e^{-qL_t^{-1}}(e^{s+t} - K)1_{\{\overline{\varepsilon}_u \leq g_\epsilon(u+s)\,\forall\, u<t\}}\int_{\mathcal{E}} 1_{\{\overline{\varepsilon} > g_\epsilon(t+s)\}}e^{-q\rho_{g_\epsilon(s+t)}(\varepsilon)}n(d\varepsilon)\right] \\ &= \int_0^{\epsilon-s}(e^{s+t} - K)e^{-\Phi(q)t}\mathbb{E}\Big[e^{-qL_t^{-1}+\Phi(q)t}1_{\{\overline{\varepsilon}_u \leq g_\epsilon(u+s)\,\forall\, u<t\}}\Big]\hat{f}(g_\epsilon(t+s))\,dt, \end{aligned}$$

where in the first equality the time index runs over local times and the sum is the usual shorthand for integration with respect to the Poisson counting measure of excursions, and $\hat{f}(u) = \frac{Z^{(q)}(u)W^{(q)\prime}(u)}{W^{(q)}(u)} - qW^{(q)}(u)$ is an expression taken from Theorem 1 in [2]. Next, note that L_t^{-1} is a stopping time and hence a change of measure according to (3.10) shows that the expectation inside the integral can be written as

$$\mathbb{P}^{\Phi(q)}\big[\overline{\varepsilon}_u \leq g_\epsilon(u+s) \text{ for all } u < t\big].$$

Using the properties of the Poisson point process of excursions (indexed by local time) and with the help of (3.41) and (3.13) we may deduce

$$\begin{aligned}\mathbb{P}^{\Phi(q)}\big[\overline{\varepsilon}_u \leq g_\epsilon(u+s) \text{ for all } u < t\big] &= \exp\left(-\int_0^t n_{\Phi(q)}(\overline{\varepsilon} > g_\epsilon(u+s))\,du\right) \\ &= \exp\left(\Phi(q)t - \int_0^t \frac{W^{(q)\prime}(g_\epsilon(u+s))}{W^{(q)}(g_\epsilon(u+s))}\,du\right),\end{aligned}$$

where $n_{\Phi(q)}$ denotes the excursion measure associated with X under $\mathbb{P}^{\Phi(q)}$. By a change of variables we finally get for $\log(K) \leq s < \epsilon$,

$$\begin{aligned}&\mathbb{E}_{s,s}\big[e^{-q\tau_\epsilon^*}\big(e^{\overline{X}_{\tau_\epsilon^*}\wedge\epsilon} - K\big)1_{\{\tau_\epsilon^* < \tau_\epsilon^+\}}\big] \\ &= \int_s^\epsilon (e^t - K)\hat{f}(g_\epsilon(t))\exp\left(-\int_s^t \frac{W^{(q)\prime}(g_\epsilon(u))}{W^{(q)}(g_\epsilon(u))}\,du\right)dt.\end{aligned}$$

As for the second term in (3.42), similarly to the computation of the first term, we obtain for $\log(K) \leq s < \epsilon$,

$$\begin{aligned}&\mathbb{E}_{s,s}\big[e^{-q\tau_\epsilon^*}\big(e^{\overline{X}_{\tau_\epsilon^*}\wedge\epsilon} - K\big)1_{\{\tau_\epsilon^* = \tau_\epsilon^+\}}\big] \\ &= (e^\epsilon - K)\mathbb{E}\big[e^{-qL^{-1}_{\epsilon-s}}1_{\{\overline{\varepsilon}_t \leq g_\epsilon(t+s)\,\forall\, t<\epsilon-s\}}\big] \\ &= (e^\epsilon - K)e^{-\Phi(q)(\epsilon-s)}\mathbb{P}^{\Phi(q)}\big[\overline{\varepsilon}_t \leq g_\epsilon(t+s)\,\forall\, t < \epsilon - s\big] \\ &= (e^\epsilon - K)\exp\left(-\int_s^\epsilon \frac{W^{(q)\prime}(g_\epsilon(u))}{W^{(q)}(g_\epsilon(u))}\,du\right).\end{aligned}$$

Adding the two terms up gives the expression in Remark 3.3.

In the case that $\epsilon = \beta = \infty$ the second term on the right-hand side of (3.42) is not needed. In the case that $\beta = \log\big(K(1-\mathtt{d}/q)^{-1}\big) < \epsilon$, the cap ϵ may effectively be replaced by β in (3.42).

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CHAPTER 4

BOTTLENECK OPTION

In the spirit of [23, 29], we consider an option whose payoff corresponds to a capped American lookback option with floating-strike and solve the associated pricing problem (an optimal stopping problem) in a financial market whose price process is modelled by an exponential spectrally negative Lévy process. Despite the simple interpretation of the cap as a moderation of the payoff, it turns out that the optimal strategy to exercise the option looks very different compared to the situation without a cap. In fact, we show that the continuation region has a feature that resembles a bottleneck and hence the name "Bottleneck option".

4.1 Introduction

Consider a financial market consisting of a riskless bond and a risky asset whose price is modelled by a strictly positive stochastic process $S = \{S_t : t \geq 0\}$. A "Bottleneck option" (the name will be justified in due course) gives the holder the right to exercise at any finite time τ (a stopping time) yielding payouts

$$e^{-\alpha\tau}\left(M_0 \vee \sup_{0\leq u\leq\tau} S_u \wedge C - KS_\tau\right)^+, \quad C > M_0 \geq S_0, \alpha > 0. \tag{4.1}$$

The constant M_0 can be viewed as representing the "starting" maximum of the stock price (say, over some previous period $(-t_0, 0]$), $K > 0$ is referred to as strike, α is a discount factor and C is the cap. This type of payoff belongs to the class of so-called perpetual "lookback" options – "lookback" because it involves the term $\sup_{0\leq u\leq\tau} S_u$ and thus the holder of such an option has to look back in time in order to determine the payoff at time τ. The simplest example is a Russian option which was introduced by Shepp and Shiryaev [38, 39] and corresponds to setting $K = 0$ and $C = \infty$ above.

Another example would be an American lookback option with fixed strike which is (4.1) with $C = \infty$ and the term KS_τ replaced by K; cf. [14, 17, 30].

Assuming that $C = \infty$ and taking into account the particular form of the payoff in (4.1), one sees that it is positive at time t provided the quantity $\overline{S}_t - S_t$ is sufficiently large, where $\overline{S} = \{\overline{S}_u : u \geq 0\}$ is given by $\overline{S}_u := M_0 \vee \sup_{0 \leq v \leq u} S_v$, $u \geq 0$. We will refer to the quantity $\overline{S}_t - S_t$ as the depth of the excursion of S away from its running maximum. In view of the discounting in (4.1), this suggests that it is worth exercising the option as soon as S undertakes an excursion away from its running maximum that is deep enough. Thus a payoff of the form (4.1) could be particularly interesting for an investor interested in exploiting instances when S drops significantly after reaching new maxima. Payoffs of type (4.1) with $C = \infty$ have been known before and are sometimes called American lookback options with floating-strike, cf. [11, 14]. One additional feature here is that we allow $C < \infty$ which corresponds to a moderation of the payoff in the sense that it is bounded from above by C. We therefore refer to C as the cap. The case when $C = \infty$ simply means no moderation at all. Alternatively, the cap can be viewed as a means to limit the downside risk for an issuer of a payoff of the form (4.1).

Apart from the simple economic interpretation of the cap mentioned in the previous paragraph, we will show that its presence has a surprising effect on the optimal exercise strategy. Here optimal is understood in the sense that the expected discounted payoff is maximised. As informally described above, if $C = \infty$, it is plausible that the optimal strategy to exercise (4.1) is to wait until S undertakes an excursion away from its running maximum that is deep enough. In fact, this was proved rigorously for a Black–Scholes model in [11, 30]. Their result can be visualised by drawing the trace of a realisation of the process $t \mapsto (S_t, \overline{S}_t)$ in the positive quadrant; see Figure 4.1. The grey area corresponds to the continuation region, that is, the region where one continues to observe the evolution of $(S, \overline{S})$ and does not exercise the option. Note that the dynamics of $(S, \overline{S})$ are such that it can only climb upwards along the diagonal. The horizontal lines in Figure 4.1 are meant to schematically indicate the trace of the excursions of S away from its running maximum.

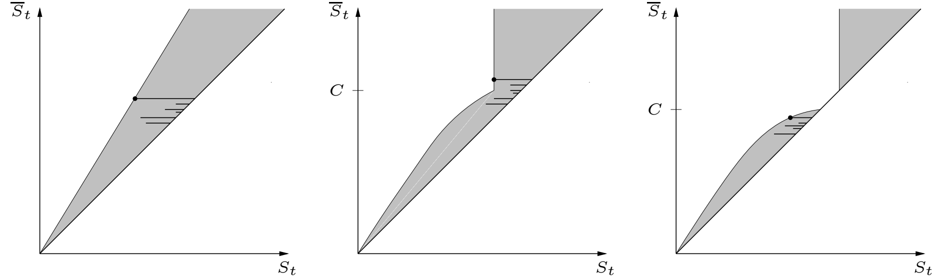

Fig. 4.1 *The expected continuation region (grey) and stopping region for the cases when $C = \infty$, $C < \infty$ and $K > 1$, and $C < \infty$ and K is small enough.*

On the other hand, if $C < \infty$ and $K > 1$, we will show that in a specific model, which includes the Black-Scholes model, the optimal strategy to exercise (4.1) is of the following form: As long as the second component of $(S_t, \overline{S}_t)$ lies below C, one waits until S undergoes an excursion away from its running maximum of depth at least $g(\overline{S}_t)$ for some function g. Once the level C is reached, the strategy consists of stopping as soon as S_t drops below a fixed value. Pictorially displaying this (see Figure 4.1), one sees that the continuation region shows a feature that resembles a bottleneck and hence the name "Bottleneck" option. Furthermore, it turns out that as one decreases K, the bottleneck becomes smaller and smaller and eventually vanishes once K drops below a critical value. The resulting continuation region then consists of two disjoint regions; see Figure 4.1.

In order to make things more rigorous, let us specify the underlying model. Suppose that $X = \{X_t : t \geq 0\}$ is a spectrally negative Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$ satisfying the natural conditions; cf. [7], Section 1.3, page 39. For $x \in \mathbb{R}$, denote by $\mathbb{P}_x$ the probability measure under which X starts at x and for simplicity write $\mathbb{P}_0 = \mathbb{P}$. The value of the bond $B = \{B_t : t \geq 0\}$ evolves deterministically such that

$$B_t = B_0 e^{rt}, \quad B_0 > 0, r \geq 0, t \geq 0, \tag{4.2}$$

and the price of the risky asset is modeled as the exponential spectrally negative Lévy process

$$S_t = S_0 e^{X_t}, \quad S_0 > 0, t \geq 0. \tag{4.3}$$

In order to guarantee that our model is free of arbitrage we will assume that $\psi(1) = r$, where ψ is the Laplace exponent of X. If $X_t = \mu t + \sigma W_t$, where $W = \{W_t : t \geq 0\}$ is a standard Brownian motion, we get the standard Black–Scholes model for the price of the asset. Of course, it is an important question whether this model of a financial market is appropriate, but we will not discuss this issue here. Nevertheless, it is worth mentioning that Carr and Wu [8] as well as Madan and Schoutens [25] offered empirical evidence to support this model in which the risky asset is driven by a spectrally negative Lévy process for appropriate market scenarios.

Finding the optimal time to exercise (4.1) and the corresponding expected payoff leads by the standard theory of pricing American-type options [41] to solving the optimal stopping problem

$$V_r(M_0, S_0, C) := B_0 \sup_{\tau} \mathbb{E}\left[B_\tau^{-1} e^{-\alpha\tau} \left(M_0 \vee \sup_{0 \leq u \leq \tau} S_u \wedge C - K S_\tau \right)^+ \right], \tag{4.4}$$

where the supremum is taken over all $[0, \infty)$-valued stopping times. In other words, we want to find a stopping time which optimises the expected discounted claim. It will be convenient to rewrite (4.4) in a slightly different way. Specifically, we associate with X the maximum process $\overline{X} = \{\overline{X}_t : t \geq 0\}$, where $\overline{X}_t := s \vee \sup_{0 \leq u \leq t} X_u$ for $t \geq 0, s \geq x$. The law under which $(X, \overline{X})$ starts at (x, s) is denoted by $\mathbb{P}_{x,s}$. Thus, summing up, the aim of this article is to solve the optimal stopping problem

$$V_\epsilon^*(x,s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s}\big[e^{-q\tau}\big(e^{\overline{X}_\tau \wedge \epsilon} - Ke^{X_\tau}\big)^+\big], \tag{4.5}$$

where $q > 0$, $K > 0$, $\epsilon \in \mathbb{R} \cup \{\infty\}$, $(x,s) \in E := \{(x_1, s_1) \in \mathbb{R}^2 \,|\, x_1 \leq s_1\}$ and $\mathcal{M}$ is the set of all $[0,\infty)$-valued $\mathbb{F}$-stopping times. In particular, note that $V_r(M_0, S_0, C) = V_\epsilon^*(x,s)$ with $x = \log(S_0)$, $s = \log(M_0)$, $\epsilon = \log(C)$, $q = \alpha + r$ and $\psi(1) = r$. When $\epsilon = \infty$ this problem was solved in [11, 30] for the case when X is a linear Brownian motion and in [14] for the case when X is a jump-diffusion. In the case when $\epsilon = \infty$ and $K = 0$ this problem is known as the Russian optimal stopping problem [2, 14, 38, 39].

Our method for solving (4.5) consists of a verification technique, that is, we heuristically derive a candidate solution and then verify that it is indeed a solution. In particular, we will establish a link to the so-called McKean optimal stopping problem [1, 26] as well as make use of the principle of smooth or continuous fit [28, 32, 33] in a similar way to [23, 29]. As one would expect from the general theory of optimal stopping [33, 42], the optimal stopping time is the first entry time of the two-dimensional Markov process $(X, \overline{X})$ into a certain subset (the stopping region) of E. Interestingly, and as already alluded to above, it turns out that depending on the different parameters, the continuation region (the complement of the stopping region) is a connected set or consists of two disjoint components. In fact, in the former case it has a feature that resembles a bottleneck; see Theorem 4.5 and Figure 4.4. Furthermore, it will also be interesting from a technical point of view to see how the fact that the payoff depends not only on $\overline{X}$ but also on X (compare with [23, 29] where the payoff only depends on $\overline{X}$) enters the solution of the optimal stopping problem.

One of the assumptions above is that the underlying Lévy process is spectrally negative, that is, a Lévy process whose trajectories have only negative discontinuities. This restriction, which can be justified from a modelling point of view [8, 25], opens the door to the theory of scale functions for spectrally negative Lévy processes [20, 21] and essentially allows us to obtain the results in the form in which we are going to present them below. However, we believe that from a qualitative point of view the results should still hold even if one allowed X to be a general Lévy process. This would lead to an interesting phenomena where the process $(S, \overline{S})$ jumps from one component of the continuation region to the other one in the case when the continuation region consists of two parts.

We conclude this section with a brief overview of this article. In Section 4.2 we introduce some more notation and provide some necessary background. In Sections 4.3 and 4.6 we explain how to heuristically derive a candidate solution for (4.5). Our main results are presented in Section 4.4 and their proofs are given in Section 4.7. Finally, some examples are considered in Section 4.5.

4.2 Preliminaries

4.2.1 Spectrally negative Lévy processes

It is well known that a spectrally negative Lévy process X is characterised by its Lévy triplet (γ, σ, Π), where $\sigma \geq 0, \gamma \in \mathbb{R}$ and Π is a measure on $(-\infty, 0)$ satisfying the condition $\int_{(-\infty,0)}(1 \wedge x^2)\,\Pi(dx) < \infty$. By the Lévy–Itô decomposition, the latter may be represented in the form

$$X_t = \sigma B_t - \gamma t + X_t^{(1)} + X_t^{(2)}, \tag{4.6}$$

where $\{B_t : t \geq 0\}$ is a standard Brownian motion, $\{X_t^{(1)} : t \geq 0\}$ is a compound Poisson process with discontinuities of magnitude bigger than or equal to one and $\{X_t^{(2)} : t \geq 0\}$ is a square integrable martingale with discontinuities of magnitude strictly smaller than one and the three processes are mutually independent. In particular, if X is of bounded variation, the decomposition reduces to

$$X_t = \mathtt{d}t - \chi_t \tag{4.7}$$

where $\mathtt{d} := -\gamma - \int_{(-1,0)} x\,\Pi(dx) > 0$ and $\{\chi_t : t \geq 0\}$ is a driftless subordinator. Further let

$$\psi(\theta) := \mathbb{E}\big[e^{\theta X_1}\big]$$

be the Laplace exponent of X for all $\theta \in \mathbb{R}$ such that the expectation exists. Since X is spectrally negative this is at least the case for $\theta \geq 0$. It is known that ψ takes the form

$$\psi(\theta) = -\gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,0)} \big(e^{\theta x} - 1 - \theta x 1_{\{x>-1\}}\big)\,\Pi(dx), \quad \theta \geq 0.$$

When X has bounded variation, that is, $\sigma = 0$ and $\int_{(-1,0)} |x|\,\Pi(dx) < \infty$, we may always write

$$\psi(\theta) = \mathtt{d}\theta - \int_{(-\infty,0)} (1 - e^{\theta x})\,\Pi(dx), \quad \theta \geq 0. \tag{4.8}$$

The right-inverse of ψ is defined by

$$\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}$$

for $q \geq 0$.

For any spectrally negative Lévy process having $X_0 = 0$ we introduce the family of martingales

$$\exp(vX_t - \psi(v)t),$$

defined for any $v \in \mathbb{R}$ for which $\psi(v) < \infty$, and further the corresponding family of measures $\{\mathbb{P}^v\}$ with Radon–Nikodym derivatives

$$\left.\frac{d\mathbb{P}^v}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp(vX_t - \psi(v)t). \tag{4.9}$$

For all such v the measure $\mathbb{P}^v_x$ will denote the translation of $\mathbb{P}^v$ under which $X_0 = x$. In particular, under $\mathbb{P}^v_x$ the process X is still a spectrally negative Lévy process; cf. Theorem 3.9 in [21].

Finally, introduce the first passage times of X below and above $k \in \mathbb{R}$,

$$\tau_k^- := \inf\{t > 0 : X_t \leq k\} \quad \text{and} \quad \tau_k^+ := \inf\{t > 0 : X_t \geq k\}.$$

4.2.2 Scale functions

A special family of functions associated with spectrally negative Lévy processes is that of scale functions (cf. [20, 21]) which are defined as follows. For $q \geq 0$, the q-scale function $W^{(q)} : \mathbb{R} \longrightarrow [0, \infty)$ is the unique function whose restriction to $(0, \infty)$ is continuous and has Laplace transform

$$\int_0^\infty e^{-\theta x} W^{(q)}(x)\, dx = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q),$$

and is defined to be identically zero for $x \leq 0$. Further, we shall use the notation $W_v^{(q)}(x)$ to mean the q-scale function associated to X under $\mathbb{P}^v$. It is possible for fixed $x \geq 0$ to extend the mapping $q \mapsto W_v^{(q)}(x)$ to the complex plane (cf. Lemma 3.6 in [20]) and we have the following relationship

$$W^{(q)}(x) = e^{vx} W_v^{(q-\psi(v))}(x) \tag{4.10}$$

for $v \in \mathbb{R}$ such that $\psi(v) < \infty$ and $q \in \mathbb{C}$; cf. Lemma 3.7 in [20]. Moreover, the following regularity properties of scale functions are known; cf. Sections 2.3 and 3.1 of [20].

Smoothness: For all $q \geq 0$,

$$W^{(q)}|_{(0,\infty)} \in \begin{cases} C^1(0,\infty), & \text{if } X \text{ is of bounded variation and } \Pi \text{ has no atoms,} \\ C^1(0,\infty), & \text{if } X \text{ is of unbounded variation and } \sigma = 0, \\ C^2(0,\infty), & \sigma > 0. \end{cases}$$

Continuity at the origin: For all $q \geq 0$,

$$W^{(q)}(0+) = \begin{cases} \mathtt{d}^{-1}, & \text{if } X \text{ is of bounded variation,} \\ 0, & \text{if } X \text{ is of unbounded variation.} \end{cases} \tag{4.11}$$

Right-derivative at the origin: For all $q \geq 0$,

$$W_+^{(q)\prime}(0+) = \begin{cases} \frac{q+\Pi(-\infty,0)}{\mathtt{d}^2}, & \text{if } \sigma = 0 \text{ and } \Pi(-\infty,0) < \infty, \\ \frac{2}{\sigma^2}, & \text{if } \sigma > 0 \text{ or } \Pi(-\infty,0) = \infty, \end{cases} \tag{4.12}$$

where we understand the second case to be $+\infty$ when $\sigma = 0$.

The second scale function is $Z_v^{(q)}$ which is defined as follows. For $v \in \mathbb{R}$ such that $\psi(v) < \infty$ and $q \geq 0$ we define $Z_v^{(q)} : \mathbb{R} \longrightarrow [1, \infty)$ by

$$Z_v^{(q)}(x) = 1 + q \int_0^x W_v^{(q)}(z)\, dz. \tag{4.13}$$

This function can also be extended to $q \in \mathbb{C}$ for fixed $x \geq 0$.

For technical reasons, we require for the rest of the paper that $W^{(q)}$ is in $C^1(0,\infty)$ [and hence $Z^{(q)} \in C^2(0,\infty)$]. This is ensured by henceforth assuming that Π is atomless whenever X is of bounded variation.

4.3 First observations and candidate solution

The overall strategy to solve (4.5) is "guess and verify", that is, we try to "guess" the solution of (4.5) and once we have a candidate solution we verify that it is indeed a solution. This section is concerned with the guessing part of our approach. We will link (4.5) to the McKean optimal stopping problem (cf. [1, 26] and Section 11.2 of [21]) as well as to the general theory of optimally stopping a maximum process [31, 33] which will provide us with a candidate solution for (4.5). Assume throughout this section that $\epsilon \in \mathbb{R}$.

First of all, observe that if $s \geq \epsilon$, then then the process $\overline{X}_t \wedge \epsilon$ equals ϵ for all $t \geq 0$ and (4.5) becomes

$$V_\epsilon^*(x,s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s}[e^{-q\tau}(e^\epsilon - Ke^{X_\tau})^+] = K \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s}[e^{-q\tau}(K^{-1}e^\epsilon - e^{X_\tau})^+].$$

Up to the factor K in front of the supremum, this is nothing else than the McKean optimal stopping problem with strike $K^{-1}e^\epsilon$. The following result then follows directly from Corollary 11.3 in [21].

Proposition 4.1. *Fix $\epsilon \in \mathbb{R}$ and assume that $s \geq \epsilon$. The solution of* (4.5) *is given by*

$$V_\epsilon^*(x,s) = e^\epsilon Z^{(q)}(x - x_\epsilon^*) - Ke^x Z_1^{(q-\psi(1))}(x - x_\epsilon^*),$$

where

$$x_\epsilon^* := \epsilon + \begin{cases} \log\left(K^{-1}\frac{q}{\Phi(q)}\frac{\Phi(q)-1}{q-\psi(1)}\right), & q \neq \psi(1), \\ \log\left(K^{-1}\frac{q}{\psi'(1)}\right), & q = \psi(1), \end{cases}$$

and corresponding optimal stopping time $\tau_\epsilon^* := \inf\{t \geq 0 : X_t < x_\epsilon^*\}$.

Next, define the quantity

$$\eta := \begin{cases} \log\left(K\frac{\Phi(q)}{q}\frac{q-\psi(1)}{\Phi(q)-1}\right), & q \neq \psi(1), \\ \log\left(K\frac{\psi'(1)}{q}\right), & q = \psi(1), \end{cases} \tag{4.14}$$

and note that $\epsilon - x_\epsilon^* = \eta$. Moreover, equation (8.4) in [21] states that

$$\mathbb{E}[e^{\underline{X}_{\mathbf{e}_q}}] = \begin{cases} \frac{q}{\Phi(q)}\frac{\Phi(q)-1}{q-\psi(1)}, & q \neq \psi(1), \\ \frac{q}{\psi'(1)}, & q = \psi(1), \end{cases} \tag{4.15}$$

where $\mathbf{e}_q$ is an exponential random variable with parameter $q > 0$ independent of X. In particular, the terms on the right-hand side of (4.15) are smaller or equal than one.

Now we want to investigate the solution of (4.5) for $s < \epsilon$. To this end, assume temporarily that $\epsilon < x_\epsilon^*$ or, equivalently, $\eta < 0$, and hence $K < 1$ which implies that $e^{-qt}(e^{\overline{X}_t \wedge \epsilon} - Ke^{X_t})^+ = e^{-qt}(e^{\overline{X}_t \wedge \epsilon} - Ke^{X_t})$ as long as $\overline{X}_t \leq \epsilon$. We are now going to argue in the same way as described in [31], Section 3, page 6: The dynamics of $(X, \overline{X})$ are such that $\overline{X}$ remains constant at times when X is undertaking an excursion away from $\overline{X}$. Although $e^{\overline{X}_t \wedge \epsilon} - Ke^{X_t}$ increases with the depth of the excursion, the payoff during an excursion is bounded above by e^s, where s is the current value of $\overline{X}$ during the excursion. Due to the exponential discounting one should therefore not allow X to drop too far below $\overline{X}$ as otherwise the time it will take X to recover and reach value s will prove costly in terms of gain. Hence, given that $\overline{X}$ is at level s, there should be a point $g_\epsilon(s) > 0$ such that if the process X reaches or jumps below the value $s - g_\epsilon(s)$ we should stop. In more mathematical terms, we expect, as long as $\overline{X} < \epsilon$, an optimal strategy of the form

$$\inf\{t \geq 0 : \overline{X}_t - X_t \geq g_\epsilon(\overline{X}_t)\} \tag{4.16}$$

for some decreasing function $g_\epsilon : (-\infty, \epsilon) \to [0, \infty)$. Once $\overline{X}$ reaches level ϵ, Proposition 4.1 says that one should stop immediately as $\epsilon < x_\epsilon^*$. This means that g_ϵ has to satisfy the additional requirement $\lim_{s \uparrow \epsilon} g_\epsilon(s) = 0$. Summing up, we expect an optimal

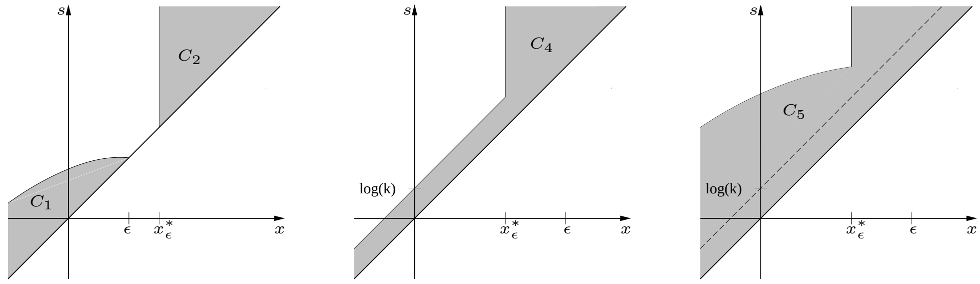

Fig. 4.2 *Left: expected continuation and stopping region when* $\epsilon < x_\epsilon^*$. *Middle: the set* C_4 *which is necessarily contained in the continuation region when* $\epsilon \geq x_\epsilon^*$ *and* $K > 1$. *Right: The expected continuation and stopping region when* $\epsilon \geq x_\epsilon^*$ *and* $K > 1$.

stopping time of the form

$$\rho_\epsilon = \inf\{t \geq 0 : (X_t, \overline{X}_t) \notin C_1 \cup C_2\}, \tag{4.17}$$

where $C_1 := \{(x,s) \in E : x \geq x_\epsilon^*\}$ and $C_2 := \{(x,s) \in E : s - x \geq g_\epsilon(s)\}$. The set $C_1 \cup C_2$ is usually called continuation region and it is shown in the drawing on the left-hand side in Figure 4.2.

Now assume that $\epsilon \geq x_\epsilon^*$ or, equivalently, $\eta \geq 0$, and that $K > 1$. Under these assumptions the situation looks quite different. Because $K > 1$ we see that $e^{-qt}(e^{\overline{X}_t} - Ke^{X_t})^+ = 0$ whenever $(X, \overline{X})$ lies in in the strip

$$C_3 := \{(x,s) \in E : s - \log(K) \leq x\}$$

and therefore it is never optimal to stop as long as the process $(X, \overline{X})$ lies in C_3. Combining this with Proposition 4.1, we see that the continuation region must at least contain the set $C_4 := C_3 \cup \{(x,s) \in E : x \geq x_\epsilon^*\}$; see middle drawing in Figure 4.2. The whole discussion in the previous paragraph applies here as well, except that one has to take into account the strip C_3. In other words, we look again for stopping strategies of the form (4.16) as long as $\overline{X} < \epsilon$, but the boundary condition $\lim_{s\uparrow\epsilon} g_\epsilon(s) = 0$ should be replaced by $\lim_{s\uparrow\epsilon} g_\epsilon(s) = \eta = \epsilon - x_\epsilon^* \geq 0$. The expected continuation region

$$C_5 := \{(x,s) \in E \,|\, s \leq \epsilon \text{ and } s - g_\epsilon(s) < x \text{ or } x \geq x_\epsilon^*\}$$

is pictorially displayed on the right-hand side in Figure 4.2. Finally, if $\epsilon \geq x_\epsilon^*$ and $K \leq 1$ a similar reasoning applies except that there will be no strip C_3.

The discussion so far leaves us with two questions:

- How to choose g_ϵ?
- Given g_ϵ, what is $\mathbb{E}_{x,s}\big[e^{-q\rho_\epsilon}\big(e^{\overline{X}_{\rho_\epsilon}\wedge\epsilon} - Ke^{X_{\rho_\epsilon}}\big)^+\big]$, where ρ_ϵ is either as in (4.17) or $\rho_\epsilon = \inf\{t \geq 0 : (X_t, \overline{X}_t) \notin C_5\}$?

These questions can be answered with the help of the so-called principle of smooth or continuous fit [28, 32, 33] which will provide an ordinary differential equation characterising g_ϵ and a candidate value function. The details are given in Section 4.6.

4.4 Main results

This section is the verification part of our "guess and verify" approach. Given the candidate solution derived in Sections 4.3 and 4.6, we now verify that it is indeed a solution. The proofs of all the results presented in this section are deferred to Section 4.7.

We begin by introducing an auxiliary function $f : (0, \infty) \to \mathbb{R}$ which is defined by

$$f(z) := Z^{(q)}(z) - \big(q - K(q - \psi(1))e^{-z}\big)W^{(q)}(z).$$

This function will play an important role throughout the remainder of this article and hence we spend some time investigating some of its properties.

Lemma 4.2. *Suppose that $q > \psi(1)$.*

(a) If $0 < K < q/(q - \psi(1))$, then f is strictly decreasing on $(0, \infty)$.

(b) If $K \geq q/(q - \psi(1))$, then f is strictly increasing on $(0, \beta_0]$ and strictly decreasing on (β_0, ∞), where $\beta_0 := \log(K(q - \psi(1))/q) \geq 0$.

In both cases f tends to $-\infty$ as $z \to \infty$.

Next, denote by $\mathcal{G}$ be the general class of spectrally negative Lévy processes and define the subclass

$$\begin{aligned} \mathcal{H}_{q,K} \; := \; & \{X \in \mathcal{G} : \; X \text{ is of unbounded variation or } X \text{ is} \\ & \text{of bounded variation with } \mathtt{d} > q - K(q - \psi(1))\}. \end{aligned}$$

Furthermore, define the quantity

$$k^* := \inf\{z > \eta \vee 0 : f(z) \leq 0\} \in [0, \infty], \tag{4.18}$$

where η was defined in (4.14) and we understand $\inf \varnothing = \infty$.

Lemma 4.3.

(a) If $q > \psi(1)$ and $X \in \mathcal{H}_{q,K}$, then $k^* \in (\eta \vee 0, \infty)$.

(b) If $q > \psi(1)$ and $X \in \mathcal{G} \setminus \mathcal{H}_{q,K}$, then $k^* = 0$.

(c) If $q \leq \psi(1)$, then $k^* = \infty$.

We are now in a position to define the function g_ϵ which will, as we will see in due course, describe the optimal boundary of (4.5).

Lemma 4.4. *Fix $\epsilon \in \mathbb{R}$. Moreover, suppose that $q > \psi(1)$ and $X \in \mathcal{H}_{q,K}$ or $q \leq \psi(1)$. Then there exists a unique solution $g_\epsilon : (-\infty, \epsilon) \to (\eta \vee 0, k^*)$ of the differential equation*

$$g_\epsilon'(s) = 1 - \frac{Z^{(q)}(g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))(q - K(q - \psi(1))e^{-g_\epsilon(s)})} \qquad \text{on } (-\infty, \epsilon) \tag{4.19}$$

satisfying $\lim_{s \uparrow \epsilon} g_\epsilon(s) = \eta \vee 0$. In particular, $\lim_{s \downarrow -\infty} g_\epsilon(s) = k^*$.

On the other hand, when $q > \psi(1)$ and $X \in \mathcal{G} \setminus \mathcal{H}_{q,K}$, we will adopt the convention that $g_\epsilon(s) = k^* = 0$ for $s \in (-\infty, \epsilon)$.

It is possible to say a bit more about the function g_ϵ in the case when $q > \psi(1)$ and $X \in \mathcal{H}_{q,K}$ or $q \leq \psi(1)$. Specifically, with the help of (4.11) and Lemma 3.3 in [20] one obtains

$$\lim_{s \uparrow \epsilon} g_\epsilon'(s) = 1 - \frac{Z^{(q)}(\eta \vee 0)}{W^{(q)}(\eta \vee 0)(q - K(q - \psi(1))e^{-(\eta \vee 0)})}$$

and

$$\lim_{s \downarrow -\infty} g_\epsilon'(s) = \begin{cases} 0, & \text{if } q > \psi(1) \text{ and } X \in \mathcal{H}_{q,K}, \\ 1 - \Phi(q)^{-1}, & \text{if } q \leq \psi(1). \end{cases}$$

Note in particular that $\lim_{s \uparrow \epsilon} g_\epsilon'(s) = -\infty$ whenever $\eta \leq 0$ and X is of unbounded variation and that this cannot happen when X is of bounded variation as $W^{(q)}(0) > 0$. Put differently, the shape of g_ϵ at ϵ may change according to the path variation of X. A similar observation has already been made in [29] which treats (4.5) for $K = 0$. The differences in the behaviour of g_ϵ are illustrated in Figure 4.3.

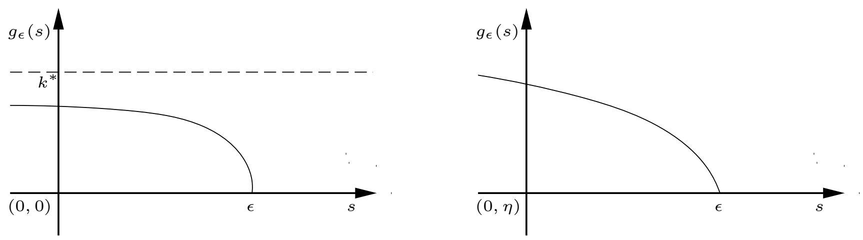

Fig. 4.3 *In both pictures it is supposed that X is of unbounded variation. However, on the left-hand side we additionally assume that $q > \psi(1)$ [and hence $k^* \in (\eta \vee 0, \infty)$] and $\eta < 0$, whereas on the right-hand side it is assumed that $q \leq \psi(1)$ (and hence $k^* = \infty$) and $\eta > 0$.*

In order to state the main result, we need some more notation. Define the continuation regions

$$C_I^* = C_{I,g_\epsilon}^* := \{(x,s) \in E \,|\, s \leq \epsilon \text{ and } s - g_\epsilon(s) < x \leq s\},$$
$$C_{II}^* = C_{II,\epsilon}^* := \{(x,s) \in E \,|\, x > x_\epsilon^*\}$$

and the stopping region $D^* = D^*_{g_\epsilon} = E \setminus (C^*_I \cup C^*_{II})$. Note that if $q > \psi(1)$ and $X \in \mathcal{G} \setminus \mathcal{H}_{q,K}$, then $C^*_I = \varnothing$.

Theorem 4.5. *Fix $\epsilon \in \mathbb{R}$. The solution of (4.5) is given by*

$$V^*_\epsilon(x,s) = \begin{cases} e^\epsilon Z^{(q)}(x - x^*_\epsilon) - Ke^x Z_1^{(q-\psi(1))}(x - x^*_\epsilon), & s \geq \epsilon, \\ e^s Z^{(q)}(x - s + g_\epsilon(s)) - Ke^x Z_1^{(q-\psi(1))}(x - s + g_\epsilon(s)), & s < \epsilon, \end{cases}$$

with corresponding optimal strategy $\rho^*_\epsilon := \inf\{t \geq 0 : (X_t, \overline{X}_t) \in D^*_{g_\epsilon}\}$ and g_ϵ as in Lemma 4.4.

Remark 4.6. *Let $\epsilon \in \mathbb{R}$ and suppose that $q > \psi(1)$ and $X \in \mathcal{H}_{q,K}$ or $q \leq \psi(1)$. Similarly to Remark 3.4 one can show that whenever $\eta \leq 0$ we have $\mathbb{P}_{x,s}[\tau^*_\epsilon = \tau^+_\epsilon] > 0$ for $(x,s) \in E$ such that $s < \epsilon$. We omit the details.*

Some examples for the stopping and continuation region are pictorially displayed in Figure 4.4. In particular, let us emphasise that the continuation region is connected if and only if $\epsilon > x^*_\epsilon$ or, equivalently, $\eta > 0$; otherwise it consists of two disjoint components. Moreover, in the case when $\epsilon > x^*_\epsilon$, one sees that the process $(X, \overline{X})$ has to squeeze through a "bottleneck" to get into the region where the second component of $(X, \overline{X})$ is larger or equal to ϵ. It is this "special" feature of the continuation region that has motivated the name "Bottleneck option" for payoffs of type (4.1). Also note that provided $X \in \mathcal{H}_{q,K}$ it follows from the definition of η in (4.14) that the critical value in order to see a bottleneck or not is given by $K = \frac{q(\Phi(q)-1)}{\Phi(q)(q-\psi(1))}$ if $q \neq \psi(1)$ and $K = \frac{q}{\psi'(1)}$ if $q = \psi(1)$.

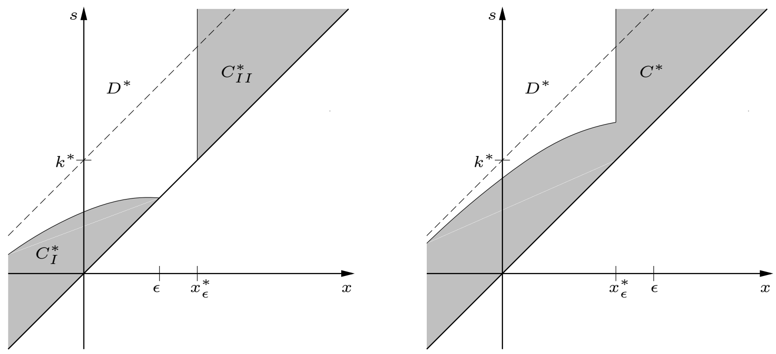

Fig. 4.4 *In both pictures it is supposed that X is of unbounded variation and $q > \psi(1)$. The difference is that on the left-hand side we have $\epsilon < x^*_\epsilon$ which leads to a continuation region consisting of two components, whereas on the right-hand side we have $\epsilon > x^*_\epsilon$ resulting in a connected continuation region.*

It is also interesting to investigate what happens if no cap is present; that is, $\epsilon = \infty$. In this case, problem (4.5) reads

$$V^*_\infty(x,s) = \sup_{\tau \in \mathcal{M}} \mathbb{E}_{x,s}[e^{-q\tau}(e^{\overline{X}_\tau} - Ke^{X_\tau})^+]. \tag{4.20}$$

By a change of measure according to (4.9) one could now reduce this problem to a one-dimensional optimal stopping problem for the reflected process $Y = \{Y_t : t \geq 0\}$, where $Y_t = \overline{X}_t - X_t$; see [2] for a very similar argument in the case when $K = 0$ in (4.20). In this case the general theory of optimal stopping [33] suggests that the optimal stopping time is an upcrossing time of the process Y at a certain constant level. This is indeed the case and one could in principle prove this by actually solving the resulting one-dimensional optimal stopping problem for Y. Here, however, we will solve (4.20) with the help of the work already done in Theorem 4.5 and a simple limiting procedure.

Corollary 4.7. *Assume that $\epsilon = \infty$.*

i) Suppose that $q > \psi(1)$. The solution of (4.5) *is given by*

$$V_\infty^*(x,s) = e^s Z^{(q)}(x - s + k^*) - K e^x Z_1^{(q-\psi(1))}(x - s + k^*)$$

with corresponding optimal strategy $\rho_\infty^* := \inf\{t \geq 0 : \overline{X}_t - X_t \geq k^*\}$, where $k^* \in [0,\infty)$ is defined in (4.18).

ii) If $q \leq \psi(1)$, then there is no solution to (4.5) *and $V_\infty^*(x,s) \equiv \infty$.*

Observe that if $q \leq \psi(1)$ then the value function is equal to infinity. Of course, this is not possible in the presence of a cap $\epsilon \in \mathbb{R}$.

4.5 Example

The solution of (4.5) in Theorems 4.5 and 4.7 is given semi-explicitly in terms of scale functions and a specific solution g_ϵ of the ordinary differential equation (4.19). A first step towards more explicit solutions of (4.5) is looking at processes X where explicit expressions for $W^{(q)}$ and $Z^{(q)}$ are available. In recent years various authors have found several processes whose scale functions are explicitly known; for instance, see Example 1.3 as well as Chapters 4 and 5 in [20]. Here we will consider one example where X has jumps. Specifically, suppose that X is an α-stable process, where $\alpha \in (1,2]$ with Laplace exponent $\psi(\theta) = \theta^\alpha, \theta \geq 0$. Moreover, suppose that $q > \psi(1)$ which in this case means that $q > 1$. It is known from Example 4.17 of [20] and Subsection 8.3 of [2] that, for $x \geq 0$,

$$W^{(q)}(x) = x^{\alpha-1} E_{\alpha,\alpha}(q x^\alpha) \quad \text{and} \quad Z^{(q)}(x) = E_{\alpha,1}(q x^\alpha),$$

where $E_{\alpha,\beta}$ is the two-parameter Mittag–Leffler function which is defined for $\alpha, \beta > 0$ as

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)}.$$

By definition of $Z_1^{(q)}$ [see (4.13)] and (4.10) we obtain

$$Z_1^{(q-\psi(1))}(x) = 1 + (q-\psi(1)) \int_0^x e^{-y} W^{(q)}(y)\, dy, \qquad x \geq 0.$$

In order to compute the boundary one might try to solve (4.19) numerically, but this is not straightforward as there might be no initial point to start a numerical scheme from and, moreover, the possibility of g_ϵ having infinite gradient at ϵ might lead to inaccuracies in the numerical scheme. Therefore, we follow a different route which avoids these difficulties. Instead of looking at g_ϵ, we rather focus on its inverse

$$H(s) = \epsilon - \int_{\eta\vee 0}^{s} \frac{(q - K(q-\psi(1))e^{-u})W^{(q)}(u)}{Z^{(q)}(u) - (q-K(q-\psi(1))e^{-u})W^{(q)}(u)}\, du, \quad s \in (\eta \vee 0, k^*), \quad (4.21)$$

where $k^* \in (0,\infty)$ is the unique root of

$$Z^{(q)}(z) - (q - K(q-\psi(1))e^{-z})W^{(q)}(z) = 0.$$

In fact, passing to the inverse is a standard trick in this setting and, for instance, used in [33]. As H is the inverse of g_ϵ, plotting $(H(y), y)$ for $y \in (\eta \vee 0, k^*)$ gives graphical representations of $s \mapsto g_\epsilon(s)$, $s \in (-\infty, \epsilon)$; see Figure 4.5. Similarly, plotting $(H(y) - y, H(y))$ for $y \in (\eta \vee 0, k^*)$ produces visualisations of the optimal stopping boundary in the (x,s)-plane; see Figure 4.5. Further, in order to obtain the continuation and stopping region for the original problem involving the processes S and $\overline{S}$, one only needs to plot $\big(\exp(H(y)-y), \exp(H(y))\big)$ for $y \in (\eta \vee 0, k^*)$; see Figure 4.5. Because we are unable to compute the integral in (4.21) explicitly, we use numerical integration in Matlab to obtain an approximation of the integral. We also use Matlab to compute the Mittag–Leffler function (cf. [35]) and to solve the equation for k^*.

Of course, once one starts to compute things numerically there are many more examples that could be looked at. For instance, the case Black–Scholes case when X corresponds to a linear Brownian motion or when X is jump-diffusion. Similar results in this direction for a slightly different problem have been considered in [29] and could be carried over to the setting here in a straightforward way.

4.6 Guess via principle of smooth or continuous fit

The goal of this section is to answer the two questions raised at the end of Section 4.3. The argument presented here is an adaptation of [31] to our setting. It has already been successfully applied in [23, 29] in similar/related situations. The difference to [23, 29], however, is that here the payoff also depends on X and not only $\overline{X}$. As we will see in due course, this can be dealt with by a change of measure which essentially puts oneself back into the situation where the payoff only depends on $\overline{X}$. Throughout this

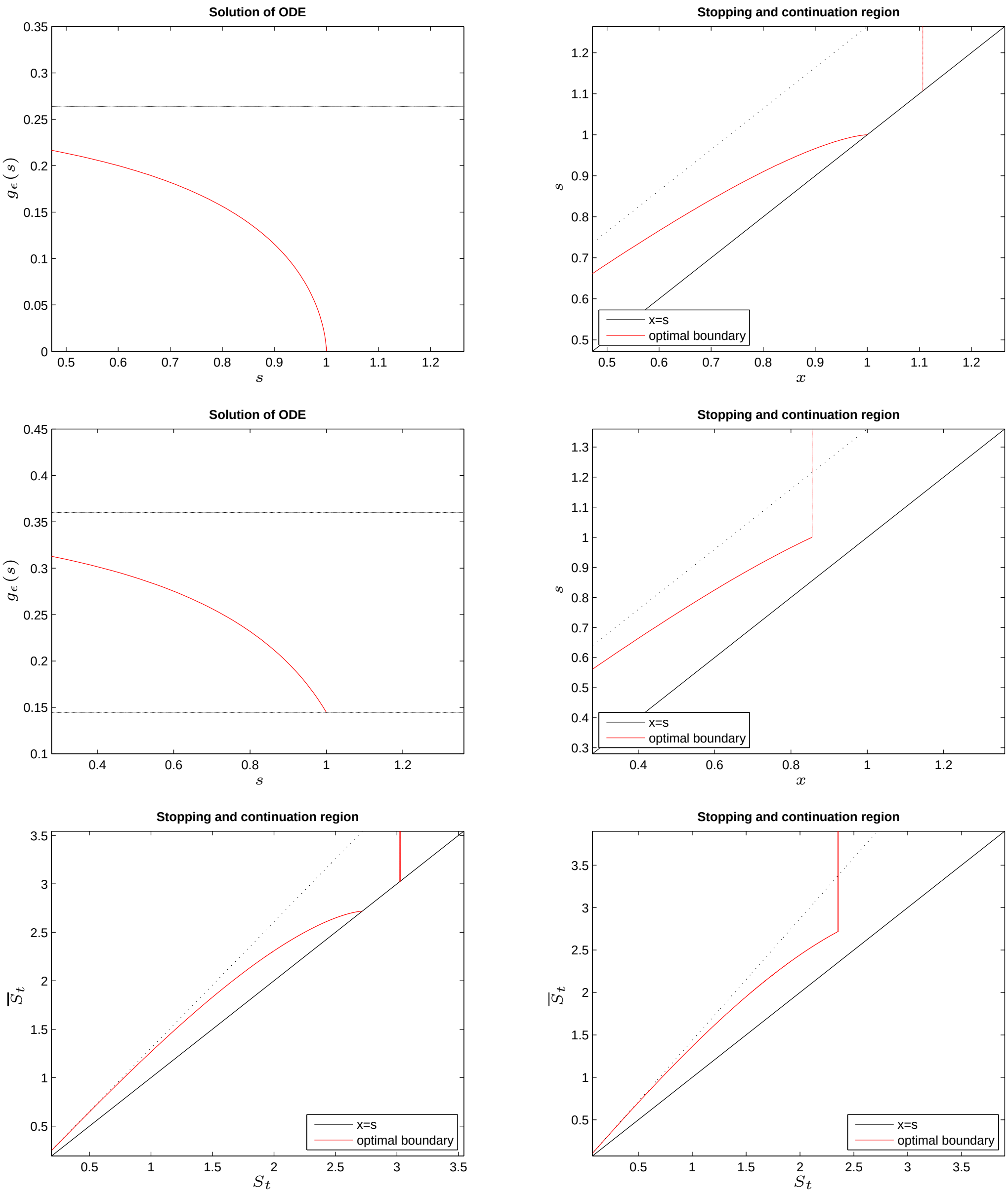

Fig. 4.5 *Top two pictures: A visualisation of* $s \mapsto g_\epsilon(s)$ *and the resulting optimal boundary when* $q = 3$, $\epsilon = 1$, $K = 0.7$ *and* $\alpha = 1.5$. *It follows that* $x^*_\epsilon \approx 1.11$, $\eta \approx -0.11$ *and* $k^* \approx 0.26$. *Middle two pictures: A visualisation of* $s \mapsto g_\epsilon(s)$ *and the resulting optimal boundary when* $q = 3$, $\epsilon = 1$, $K = 0.9$ *and* $\alpha = 1.5$. *It follows that* $x^*_\epsilon \approx 0.86$, $\eta \approx 0.14$ *and* $k^* \approx 0.36$. *Bottom two pictures: The corresponding continuation and stopping regions for the original problem for* $(S, \overline{S})$ *with* $C = e$.

section we will assume that $s < \epsilon$. Moreover, for simplicity, suppose that $q > \psi(1)$.

To begin with assume that X is of unbounded variation. We will deal with the bounded variation case later; see page 89. From the general theory of optimal stopping (cf. Section 13 of [33]) we informally expect the value function

$$U_\epsilon(x,s) := \mathbb{E}_{x,s}\big[e^{-q\rho_\epsilon}\big(e^{\overline{X}_{\rho_\epsilon}\wedge\epsilon} - Ke^{X_{\rho_\epsilon}}\big)^+\big],$$

where ρ_ϵ was defined in Section 4.3, to satisfy the system

$$\begin{array}{rcll} \Gamma U_\epsilon(x,s) &=& qU_\epsilon(x,s) & \text{for } s-g_\epsilon(s)<x<s \text{ with } s \text{ fixed}, \\ \frac{\partial U_\epsilon}{\partial s}(x,s)\big|_{x=s-} &=& 0 & \text{(normal reflection)}, \\ U_\epsilon(x,s)|_{x=(s-g_\epsilon(s))+} &=& e^s - Ke^{s-g_\epsilon(s)} & \text{(instantaneous stopping)}, \end{array} \tag{4.22}$$

where Γ is the infinitesimal generator of the process X under $\mathbb{P}_0$. Moreover, the principle of smooth fit [28, 33] suggests that this system should be complemented by

$$\lim_{x\downarrow s-g_\epsilon(s)} \frac{\partial U_\epsilon}{\partial x}(x,s) = -Ke^{s-g_\epsilon(s)} \quad \text{(smooth fit)}. \tag{4.23}$$

Note that, although the smooth fit condition is not necessarily part of the general theory, it is imposed since by the "rule of thumb" outlined in Section 7 in [1] it should hold in this setting because of path regularity. This belief will be vindicated when we show that system (4.22) together with (4.23) leads to the solution of (4.5).

Next, splitting over the events $\{\rho_\epsilon < \tau_s^+\}$ and $\{\rho_\epsilon > \tau_s^+\}$ in the first equality and applying the strong Markov property at τ_s^+ and a change of measure according to (4.9) in the second equality gives

$$\begin{aligned} U_\epsilon(x,s) &= e^s\mathbb{E}_{x,s}\big[e^{-q\tau^-_{s-g_\epsilon(s)}}1_{\{\tau^-_{s-g_\epsilon(s)}<\tau_s^+\}}\big] - K\mathbb{E}_{x,s}\big[e^{-q\tau^-_{s-g_\epsilon(s)}+X_{\tau^-_{s-g(s)}}}1_{\{\tau^-_{s-g_\epsilon(s)}<\tau_s^+\}}\big] \\ &\quad +\mathbb{E}_{x,s}\big[e^{-q\rho_\epsilon}\big(e^{\overline{X}_{\rho_\epsilon}\wedge\epsilon} - Ke^{X_{\rho_\epsilon}}\big)^+1_{\{\tau^-_{s-g(s)}>\tau_s^+\}}\big] \\ &= e^s\mathbb{E}_{x,s}\big[e^{-q\tau^-_{s-g_\epsilon(s)}}1_{\{\tau^-_{s-g_\epsilon(s)}<\tau_s^+\}}\big] - Ke^x\mathbb{E}^1_{x,s}\big[e^{-(q-\psi(1))\tau^-_{s-g_\epsilon(s)}}1_{\{\tau^-_{s-g_\epsilon(s)}<\tau_s^+\}}\big] \\ &\quad +\mathbb{E}_{x,s}\big[e^{-q\tau_s^+}1_{\{\tau^-_{s-g_\epsilon(s)}>\tau_s^+\}}\big]U_\epsilon(s,s). \end{aligned}$$

Furthermore, using Proposition 1 of [2] and rearranging terms in the first equality and applying (4.10) in the second equality shows that

$$\begin{aligned} U_\epsilon(x,s) &= e^sZ^{(q)}(x-s+g_\epsilon(s)) - Ke^xZ_1^{(q-\psi(1))}(x-s+g_\epsilon(s)) \\ &\quad -e^sW^{(q)}(x-s+g_\epsilon(s))\frac{Z^{(q)}(g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))} + \frac{W^{(q)}(x-s+g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))}U_\epsilon(s,s) \end{aligned}$$

$$
\begin{aligned}
&+Ke^x W_1^{(q-\psi(1))}(x-s+g_\epsilon(s))\frac{Z_1^{(q-\psi(1))}(g_\epsilon(s))}{W_1^{(q-\psi(1))}(g_\epsilon(s))}\\
= \quad & e^s Z^{(q)}(x-s+g_\epsilon(s)) - Ke^x Z_1^{(q-\psi(1))}(x-s+g_\epsilon(s))\\
&-e^s W^{(q)}(x-s+g_\epsilon(s))\frac{Z^{(q)}(g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))} + \frac{W^{(q)}(x-s+g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))}U_\epsilon(s,s)\\
&+Ke^s W^{(q)}(x-s+g_\epsilon(s))\frac{Z_1^{(q-\psi(1))}(g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))}.
\end{aligned}
$$

The smooth fit condition in (4.23) now implies that

$$
\frac{W^{(q)\prime}(x-s+g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))}\Big[e^s Z^{(q)}(g_\epsilon(s)) - U_\epsilon(s,s) - Ke^s Z_1^{(q-\psi(1))}(g_\epsilon(s))\Big] \to 0
$$

as $x \downarrow s - g_\epsilon(s)$. However, by (4.12) the first factor tends to a strictly positive value or infinity which shows that

$$
U_\epsilon(s,s) = e^s Z^{(q)}(g_\epsilon(s)) - Ke^s Z_1^{(q-\psi(1))}(g_\epsilon(s)).
$$

This would mean that for $(x,s) \in E$ such that $s - g_\epsilon(s) < x < s$ we have

$$
U_\epsilon(x,s) = e^s Z^{(q)}(x-s+g_\epsilon(s)) - Ke^x Z_1^{(q-\psi(1))}(x-s+g_\epsilon(s)). \tag{4.24}
$$

Having derived the form of a candidate optimal value function U_ϵ, we still need to do the same for g_ϵ. Using the normal reflection condition (4.22) shows that our candidate function g_ϵ should satisfy the differential equation

$$
g_\epsilon'(s) = 1 - \frac{Z^{(q)}(g_\epsilon(s))}{W^{(q)}(g_\epsilon(s))(q - K(q-\psi(1))e^{-g_\epsilon(s)})} \qquad \text{on } (-\infty, \epsilon) \tag{4.25}
$$

If X is of bounded variation, we informally expect from the general theory that U_ϵ satisfies the first two equations of (4.22). Additionally, the principle of continuous fit [1, 32] suggests that the system should be complemented by

$$
\lim_{x \downarrow s - g_\epsilon(s)} U_\epsilon(x,s) = e^s - Ke^{s-g_\epsilon(s)} \quad \text{(continuous fit)}.
$$

A very similar argument as above produces the same candidate value function and the same ordinary differential equation for g_ϵ.

It remains to check that the heuristic argument presented above leads to the solution of (4.5) – this is essentially the content of Theorem 4.5.

4.7 Proofs

Proof of Lemma 4.2. Using the assumed regularity of $W^{(q)}$ and relation (4.10) in the second equality one sees that

$$\begin{aligned} f'(z) &= (q - K(q-\psi(1))e^{-z})(W^{(q)}(z) - W^{(q)\prime}(z)) \\ &= \big(q - K(q-\psi(1))e^{-z}\big)e^{\Phi(q)z}\big(W_{\Phi(q)}(z)(1-\Phi(q)) - W'_{\Phi(q)}(z)\big). \end{aligned}$$

Since $\Phi(q) > 1$, it holds that $W_{\Phi(q)}(z)(1-\Phi(q)) - W'_{\Phi(q)}(z) < 0$ for $z > 0$ and hence the stated monotonicity properties of f follow from the monotonicity properties of the map $z \mapsto q - K(q-\psi(1))e^{-z}$. As for the behaviour of $f(z)$ for large z, we infer from Lemma 3.3 in [20] that

$$\lim_{z\to\infty} f(z)/(qW^{(q)}(z)) = \Phi(q)^{-1} - 1. \tag{4.26}$$

Again using (4.10), we have $W^{(q)}(z) = e^{\Phi(q)z}W_{\Phi(q)}(z)$ which tends to infinity as $z \to \infty$. As $\Phi(q) > 1$, we conclude that $\lim_{z\to\infty} f(z) = -\infty$. $\square$

Proof of Lemma 4.3.

(a) First suppose that X has paths of unbounded variation. By (4.11) this necessarily means that $W^{(q)}(0+) = 0$. Thus we see that $f(0+) = 1$ and the existence of a unique root $k^* > 0$ of $f(z) = 0$ is guaranteed by Lemma 4.2 and the intermediate value theorem. Moreover, one needs to check whether $k^* > \eta$ whenever $\eta > 0$. Since k^* is a root of $f(z) = 0$, we have

$$\frac{Z^{(q)}(k^*)}{W^{(q)}(k^*)} = q - K(q-\psi(1))e^{-k^*}. \tag{4.27}$$

Since the map $z \mapsto Z^{(q)}(z)/W^{(q)}(z), z > 0$, is decreasing (cf. equation (45) of [20]) and because of Lemma 3.3 in [20], the left-hand side of (4.27) is (strictly) bounded below by $q/\Phi(q)$. Hence, after some algebra, one sees that

$$k^* > \log\left(K\frac{\Phi(q)}{q}\frac{q-\psi(1)}{\Phi(q)-1}\right) = \eta.$$

Now suppose that X has paths of finite variation and $\mathtt{d} > q - K(q-\psi(1))$. In this case we see that $f(0+) > 0$. Using Lemma 4.2 in conjunction with the intermediate value theorem shows again that there exists a unique root $k^* > 0$ of $f(z) = 0$. The fact that $k^* > \eta$ whenever $\eta > 0$ follows as above.

(b) The fact that $0 < \mathtt{d} \le q - K(q-\psi(1))$ implies on the one hand that $f(0+) \le 0$ and on the other hand that $K < q/(q-\psi(1))$. By Lemma 4.2 we therefore have $f(z) < 0$ for $z > 0$. To conclude that $k^* = 0$ it remains to check that $\eta \le 0$. Since

$\mathtt{d} \leq q - K(q - \psi(1))$ we have $K \leq (q - \mathtt{d})/(q - \psi(1))$. Combining this with $\psi(1) < \mathtt{d}$ [see (4.8)] we get

$$\eta = \log\left(K\frac{\Phi(q)}{q}\frac{q-\psi(1)}{\Phi(q)-1}\right) \leq \log\left(\frac{\Phi(q)}{q}\frac{q-\mathtt{d}}{\Phi(q)-1}\right) \leq \log\left(\frac{\Phi(q)}{q}\frac{q-\psi(1)}{\Phi(q)-1}\right).$$

It now follows by (4.15) that $\eta \leq 0$.

(c) First assume that $q < \psi(1)$ and assume for a contradiction that there exists a $z_0 > \eta \vee 0$ such that $f(z_0) \leq 0$. Since $Z^{(q)}(z_0)/W^{(q)}(z_0)$ is bounded below by $q/\Phi(q)$ [as explained in (a)], it follows that

$$\frac{q}{\Phi(q)} < q - K(q - \psi(1))e^{-z_0}$$

or, after some straightforward algebra and using that $q < \psi(1)$,

$$z_0 < \log\left(K\frac{\Phi(q)}{q}\frac{q-\psi(1)}{\Phi(q)-1}\right) = \eta.$$

This is a contraction to $z_0 \geq \eta \vee 0$ and hence $f(z) > 0$ for $z > \eta \vee 0$. In other words, $k^* = \infty$. Finally, if $q = \psi(1)$, we have $f(z) = Z^{(q)}(z) - qW^{(q)}(z) > 0$ for $z > 0$ by equation (42) of [20] and hence again $k^* = \infty$. $\square$

Proof of Lemma 4.4. The proof is very similar to the proof of Lemma 4.1 in [29]. The idea is to construct the solution g_ϵ by defining a suitable bijection from $(\eta \vee 0, k^*)$ to $(-\infty, \epsilon)$ whose inverse satisfies the differential equation and the boundary conditions. We will present the case when $q > \psi(1)$ and $X \in \mathcal{H}_{q,K}$. The case when $q \leq \psi(1)$ follows analogously to the proof of Lemma 4.1 in [29].

Assume that $q > \psi(1)$ and $X \in \mathcal{H}_{q,K}$. It follows from Lemma 4.2 and 4.3 that $k^* \in (\eta \vee 0, \infty)$ and that the function

$$s \mapsto h(s) := 1 - \frac{Z^{(q)}(s)}{W^{(q)}(s)(q - K(q - \psi(1))e^{-s})}$$

is strictly negative on $(\eta \vee 0, k^*)$. Moreover, $\lim_{s \downarrow \eta \vee 0} h(s) \in [-\infty, 0)$ and $\lim_{s \uparrow k^*} h(s) = 0$. These properties imply that the function $H : (\eta \vee 0, k^*) \to (-\infty, \epsilon)$ defined by

$$H(s) := \epsilon + \int_{\eta \vee 0}^{s} \frac{1}{h(u)}\, du = \epsilon - \int_{\eta \vee 0}^{s} \frac{W^{(q)}(u)(q - K(q - \psi(1))e^{-u})}{f(u)}\, du$$

is strictly decreasing. If we can show that the integral tends to ∞ as s approaches k^*, we could deduce that H is a bijection from $(\eta \vee 0, k^*)$ to $(-\infty, \epsilon)$. Indeed, by l'Hôspital's rule and due to the fact that $f'(k^*) < 0$ we have

$$\lim_{s \uparrow k^*} \frac{k^* - s}{f(s)} = \frac{-1}{f'(k^*)} =: c > 0.$$

Hence there exists a $\delta > 0$ and $s_0 > \eta \vee 0$ such that $c - \delta > 0$ and

$$\frac{1}{f(s)} > \frac{c-\delta}{k^* - s} \quad \text{for } s_0 < s < k^*.$$

Thus it follows that

$$\lim_{s \uparrow k^*} H(s) \leq \epsilon - (c - \delta) \lim_{s \uparrow k^*} \int_{s_0}^{s} \frac{W^{(q)}(u)(q - K(q - \psi(1))e^{-u})}{k^* - u} \, du = -\infty.$$

The discussion above permits us to define $g_\epsilon := H^{-1} \in C^1((-\infty, \epsilon); (\eta \vee 0, k^*))$. In particular, differentiating g_ϵ gives

$$g_\epsilon'(s) = \frac{1}{H'(g_\epsilon(s))} = 1 - \frac{Z^{(q)}(g_\epsilon(s))}{qW^{(q)}(g_\epsilon(s))(q - K(q - \psi(1))e^{-g_\epsilon(s)})}$$

for $s \in (-\infty, \epsilon)$, and g_ϵ satisfies $\lim_{s \to -\infty} g_\epsilon(s) = k^*$ and $\lim_{s \uparrow \epsilon} g_\epsilon(s) = \eta \vee 0$ by construction. Finally, uniqueness follows as in the last part of the proof of Lemma 4.1 in [29]. $\square$

Proof of Theorem 4.5. Define for $(x, s) \in E$ the function

$$V_\epsilon(x,s) := \begin{cases} e^{\epsilon} Z^{(q)}(x - x_\epsilon^*) - K e^x Z_1^{(q-\psi(1))}(x - x_\epsilon^*), & s \geq \epsilon, \\ e^{s} Z^{(q)}(x - s + g_\epsilon(s)) - K e^x Z_1^{(q-\psi(1))}(x - s + g_\epsilon(s)), & s < \epsilon. \end{cases}$$

Because of the infinite horizon and Markovian claim structure of (4.5) it is enough to establish the following three results whose proofs are given below:

Lemma 4.8. *We have $V_\epsilon(x,s) \geq (e^{s \wedge \epsilon} - K e^x)^+$ for all $(x,s) \in E$.*

Lemma 4.9. *The process $e^{-qt} V_\epsilon(X_t, \overline{X}_t)$, $t \geq 0$, is a right-continuous $\mathbb{P}_{x,s}$-supermartingale for $(x,s) \in E$.*

Lemma 4.10. *We have $V_\epsilon(x,s) = \mathbb{E}_{x,s}\big[e^{-q\rho_\epsilon^*}\big(e^{\overline{X}_{\rho_\epsilon^*} \wedge \epsilon} - K e^{X_{\rho_\epsilon^*}}\big)^+\big]$ for all $(x,s) \in E$.*

To see why these three results suffice, note that Lemmas 4.8 and 4.9 together with Fatou's lemma in the second inequality and Doob's stopping theorem in the third inequality show that for $\tau \in \mathcal{M}$ and $(x,s) \in E$,

$$\begin{aligned} \mathbb{E}_{x,s}\big[e^{-q\tau}\big(e^{\overline{X}_\tau \wedge \epsilon} - K e^{X_\tau}\big)^+\big] &\leq \mathbb{E}_{x,s}\big[e^{-q\tau} V_\epsilon(X_\tau, \overline{X}_\tau)\big] \\ &\leq \liminf_{t \to \infty} \mathbb{E}_{x,s}\big[e^{-q(t \wedge \tau)} V_\epsilon(X_{t \wedge \tau}, \overline{X}_{t \wedge \tau})\big] \\ &\leq V_\epsilon(x,s). \end{aligned}$$

In view of Lemma 4.10 this implies $V_\epsilon^* = V_\epsilon$ and that ρ_ϵ^* is optimal. $\square$

Proof of Lemma 4.8. Choosing $\tau = 0$ in Proposition 4.1 shows that

$$V_\epsilon(x,s) \geq (e^\epsilon - Ke^x)^+$$

for $(x,s) \in E$ such that $s \geq \epsilon$. Hence, we can restrict ourselves to proving the assertion for $x \leq s < \epsilon$.

As for a first step, we claim that

$$g_\epsilon(s) \geq \eta \vee 0 \geq \log(K) \vee 0, \qquad s \in (-\infty, \epsilon). \tag{4.28}$$

If $q > \psi(1)$ and $X \in \mathcal{H}_{q,K}$ or $q \leq \psi(1)$ then the first inequality in (4.28) holds by construction of g_ϵ; see Lemma 4.4. On the other hand, if $q > \psi(1)$ and $X \in \mathcal{G} \setminus \mathcal{H}_{q,K}$, we need to show that $\eta \leq 0$ for the first inequality to be true, and this was done in the proof of part (b) of Lemma 4.3. The second inequality follows by definition of η and (4.15).

Next, using (4.10) in the first equality and a change of variables in the second equality, we may rewrite $V_\epsilon(x,s)$ as

$$\begin{aligned}
&e^s Z^{(q)}(x-s+g_\epsilon(s)) - Ke^x Z_1^{(q-\psi(1))}(x-s+g_\epsilon(s)) \qquad (4.29)\\
&= e^s - Ke^x + qe^s \int_0^{x-s+g_\epsilon(s)} W^{(q)}(y)\,dy\\
&\quad -Ke^x(q-\psi(1)) \int_0^{x-s+g_\epsilon(s)} e^{-y} W^{(q)}(y)\,dy\\
&= e^s - Ke^x + e^s \int_{s-x}^{g_\epsilon(s)} W^{(q)}(y+x-s)\big(q - K(q-\psi(1))e^{-y}\big)\,dy,
\end{aligned}$$

where we understand the integral on the right-hand side not to be present whenever $s - x \geq g_\epsilon(x)$. In order to prove the assertion, we need some more preparation. The function $y \mapsto q - K(q-\psi(1))e^{-y}$, $y \geq 0$, is strictly negative on $[0, z^* \vee 0)$ and positive on $[z^* \vee 0, \infty)$, where

$$z^* = \log(K) + \log\left(\frac{(q-\psi(1)) \vee 0}{q}\right).$$

Here we understand $\log(0) = -\infty$. Moreover, observe that

$$z^* \vee 0 \leq \eta \vee 0. \tag{4.30}$$

This is clear if $q \leq \psi(1)$, and if $q > \psi(1)$ we need to show that $z^* \geq 0$ implies $\eta \geq z^*$. Indeed, by definition of η, we have

$$q - K(q-\psi(1))e^{-\eta} = q/\Phi(q) > 0$$

and therefore, by definition of z^*, it follows that $\eta > z^*$.

We can finally prove the statement of the lemma, namely that the right-hand side of (4.29) is greater or equal to $(e^s - Ke^x)^+$. If $\eta \vee 0 \le s - x$, we see from (4.28) that $s - x \ge \log(K) \vee 0$ which together with (4.30) implies that

$$V_\epsilon(x,s) \ge e^s - Ke^x = (e^s - Ke^x)^+.$$

On the other hand, if $0 \le s - x < \eta$ (whenever $\eta > 0$), the situation is slightly more complicated as the integrand on the right-hand side of (4.29) might change sign (if $0 < z^* < \eta$) and it is not clear how much the negative and positive parts contribute. To resolve this difficulty, we reduce the problem to an estimate obtained from Proposition 4.1. Specifically, it follows from Proposition 4.1 that

$$\begin{aligned} V_\epsilon^*(\hat{x},\epsilon) &= e^\epsilon Z^{(q)}(\hat{x} - x_\epsilon^*) - Ke^{\hat{x}} Z_1^{(q-\psi(1))}(\hat{x} - x_\epsilon^*) \\ &= e^\epsilon - Ke^{\hat{x}} + e^\epsilon \int_0^{\hat{x}-x_\epsilon^*} W^{(q)}(y)(q - K(q-\psi(1))e^{\hat{x}-\epsilon-y}\, dy \qquad (4.31) \\ &\ge (e^\epsilon - Ke^{\hat{x}})^+ \end{aligned}$$

for $x_\epsilon^* \le \hat{x} \le \epsilon$. Now define $\delta := \epsilon - s$ and $\tilde{x} := x + \delta$. In particular, note that $0 \le s - x < \eta = \epsilon - x_\epsilon^*$ implies $x_\epsilon^* < \tilde{x} \le \epsilon$. Then, using the fact that $g_\epsilon(s) \ge \eta \ge z^* \vee 0$ in the first and (4.31) with $\hat{x} = \tilde{x}$ in the second inequality we obtain

$$\begin{aligned} &V_\epsilon(x,s) \\ &= e^s - Ke^x + e^s \int_0^{x-s+g_\epsilon(s)} W^{(q)}(y)\big(q - K(q-\psi(1))e^{x-s-y}\big)\, dy \\ &= e^{-\delta}\left(e^\epsilon - Ke^{\tilde{x}} + e^\epsilon \int_0^{\tilde{x}-\epsilon+g_\epsilon(s)} W^{(q)}(y)\big(q - K(q-\psi(1))e^{\tilde{x}-\epsilon-y}\big)\, dy\right) \\ &\ge e^{-\delta}\left(e^\epsilon - Ke^{\tilde{x}} + e^\epsilon \int_0^{\tilde{x}-\epsilon+\eta} W^{(q)}(y)\big(q - K(q-\psi(1))e^{\tilde{x}-\epsilon-y}\big)\, dy\right) \\ &= e^{-\delta}\left(e^\epsilon - Ke^{\tilde{x}} + e^\epsilon \int_0^{\tilde{x}-x_\epsilon^*} W^{(q)}(y)\big(q - K(q-\psi(1))e^{\tilde{x}-\epsilon-y}\big)\, dy\right) \\ &\ge e^{-\delta}(e^\epsilon - Ke^{\tilde{x}})^+ = (e^s - Ke^x)^+. \end{aligned}$$

This completes the proof. ☐

Proof of Lemma 4.9. We only prove the result in detail when X has paths of unbounded variation. If it has paths of bounded variation the proof is similar and we restrict ourselves to only pointing out major changes.

Unbounded variation case: As a first step we prove that

$$e^{-q(t\wedge\tau_\epsilon^+)}V_\epsilon(X_{t\wedge\tau_\epsilon^+},\overline{X}_{t\wedge\tau_\epsilon^+}), \qquad t\geq 0, \tag{4.32}$$

is a right-continuous $\mathbb{P}_{x,s}$-supermartingale for all $(x,s)\in E$ such that $s<\epsilon$. Note that in this case $Z^{(q)}\in C^1(\mathbb{R})\cap C^2(\mathbb{R}\setminus\{0\})$ and hence

$$h_v(x):=e^{vx}Z_v^{(q-\psi(v))}(x)$$

is in $C^1(\mathbb{R})\cap C^2(\mathbb{R}\setminus\{0\})$, where $v\geq 0$. Now let Γ be the infinitesimal generator of X under $\mathbb{P}_0$ and formally define the function $\Gamma h_v:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ by

$$\begin{aligned}\Gamma h_v(x) \quad &:= \quad -\gamma h_v'(x)+\frac{\sigma^2}{2}h_v''(x)\\ &\qquad +\int_{(-\infty,0)}\big(h_v(x+y)-h_v(x)-yh_v'(x)1_{\{y\geq -1\}}\big)\,\Pi(dy).\end{aligned}$$

The regularity of h_v together with Taylor's theorem allows one to show that the quantity $\Gamma h_v(x)$ is well defined for $x>0$. Moreover, for $x<0$, we have $h_v(x)=e^{vx}$ and hence $\Gamma h_v(x)$ is well defined too. Applying an appropriate version of the Itô–Meyer formula (cf. Theorem 71, Chapter IV of [36]) to $e^{-qt}h_v(X_t)$, we find that

$$e^{-q(t\wedge\tau_0^-\wedge\tau_b^+)}h_v(X_{t\wedge\tau_0^-\wedge\tau_b^+})-\int_0^{t\wedge\tau_0^-\wedge\tau_b^+}e^{-qu}(\Gamma-q)h_v(X_u)\,du$$

is a $\mathbb{P}_x$-martingale for $x\in(0,b)$. The martingale property of the first term (see Section 4.8) then implies that

$$(\Gamma-q)h_v(x)=0, \qquad x\in(0,b).$$

Moreover, one may show that $\Gamma e^{vy}=\psi(v)e^{vy}$ for $y\in\mathbb{R}$ by taking Laplace transforms on both sides. Hence it follows for $x<0$ that

$$(\Gamma-q)h_v(x)=(\Gamma-q+\psi(v)-\psi(v))e^{vx}=-(q-\psi(v))e^{vx}.$$

Next, fix $(x,s)\in E$ such that $x\leq s<\epsilon$ and define $Y_t:=X_t-\overline{X}_t+g_\epsilon(\overline{X}_t)$, $t\geq 0$, which is a semimartingale. We then have

$$\begin{aligned}&e^{-q(t\wedge\tau_\epsilon^+)}V_\epsilon(X_{t\wedge\tau_\epsilon^+},\overline{X}_{t\wedge\tau_\epsilon^+})\\ &=e^{-q(t\wedge\tau_\epsilon^+)}\bigg(e^{\overline{X}_{t\wedge\tau_\epsilon^+}}h_0(Y_{t\wedge\tau_\epsilon^+})-Ke^{\overline{X}_{t\wedge\tau_\epsilon^+}-g_\epsilon(\overline{X}_{t\wedge\tau_\epsilon^+})}h_1(Y_{t\wedge\tau_\epsilon^+})\bigg)\end{aligned}$$

Applying an appropriate version of the Itô–Meyer formula (cf. Theorem 71, Chapter IV of [36]) to $h_0(Y_{t\wedge\tau_\epsilon^+})$ and $h_1(Y_{t\wedge\tau_\epsilon^+})$ (see [23, 29] for a similar argument) and then using stochastic integration by parts for semimartingales (cf. Corollary 2 of Theorem 22, Chapter II of [36]) one obtains, $\mathbb{P}_{x,s}$-a.s.,

$$\begin{aligned}
& e^{-q(t\wedge\tau_\epsilon^+)}V_\epsilon(X_{t\wedge\tau_\epsilon^+},\overline{X}_{t\wedge\tau_\epsilon^+}) = V_\epsilon(x,s)+\tilde{M}_{t\wedge\tau_\epsilon^+} \\
& +\int_0^{t\wedge\tau_\epsilon^+} e^{-qu+\overline{X}_u}\times \\
& \Big(\Gamma h_0(Y_u)-qh_0(Y_u)-Ke^{-g_\epsilon(\overline{X}_u)}\big(\Gamma h_1(Y_u)-qh_1(Y_u)\big)\Big)\,du \\
& +\int_0^{t\wedge\tau_\epsilon^+} e^{-qu+\overline{X}_u}\Big[h_0(Y_u)+h_0'(Y_u)(g_\epsilon'(\overline{X}_u)-1) \\
& -Ke^{-g_\epsilon(X_u)}\big(-h_1(Y_u)+h_1'(Y_u)\big)(g_\epsilon'(\overline{X}_u)-1)\Big]\,d\overline{X}_u,
\end{aligned} \tag{4.33}$$

for some martingale $\tilde{M}$ whose specific form is irrelevant. We claim that the first integral in (4.33) is a decreasing process. Indeed, for $x>0$ we have $\Gamma h_0(x)-qh_0(x)=0$ and $\Gamma h_1(x)-qh_1(x)=0$. Moreover, for $x<0$, it holds $\Gamma h_0(x)-qh_0(x)=-q$ and $\Gamma h_1(x)-qh_1(x)=-(q-\psi(1))e^x$. Hence the first integral is nonpositive provided that

$$-q+Ke^{-g_\epsilon(\overline{X}_t)}(q-\psi(1))e^{Y_t}\le 0 \quad \text{on } \{Y_t\le 0\}.$$

This is clear if $q\le\psi(1)$. When $q>\psi(1)$, recall from (4.28) that $g_\epsilon(s)\ge\eta\vee 0$ and thus

$$-q+Ke^{-g_\epsilon(\overline{X}_t)}(q-\psi(1))e^{Y_t}\le -q+Ke^{-(0\vee\eta)}(q-\psi(1)).$$

By (4.30) the right-hand side is smaller than zero and hence the first integral in (4.33) is a decreasing process.

The second integral in (4.33) vanishes since the process $\overline{X}_u$ only increments when $\overline{X}_u=X_u$ and by definition of g_ϵ. Thus, the process $e^{-q(t\wedge\tau_\epsilon^+)}V_\epsilon(X_{t\wedge\tau_\epsilon^+},\overline{X}_{t\wedge\tau_\epsilon^+})$, $t\ge 0$, can be written as the sum of an initial value, a martingale and a decreasing process. In other words, it is a $\mathbb{P}_{x,s}$-supermartingale.

Finally, with all the preparation done, we can now prove the assertion, that is, show that the process $e^{-qt}V_\epsilon(X_t,\overline{X}_t)$, $t\ge 0$, is a right-continuous $\mathbb{P}_{x,s}$-supermartingale for $(x,s)\in E$. In view of Proposition 4.1 it suffices to assume that $(x,s)\in E$ such that $s<\epsilon$. Moreover, by the Markov property (see [23, 29] for a similar argument) it is enough to show that

$$\mathbb{E}_{x,s}[e^{-qt}V_\epsilon(X_t,\overline{X}_t)]\le V_\epsilon(x,s). \tag{4.34}$$

Using the strong Markov property and Proposition 4.1 we now obtain

$$\begin{aligned}
\mathbb{E}_{x,s}\big[e^{-qt}V_\epsilon(X_t,\overline{X}_t)\big|\mathcal{F}_{\tau_\epsilon^+}\big] &= e^{-qt}V_\epsilon(X_t,\overline{X}_t)1_{\{t<\tau_\epsilon^+\}} \\
&\quad +E_{x,s}\big[e^{-qt}V_\epsilon(X_t,\overline{X}_t)\big|\mathcal{F}_{\tau_\epsilon^+}\big]1_{\{t\ge\tau_\epsilon^+\}}
\end{aligned}$$

$$
\begin{aligned}
&= e^{-qt}V_\epsilon(X_t,\overline{X}_t)1_{\{t<\tau_\epsilon^+\}} \\
&\quad +e^{-q\tau_\epsilon^+}E_{\epsilon,\epsilon}\big[e^{-qt}V_\epsilon(X_t,\overline{X}_t)\big]1_{\{t\geq\tau_\epsilon^+\}} \\
&\leq e^{-q(t\wedge\tau_\epsilon^+)}V_\epsilon(X_{t\wedge\tau_\epsilon^+},\overline{X}_{t\wedge\tau_\epsilon^+}).
\end{aligned}
$$

Taking expectations on both sides and using that the process in (4.32) is a $\mathbb{P}_{x,s}$- supermartingale we get

$$
\mathbb{E}_{x,s}[e^{-qt}V_\epsilon(X_t,\overline{X}_t)]\leq\mathbb{E}_{x,s}\big[e^{-q(t\wedge\tau_\epsilon^+)}V_\epsilon(X_{t\wedge\tau_\epsilon^+},\overline{X}_{t\wedge\tau_\epsilon^+})\big]\leq V_\epsilon(x,s).
$$

This completes the proof in the unbounded variation case.

Bounded variation case: If X has bounded variation, then the Itô–Meyer formula is nothing more than an appropriate version of the change of variable formula for Stieltjes integrals and the rest of the proof follows the same line of reasoning as above. The only change worth mentioning is that the generator of X takes a different form. Specifically, one has to work with

$$
\Gamma\tilde{f}(x)=\mathtt{d}\tilde{f}'(x)+\int_{(-\infty,0)}\big(\tilde{f}(x+y)-\tilde{f}(x)\big)\Pi(dy)
$$

for appropriate $\tilde{f}$. □

Proof of Lemma 4.10. The assertion is again true for $(x,s)\in E$ such that $s\geq\epsilon$ by Proposition 4.1. Thus, let $(x,s)\in E$ such that $s<\epsilon$. The assertion is clear if $x-s+g_\epsilon(s)\leq 0$. Hence, suppose that $s<\epsilon$ and $x-s+g_\epsilon(s)>0$. Replacing $t\wedge\tau_\epsilon^+$ by $t\wedge\tau_\epsilon^+\wedge\rho_\epsilon^*$ in (4.33) and recalling that $\Gamma h_0(y)=qh_0(y)$ and $\Gamma h_1(y)=qh_1(y)$ for $y>0$ shows that

$$
\mathbb{E}_{x,s}\big[e^{-q(t\wedge\tau_\epsilon^+\wedge\rho_\epsilon^*)}V_\epsilon(X_{t\wedge\tau_\epsilon^+\wedge\rho_\epsilon^*},\overline{X}_{t\wedge\tau_\epsilon^+\wedge\rho_\epsilon^*})\big]=V_\epsilon(x,s)
$$

and hence by dominated convergence

$$
\mathbb{E}_{x,s}\big[e^{-q(\tau_\epsilon^+\wedge\rho_\epsilon^*)}V_\epsilon(X_{\tau_\epsilon^+\wedge\rho_\epsilon^*},\overline{X}_{\tau_\epsilon^+\wedge\rho_\epsilon^*})\big]=V_\epsilon(x,s). \tag{4.35}
$$

Using the strong Markov property one may now deduce that

$$
\mathbb{E}_{x,s}\big[e^{-q\rho_\epsilon^*}V_\epsilon(X_{\rho_\epsilon^*},\overline{X}_{\rho_\epsilon^*})\big|\mathcal{F}_{\tau_\epsilon^+}\big]=e^{-q(\tau_\epsilon^+\wedge\rho_\epsilon^*)}V_\epsilon(X_{\tau_\epsilon^+\wedge\rho_\epsilon^*},\overline{X}_{\tau_\epsilon^+\wedge\rho_\epsilon^*})
$$

and thus taking expectations on both sides and using (4.35) gives the desired result. □

Proof of Corollary 4.7.

(i) Since $q > \psi(1)$, Lemma A.1 of [23] implies that

$$\mathbb{E}_{x,s}\Big[\sup_{0\le t<\infty} e^{-qt}\big(e^{\overline{X}_t} - Ke^{X_t}\big)^+\Big] \le \mathbb{E}_{x,s}\Big[\sup_{0\le t<\infty} e^{-qt+\overline{X}_t}\Big] < \infty \tag{4.36}$$

for $(x,s) \in E$.

For $\epsilon \in \mathbb{R}$, let V^*_ϵ, ρ^*_ϵ and g_ϵ be as in Theorem 4.5 and V^*_∞ and ρ^*_∞ as stated in Corollary 4.7. It follows by construction of g_ϵ that $\lim_{\epsilon\uparrow\infty} g_\epsilon(s) = k^* \in [0,\infty)$ for $s \in \mathbb{R}$ which in turn implies that $\lim_{\epsilon\uparrow\infty}\rho^*_\epsilon = \rho^*_\infty$, $\mathbb{P}_{x,s}$-a.s., for all $(x,s) \in E$. Moreover, it is clear that $\lim_{\epsilon\uparrow\infty} V^*_\epsilon(x,s) = V^*_\infty(x,s)$ due to the continuity of scale functions. Next, we claim that:

(i) $V^*_\infty(x,s) \ge (e^s - Ke^x)^+$ for $(x,s) \in E$;

(ii) $e^{-qt}V^*_\infty(X_t,\overline{X}_t)$, $t \ge 0$, is a $\mathbb{P}_{x,s}$-supermartingale for $(x,s) \in E$;

(iii) $V^*_\infty(x,s) = \mathbb{E}_{x,s}\big[e^{-q\rho^*_\infty}\big(e^{\overline{X}_{\rho^*_\infty}} - Ke^{X_{\rho^*_\infty}}\big)^+\big]$ for $(x,s) \in E$.

Condition (i) is satisfied since $V^*_\epsilon(x,s) \ge (e^s - Ke^x)^+$ for $(x,s) \in E$ by Theorem 4.5 and the inequality remains valid in the limit. To prove (ii), use Fatou's lemma and Lemma 4.9 to show that

$$\begin{aligned}\mathbb{E}_{x,s}[e^{-qt}V^*_\infty(X_t,\overline{X}_t)] &\le \liminf_{\epsilon\to\infty}[e^{-qt}V^*_\epsilon(X_t,\overline{X}_t)]\\ &\le \liminf_{\epsilon\to\infty} V^*_\epsilon(x,s)\\ &= V^*_\infty(x,s)\end{aligned}$$

for $(x,s) \in E$. By the Markov property, this inequality implies the desired $\mathbb{P}_{x,s}$-supermartingale property (see [23, 29] for a similar argument). As for (iii), using (4.36) and dominated convergence we deduce that

$$\begin{aligned}V^*_\infty(x,s) &= \lim_{\epsilon\to\infty} V^*_\epsilon(x,s)\\ &= \lim_{\epsilon\to\infty}\mathbb{E}_{x,s}\big[e^{-q\rho^*_\epsilon}\big(e^{\overline{X}_{\rho^*_\epsilon\wedge\epsilon}} - Ke^{X_{\rho^*_\epsilon}}\big)^+\big]\\ &= \mathbb{E}_{x,s}\big[e^{-q\rho^*_\infty}\big(e^{\overline{X}_{\rho^*_\infty}} - Ke^{X_{\rho^*_\infty}}\big)^+\big]\end{aligned}$$

for $(x,s) \in E$. The proof of the corollary is now completed by using (i)–(iii) in the same way as in the proof of Theorem 4.5.

(ii) For $\epsilon \in \mathbb{R}$, let V^*_ϵ, ρ^*_ϵ and g_ϵ be as in Theorem 4.5. It follows by construction of g_ϵ that $\lim_{\epsilon\uparrow\infty} g_\epsilon(s) = \infty$ and $\lim_{\epsilon\uparrow\infty} V^*_\epsilon(x,s) = \infty$ for $x \le s$ and hence

$$\lim_{\epsilon\uparrow\infty}\mathbb{E}_{x,s}\big[e^{-q\tau^*_\epsilon}\big(e^{\overline{X}_{\tau^*_\epsilon}} - Ke^{X_{\tau^*_\epsilon}}\big)\big] = \lim_{\epsilon\uparrow\infty} V_\epsilon(x,s) = \infty.$$

This completes the proof. $\square$

4.8 Appendix

The goal of this section is to prove an auxiliary result that was used in the proof of Lemma 4.9. More precisely, for $q, v \geq 0$, we claim that the process

$$e^{-q(t\wedge\tau_0^-\wedge\tau_b^+)} h_v(X_{t\wedge\tau_0^-\wedge\tau_b^+}), \qquad t \geq 0,$$

is a $\mathbb{P}_x$-martingale for $x \in (0,b)$. To see this, we need to recall from Section 3.3. of [20] the identity

$$f_1(x) \quad := \quad \mathbb{E}_x\left[e^{-q\tau_0^- + vX_{\tau_0^-}} 1_{\{\tau_0^-<\infty\}}\right] = e^{vx}\left(Z_v^{(q-\psi(v))}(x) - \frac{q-\psi(v)}{\Phi(q)-v} W_v^{(q-\psi(v))}(x)\right),$$

where $x \in \mathbb{R}$. Applying the same technique (analytic extension) as in Section 3.3 of [20], one may also show that, for $q, v \geq 0$ and $x \in (0,b)$,

$$f_2(x) := \mathbb{E}_x\left[e^{-q\tau_b^+ + vX_{\tau_b^+}} 1_{\{\tau_b^+<\tau_0^-\}}\right] = e^{vx}\frac{W_v^{(q-\psi(v))}(x)}{W_v^{(q-\psi(v))}(b)}.$$

An application of the Markov property yields for $t \geq 0$,

$$\begin{aligned}
\mathbb{E}_x\left[e^{-q\tau_0^-} f_1(X_{\tau_0^-}) 1_{\{\tau_0^-<\infty\}} | \mathcal{F}_t\right] &= e^{-q\tau_0^-} f_1(X_{\tau_0^-}) 1_{\{\tau_0^-<t\}} \\
&\quad + e^{-qt}\mathbb{E}_{X_t}\left[e^{-q\tau_0^-} f_1(X_{\tau_0^-}) 1_{\{\tau_0^-<\infty\}}\right] 1_{\{\tau_0^->t\}} \\
&= e^{-q\tau_0^-} f_1(X_{\tau_0^-}) 1_{\{\tau_0^-<t\}} \\
&\quad + e^{-qt}\mathbb{E}_{X_t}\left[e^{-q\tau_0^- + vX_{\tau_0^-}} 1_{\{\tau_0^-<\infty\}}\right] 1_{\{\tau_0^->t\}} \\
&= e^{-q(t\wedge\tau_0^-)} f_1(X_{t\wedge\tau_0^-}),
\end{aligned}$$

which shows that the process $e^{-q(t\wedge\tau_0^-)} f_1(X_{t\wedge\tau_0^-})$, $t \geq 0$, is a $\mathbb{P}_x$-martingale for $x > 0$. By Doob's optional stopping theorem it then follows that $e^{-q(t\wedge\tau_0^-\wedge\tau_b^+)} f_1(X_{t\wedge\tau_0^-\wedge\tau_b^+})$, $t \geq 0$, is a $\mathbb{P}_x$-martingale for $x \in (0,b)$. A similar argument as above shows that $e^{-q(t\wedge\tau_0^-\wedge\tau_b^+)} f_2(X_{t\wedge\tau_0^-\wedge\tau_b^+})$, $t \geq 0$, is a $\mathbb{P}_x$-martingale for $x \in (0,b)$ as well and hence appropriately combining the two martingales completes the proof.

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CHAPTER 5

PREDICTION OF GLOBAL EXTREMA

Motivated by the recent results in [15], we study the problem of predicting the time at which a positive self-similar Markov process X attains its pathwise global supremum or infimum. In particular, we show that the simple solution of the prediction problem found in [15] for the case when X is a d-dimensional Bessel process for $d > 2$ can be seen as a consequence of the self-similarity of X.

5.1 Introduction

This chapter addresses the question of predicting the time when a positive self-similar Markov process (pssMp) attains its pathwise global supremum or infimum. We shall spend some time to set up some notation in order to formulate the problem rigorously.

A positive self-similar Markov process $X = \{X_t : t \geq 0\}$ with self-similarity index $\alpha > 0$ is a $[0, \infty)$-valued standard Markov process defined on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{G} := \{\mathcal{G}_t : t \geq 0\}, \{P_x : x > 0\})$, which has 0 as an absorbing state and which satisfies the scaling property: for every $x, c > 0$,

$$\text{the law of } \{cX_{c^{-\alpha}t} : t \geq 0\} \text{ under } P_x \text{ is equal to the law of } X \text{ under } P_{cx}.$$

Here, we mean "standard" in the sense that $\mathbb{G}$ satisfies the natural conditions (cf. [7], Section 1.3, page 39) and X is strong Markov with càdlàg and quasi-left-continuous paths. Lamperti [24] proved in a seminal paper that the set of pssMps splits into three exhaustive classes which can be distinguished from each other by comparing their hitting time of 0, that is, $\zeta := \inf\{t > 0 : X_t = 0\}$. The classification reads as follows:

(i) $P_x[\zeta = \infty] = 1$ for all starting points $x > 0$,

(ii) $P_x[\zeta < \infty, X_{\zeta-} = 0] = 1$ for all starting points $x > 0$,

(iii) $P_x[\zeta < \infty, X_{\zeta-} > 0] = 1$ for all starting points $x > 0$.

In other words, a pssMp X starting at $x > 0$ either never hits zero, hits zero continuously or hits zero by jumping onto it. The two subclasses of pssMps that are used here are

$$\begin{aligned}
\mathcal{C}_+ \quad &:= \quad \big\{X \text{ is spectrally negative with non-monotone paths and} \\
&\qquad\quad \text{either of type (ii) or (iii)}\big\}, \\
\mathcal{C}_- \quad &:= \quad \big\{X \text{ is spectrally positive with non-monotone paths and} \\
&\qquad\quad \text{either of type (i) and drifting to } \infty \text{ or of type (iii)}\big\}.
\end{aligned}$$

By spectrally negative and spectrally positive we mean that the trajectories of X only have downward or upward jumps respectively.

One of the aims here is to answer the following question: Given $X \in \mathcal{C}_+$, is it possible to stop "as close as possible" to the time at which X "attains" its supremum? In more mathematical terms, define

$$\Theta := \sup\{t \geq 0 : X_t = \overline{X}_\infty\} = \sup\{0 \leq t < \zeta : X_t = \overline{X}_\infty\},$$

where $\overline{X} = \{\overline{X}_t : t \geq 0\}$ is the running maximum process $\overline{X}_t := \sup_{0 \leq u \leq t} X_u$, $t \geq 0$. By definition of $\mathcal{C}_+$, it follows that the set $\{t \geq 0 : X_t = \overline{X}_\infty\}$ is a singleton; see Subsection 5.2.3 for details. We are interested in the optimal stopping problem

$$\inf_\tau E_x[|\Theta - \tau| - \Theta], \tag{5.1}$$

where $x > 0$ and the infimum is taken over a certain set of $\mathbb{G}$-stopping times τ which is specified later. The term "attains" is used in a loose sense here. Indeed, if X has negative jumps it might happen that the supremum is never attained. However, the above definition ensures that we have $X_\Theta = \overline{X}_\zeta$ on the event $\{X_\Theta \geq X_{\Theta-}\}$ while $X_{\Theta-} = \overline{X}_\zeta$ on the event $\{X_\Theta < X_{\Theta-}\}$.

Analogously, one may try to stop "as close as possible" to the time at which a process $X \in \mathcal{C}_-$ "attains" its infimum before hitting zero (if at all). To this end, let

$$\hat{\Theta} := \sup\{0 \leq t < \zeta : X_t = \underline{X}_t\},$$

where $\underline{X} = \{\underline{X}_t : t \geq 0\}$ the running minimum process $\underline{X}_t := \inf_{0 \leq u \leq t} X_u$, $t \geq 0$. Again, by definition of $\mathcal{C}_-$, the set $\{0 \leq t < \zeta : X_t = \underline{X}_t\}$ a singleton; see Subsection 5.2.3 for details. If X has positive jumps, the word "attains" is used in a loose sense analogously to above. Stopping as close as possible to $\hat{\Theta}$ then leads to solving the optimal stopping problem

$$\inf_{\tau} E_x[|\hat{\Theta} - \tau| - \hat{\Theta}], \tag{5.2}$$

where $x > 0$ and the infimum is taken over a certain set of $\mathbb{G}$-stopping times τ which is specified later.

Our interest in (5.1) and (5.2) was raised due to [15] in which the authors solve (5.2) under the assumption that X is a diffusion in $(0, \infty)$ such that $\lim_{t\to\infty} X_t = \infty$. Their result states that the optimal stopping time is given by

$$\rho_1^* = \inf\{t \geq 0 : X_t \geq f^*(\underline{X}_t)\}, \tag{5.3}$$

where f^* is the minimal solution to a certain differential equation. In particular, when X is a d-dimensional Bessel process with $d > 2$, it is shown that $f^*(z) = \lambda_1^* z$, $z \geq 0$, for some constant $\lambda_1^* > 1$, which is a root of some polynomial. Due to the fact that the class of Bessel processes for $d > 2$ belongs to the class of pssMps with $\alpha = 2$, it is possible to express the optimal stopping time (5.3) (up to a time-change) in terms of the underlying Lamperti representation ξ (of X) reflected at its infimum. This raises the suspicion that the simple form of (5.3) in the Bessel case could be a consequence of the self-similarity of X and suggests that (5.2) (or an analogue of it) can also be solved for the class of pssMps.

In this chapter we show that the speculations in the previous paragraph are indeed true. Specifically, we prove that the optimal stopping times in (5.1) and (5.2) are of the simple form

$$\tau^* = \inf\{t \geq 0 : X_t \geq K^* \overline{X}_t\} \quad \text{and} \quad \hat{\tau}^* = \inf\{t \geq 0 : X_t \leq \hat{K}^* \underline{X}_t\}$$

for some constants $0 < K^* < 1$ and $\hat{K}^* > 1$ respectively. As alluded to above, the key step is to reduce (5.1) and (5.2) to a one-dimensional problem with the help of the so-called Lamperti transformation [24] which links pssMps to Lévy processes.

Finally, to conclude we discuss two issues. Firstly, how one might get rid of the assumption of one-sided jumps (see Section 5.8) and, secondly, we explain how the two prediction problems fit into the general context of this thesis; see Section 5.9.

5.2 Preliminaries

5.2.1 Killed Lévy processes

A process ξ with values in $\mathbb{R} \cup \{-\infty\}$ is called a Lévy process killed at rate $q \geq 0$ if ξ starts at 0, has stationary and independent increments and $\mathbf{k} := \inf\{t > 0 : \xi_t = -\infty\}$ has an exponential distribution with parameter $q \geq 0$. In the case $q = 0$ it is understood that $\mathbb{P}[\mathbf{k} = \infty] = 1$, that is, no killing. It is well known that a Lévy process X killed at rate q is characterised by its Lévy triplet (γ, σ, Π) and the killing rate q, where $\sigma \geq 0, \gamma \in \mathbb{R}$ and Π is a measure on $\mathbb{R}$ satisfying the condition $\int_{\mathbb{R}}(1 \wedge x^2)\,\Pi(dx) < \infty$. The Laplace exponent of ξ under $\mathbb{P}$ is defined by

$$\psi(\theta) := \log(\mathbb{E}[e^{\theta \xi_1}])$$

for any $\theta \in \mathbb{R}$ such that $\psi(\theta) < \infty$. It is known that (cf. Theorem 3.6 in [21]), for $\theta \in \mathbb{R}$,

$$\mathbb{E}[e^{\theta \xi_t}] < \infty \text{ for all } t \geq 0 \qquad \Longleftrightarrow \qquad \int_{|x|\geq 1} e^{\theta x}\,\Pi(dx) < \infty, \tag{5.4}$$

and in this case we have

$$\psi(\theta) = -q - \gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left(e^{\theta x} - 1 - \theta x 1_{\{|x|<1\}}\right) \Pi(dx). \tag{5.5}$$

In particular, if ξ is of bounded variation, (5.5) may be written as

$$\psi(\theta) = \mathtt{d}\theta - \int_{\mathbb{R}} (1 - e^{\theta x})\,\Pi(dx)$$

for some $\mathtt{d} \in \mathbb{R}$.

Finally, for any killed Lévy process (starting at zero) and any $v \in \mathbb{R}$ with $\psi(v) < \infty$ the process

$$\exp(v\xi_t - \psi(v)t)1_{\{t<\mathbf{k}\}}, \qquad t \geq 0,$$

is a $\mathbb{P}$-martingale. Hence, we may further define the family of measures $\{\mathbb{P}^v\}$ with Radon-Nikodym derivatives

$$\left.\frac{d\mathbb{P}^v}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp(v\xi_t - \psi(v)t)1_{\{t<\mathbf{k}\}}. \tag{5.6}$$

In particular, under $\mathbb{P}^v$ the process ξ is a Lévy process and its Laplace exponent is given by $\psi_v(\theta) = \psi(v+\theta) - \psi(v)$ and infinite lifetime, that is, $\mathbb{P}^v[\mathbf{k} = \infty] = 1$; cf. Theorem 3.9 in [21].

5.2.2 Scale functions

Suppose throughout this subsection that ξ is an unkilled spectrally negative Lévy process ($q = 0$). Spectrally negative means that Π is concentrated on $(-\infty, 0)$ and thus ξ only exhibits downward jumps. Observe that in this case, the Laplace exponent $\psi(\theta)$ exists at least for $\theta \geq 0$ by (5.4). Its right-inverse is defined by

$$\Phi(\lambda) := \sup\{\theta \geq 0 : \psi(\theta) = \lambda\}, \qquad \lambda \geq 0.$$

A special family of functions associated with unkilled spectrally negative Lévy processes is that of scale functions (cf. [20, 21]) which are defined as follows. For $\eta \geq 0$, the η-scale function $W^{(\eta)} : \mathbb{R} \to [0, \infty)$ is the unique function whose restriction to $(0, \infty)$ is continuous and has Laplace transform

$$\int_0^\infty e^{-\theta x} W^{(\eta)}(x)\, dx = \frac{1}{\psi(\theta) - \eta}, \quad \theta > \Phi(\eta),$$

and is defined to be identically zero for $x \leq 0$. Further, we shall use the notation $W_v^{(\eta)}(x)$ to mean the η-scale function associated to X under $\mathbb{P}^v$. For fixed $x \geq 0$, it is also possible to analytically extend $\eta \mapsto W^{(\eta)}(x)$ to $\eta \in \mathbb{C}$. A useful relation that links the different scale functions is (cf. Lemma 3.7 in [20])

$$W^{(\eta)}(x) = e^{vx} W_v^{(\eta - \psi(v))}(x) \tag{5.7}$$

for $v \in \mathbb{R}$ such that $\psi(v) < \infty$ and $\eta \in \mathbb{C}$. Moreover, the following regularity properties of scale functions are known; cf. Sections 2.3 and 3.1 of [20].

Smoothness: For all $\eta \geq 0$,

$$W^{(\eta)}|_{(0,\infty)} \in \begin{cases} C^1(0,\infty), & \text{if } X \text{ is of bounded variation and } \Pi \text{ has no atoms,} \\ C^1(0,\infty), & \text{if } X \text{ is of unbounded variation and } \sigma = 0, \\ C^2(0,\infty), & \sigma > 0. \end{cases} \tag{5.8}$$

Continuity at the origin: For all $\eta \geq 0$,

$$W^{(q)}(0+) = \begin{cases} \mathtt{d}^{-1}, & \text{if } X \text{ is of bounded variation,} \\ 0, & \text{if } X \text{ is of unbounded variation.} \end{cases} \tag{5.9}$$

Right-derivative at the origin: For all $q \geq 0$,

$$W_+^{(q)\prime}(0+) = \begin{cases} \frac{q + \Pi(-\infty,0)}{\mathtt{d}^2}, & \text{if } \sigma = 0 \text{ and } \Pi(-\infty,0) < \infty, \\ \frac{2}{\sigma^2}, & \text{if } \sigma > 0 \text{ or } \Pi(-\infty,0) = \infty, \end{cases} \tag{5.10}$$

where we understand the second case to be $+\infty$ when $\sigma = 0$.

The second scale function is $Z_v^{(\eta)}$ and defined as follows. For $v \in \mathbb{R}$ such that $\psi(v) < \infty$ and $\eta \geq 0$ we define $Z_v^{(\eta)} : \mathbb{R} \longrightarrow [1, \infty)$ by

$$Z_v^{(\eta)}(x) = 1 + \eta \int_0^x W_v^{(\eta)}(z)\, dz. \tag{5.11}$$

5.2.3 The Lamperti transformation

Lamperti's main result in [24] asserts that any pssMp X may, up to its first hitting time of zero, be expressed as the exponential of a time-changed Lévy process. We will now explain this in more detail. Instead of writing (X, P_x) to denote the positive self-similar Markov process starting at $x > 0$, we shall sometimes write $X^{(x)} = \{X_t^{(x)} : t \geq 0\}$. Similarly, we write $\zeta^{(x)} = \inf\{t > 0 : X_t^{(x)} = 0\}$.

For fixed $x > 0$ define

$$\varphi(t) := \int_0^{x^\alpha t} (X_s^{(x)})^{-\alpha} \, ds, \qquad t < x^{-\alpha}\zeta^{(x)}.$$

It will be important to understand the behaviour of $\varphi(x^{-\alpha}\zeta-) := \lim_{t\uparrow\zeta} \varphi(x^{-\alpha}t)$. In particular, note that the distribution of $\varphi(x^{-\alpha}\zeta-)$ does not depend on $x > 0$. Moreover, the following result is known; see Lemma 13.3 in [21].

Lemma 5.1. *In the case that $\zeta = \infty$ or that $\{\zeta < \infty$ and $X_{\zeta-} = 0\}$, we have $P_x[\varphi(x^{-\alpha}\zeta-) = \infty] = 1$, for all $x > 0$. In the case that $\zeta < \infty$ and $X_{\zeta-} > 0$, we have that, under P_x, $\varphi(x^{-\alpha}\zeta-)$ is exponentially distributed with a parameter that does not depend on the value of $x > 0$.*

As the distribution of $\varphi(x^{-\alpha}\zeta^{(x)}-)$ is independent of x, we will rename it $\mathbf{e}$. When $\mathbf{e} = \infty$ almost surely we interpret it as an exponential distribution with parameter zero. Now define the right-inverse of φ,

$$I_u := \inf\{0 < t < x^{-\alpha}\zeta^{(x)} : \varphi(t) > u\}, \quad u \geq 0.$$

Moreover, define the process $\xi := \{\xi_t : t \geq 0\}$ by setting, for $x > 0$,

$$\xi_t := \log(X_{x^\alpha I_t}/x), \qquad 0 \leq t < \mathbf{e}$$

and $\xi_t = -\infty$ for $t \geq \mathbf{e}$ (in the case that $\mathbf{e} < \infty$). The main result in [24] states that a pssMp is nothing else than a space and time-changed killed Lévy process.

Proposition 5.2 (Lamperti transformation). *If $X^{(x)}$, $x > 0$, is a positive self-similar Markov process with index of self-similarity $\alpha > 0$, then it can be represented as*

$$X_t^{(x)} = x \exp(\xi_{\varphi(x^{-\alpha}t)}), \qquad t \geq 0,$$

and either

(i) $\zeta^{(x)} = \infty$ *almost surely for all $x > 0$, in which case ξ is an unkilled Lévy process satisfying* $\limsup_{t\uparrow\infty} \xi_t = \infty$, *or*

(ii) $\zeta^{(x)} < \infty, X^{(x)}_{\zeta^{(x)}-} = 0$ *almost surely for all $x > 0$, in which case ξ is an unkilled Lévy process satisfying* $\lim_{t\uparrow\infty} \xi_t = -\infty$, *or*

(iii) $\zeta^{(x)} < \infty, X^{(x)}_{\zeta^{(x)}-} > 0$ *almost surely for all* $x > 0$, *in which case* ξ *is a killed Lévy process.*

Also note that we may identify

$$I_t = \int_0^t e^{\alpha \xi_s}\, ds, \qquad t < \mathbf{e}.$$

The version of the Lamperti transformation we have just given is Theorem 13.1 in [21], where one can also find a proof of it.

We conclude this subsection by explaining why the sets $\{t \geq 0 : X_t = \overline{X}_\infty\}$ and $\{0 \leq t < \zeta : X_t = \underline{X}_t\}$ mentioned in the introduction are singletons. By definition of $\mathcal{C}_+$ and $\mathcal{C}_-$ it is clear that both sets are non-empty, but they could potentially contain more than one element. In view of the Lamperti transformation we see that the aforementioned sets contain only a single element provided the same is true for the sets $\{t \geq 0 : \xi_t = \sup_{0 \leq u < \infty} \xi_u\}$ and $\{0 \leq t < \mathbf{e} : \xi_t = \inf_{0 \leq u < t} \xi_u\}$, where ξ is the underlying Lamperti representation of X in $\mathcal{C}_+$ and $\mathcal{C}_-$ respectively. However, it is known that local extrema (and hence global extrema) of Lévy processes are distinct except for compound Poisson processes, see Proposition 4 in [6]. But for X in $\mathcal{C}_+$ or $\mathcal{C}_-$ the Lamperti transformation can never be a compound Poisson process and thus the assertion follows.

5.3 Reformulation of problems and main results

5.3.1 Predicting the time at which the maximum is attained

Suppose throughout this subsection that $X \in \mathcal{C}_+$ with parameter of self-similarity $\alpha > 0$ and let ξ be its Lamperti representation which is a spectrally negative Lévy process killed at some rate $q \geq 0$ satisfying $\lim_{t\uparrow\infty} \xi_t = -\infty$ whenever $q = 0$. For $\theta \geq 0$, let $\psi(\theta)$ be the Laplace exponent of ξ and $\phi(\theta) = q + \psi(\theta)$ the Laplace exponent of ξ unkilled. Denote by Φ the right-inverse of ϕ and note that $\Phi(q) > 0$.

We begin our analysis with two steps that are almost identical to Lemmas 1 and 2 of [15].

Lemma 5.3. *For any* $\mathbb{G}$*-stopping time* τ *we have*

$$|\Theta - \tau| = \Theta + \int_0^\tau \left(2 1_{\{\Theta \leq t\}} - 1\right) dt.$$

Lemma 5.4. *For* $x > 0$ *and any* $\mathbb{G}$*-stopping time* τ *with finite mean we have*

$$E_x[|\Theta - \tau| - \Theta] = E_x\Big[\int_0^{\tau\wedge\zeta} F(\overline{X}_t/X_t)\, dt] + E_x[(\tau - \zeta)1_{\{\tau > \zeta\}}], \tag{5.12}$$

where $F(y) = 1 - 2y^{-\Phi(q)}$, $y \geq 1$.

We are interested in minimising the expectation on the left-hand side of (5.12) over the set $\mathcal{M}$ of all integrable $\mathbb{G}$-stopping times τ. The requirement that τ is integrable ensures that (5.12) is well defined. Taking into account the specific form of the right-hand side of (5.12), one sees that for $x > 0$,

$$\inf_{\tau \in \mathcal{M}} E_x[|\Theta - \tau| - \Theta] = \inf_{\tau \in \mathcal{M}} E_x\Big[\int_0^{\tau \wedge \zeta} F(\overline{X}_t / X_t)\, dt\Big].$$

Although the stopping time ζ, which corresponds to waiting until X hits zero, might not be optimal, it is a very natural stopping strategy and should belong to $\mathcal{M}$. However, not every $X \in \mathcal{C}_+$ is such that $E_x[\zeta] < \infty$. To see this, use the Lamperti transformation together with Fubini's theorem to obtain

$$\begin{aligned} E_x[\zeta] &= x^\alpha E_x\Big[\int_0^{\mathbf{e}} e^{\alpha \xi_t}\, dt\Big] \\ &= x^\alpha \int_0^\infty E_x[e^{\alpha \xi_t} 1_{\{t < \mathbf{e}\}}]\, dt \\ &= x^\alpha \int_0^\infty e^{\psi(\alpha) t}\, dt \\ &= x^\alpha \int_0^\infty e^{-qt + \phi(\alpha) t}\, dt. \end{aligned}$$

Hence $X \in \mathcal{C}_+$ satisfies $E_x[\zeta] < \infty$ if and only if the underlying Lamperti representation ξ of X is such that $\psi(\alpha) < 0$ or, equivalently, $q > \phi(\alpha)$. Consequently, we need to adapt $\mathcal{C}_+$ and define

$$\begin{aligned} \mathcal{C}_+^1 &:= \{X \text{ is spectrally negative with non-monotone paths, and} \\ &\qquad \text{either of type (ii) or (iii) and such that } \zeta \text{ is integrable}\}. \end{aligned}$$

The criterion in terms of the Lamperti transformation above will be useful at a later point when we consider examples and it is necessary to check whether a specific X actually lies in $\mathcal{C}_+^1$ or not; see Section 5.6.

Remark 5.5. At this point one might wonder why we try to minimise $\mathbb{E}[|\Theta - \tau| - \Theta]$ rather than $\mathbb{E}[|\Theta - \tau|]$. As our assumptions require ζ to be integrable, it follows that $\Theta \leq \zeta$ is also integrable. Hence it does not really depend which of the two quantities above we minimise. However, in order to be consistent with what follows in Subsection 5.3.2, we chose to minimise $\mathbb{E}[|\Theta - \tau| - \Theta]$.

Summing up, for $X \in \mathcal{C}_+^1$ we are led to the optimal stopping problem

$$v(x, s) = \inf_\tau E_x\Big[\int_0^{\tau \wedge \zeta} F((s \vee \overline{X}_t)/X_t)\, dt\Big], \qquad 0 < x \leq s, \tag{5.13}$$

where the infimum is taken over all $\mathbb{G}$-stopping times τ. We are now in a position to state our first main result.

Theorem 5.6. *Let $X \in \mathcal{C}_+^1$ with index of self-similarity $\alpha > 0$, in which case its Lamperti representation ξ is a spectrally negative Lévy process killed at rate $q \geq 0$. Assume that ξ is such that the Lévy measure associated with it has no atoms whenever ξ is of bounded variation. Moreover, recall that ϕ is the Laplace exponent of ξ unkilled and Φ its right-inverse. Let $W^{(\cdot)}(z)$ be the scale function associated with ϕ. Then the solution of* (5.13) *is given by*

$$v(x,s) = -\int_{K^*s}^{x} z^{\alpha-1}\Big(1-2\Big(\frac{z}{s}\Big)^{\Phi(q)}\Big)W^{(q)}(\log(x/z))\,dz$$

and $\tau^* := \inf\{t \geq 0 : X_t \leq K^*(s \vee \overline{X}_t)\}$, where $K^* \in (0, 2^{-\frac{1}{\Phi(q)}})$ is the unique solution to the equation (in K)

$$\int_0^{\log(1/K)} (1-2e^{-\Phi(q)z})W_\alpha^{(q-\phi(\alpha))\prime}(z)\,dz = W^{(q)}(0) \quad \textit{on } (0,1). \tag{5.14}$$

Remark 5.7.

i) The right-hand side of (5.14) is equal to zero unless ξ is of bounded variation; see (5.9).

ii) The assumption on the Lévy measure of ξ is purely technical and ensures that the scale functions associated with ξ are continuously differentiable on $(0,\infty)$; see (5.8).

This result is a consequence of the analysis in Sections 5.4 and 5.5. An explicit example is provided in Section 5.6.

5.3.2 Predicting the time at which the minimum is attained

Suppose throughout this subsection that $X \in \mathcal{C}_-$ with parameter of self-similarity $\alpha > 0$ and let ξ again be its Lamperti representation which is a spectrally positive Lévy process killed at rate $q \geq 0$ satisfying $\lim_{t\uparrow\infty} \xi_t = \infty$ whenever $q = 0$. Introduce the dual $\hat{\xi} = \{\hat{\xi}_t : t \geq 0\}$ of ξ which is defined as

$$\hat{\xi}_t := \begin{cases} -\xi_t, & t < \mathbf{e}, \\ -\infty, & t \geq \mathbf{e}, \end{cases}$$

where $\mathbf{e} = \inf\{t > 0 : \xi_t = -\infty\}$. It follows that $\hat{\xi}$ is a spectrally negative Lévy process killed at rate $q \geq 0$ satisfying $\lim_{t\uparrow\infty} \hat{\xi}_t = -\infty$ whenever $q = 0$. For $\theta \geq 0$, let $\hat{\psi}$ be the Laplace exponent of $\hat{\xi}$ and $\hat{\phi}(\theta) = q + \hat{\psi}(\theta)$ the Laplace exponent of $\hat{\xi}$ unkilled. Finally, denote by $\hat{\Phi}$ the right-inverse of $\hat{\phi}$ and note that $\hat{\Phi}(q) > 0$.

Analogously to Lemma 5.3 and 5.4, one can prove the following result.

Lemma 5.8. *For $x > 0$ and any $\mathbb{G}$-stopping time τ with finite mean we have*

$$E_x[|\hat{\Theta} - \tau| - \hat{\Theta}] = E_x\Big[\int_0^{\tau\wedge\zeta} \hat{F}(X_t/\underline{X}_t)\,dt\Big] + E_x[(\tau-\zeta)1_{\{\tau>\zeta\}}], \tag{5.15}$$

where $\hat{F}(y) := 1 - 2y^{-\hat{\Phi}(q)}$, $y \geq 1$.

Now assume temporarily that X is of type (iii). In this case, the specific form of the right-hand side of (5.15) shows again that for $x > 0$,

$$\inf_{\tau\in\mathcal{M}} E_x[|\hat{\Theta} - \tau| - \hat{\Theta}] = \inf_{\tau\in\mathcal{M}} E_x\Big[\int_0^{\tau\wedge\zeta} \hat{F}(X_t/\underline{X}_t)\,dt\Big],$$

where $\mathcal{M}$ is the set of all integrable $\mathbb{G}$-stopping times τ. As in Subsection 5.3.1, it is natural to require that $\zeta \in \mathcal{M}$ and we have a criterion in terms of the dual of the Lamperti representation to check whether $\zeta \in \mathcal{M}$. Specifically, by the Lamperti transformation and Fubini's theorem we have

$$E_x[\zeta] = x^\alpha \int_0^\infty E_x[e^{\alpha\xi_t}1_{\{t<\mathbf{e}\}}]\,dt = x^\alpha \int_0^\infty E_x[e^{-\alpha\hat{\xi}_t}1_{\{t<\mathbf{e}\}}]\,dt.$$

It follows that $X \in \mathcal{C}_-$ satisfies $E_x[\zeta] < \infty$ if and only if the Laplace exponent $\hat{\psi}$ exists at $-\alpha$ and $\hat{\psi}(-\alpha) < 0$. This in turn is equivalent to saying that $\hat{\phi}$ exists at $-\alpha$ and $q > \hat{\phi}(-\alpha)$.

Remark 5.9. Note that in Subsection 5.3.1 the integrability condition was used to deduce that $\psi(\alpha) < 0$, but existence of $\psi(\alpha)$ was not an issue as ξ was spectrally negative and $\alpha > 0$. Here, however, the integrability condition implies existence of $\hat{\psi}$ at $-\alpha$ and $\hat{\psi}(-\alpha) < 0$.

On the other hand, if X is of type (i) with $\lim_{t\uparrow\infty} X_t = \infty$, then $\zeta = \infty$ and the integrability condition makes no sense. In this case we understand the minimisation of (5.15) over integrable $\tau \in \mathcal{M}$. Further, it is necessary to impose an additional condition on X, namely that it is such that Laplace exponent of $\hat{\xi}$ at $-\alpha$ exists. However, note that in contrast to all the other cases $\hat{\psi}(-\alpha) = \hat{\phi}(-\alpha) > 0$.

Following the discussion in the previous two paragraphs, we need to adapt $\mathcal{C}_-$ and define

$$\begin{aligned}\mathcal{C}_-^1 \;:=\; &\{X \text{ is spectrally positive and of type (i) with } \lim_{t\uparrow\infty} X_t = \infty \text{ such} \\ &\quad \text{that } \hat{\psi} \text{ exists at } -\alpha \text{ or of type (iii) such that } \zeta \text{ is integrable}\}.\end{aligned}$$

Remark 5.10. In contrast to Subsection 5.3.1, if $X \in \mathcal{C}_-^1$ of type (i) such that $\lim_{t\uparrow\infty} X_t = \infty$, it is not necessarily the case that $\hat{\Theta}$ is integrable; see [15]. In that case $\mathbb{E}[|\hat{\Theta} - \tau|] = \infty$ for all integrable $\mathbb{G}$-stopping times τ and hence in order to obtain a sensible problem one has to consider $\mathbb{E}[|\hat{\Theta} - \tau| - \hat{\Theta}]$.

For $X \in \mathcal{C}_-^1$, we are led to the optimal stopping problem

$$\hat{v}(x,i) := \inf_{\tau} E_x[\int_0^{\tau\wedge\zeta} \hat{F}(X_t/(i \wedge \underline{X}_t))\, dt], \qquad 0 < i \leq x, \tag{5.16}$$

where the infimum is taken respectively with the two cases over all $\mathbb{G}$-stopping times τ or all integrable $\mathbb{G}$-stopping times τ. We can now state the analogue of Theorem 5.6.

Theorem 5.11. *Assume that $X \in \mathcal{C}_-^1$ with index of self-similarity $\alpha > 0$, in which case the dual $\hat{\xi}$ of the Lamperti representation of X is a spectrally negative Lévy process killed at rate $q \geq 0$. Assume that the Lévy measure associated with $\hat{\xi}$ has no atoms whenever $\hat{\xi}$ is of bounded variation. Moreover, recall that $\hat{\phi}$ is the Laplace exponent of the dual $\hat{\xi}$ unkilled and $\hat{\Phi}$ its right-inverse. Let $\hat{W}^{(\cdot)}(z)$ be the scale function associated with $\hat{\phi}$. Then the solution of (5.16) is given by*

$$\hat{v}(x,i) = -\int_x^{\hat{K}^* i} z^{\alpha-1}\Big(1 - 2\Big(\frac{i}{z}\Big)^{\hat{\Phi}(q)}\Big)\hat{W}^{(q)}(\log(z/x))\, dz$$

and $\hat{\tau}^* := \inf\{t \geq 0 : X_t \geq \hat{K}^*(i \wedge \underline{X}_t)\}$, where $\hat{K}^* > 2^{1/\hat{\Phi}(q)}$ is the unique solution to the equation (in K)

$$\int_0^{\log(K)} (1 - 2e^{-\hat{\Phi}(q)z})\hat{W}_{-\alpha}^{(q-\hat{\phi}(-\alpha))\prime}(z)\, dz = \hat{W}^{(q)}(0) \qquad \text{on } (1,\infty). \tag{5.17}$$

This result is again a consequence of the analysis of Sections 5.4 and 5.5 and the analogue of Remark 5.7 applies here as well. An example including the case when X is a d-dimensional Bessel process for $d > 2$ is provided in Section 5.6.

5.4 Reduction to a one-dimensional problem

5.4.1 Reduction of problem (5.13)

The aim in this subsection is to reduce (5.13) to a one-dimensional optimal stopping problem.

We begin by reducing (5.13) to an optimal stopping problem in which X starts at one. More precisely, the self-similarity of X implies that the process

$$\int_0^{t\wedge\zeta^{(x)}} F((s \vee \overline{X}_u^{(x)})/X_u^{(x)})\, du, \qquad t \geq 0, \tag{5.18}$$

is equal in law to the process

$$x^{\alpha} \int_0^{(x^{-\alpha}t)\wedge\zeta^{(1)}} F(((s/x) \vee \overline{X}_u^{(1)})/X_u^{(1)})\, du, \qquad t \geq 0. \tag{5.19}$$

Note that the process in (5.18) is adapted to $\mathbb{G}$, whereas the process in (5.19) is adapted to $\tilde{\mathbb{G}}^{(x)} = \{\tilde{\mathcal{G}}_u^{(x)} : u \geq 0\}$, where $\tilde{\mathcal{G}}_u^{(x)} := \mathcal{G}_{x^{-\alpha}u}$. Using the general theory of optimal stopping [to deduce that the optimal time is the first hitting time of a closed set (we omit the details)] and this equality in law, we conclude that for $0 < x \leq s$,

$$\begin{aligned} v(x,s) &= \inf_{\tau} E_x[\int_0^{\tau\wedge\zeta} F((s \vee \overline{X}_t)/X_t)\, dt] \\ &= x^{\alpha} \inf_{\tau'} E_1[\int_0^{(x^{-\alpha}\tau')\wedge\zeta} F(((s/x) \vee \overline{X}_t)/X_t)\, dt], \end{aligned}$$

where the first infimum is taken over $\mathbb{G}$-stopping times τ and the second over $\tilde{\mathbb{G}}^{(x)}$-stopping times τ'. Before we can continue with the reduction of (5.13), we need to introduce a new filtration $\mathbb{H} := \{\mathcal{H}_t : t \geq 0\}$ in $\mathcal{G}$. Recall that the process

$$\varphi(t) = \int_0^t (X_u^{(1)})^{-\alpha}\, du, \qquad t < \zeta^{(1)},$$

is right-continuous and adapted to $\mathbb{G}$. Then

$$I_u = \inf\{0 < t < \zeta^{(1)} : \varphi(t) > u\}, \qquad u \geq 0,$$

is a right-continuous process which is strictly increasing on $[0, \varphi(\zeta^{(1)}-))$. In particular, I_u is a $\mathbb{G}$-stopping time for each $u \geq 0$. We now use I_u, $u \geq 0$, to time-change the filtration $\mathbb{G}$ according to

$$\mathcal{H}_u := \mathcal{G}_{I_u}, \qquad u \geq 0. \tag{5.20}$$

By Lemma 7.3 in [19] it follows that $\mathbb{H}$ is right-continuous. Also observe that the Lamperti representation ξ is adapted to $\mathbb{H}$. Finally, denote by $\mathcal{M}_1^{(x)}$ the set of all $\tilde{\mathbb{G}}^{(x)}$-stopping times and by $\mathcal{M}_2$ the set of all $\mathbb{H}$-stopping times. We can now formulate the main result of this subsection.

Lemma 5.12. *Let $f(z) = 1 - 2e^{-\Phi(q)z}$, $z \geq 0$, where Φ and q are as at the beginning of Subsection 5.3.1. Moreover, define the measure P^{α} by*

$$\left.\frac{dP^{\alpha}}{dP_1}\right|_{\mathcal{H}_t} = e^{\alpha\xi_t - \psi(\alpha)t} 1_{\{t<\mathbf{e}\}}. \tag{5.21}$$

For $0 < x \leq s$, we have

$$\begin{aligned} v(x,s) &= x^{\alpha} \inf_{\tau' \in \mathcal{M}_1^{(x)}} E_1[\int_0^{(x^{-\alpha}\tau')\wedge\zeta} F(((s/x)\vee\overline{X}_t)/X_t)\,dt] && (5.22)\\ &\geq x^{\alpha} \inf_{\nu\in\mathcal{M}_2} E^{\alpha}[\int_0^{\nu} e^{\psi(\alpha)u} f(Y_u^{\log(y)})\,du], && (5.23)\end{aligned}$$

where $y = s/x$, $Y_u^{\log(y)} := \log(y)\vee\overline{\xi}_u - \xi_u$ and $\overline{\xi}_u := \sup_{0\leq t\leq u}\xi_t$ for $u \geq 0$. In particular, under P^{α} the spectrally negative Lévy process ξ is not killed.

Despite the inequality in (5.23), it will be enough to deduce the solution of (5.13). To see why, suppose that the optimal stopping time for (5.23) is given by

$$\nu^* = \inf\{t \geq 0 : Y_t^{\log(y)} \geq k^*\}$$

for some $k^* > 0$. Additionally, setting $K^* := e^{-k^*}$, define

$$\begin{aligned}\tau^* &= \inf\{t \geq 0 : X_t \leq K^*(s\vee\overline{X}_t)\},\\ \tau' &= \inf\{t\geq 0 : X_{x^{-\alpha}t} \leq K^*((s/x)\vee\overline{X}_{x^{-\alpha}t})\}.\end{aligned}$$

It then holds that

$$\begin{aligned} E_x\int_0^{\tau^*} F((s\vee\overline{X}_t)/X_t)\,dt] &= x^{\alpha}E_1[\int_0^{x^{-\alpha}\tau'} F(((s/x)\vee\overline{X}_t)/X_t)\,dt]\\ &= x^{\alpha}E^{\alpha}[\int_0^{\nu^*} e^{\psi(\alpha)t} f(Y_t^{\log(s/x)})\,dt]\end{aligned}$$

and thus τ^* is optimal for (5.13). Hence it remains to show that the optimal stopping time for (5.23) is indeed of the assumed form. This is done in Section 5.5.

5.4.2 Reduction of problem (5.16)

Analogously to the previous subsection, we want to reduce (5.16) to a one-dimensional optimal stopping problem.

Let $\mathcal{M}_1^{(x)}$ be the set of all $\tilde{\mathbb{G}}^{(x)}$-stopping times and $\mathcal{M}_2$ the set of all $\mathbb{H}$-stopping times whenever $X \in \mathcal{C}_-^1$ is of type (iii). On the other hand, if $X \in \mathcal{C}_-^1$ is of type (i), then denote by $\mathcal{M}_1^{(x)}$ the set of all integrable $\tilde{\mathbb{G}}^{(x)}$-stopping times and by $\mathcal{M}_2$ the set of all $\mathbb{H}$-stopping times ν such that

$$\hat{E}^{-\alpha}[\int_0^{\nu} e^{\hat{\psi}(-\alpha)t}\,dt] < \infty,$$

where the precise definition of $\hat{P}^{-\alpha}$ and $\hat{E}^{-\alpha}$ respectively is given in Lemma 5.13 below. Following the same line of reasoning as in Subsection 5.4.1, one may obtain the analogue of Lemma 5.12; see Lemma 5.13 below. The only difference is that we express all in terms of the dual process $\hat{\xi}$ so that we obtain a one-dimensional optimal stopping problem in (5.26) that is of the same type as in (5.23) (a one-dimensional optimal stopping problem for a spectrally negative Lévy process reflected at its supremum). The advantage of this is that once the one-dimensional problem is solved, we can deduce the solution for both (5.13) and (5.16). Moreover, the fact that (5.23) and (5.26) only differ by switching to the dual essentially says that the problem of predicting the time at which the maximum or minimum is attained is, at least on the level of Lamperti representations, essentially the same.

Lemma 5.13. *Let $\hat{f}(z) = 1 - 2e^{-\hat{\Phi}(q)z}$, $z \geq 0$, where $\hat{\Phi}$ and q are as at the beginning of Subsection 5.3.2. Moreover, define the measure $\hat{P}^{-\alpha}$ by*

$$\frac{d\hat{P}^{-\alpha}}{dP_1}\bigg|_{\mathcal{H}_t} = e^{-\alpha\hat{\xi}_t - \hat{\psi}(-\alpha)t}1_{\{t<\mathbf{e}\}}. \tag{5.24}$$

For $0 < i \leq x$, we have

$$\begin{aligned} \hat{v}(x,i) &= x^{\alpha} \inf_{\tau' \in \mathcal{M}_1^{(x)}} E_1\Big[\int_0^{x^{-\alpha}\tau' \wedge \zeta} \hat{F}(X_t/(i \wedge \underline{X}_t))\, dt\Big] & (5.25)\\ &\geq x^{\alpha} \inf_{\nu \in \mathcal{M}_2} \hat{E}^{-\alpha}\Big[\int_0^{\nu} e^{\hat{\psi}(-\alpha)u} \hat{f}(\hat{Y}_t^{\log(\hat{y})})\, du\Big], & (5.26)\end{aligned}$$

where $\hat{y} = x/i$, $\hat{Y}_u^{\log(y)} := \log(y) \vee \overline{\hat{\xi}_u} - \hat{\xi}_u$ and $\overline{\hat{\xi}_u} := \sup_{0\leq t\leq u} \hat{\xi}_t$ for $u \geq 0$. In particular, under $\hat{P}^{-\alpha}$ the spectrally negative Lévy process $\hat{\xi}$ is not killed.

Analogously to Subsection 5.4.1, it follows that if the optimal stopping time for (5.26) is given by $\nu^* = \inf\{t \geq 0 : Y_t^{\log(y)} \geq \hat{k}^*\}$ for some $\hat{k}^* > 0$, then

$$\hat{\tau}^* = \inf\{t \geq 0 : X_t \geq \hat{K}^*(i \wedge \underline{X}_t)\}$$

is optimal in (5.16), where $\hat{K}^* := e^{\hat{k}^*}$. The remaining task is again to solve (5.26) and show that the optimal stopping time is indeed given by ν^*. This is done in Section 5.5.

5.5 The one-dimensional optimal stopping problem

In this section we solve a separate optimal stopping problem which is set up in such a way that once it is solved one can use it to deduce the solution of (5.23) and (5.26) and hence the solution of (5.13) and (5.16) respectively. This section is self-contained and can be read completely independently of Sections 5.3 and 5.4. Therefore, for convenience we will reuse some of the notation – there should be no confusion.

5.5.1 Setting and formulation of one-dimensional problem

Let us spend some time introducing the notation and formulating the problem. Suppose that $\Xi = \{\Xi_t : t \geq 0\}$ is an (unkilled) spectrally negative Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t : t \geq 0\}, \tilde{\mathbb{P}})$ satisfying the natural conditions; cf. [7], Section 1.3, p.39. For convenience we will assume without loss of generality that $(\Omega, \mathcal{F}) = (\mathbb{R}^{[0,\infty)}, \mathcal{B}^{[0,\infty)})$, where $\mathcal{B}$ is the Borel-σ-field on $\mathbb{R}$. The coordinate process on $(\Omega, \mathcal{F})$ is denoted by $Y = \{Y_t : t \geq 0\}$. Further, let $q \geq 0$ and suppose that Ξ under $\tilde{\mathbb{P}}$ is such that $\lim_{t\uparrow\infty} \Xi_t = -\infty$ whenever $q = 0$. Also assume that the Lévy measure associated with Ξ has no atoms whenever Ξ is of bounded variation. This is a purely technical condition which ensures that the q-scale functions $W^{(q)}$ associated with Ξ are continuously differentiable on $(0, \infty)$; see (5.8). Next, let $\beta \in \mathbb{R} \setminus \{0\}$ such that $\tilde{\mathbb{E}}[e^{\beta\Xi_1}] < \infty$. This condition is automatically satisfied if $\beta > 0$ due to the spectral negativity of Ξ and hence it is only an additional assumption when $\beta < 0$. The Laplace exponent is given by

$$\phi(\theta) := \log(\tilde{\mathbb{E}}[e^{\theta\Xi_1}]), \qquad \theta \geq 0 \wedge \beta,$$

and its right-inverse is defined as

$$\Phi(\lambda) := \sup\{\theta \geq 0 : \phi(\theta) = \lambda\}, \qquad \lambda \geq 0.$$

In particular, note that $\Phi(q) > 0$ and define

$$f(y) := 1 - 2e^{-\Phi(q)y}, \qquad y \geq 0.$$

Moreover, denote by $\tilde{\mathbb{P}}^\beta$ the measure obtained by the change of measure

$$\left.\frac{d\tilde{\mathbb{P}}^\beta}{d\tilde{\mathbb{P}}}\right|_{\mathcal{F}_t} = e^{\beta\Xi_t - \phi(\beta)t}, \qquad t \geq 0.$$

Finally, for $y \geq 0$, let $\mathbb{P}_y^\beta$ be the law of

$$y \vee \sup_{0\leq u\leq t} \Xi_u - \Xi_t, \qquad t \geq 0,$$

under $\tilde{\mathbb{P}}^\beta$.

We are interested in the optimal stopping problem

$$V^*(y) := \inf_{\tau\in\mathcal{M}} \mathbb{E}_y^\beta\Big[\int_0^\tau e^{-qt+\phi(\beta)t} f(Y_t)\, dt\Big] \tag{5.27}$$

for $y \geq 0$ and $(q, \beta) \in \mathcal{A}$, where

$$\mathcal{A} := \{(q, \beta) \in [0, \infty) \times \mathbb{R} \setminus \{0\} : q > \phi(\beta) \text{ or } q = 0 \text{ and } \beta < 0\},$$

and the set $\mathcal{M}$ denotes the set of $\mathbb{F}$-stopping times such that

$$\mathbb{E}_y^\beta[\int_0^\tau e^{-qt+\phi(\beta)t}\,dt] < \infty. \tag{5.28}$$

Note that $\mathcal{M}$ is the set of all $\mathbb{F}$-stopping times except when $q = 0$ and $\beta < 0$ in which case (5.28) is indeed a restriction because $\phi(\beta) > 0$ due to the assumption that $\lim_{t\uparrow\infty} \Xi_t = -\infty$.

5.5.2 Solution of one-dimensional problem

Given the underlying Markovian structure of (5.27), it is reasonable to look for an optimal stopping time of the form

$$\tau_k = \inf\{t \geq 0 : Y_t \geq k\}, \qquad k > 0.$$

However, when $q = 0$ and $\beta < 0$, we need to check whether $\tau_k \in \mathcal{M}$.

Lemma 5.14. *Let $k > 0$. If $q = 0$ and $\beta < 0$ (and hence $\phi(\beta) > 0$), it holds that $\mathbb{E}_y^\beta[\int_0^{\tau_k} e^{\phi(\beta)t}\,dt] < \infty$ for all $y \geq 0$.*

The next question we address is what the value function associated with the stopping times τ_k looks like. To this end, introduce the quantity

$$V_k(y) := \mathbb{E}_y^\beta[\int_0^{\tau_k} e^{-qt+\phi(\beta)t} f(Y_t)\,dt], \qquad y \geq 0.$$

The next result gives an expression for V_k in terms of scale functions.

Lemma 5.15. *For $k \geq 0$, we have*

$$\begin{aligned} V_k(y) &= -\int_y^k f(z) W_\beta^{(q-\phi(\beta))}(z-y)\,dz \\ &\quad + \frac{W_\beta^{(q-\phi(\beta))}(k-y)}{W_\beta^{(q-\psi(\beta))\prime}(k)}\Big(\int_0^k f(z) W_\beta^{(q-\phi(\beta))\prime}(z)\,dz - W_\beta^{(q-\phi(\beta))}(0)\Big). \end{aligned} \tag{5.29}$$

Having this semi-explicit form for V_k, the next step is to find the "good" threshold $k > 0$. This is done using the principle of smooth or continuous fit (cf. [28, 32, 33]) which suggests to choose k such that $\lim_{y\uparrow k} V_k'(y) = 0$ if Ξ is of unbounded variation and $\lim_{y\uparrow k} V_k(y) = 0$ if Ξ is of bounded variation. Note that, although the smooth or continuous fit condition is not necessarily part of the general theory of optimal stopping, it is imposed by the "rule of thumb" outlined in Section 7 of [1].

First assume that Ξ is of unbounded variation. Using (5.7) and (5.9), it follows that

$$V_k'(y) \quad = \quad \int_y^k f(z) W_\beta^{(q-\phi(\beta))\prime}(z-y)\,dz - \frac{W_\beta^{(q-\phi(\beta))\prime}(k-y)}{W_\beta^{(q-\phi(\beta))\prime}(k)} \int_0^k f(z) W_\alpha^{(q-\phi(\beta))\prime}(z)\,dz.$$

Letting y tend to k yields

$$0 = \lim_{y\uparrow k} \frac{W_\beta^{(q-\phi(\beta))\prime}(k-y)}{W_\beta^{(q-\phi(\beta))\prime}(k)} \int_0^k f(z) W_\beta^{(q-\phi(\beta))\prime}(z)\,dz. \tag{5.30}$$

Now note that by (5.7) and (5.10) we have

$$\lim_{y\uparrow k} W_\beta^{(q-\phi(\beta))\prime}(k-y) = \lim_{y\uparrow k} e^{-\beta(k-y)}(W^{(q)\prime}(k-y) - \beta W^{(q)}(k-y)) \in (0,\infty].$$

Similarly, $W_\beta^{(q-\phi(\beta))\prime}(k) = e^{-\beta k}(W^{(q)\prime}(k) - \beta W^{(q)}(k))$ which is clearly positive if $\beta < 0$. If $\beta > 0$, this is still true because $W^{(q)\prime}(z)/W^{(q)}(z) > \Phi(q)$ for $z > 0$ and $\Phi(q) > \beta$. In view of (5.30), we are forced to conclude that

$$\int_0^k f(z) W_\beta^{(q-\phi(\beta))\prime}(z)\,dz = 0.$$

Similarly, if Ξ is of bounded variation, we get

$$0 = \frac{W_\beta^{(q-\phi(\beta))}(0)}{W_\beta^{(q-\phi(\beta))\prime}(k)} \Big(\int_0^k f(z) W_\beta^{(q-\phi(\beta))\prime}(z)\,dz - W_\beta^{(q-\phi(\beta))}(0) \Big)$$

and hence, using (5.7) and (5.9), we infer

$$\int_0^k f(z) W_\beta^{(q-\phi(\beta))\prime}(z)\,dz = W^{(q)}(0). \tag{5.31}$$

Summing up, irrespective of the path variation of Ξ, we expect the optimal $k > 0$ to solve (5.31) and therefore we need to investigate the equation more closely.

Lemma 5.16. *The equation*

$$h(k) := \int_0^k f(z) W_\beta^{(q-\phi(\beta))\prime}(z)\,dz - W^{(q)}(0) = 0 \tag{5.32}$$

has a unique solution k^* on $(0,\infty)$. In particular, $k^* > \log(2)/\Phi(q)$.

We are now in a position to formulate our main result of this section.

Theorem 5.17. *The solution to* (5.27) *is given by*

$$V^*(y) \;=\; -\int_y^{k^*} f(z)e^{-\beta(z-y)}W^{(q)}(z-y)\,dz, \qquad y \geq 0, \tag{5.33}$$

with optimal stopping time τ_{k^*}, *where* k^* *is as in Lemma 5.16.*

5.6 Examples

In this section we present two examples, one of which shows that our results are consistent with the existing literature.

Corollary 5.18. *Let* X *be a pssMp with index of self-similarity* $\alpha > 0$ *such that its Lamperti representation is given by* $\xi_t = \sigma W_t - \mu t$, $t \geq 0$, *where* $\sigma > 0$, $\mu > 0$ *and* W_t, $t \geq 0$, *is a standard Brownian motion. In other words,* X *is of type* (ii) *such that* $\lim_{t\uparrow\infty} X_t = -\infty$. *Moreover, suppose that* $\alpha < 2\mu/\sigma^2$ *(this ensures that* $X \in \mathcal{C}^1_+$*). Then we have*

$$v(x,s) = \frac{1}{\mu}\Bigg[x^\alpha\left(1-\left(\frac{K^*s}{x}\right)^\alpha\right)\left(\frac{1}{\alpha}+\frac{2}{\alpha}\left(\frac{x}{s}\right)^{\Phi(0)}\right)$$
$$-\frac{x^\alpha}{\alpha-\Phi(0)}\left(1-\left(\frac{K^*s}{x}\right)^{\alpha-\Phi(0)}\right)+\frac{2s^\alpha(K^*)^{\alpha+\Phi(0)}}{\alpha+\Phi(0)}\left(1-\left(\frac{K^*s}{x}\right)^{-\Phi(0)-\alpha}\right)\Bigg],$$

where $\Phi(0) = 2\mu/\sigma^2$, *and* K^* *is the unique solution to*

$$K^{\alpha-\Phi(0)} + \frac{2\Phi(0)-3\alpha}{\alpha}K^\alpha + \frac{2\alpha}{\alpha+\Phi(0)}K^{\alpha+\Phi(0)} - \frac{2\Phi(0)^2}{\alpha(\alpha+\Phi(0))} = 0$$

on $(0,1)$. *In particular,* $K^* \in (0, 2^{-1/\Phi(0)})$.

Corollary 5.19. *Let* X *be a pssMp with index of self-similarity* $\alpha > 0$ *such that its Lamperti representation is given by* $\xi_t = \sigma W_t + \mu t$, $t \geq 0$, *where* $\sigma > 0$, $\mu > 0$ *and* W_t, $t \geq 0$, *is a standard Brownian motion. In other words,* X *is of type* (i) *such that* $\lim_{t\uparrow\infty} X_t = \infty$.

1. *If* $\alpha \neq 2\mu/\sigma^2$, *we have*

$$\hat{v}(x,i) = \frac{1}{\mu}\Bigg[x^\alpha\left(\left(\frac{\hat{K}^*i}{x}\right)^\alpha - 1\right)\left(\frac{1}{\alpha}+\frac{2}{\alpha}\left(\frac{i}{x}\right)^{\hat{\Phi}(0)}\right)$$
$$-\frac{x^\alpha}{\alpha+\hat{\Phi}(0)}\left(\left(\frac{\hat{K}^*i}{x}\right)^{\alpha+\hat{\Phi}(0)}-1\right)-\frac{2i^\alpha(\hat{K}^*)^{\alpha-\hat{\Phi}(0)}}{\hat{\Phi}(0)-\alpha}\left(\left(\frac{\hat{K}^*i}{x}\right)^{\hat{\Phi}(0)-\alpha}-1\right)\Bigg],$$

where $\hat{\Phi}(0) = 2\mu/\sigma^2$, *and* $\hat{K}^*$ *is the unique solution to*

$$K^{\hat{\Phi}(0)+\alpha} - \frac{3\alpha + 2\hat{\Phi}(0)}{\alpha}K^{\alpha} + \frac{2\alpha}{\alpha - \hat{\Phi}(0)}K^{\alpha-\hat{\Phi}(0)} - \frac{2\hat{\Phi}(0)^2}{\alpha(\alpha - \hat{\Phi}(0))} = 0$$

on $(1,\infty)$. *In particular,* $\hat{K}^* > 2^{1/\hat{\Phi}(0)}$.

2. *If* $\alpha = 2\mu/\sigma^2$, *we have*

$$\begin{aligned} \hat{v}(x,i) &= \frac{1}{\mu}\Bigg[x^{\alpha}\left(\frac{1}{\alpha} + \frac{2}{\alpha}\left(\frac{i}{x}\right)^{\alpha}\right)\left(\left(\frac{\hat{K}^* i}{x}\right)^{\alpha} - 1\right) \\ &\quad -\frac{x^2}{2\alpha}\left(\left(\frac{\hat{K}^* i}{x}\right)^{2\alpha} - 1\right) - 2i^{\alpha}\log(\hat{K}^* i/x)\Bigg], \end{aligned}$$

and $\hat{K}^*$ *is the unique solution to*

$$K^{2\alpha} - 5K^{\alpha} + 2\alpha\log(K) + 4 = 0$$

on $(1,\infty)$. *In particular,* $\hat{K}^* > 2^{1/\hat{\Phi}(0)}$.

Remark 5.20. Note that in contrast to Corollary 5.18, in Corollary 5.19 there is no condition required to ensure that $X \in \mathcal{C}^1_-$, since in this case X is of type (i) and then the only requirement is that the Laplace exponent of the Lamperti transformation of X exists. This is clearly the case in Corollary 5.19.

Remark 5.21. If X is a d-dimensional Bessel process with $d > 2$, then X is a pssMp with index of self-similarity $\alpha = 2$ and of type (i) with $\lim_{t\uparrow\infty} X_t = \infty$. It is known that its Lamperti representation is given by $\xi_t = W_t + \frac{(d-2)}{2}t$. Setting $\sigma = 1$ and $\mu = \frac{d-2}{2}$ in Corollary 5.19, one recovers Theorem 4 of [15]. In particular, if $d = 3$ one sees that $\hat{K}^*$ is the unique solution to

$$K^3 - 4K^2 + 4K - 1 = (K-1)(K^2 - 3K + 1) = 0$$

on $(1,\infty)$. Solving this equation shows that $\hat{K}^* = (3+\sqrt{5})/2$. The corresponding optimal stopping time can then be expressed as

$$\hat{\tau}^* = \inf\{t \geq 0 : X_t \geq \hat{K}^*(i \wedge \underline{X}_t)\} = \inf\{t \geq 0 : (X_t - (i \wedge \underline{X}_t))/(i \wedge \underline{X}_t) \geq \varphi\},$$

where $\varphi := \hat{K}^* - 1$ is the golden ratio. This was first observed and proved in [15].

5.7 Proofs

Proof of Lemma 5.3. For any $\mathbb{G}$-stopping time τ, we have

$$\begin{aligned}
|\Theta-\tau| &= (\Theta-\tau)^+ + (\tau-\Theta)^+ \\
&= \int_0^\Theta 1_{\{\tau\le t\}}\,dt + \int_0^\tau 1_{\{\Theta\le t\}}\,dt \\
&= \int_0^\Theta \big(1-1_{\{\tau>t\}}\big)\,dt + \int_0^\tau 1_{\{\Theta\le t\}}\,dt \\
&= \Theta - \int_0^\tau 1_{\{\Theta>t\}}\,dt + \int_0^\tau 1_{\{\Theta\le t\}}\,dt \\
&= \Theta - \int_0^\tau \big(1-1_{\{\Theta\le t\}}\big)\,dt + \int_0^\tau 1_{\{\Theta\le t\}}\,dt \\
&= \Theta + \int_0^\tau \big(2 1_{\{\Theta\le t\}}-1\big)\,dt.
\end{aligned}$$

□

Proof of Lemma 5.4. For any $\mathbb{G}$-stopping time τ with finite mean we have by Fubini's theorem,

$$\begin{aligned}
E_x\Big[\int_0^\tau (2 1_{\{\Theta\le t\}}-1)\,dt\Big] &= E_x\Big[\int_0^\infty (2 1_{\{\Theta\le t\}}-1)1_{\{t<\tau\}}\,dt\Big] \\
&= \int_0^\infty E_x\big[1_{\{t<\tau\}}\mathbb{E}_x\big[2 1_{\{\Theta\le t\}}-1|\mathcal{G}_t\big]\big]\,dt \\
&= E_x\Big[\int_0^\tau (2\mathbb{P}_x[\Theta\le t|\mathcal{G}_t]-1)\,dt\Big] \\
&= E_x\Big[\int_0^\tau (1-2\mathbb{P}_x[\Theta>t|\mathcal{G}_t])1_{\{t<\zeta\}}\,dt\Big] \\
&\quad +E_x[(\tau-\zeta)1_{\{\tau>\zeta\}}].
\end{aligned} \tag{5.34}$$

Using the strong Markov property of X we obtain on $\{t<\zeta\}$,

$$\begin{aligned}
P_x[\Theta>t|\mathcal{G}_t] &= P_x\Big[\overline{X}_t < \sup_{t\le u<\zeta} X_u\Big|\mathcal{G}_t\Big] \\
&= P_x\Big[s < \sup_{t\le u<\zeta} X_u\Big|\mathcal{G}_t\Big]\Big|_{s=\overline{X}_t} \\
&= P_{X_t}\Big[s < \sup_{0\le u<\zeta} X_u\Big]\Big|_{s=\overline{X}_t}.
\end{aligned}$$

Hence, using the Lamperti transformation we obtain for $0<x\le s$,

$$P_x\Big[s<\sup_{0\le u<\zeta} X_u\Big] = P_x\Big[\log(s/x) < \sup_{0\le u<\mathbf{e}} \xi_u\Big] = e^{-\Phi(q)\log(s/x)}.$$

Plugging this into (5.34) gives the result. □

Proof of Lemma 5.12. Using the fact that φ is strictly increasing on $[0,\zeta)$ and the Lamperti transformation shows that for $\tau' \in \mathcal{M}_1^{(x)}$,

$$\begin{aligned} &E_1[\int_0^{(x^{-\alpha}\tau')\wedge\zeta} F((y\vee \overline{X}_t)/X_t)\,dt] \qquad (5.35)\\ &= E_1[\int_0^{(x^{-\alpha}\tau')\wedge\zeta} F((y\vee \overline{X}_t)/X_t)1_{\{t<\zeta\}}\,dt]\\ &= E_1\Big[\int_0^{(x^{-\alpha}\tau')\wedge\zeta} f\big(\log(y)\vee\overline{\xi}_{\varphi(t)}-\xi_{\varphi(t)}\big)1_{\{\varphi(t)<\varphi(\zeta)\}}\,dt\Big]. \end{aligned}$$

Next, note that $\varphi'(t) = (X_t^{(1)})^{-\alpha} = e^{-\alpha\xi_{\varphi(t)}}$ for $t < \zeta^{(1)}$. Hence, changing variables according to $u = \varphi(t)$ shows that the right-hand side of (5.35) is equal to

$$E_1[\int_0^{\varphi((x^{-\alpha}\tau')\wedge\zeta)} e^{\alpha\xi_u} f\big(\log(y)\vee\overline{\xi}_u-\xi_u\big)1_{\{u<\mathbf{e}\}}\,du].$$

As $\tau' \in \mathcal{M}_1^{(x)}$, it follows that $\varphi((x^{-\alpha}\tau')\wedge\zeta)$ is a $\mathbb{H}$-stopping time that is less or equal than $\mathbf{e}$, and hence we conclude that

$$v(x,s) \geq x^{-\alpha}\inf_{\nu\in\mathcal{M}_2} E_1[\int_0^{\nu} e^{\alpha\xi_u} f\big(\log(y)\vee\overline{\xi}_u-\xi_u\big)1_{\{u<\mathbf{e}\}}\,du]. \qquad (5.36)$$

In other words, we have found a lower bound for $v(x,s)$ in terms of an optimal stopping problem for the Lamperti representation ξ reflected at its maximum. Using Fubini's theorem and a change of measure according to (5.21) yields for $\nu \in \mathcal{M}_2$,

$$\begin{aligned} &E_1[\int_0^{\nu} e^{\alpha\xi_u} f\big(\log(y)\vee\overline{\xi}_u-\xi_u\big)1_{\{u<\mathbf{e}\}}\,du]\\ &= \int_0^\infty E_1[e^{\alpha\xi_u} f\big(\log(y)\vee\overline{\xi}_u-\xi_u\big)1_{\{u<\mathbf{e}\}}1_{\{u<\nu\}}]\,du\\ &= \int_0^\infty E^{\alpha}[e^{\psi(\alpha)u} f\big(\log(y)\vee\overline{\xi}_u-\xi_u\big)1_{\{u<\nu\}}]\,du\\ &= E^{\alpha}[\int_0^{\nu} e^{\psi(\alpha)u} f(Y_u^{\log(y)})\,du]. \end{aligned}$$

Finally, note that the Laplace exponent of ξ under P^α is given by the expression $\psi_\alpha(\theta) = \psi(\theta+\alpha)-\psi(\alpha)$, $\theta \geq 0$. In particular, $\psi_\alpha(0) = 0$ and hence ξ is not killed under P^α. □

Proof of Lemma 5.14. Throughout this proof, let $\overline{\Xi}_t := \sup_{0\leq u\leq t}\Xi_u$, $t \geq 0$, and write $\tau_{k,y} := \inf\{t\geq 0 : y\vee\overline{\Xi}_t-\Xi_t \geq k\}$ for $y \geq 0$. If $y \geq k$ the assertion is clearly true and hence suppose that $y < k$. Using the fact that $\beta < 0$ in the second inequality, we have

$$\mathbb{E}_y^{\beta}[\int_0^{\tau_k} e^{\phi(\beta)t}\,dt] \;=\; \mathbb{E}_y^{\beta}\left[\frac{e^{\tau_k\phi(\beta)}}{\phi(\beta)}\right]-\frac{1}{\phi(\beta)}$$

$$
\begin{aligned}
&\le \phi(\beta)^{-1}\mathbb{E}_y^{\beta}[e^{\tau_k\phi(\beta)}] \\
&= \phi(\beta)^{-1}\tilde{\mathbb{E}}[e^{\beta\Xi_{\tau_{k,y}}}] \\
&= \phi(\beta)^{-1}\tilde{\mathbb{E}}\big[e^{-\beta(y\vee\overline{\Xi}_{\tau_{k,y}}-\Xi_{\tau_k,y})+\beta(y\vee\overline{\Xi}_{\tau_{k,y}})}\big] \\
&\le \phi(\beta)^{-1}\tilde{\mathbb{E}}\big[e^{-\beta(y\vee\overline{\Xi}_{\tau_{k,y}}-\Xi_{\tau_k,y})}\big].
\end{aligned}
$$

It is now shown in Theorem 1 in [2] that the expression on the right-hand side is finite. □

Proof of Lemma 5.15. Define for $\eta \ge 0$ the functions

$$\tilde{V}_k(y) := \mathbb{E}_y^{\beta}[\int_0^{\tau_k} e^{-\eta t} f(Y_t)\, dt].$$

Now recall from Theorem 8.11 in [21] that the density of the η-potential measure of Y upon leaving $[0,k)$ under $\mathbb{P}_y^{\beta}$ is, for $y, z \in [0,k]$, given by

$$
\begin{aligned}
U^{(\eta)}(y,dz) \;=\; & \left(W_\beta^{(\eta)}(k-y)\frac{W_\beta^{(\eta)\prime}(z)}{W_\beta^{(\eta)\prime}(k)} - W_\beta^{(\eta)}(z-y)\right)dz \\
& + W_\beta^{(\eta)}(k-y)\frac{W_\beta^{(\eta)}(0)}{W_\beta^{(\eta)\prime}(k)}\delta_0(dz).
\end{aligned}
$$

Using this expression, we see that for $y \ge 0$,

$$
\begin{aligned}
\tilde{V}_k(y) \;=\; & \int_0^k f(z)\left(W_\beta^{(\eta)}(k-y)\frac{W_\beta^{(\eta)\prime}(z)}{W_\beta^{(\eta)\prime}(k)} - W_\beta^{(\eta)}(z-y)\right)dz \\
& - W_\beta^{(\eta)}(k-y)\frac{W_\beta^{(\eta)}(0)}{W_\beta^{(\eta)\prime}(k)}. \qquad (5.37)
\end{aligned}
$$

If $(q,\beta) \in \mathcal{A}$ is such that $q > \phi(\beta)$ the result follows by setting $\eta = q - \phi(\beta)$. Hence, the remaining case is when $q = 0$ and $\beta < 0$ (and hence $\phi(\beta) > 0$). In this case, note that by Lemma 5.14 we have for any $w \in U := \{z \in \mathbb{C} : \mathfrak{Re}(z) > -\phi(\beta)\}$,

$$|\mathbb{E}_y^{\beta}[\int_0^{\tau_k} e^{-wt} f(Y_t)\, dt]| \le \mathbb{E}_y^{\beta}[\int_0^{\tau_k} e^{\phi(\beta)t}\, dt] < \infty.$$

Now define for $w \in U$ the functions

$$
\begin{aligned}
g(w) &:= \mathbb{E}_y^{\beta}[\int_0^{\tau_k} e^{-wt} f(Y_t)\, dt] \quad \text{and} \\
g_n(w) &:= \mathbb{E}_y^{\beta}[\int_0^{\tau_k} e^{-wt} f(Y_t)\, dt 1_{\{\tau_k \le n\}}], \qquad n \ge 0.
\end{aligned}
$$

The functions g_n are analytic in U since one can differentiate under the integral sign.

Moreover, for $w \in U$ we have the estimate

$$|g(w) - g_n(w)| \leq \mathbb{E}_y^\beta[\int_0^{\tau_k} e^{\phi(\beta)t}\, dt 1_{\{\tau_k > n\}}]$$

which together with the fact that the right-hand side tends to zero as $n \uparrow \infty$ implies that g_n converges uniformly to g in U. Thus, Weierstrass' theorem shows that g is analytic in U. Next, we deal with the right-hand side of (5.37). From the series representation of $W^{(q)}(x)$ provided in the proof of Lemma 3.6 in [20], it is possible to show that (after some work) the right-hand side of (5.37) is also analytic (on the whole of $\mathbb{C}$). By the identity theorem it then follows that (5.37) holds for $\eta \in U$, in particular for real η such that $\eta > -\phi(\beta)$. Finally, to obtain the result for $\eta = -\phi(\beta)$, take limits on both sides of (5.37) and use dominated convergence on the left-hand side and analyticity on the right-hand side. This completes the proof. $\square$

Proof of Lemma 5.16. Using (5.7), it follows that

$$h'(k) = f(k)e^{-\beta k}(W^{(q)\prime}(k) - \beta W^{(q)}(k))$$

for $k > 0$. If $(q, \beta) \in \mathcal{A}$ such that $\beta > 0$, then $\Phi(q) > \beta$ and, using (5.7),

$$\frac{W^{(q)\prime}(z)}{W^{(q)}(z)} = \frac{W'_{\Phi(q)}(z)}{W_{\Phi(q)}(z)} + \Phi(q) > \Phi(q) > \beta$$

for $z > 0$. Therefore, we see that $h'(k) < 0$ on $(0, k_0)$, $h'(k_0) = 0$ and $h'(k) > 0$ on (k_0, ∞), where $k_0 = \log(2)/\Phi(q)$. The same is of course true if $(q, \beta) \in \mathcal{A}$ and $\beta < 0$. Additionally, it holds that $\lim_{k\uparrow\infty} h(k) > 0$. Indeed, let $z_0 > k_0$ such that $f(z) \geq 1/2$ for $z \geq z_0$ and hence for $k > z_0$,

$$\begin{aligned}
h(k) &= h(k_0) + \int_{k_0}^k f(z) W_\beta^{(q-\phi(\beta))\prime}(k)\, dz - W^{(q)}(0) \\
&\geq h(k_0) + \frac{1}{2}\int_{z_0}^k W_\beta^{(q-\phi(\beta))\prime}(z)\, dz - W^{(q)}(0) \\
&= h(k_0) + \frac{1}{2}(e^{-\beta k}W^{(q)}(k) - e^{-\beta z_0}W^{(q)}(z_0)) - W^{(q)}(0),
\end{aligned}$$

where in the last equality we have used (5.7). Again by (5.7), $W^{(q)}(k) = e^{\Phi(q)k}W_{\Phi(q)}(k)$ which together with the fact that $\Phi(q) > \beta$ implies that the right-hand side tends to infinity as $k \uparrow \infty$. Combining this with the fact that $\lim_{k\downarrow 0} f(k) \leq 0$ and the intermediate value theorem shows that there is a unique $k^* > k_0$ such that $f(k^*) = 0$. This completes the proof. $\square$

Proof of Theorem 5.17. Let V be defined as the right-hand side of (5.33). It is enough to check the following conditions:

(i) $V(y) \leq 0$ for all $y \geq 0$;

(ii) the process

$$e^{-(q-\phi(\beta))t}V(Y_t) + \int_0^t e^{-(q-\phi(\beta))u} f(Y_u)\,du, \qquad t \geq 0,$$

is a $\mathbb{P}_y^\beta$-submartingale for all $y \geq 0$.

To see why these are sufficient conditions, note that (i) and (ii) together with Fatou's lemma in the second inequality and Doob's stopping theorem in the third inequality show that for $\tau \in \mathcal{M}$,

$$\begin{aligned}
\mathbb{E}_y^\beta[\int_0^\tau e^{-(q-\phi(\beta))t} f(Y_u)\,du] &\geq \mathbb{E}_y^\beta[V(Y_\tau) + \int_0^\tau e^{-(q-\phi(\beta))t} f(Y_u)\,du] \\
&\geq \limsup_{t\uparrow\infty} \mathbb{E}_y^\beta[V(Y_{t\wedge\tau}) + \int_0^{t\wedge\tau} e^{-(q-\phi(\beta))u} f(Y_u)\,du] \\
&\geq V(y).
\end{aligned}$$

Since these inequalities are all equalities for $\tau = \tau_{k^*}$ the result follows.

The remainder of this proof is devoted to checking conditions (i) and (ii).

Verification of condition (i)**:** Recall that $k^* > k_0 = \log(2)/\Phi(q)$ and that $f(z) \leq 0$ on $(0, k_0]$ and $f(z) > 0$ on (k_0, ∞). It follows that $\tau_{k^*} \geq \tau_{k_0}$ and hence, using the strong Markov property, we see that

$$\begin{aligned}
V(y) &= \mathbb{E}_y^\beta[\int_0^{\tau_{k_0}} e^{-(q-\phi(\beta))t} f(Y_t)\,dt] + \mathbb{E}_y^\beta[\int_{\tau_{k_0}}^{\tau_{k^*}} e^{-(q-\phi(\beta))t} f(Y_t)\,dt] \\
&= \mathbb{E}_y^\beta[\int_0^{\tau_{k_0}} e^{-(q-\phi(\beta))t} f(Y_t)\,dt] + \mathbb{E}_y^\beta[e^{-(q-\phi(\beta))\tau_{k_0}} V(Y_{\tau_{k_0}})] \\
&\leq 0,
\end{aligned}$$

where the last inequality follows from the fact that $f(z) \leq 0$ on $(0, k_0]$ and $V(y) \leq 0$ on $[k_0, \infty)$. This completes the proof of (i).

Verification of condition (ii)**:** The proof of this is similar to the previous chapters when we established the supermartingale property of certain processes. Hence, we just outline the main steps and omit the details.

As for a first step, one may use the Markov property to show that the process

$$Z_t := e^{-(q-\phi(\beta))(t\wedge\tau_{k^*})}V(Y_{t\wedge\tau_{k^*}}) + \int_0^{t\wedge\tau_{k^*}} e^{-(q-\phi(\beta))u} f(Y_u)\,du, \qquad t \geq 0,$$

is a $\mathbb{P}_y^\beta$-martingale for $0 < y < k^*$. Indeed, for $t \geq 0$, the strong Markov property gives

$$\begin{aligned}
\mathbb{E}_y^\beta[Z_{\tau_{k^*}}|\mathcal{F}_t] &= Z_{\tau_{k^*}} 1_{\{\tau_{k^*}<t\}} + \mathbb{E}_y^\beta[Z_{\tau_{k^*}}|\mathcal{F}_t] 1_{\{t\leq\tau_{k^*}\}} \\
&= Z_{\tau_{k^*}} 1_{\{\tau_{k^*}<t\}} + \int_0^t e^{-(q-\phi(\beta))u} f(Y_u)\, du 1_{\{t\leq\tau_{k^*}\}} \\
&\quad + e^{-(q-\phi(\beta))t} V(Y_t) 1_{\{t\leq\tau_{k^*}\}} \\
&= Z_{t\wedge\tau_{k^*}}
\end{aligned}$$

from which the desired martingale property follows.

As for a second step, use Doob's optional stopping theorem to deduce that for $0 < k < k^*$ the process $e^{-(q-\phi(\beta)t)(t\wedge\tau_k)} V(Y_{t\wedge\tau_k})$, $t \geq 0$, is a $\mathbb{P}_y^\beta$-martingale for $0 \leq y < k$. Using this in conjunction with the appropriate version of Itô's formula (cf. Theorem 71, Chapter IV of [36]) implies that

$$(\hat{\Gamma}^\beta V)(y) - (q - \phi(\beta))V(y) = -f(y), \qquad y \in [0, k^*), \tag{5.38}$$

where $\hat{\Gamma}^\beta$ is the generator of $-\Xi$ under $\tilde{\mathbb{P}}^\beta$.

Finally, applying the appropriate version of Itô's formula one more time to the process $e^{-(q-\phi(\beta)t)} V(Y_t)$, $t \geq 0$, and using (5.38) shows that

$$e^{-(q-\phi(\beta))t} V(Y_t) + \int_0^t e^{-(q-\phi(\beta))u} f(Y_u)\, du, \qquad t \geq 0,$$

is a $\mathbb{P}_y^\beta$-submartingale for all $y \geq 0$. This finishes the sketch of the proof of (ii). $\square$

Proof of Theorem 5.6. The result follows by Lemma 5.12 (and what was said just after it) and Theorem 5.17. Specifically, using Theorem 5.17 with Ξ equal to ξ unkilled, $\beta = \alpha$, $y = \log(s/x)$ and then setting $K^* := e^{-k^*}$ gives

$$\begin{aligned}
v(x, s) &= -x^\alpha \int_{\log(s/x)}^{-\log(K^*)} (1 - 2e^{-\Phi(q)u}) W_\alpha^{(q-\phi(\alpha))}(u - \log(s/x))\, du \\
&= -x^\alpha \int_{K^* s}^{x} z^{-1}(1 - 2e^{-\Phi(q)\log(s/z)}) W_\alpha^{(q-\phi(\alpha))}(\log(x/z))\, dz,
\end{aligned}$$

where in the second equality we changed variables according to $u = \log(s/z)$. The expression for $v(x, s)$ in the theorem now follows after an application of (5.7). As for the optimal constant K^*, we see that K^* satisfies the equation

$$\int_0^{\log(1/K)} (1 - 2e^{-\Phi(q)z}) W_\alpha^{(q-\phi(\alpha))\prime}(z)\, dz = W^{(q)}(0) \quad \text{on } (0, 1).$$

$\square$

Proof of Theorem 5.11. The result follows by Lemma 5.13 (and what was said just after it) and Theorem 5.17 with Ξ. Specifically, using Theorem 5.17 with Ξ equal to $\hat{\xi}$ unkilled, $\beta = -\alpha$, $y = \log(x/i)$ and then setting $\hat{K}^* := e^{k^*}$ gives

$$\begin{aligned}
\hat{v}(x,i) &= -x^\alpha \int_{\log(x/i)}^{\log(\hat{K}^*)} (1 - 2e^{-\hat{\Phi}(q)u}) \hat{W}_{-\alpha}^{(q-\hat{\phi}(-\alpha))}(u - \log(x/i))\, du \\
&= -x^\alpha \int_x^{\hat{K}^* i} z^{-1}(1 - 2e^{-\hat{\Phi}(q)\log(z/i)}) W_{-\alpha}^{(q-\hat{\phi}(-\alpha))}(\log(z/x))\, dz,
\end{aligned}$$

where in the second equality we changed variables according to $u = \log(z/i)$. The expression for $\hat{v}(x,i)$ in the theorem now follows after an application of (5.7). As for the optimal constant $\hat{K}^*$, we see that $\hat{K}^*$ satisfies the equation

$$\int_0^{\log(K)} (1 - 2e^{-\hat{\Phi}(q)z}) \hat{W}_{-\alpha}^{(q-\hat{\phi}(-\alpha))'}(z)\, dz = \hat{W}^{(q)}(0) \qquad \text{on } (1,\infty).$$

$\square$

Proof of Corollary 5.18. It is easy to check that $\psi(\theta) = \frac{\sigma^2}{2}\theta^2 - \mu\theta$, $\Phi(0) = \frac{2\mu}{\sigma^2}$ and $W^{(0)}(x) = \frac{e^{x\Phi(0)}-1}{\mu}$. Also note that $\alpha < \Phi(0)$ by assumption. For convenience, write $k = K^*$. It now follows from Theorem 5.6 that

$$\begin{aligned}
v(x,s) &= -\int_{ks}^x \left(1 - 2(z/s)^{\Phi(0)}\right) z^{\alpha-1} \frac{(x/z)^{\Phi(0)} - 1}{\mu}\, dz \\
&= \frac{1}{\mu}\Bigg[-x^{\Phi(0)} \int_{ks}^x z^{\alpha-1-\Phi(0)}\, dz + \int_{ks}^x z^{\alpha-1}\, dz \\
&\quad +2(x/s)^{\Phi(0)} \int_{ks}^x z^{\alpha-1}\, dz - 2s^{-\Phi(0)} \int_{ks}^x z^{\alpha-1+\Phi(0)}\, dz \Bigg] \\
&= \frac{1}{\mu}\Bigg[-x^{\Phi(0)} \left(\frac{x^{\alpha-\Phi(0)}}{\alpha - \Phi(0)} - \frac{(ks)^{\alpha-\Phi(0)}}{\alpha-\Phi(0)} \right) + \frac{x^\alpha}{\alpha} - \frac{(ks)^\alpha}{\alpha} \\
&\quad +2(x/s)^{\Phi(0)} \left(\frac{x^\alpha}{\alpha} - \frac{(ks)^\alpha}{\alpha} \right) - 2s^{-\Phi(0)} \left(\frac{x^{\alpha+\Phi(0)}}{\alpha+\Phi(0)} - \frac{(ks)^{\alpha+\Phi(0)}}{\alpha+\Phi(0)} \right) \Bigg] \\
&= \frac{1}{\mu}\Bigg[\frac{x^\alpha}{\alpha - \Phi(0)} \left(\left(\frac{ks}{x}\right)^{\alpha-\Phi(0)} - 1 \right) - \frac{x^\alpha}{\alpha} \left(\left(\frac{ks}{x}\right)^\alpha - 1 \right) \\
&\quad - \frac{2s^{-\Phi(0)} x^{\alpha+\Phi(0)}}{\alpha} \left(\left(\frac{ks}{x}\right)^\alpha - 1 \right) - \frac{2s^\alpha k^{\alpha+\Phi(0)}}{\alpha+\Phi(0)} \left(\left(\frac{ks}{x}\right)^{-\Phi(0)-\alpha} - 1 \right) \Bigg].
\end{aligned}$$

Adding the second and third term gives

$$v(x,s) = \frac{1}{\mu}\Bigg[x^\alpha \left(1 - \left(\frac{ks}{x}\right)^\alpha \right) \left(\frac{1}{\alpha} + \frac{2}{\alpha}\left(\frac{x}{s}\right)^{\Phi(0)} \right)$$

$$-\frac{x^\alpha}{\alpha-\Phi(0)}\left(1-\left(\frac{ks}{x}\right)^{\alpha-\Phi(0)}\right)+\frac{2s^\alpha k^{\alpha+\Phi(0)}}{\alpha+\Phi(0)}\left(1-\left(\frac{ks}{x}\right)^{-\Phi(0)-\alpha}\right)\Bigg].$$

Next, let us derive the equation for K^*. Using (5.7) and changing variables according to $u=e^z$ shows that K^* is the unique root of

$$\int_1^{1/K} u^{-\alpha-1}(1-2u^{-\Phi(0)})(\Phi(0)u^{\Phi(0)}-\alpha u^{\Phi(0)}+\alpha)\,du=0 \qquad \text{on } (0,1). \tag{5.39}$$

Solving the integral and rearranging gives the claim. ☐

Proof of Corollary 5.19. Clearly, $-\xi_t=\sigma W_t-\mu t$ and it is straightforward to check that $\hat{\psi}(\theta)=\frac{\sigma^2}{2}\theta^2-\mu\theta$, $\hat{\Phi}(0)=\frac{2\mu}{\sigma^2}$ and $\hat{W}^{(0)}(x)=\frac{e^{x\hat{\Phi}(0)}-1}{\mu}$. We derive the result for $\alpha\neq\hat{\Phi}(0)$, the case when $\alpha=\hat{\Phi}(0)$ is similar and we omit the details. For convenience, write $k=\hat{K}^*$. By Theorem 5.11 we have

$$\begin{aligned}
\hat{v}(x,i) &= -\int_x^{ki}\big(1-2(i/z)^{\hat{\Phi}(0)}\big)z^{\alpha-1}\frac{(z/x)^{\hat{\Phi}(0)}-1}{\mu}\,dz\\
&= \frac{1}{\mu}\Bigg[-x^{-\hat{\Phi}(0)}\int_x^{ki} z^{\alpha-1+\hat{\Phi}(0)}\,dz+\int_x^{ki} z^{\alpha-1}\,dz\\
&\quad +2(i/x)^{\hat{\Phi}(0)}\int_x^{ki} z^{\alpha-1}\,dz-2i^{\hat{\Phi}(0)}\int_x^{ki} z^{\alpha-1-\hat{\Phi}(0)}\,dz\Bigg]\\
&= \frac{1}{\mu}\Bigg[-x^{-\hat{\Phi}(0)}\left(\frac{(ki)^{\alpha+\hat{\Phi}(0)}}{\alpha+\hat{\Phi}(0)}-\frac{x^{\alpha+\hat{\Phi}(0)}}{\alpha+\hat{\Phi}(0)}\right)+\frac{(ki)^\alpha}{\alpha}-\frac{x^\alpha}{\alpha}\\
&\quad +2(i/x)^{\hat{\Phi}(0)}\left(\frac{(ki)^\alpha}{\alpha}-\frac{x^\alpha}{\alpha}\right)-2i^{\hat{\Phi}(0)}\left(\frac{(ki)^{\alpha-\hat{\Phi}(0)}}{\alpha-\hat{\Phi}(0)}-\frac{x^{\alpha-\hat{\Phi}(0)}}{\alpha-\hat{\Phi}(0)}\right)\Bigg]\\
&= \frac{1}{\mu}\Bigg[\frac{-x^\alpha}{\alpha+\hat{\Phi}(0)}\left(\left(\frac{ki}{x}\right)^{\alpha+\hat{\Phi}(0)}-1\right)+\frac{x^\alpha}{\alpha}\left(\left(\frac{ki}{x}\right)^{\alpha}-1\right)\\
&\quad +\frac{2i^{\hat{\Phi}(0)}x^{\alpha-\hat{\Phi}(0)}}{\alpha}\left(\left(\frac{ki}{x}\right)^{\alpha}-1\right)-\frac{2i^\alpha k^{\alpha-\hat{\Phi}(0)}}{\hat{\Phi}(0)-\alpha}\left(\left(\frac{ki}{x}\right)^{\hat{\Phi}(0)-\alpha}-1\right)\Bigg].
\end{aligned}$$

Adding the second and third term gives

$$\begin{aligned}
\hat{v}(x,i) &= \frac{1}{\mu}\Bigg[x^\alpha\left(\left(\frac{ki}{x}\right)^{\alpha}-1\right)\left(\frac{1}{\alpha}+\frac{2}{\alpha}\left(\frac{i}{x}\right)^{\hat{\Phi}(0)}\right)\\
&\quad -\frac{x^\alpha}{\alpha+\hat{\Phi}(0)}\left(\left(\frac{ki}{x}\right)^{\alpha+\hat{\Phi}(0)}-1\right)-\frac{2i^\alpha k^{\alpha-\hat{\Phi}(0)}}{\hat{\Phi}(0)-\alpha}\left(\left(\frac{ki}{x}\right)^{\hat{\Phi}(0)-\alpha}-1\right)\Bigg].
\end{aligned}$$

Next, let us derive the equation for $\hat{K}^*$. Using (5.7) and changing variables according to $u = e^z$ shows that $\hat{K}^*$ has to satisfy the equation

$$\int_1^K u^{\alpha-1}(1-2u^{-\hat{\Phi}(0)})(\alpha u^{\hat{\Phi}(0)} - \alpha + \hat{\Phi}(0)u^{\hat{\Phi}(0)})\, du = 0 \qquad \text{on } (1,\infty).$$

Solving the integral and rearranging gives the claim. □

5.8 Outlook/Future work

Making assumptions to obtain certain results leads naturally to the question whether one could weaken or remove them. Looking at the assumptions we made to obtain our main results in this chapter, two questions arise immediately:

- What happens if one drops the assumption on one-sided jumps?
- Can we remove the assumption on the integrability of ζ?

Let us begin with the first question and assume that we are given an X in $\mathcal{C}_+$ or $\mathcal{C}_-$, but without the restriction of one-sided jumps. Provided the Lamperti representation ξ of X is not a compound Poisson process and provided that the Laplace exponent of ξ exists where necessary, one sees that all the arguments up to Section 5.5 still go through and as a result one is led to solving (5.27), but with Y being the reflection of a general Lévy process Ξ. One special case of this, namely when $q = \phi(\beta)$ and Ξ drifts to $-\infty$ under $\tilde{\mathbb{P}}^\beta$ (and hence $\beta < 0$), is treated in [4]. Moreover, when $q > \phi(\beta)$, then (5.27) for a general Ξ is nothing else than a killed version of the problem studied in [4] and therefore one should in principle be able to solve the prediction problem under the assumption that $q > \phi(\beta)$ and with no restrictions on the jumps other than that the Laplace exponent has to exist where necessary. On the other hand, when $q \geq \phi(\beta)$, then it is not yet clear whether the proof in [4] can be modified to also provide a solution in this case.

As for the second question, suppose for simplicity that $X \in \mathcal{C}_+^1$ of type (ii) such that ζ is not integrable. In this case one can still formulate the corresponding prediction problem except that one cannot allow ζ as a potential stopping strategy in (5.13). In particular, it follows that the Laplace exponent of the Lamperti representation satisfies $\psi(\alpha) \geq 0$. As a result, one is led to (5.27), but with $q = 0$ and $\phi(\beta) > 0$. Analysing (5.27) carefully then suggests that it becomes degenerate in the sense that the value function can be made arbitrarily small. There is numerical evidence for this, but unfortunately we have so far not been able to prove this rigorously.

Finally, let us conclude this section with a further issue that could be improved. In Section 5.6 we provide two explicitly solvable examples, but the processes X we consider have continuous paths. Of course, it would be desirable to have explicitly solvable examples with discontinuous X, but, although many scale functions are known explicitly (cf. [20]), it seems difficult to find tractable examples.

5.9 Connection to previous chapters

The aim of this section is to briefly explain how the prediction problem studied in this chapter fits into the general context described in Chapter 1.

The reasoning in Subsection 5.4.1 up to equation (5.36) shows that solving (5.13) essentially means solving

$$u(x,s) := x^{-\alpha} \inf_{\nu} E_1[\int_0^{\nu} e^{\alpha \xi_u} f\big(\log(s/x) \vee \overline{\xi}_u - \xi_u\big) 1_{\{u<\mathbf{e}\}}\, du], \tag{5.40}$$

where $0 < x \le s$ and the infimum is taken over all $\mathbb{H}$-stopping times. Instead of continuing with a change of measure that reduces (5.40) to a one-dimensional problem, one could stay in this two-dimensional setting. In order to reflect this, we may rewrite (5.40) as

$$u(x,s) = x^{\alpha} \inf_{\nu} E_1[\int_0^{\nu} \tilde{c}(\xi_u, \log(s/x) \vee \overline{\xi}_u) 1_{\{u<\mathbf{e}\}}\, du], \tag{5.41}$$

where $\tilde{c}(\tilde{x},\tilde{s}) = e^{\alpha\tilde{x}} f(\tilde{s}-\tilde{x})$ for $0 < \tilde{x} \le \tilde{s}$. Since $\mathbf{e}$ is exponentially distributed with some parameter $q \ge 0$, we may change measure according to

$$\frac{dQ}{dP_1}\bigg|_{\mathcal{H}_t} = e^{qt} 1_{\{t<\mathbf{e}\}},$$

which allows us to write

$$u(x,s) = x^{\alpha} \inf_{\nu} E^Q[\int_0^{\nu} e^{-qu} \tilde{c}(\xi_u, \log(s/x) \vee \overline{\xi}_u)\, du].$$

The process ξ under P^Q is an unkilled spectrally negative Lévy process satisfying $\lim_{t\uparrow\infty} \xi_t = \infty$ whenever $q = 0$. The Laplace exponent of ξ (under P^Q) is denoted by ϕ. Setting $f \equiv 0$ and $c = \tilde{c}$ in (1.5) gives that

$$u(x,s) = -x^{\alpha} V(0, \log(s/x)).$$

Note that here the function $c = \tilde{c}$ is not positive as assumed in (1.5). However, it is clear that the method in Subsection 1.2.1 still works. Hence, denoting by $W^{(q)}$ the q-scale function associated with ξ under P^Q, one expects

$$u(x,s) = x^{\alpha} \int_{\log(s/x)-g(\log(s/x))}^{0} e^{\alpha y} f(\log(s/x) - y) W^{(q)}(-y)\, dy,$$

where g is the maximal solution of

$$g'(s) = 1 - \frac{\int_0^{g(s)} e^{\alpha(s-y)} f'(y) W^{(q)}(y)\, dy}{e^{\alpha(s-g(s))} f(g(s)) W^{(q)}(g(s))}.$$

Making use of (5.7), the equation for g may be written as

$$(1 - g'(s)) f(g(s)) W_\alpha^{(q-\phi(\alpha))}(g(s)) = \int_0^{g(s)} f'(y) W_\alpha^{(q-\phi(\alpha))}(y)\, dy.$$

Further, integration by parts and rearranging gives

$$g'(s) f(g(s)) W_\alpha^{(q-\phi(\alpha))}(g(s)) = -W_\alpha^{(q-\phi(\alpha))}(0) + \int_0^{g(s)} f(y) W_\alpha^{(q-\phi(\alpha))\prime}(y)\, dy.$$

We are now interested in the maximal solution g of this equation. However, seeing what the maximal solution is seems difficult. Naively, one could try a straight line as it is a simple choice and it has proved to be the maximal solution on a couple of occasions; see in [38] for instance. Thus, we set $g(s) \equiv k^*$ for some $0 < k^* < 1$. Using the explicit form of g and changing variables according to $y = \log(z/x)$ yields

$$u(x,s) = -\int_{se^{-k^*}}^{x} z^{\alpha-1} f(\log(s/z)) W^{(q)}(\log(x/z))\, dz. \tag{5.42}$$

Hence, we have found a candidate value function for (5.40) and a candidate optimal stopping time of the form $\inf\{t \geq 0 : \overline{\xi}_t - \xi_t \geq k^*\}$. One could now proceed with a classical verification argument and prove that this is indeed the solution and hence obtain Theorem 5.6. However, we did not do so because the reduction to one-dimension reveals why the solutions (the optimal stopping times) are of such a simple form.

Of course, a very similar argument could also be used to obtain Theorem 5.11. In fact, this was the original method which was used to derive Theorem 3 in [15].

To conclude, let us emphasise that problems (5.1) and (5.2) are of type (1.5) (modulo an application of the Lamperti transformation) and in this sense Chapter 5 is related to the other chapters in this thesis.

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