Contributions to Determinantally Equivalent Functions & Self-Similar Markov Processes

by

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A thesis presented for the degree of
Doctor of Philosophy

Department of Statistics
University of Warwick
September 2025

Contents

To my mother.

Acknowledgements

I would like to express my gratitude to my supervisors Andreas Kyprianou and Víctor Rivero; for this thesis would not have came to be without their initial vision and guidance.

Part of this thesis is based on the publication [53]. I am grateful to the anonymous referee for their numerous invaluable comments and suggestions that significantly improved the clarity and exposition of the work.

I would also like to thank my thesis examiners, Aleksandar Mijatović and Alex Watson, for their careful reading of the thesis and for their helpful comments and suggestions.

Last, but in no way least, I want to thank my mother, who has always been there for me; and whose unwavering support and encouragement have sustained me throughout the challenges of my unique doctoral journey. All the predicaments – even if I had the power to – I would not change, for one can only acquire strength, wisdom and invaluable life skills through them.

Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree. Except where it is stated otherwise, all the results of this thesis are, to the best of my knowledge, original and my own work.

- Chapters one and two provide introductory and preliminary material for the mathematical content of the thesis.
- Chapter three comprises the published article [53], supplemented by a few connections to probability theory.
- The content of chapters three to seven are based on the preprint [38], in collaboration with my supervisors Andreas Kyprianou and Víctor Rivero.

Abstract

The thesis lies at the intersection of discrete mathematics, linear algebra, and probability theory. It is structured in two parts, each centred essentially on a structural equivalence problem: the first concerns functions (and matrices), while the second concerns stochastic processes. Although originating in different mathematical areas, at their very core both parts investigate some form of structural-equivalence problem and share the common theme of understanding when two mathematical objects, be it functions, matrices or stochastic processes, should be regarded as equivalent, under natural transformations.

Chapter 1

Introduction

The thesis lies at the intersection of discrete mathematics, linear algebra, and probability theory. It is structured in two parts, each centred essentially on a structural equivalence problem: the first concerns functions (and matrices), while the second concerns stochastic processes. Although originating in different mathematical areas, at their very core both parts investigate some form of structural-equivalence problem and share the common theme of understanding when two mathematical objects, be it functions, matrices or stochastic processes, should be regarded as equivalent, under natural transformations.

The first part of the thesis studies a functional extension of the classical principal minors and diagonal similarity problem from linear algebra (cf., [44]). The second part develops a norm-dependent correspondence between self-similar Markov processes and Markov additive processes in a general Banach spaces, together with several applications to multidimensional stable processes and reflected diffusions. The results obtained in each part extend existing theories beyond their previously understood settings as well as reveal new mathematical structures.

The classical principal minors and diagonal similarity problem plainly asks to what extent a square matrix is determined, up to diagonal similarity, by its principal minors. This question has a long history in linear algebra and has been studied from various perspectives. A complete understanding of the transformations that preserve principal minors has been achieved in several important cases.

More recently, functional analogues of this problem have emerged, motivated by applications in probability theory and random matrix theory. In this setting, one replaces square matrices by bivariate functions defined on an abstract set and introduces the notion of *determinantal equivalence*. This relation generalises the matrix problem while introducing new challenges, particularly when symmetry assumptions are removed. Indeed, previous work (cf., [59]) established a classification of determinantal-equivalence-preserving transformations in the symmetric setting, where the underlying functions satisfy a natural symmetry condition. In that case, the classification relies on algebraic identities that follow directly from the determinantal structure. It was left as an open problem to extend this classification in the absence of symmetry. The first main contribution of the thesis is a resolution to this question. A counterexample disproving the conjecture proposed in [59] regarding the general case is first constructed, and subsequently we solve the conjecture under additional minor assumptions that preclude such counterexamples. We note that when restricted to finite domains, the result actually recovers the classical matrix problem as a special case.

A distinctive feature of the above work is the methodology. The proof uses novel techniques which avoid linear algebra entirely; instead it relies on elementary combinatorial arguments together with a novel analysis of algebraic identities associated with cycles in graphs. In particular, by exploiting structural relations between 3-cycles and 4-cycles, the argument reveals a hidden combinatorial structure underlying the problem. This leads to a proof that is both conceptually transparent and broadly accessible, while addressing a problem that has been traditionally and previously been approached using technically-heavy matrix-theoretic/linear-algebraic methods. This piece of mathematics therefore contributes to the understanding of a classical problem in matrix theory in a broader functional setting.

The second part of the thesis concerns self-similar Markov processes (ssMps) and their relationship with Markov additive processes (MAPs). In dimension one, Lamperti's transform (cf., [41]) establishes a bijection between positive self-similar Markov processes and Lévy processes, providing a powerful structural description that has been widely exploited in probability theory.

This correspondence has since been extended in several directions, including real-valued ssMps (cf., [13]) and Euclidean multidimensional settings (cf., [1]). However, existing results are typically tied to specific norms or geometric structures, such as the Euclidean norm on $\mathbb{R}^d$. This raises a natural question: to what extent does the Lamperti-type correspondence depend on the choice of norm, and can it be formulated in greater generality? Another contribution of this thesis is then the establishment of a norm-dependent correspondence between ssMps taking values in an arbitrary Banach space and MAPs. We show that for any choice of norm on the state space, one obtains a one-to-one correspondence between ssMps and MAPs. Besides formally proving this correspondence, the thesis provides a detailed study of several concrete examples that illustrate its consequences and which extend known theory naturally to the higher dimension. In particular, we analyse multidimensional reflected diffusions and multidimensional stable processes that are killed upon exiting cones or reflected at the boundary of the positive orthant. In each case, we characterise the associated underlying MAP and investigate how its properties depend on the choice of norm. These examples reveal new phenomena that do not arise in one-dimensional or Euclidean settings, and include MAPs with boundary behaviour encoded through reflecting conditions in their generator, as well as killing rates that are path-dependent. These applications, developed in the later chapters of the thesis, serve both as illustrations of the general norm-dependent ssMp–MAP correspondence, as well as independent contributions to the theory of stochastic processes.

We first consider multidimensional stable processes killed upon exiting the positive orthant or more general cones. Unlike the one-dimensional setting where killing occurs at a constant rate, the multidimensional case exhibits position-dependent killing rates. We derive explicit formulae for these rates and characterise the jump structure of the associated MAPs, with particular emphasis on norms that yield 'nice' closed-form expressions. Specifically, this study provides a natural high-dimensional extension to the theory developed in [10].

We then study reflected stable processes in the orthant, both with jump-reflection and continuous (Skorokhod) reflection. The associated MAPs exhibit structural features that have not previously been analysed, including boundary conditions. We provide a detailed characterisation of these MAPs and we also conjecture a decomposition principle that extends a well-known result in dimension one (cf., [39]) to higher dimensions.

Finally, we examine the case of reflected Brownian motion in the orthant. Although Brownian motion represents a classical example of a self-similar process, the analysis of its underlying MAP in a norm-dependent framework leads to substantial computational challenges. We overcome these difficulties and, in a special low-dimensional case, derive an explicit (modulated) stochastic differential equation satisfied by the MAP. This example highlights the flexibility of the general theory to not just the archetypal pure-jump stable-process setting but to the diffusion setting as well.

The remainder of the thesis is organised as follows. Chapter 2 collects preliminary material and notation used throughout the thesis. This includes background results on determinantal equivalence, graph-theoretic notions, self-similar Markov processes, Markov additive processes, and relevant tools from the theory of Lévy process and stochastic calculus. Chapter 3 is devoted to the study of determinantal equivalence for functions. We establish our main classification theorem, discuss its relationship with previous work, and present counterexamples illustrating the necessity of the hypotheses. Chapter 4 develops the norm-dependent correspondence between ssMps and MAPs in Banach spaces. Chapters 5 and 6 then apply this correspondence to multidimensional stable processes killed or reflected in cones and orthants, characterising the associated MAPs and exploring their norm-dependence. Chapter 7 studies the case of reflected Brownian motion in the orthant and derives explicit descriptions of the associated MAP in special cases.

Chapter 2

Preliminaries

The function of this chapter is to provide a single, centralized reference where all the primary definitions, notations, and known background results used throughout the thesis are collected and presented together for easy review. No results in this chapter are original. Its main purpose is to provide a self-contained reference for the concepts and tools appearing in subsequent chapters.

The chapter is divided into two main parts, corresponding to the two themes of the thesis. Section 2.1 concerns determinantal equivalence and the associated graph-theoretic language. Section 2.2 introduces self-similar Markov processes, Lévy processes, Markov additive processes, and the Lamperti-type correspondences that motivate the more probabilistic part of the thesis.

2.1 Determinantal Equivalence and Graph-Theoretic Preliminaries

2.1.1 Determinants and Principal Minors

Let $\mathbb{F}$ be a field. For $n \in \mathbb{N}$, let $M_n(\mathbb{F})$ denote the set of all $n \times n$ matrices with entries in $\mathbb{F}$.

Definition 2.1.1 *For a matrix $A \in M_n(\mathbb{F})$ and a subset $I \subseteq \{1, \ldots, n\}$, the principal minor of A indexed by I is defined as $\det(A_I)$, where A_I denotes the submatrix of A obtained by restricting both rows and columns to indices in I.*

Definition 2.1.2 *Let α, β be a partition of $\{1, \ldots, n\}$, that is $\alpha, \beta \subseteq \{1, \ldots, n\}$ are disjoint subsets* *such that $\alpha \cup \beta = \{1, \ldots, n\}$. Denote by $A[\alpha|\beta]$ the submatrix of A having row indices in α and column indices in β.*

Definition 2.1.3 *For a pair of matrices $A, B \in M_n(\mathbb{F})$, we say that B is diagonally similar to A if there exists a nonsingular diagonal matrix $D \in M_n(\mathbb{F})$ such that $B = D^{-1}AD$.*

The classical principal minors problem concerns the classification of matrices up to diagonal similarity using their principal minors; see for example, [44] and references therein. To provide a bit more context, it is clear that if $A, B \in M_n(\mathbb{F})$ are matrices such that either A is diagonally similar to B or B^T, where B^T denotes the matrix B transposed, then A and B will have equal corresponding principal minors, that is, $\det(A_I) = \det(B_I)$ for every subset $I \subseteq \{1, \ldots, n\}$. The converse question and the extent of its validity is more interesting; and the following classification result is proved in [44].

Theorem 2.1.4 *Let n be a positive integer such that $n \geq 4$. Suppose that $A, B \in M_n(\mathbb{F})$ and that A is irreducible. Suppose further that A and B have equal corresponding principal minors, and that for every partition of $\{1, \ldots, n\}$ into subsets α, β such that $|\alpha| \geq 2$, $|\beta| \geq 2$ either $\operatorname{rank}(A[\alpha|\beta]) \geq 2$ or $\operatorname{rank}(A[\beta|\alpha]) \geq 2$. Then A is either diagonally similar to B or B^T.*

2.1.2 Functional Setting and Determinantal Equivalence

Let Λ be a non-empty set of arbitrary cardinality and let $f, g : \Lambda^2 \to \mathbb{F}$ be functions.

Given distinct elements $x_1, \ldots, x_n \in \Lambda$, define the associated matrices

$$F = (f(x_i, x_j))_{1 \leq i,j \leq n}, \;\; G = (g(x_i, x_j))_{1 \leq i,j \leq n} \in M_n(\mathbb{F}).$$

Definition 2.1.5 *The functions f and g are said to be determinantally equivalent if*

$$\det(F) = \det(G)$$

for all $n \geq 1$ and all choices of distinct $x_1, \ldots, x_n \in \Lambda$.

This notion extends the classical matrix problem to a functional setting and has appeared in probabilistic contexts, particularly in the study of determinantal point processes; see, for example, [59].

Two elementary types of transformations play a central role in the study of determinantal equivalence:

1. **Conjugation transformations**, where

$$f(x, y) \mapsto g(x)f(x, y)g(y)^{-1}$$

for some nowhere zero function $g : \Lambda \to \mathbb{F}$;

2. **Transposition transformations**, where

$$f(x, y) \mapsto f(y, x).$$

Both of the above transformations can easily be seen to preserve determinantal equivalence, and in symmetric settings, that is, when $f(x, y) = f(y, x)$ and $g(x, y) = g(y, x)$ for all $x, y \in \Lambda$, these two transformation are known to generate all equivalence-preserving transformations, cf., [59].

2.1.3 Graph-Theoretic Notation and Terminology

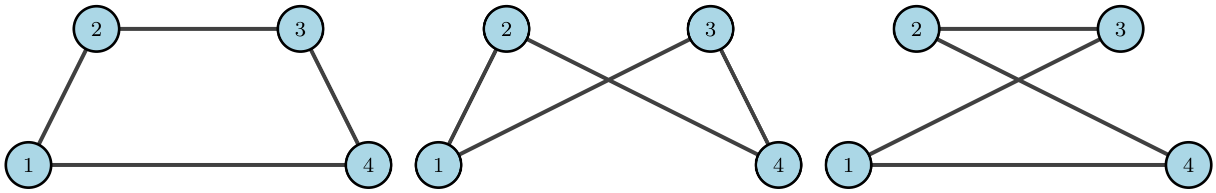
Throughout the first part of the thesis, we make systematic use of basic graph-theoretic concepts which for the most part abide to standard graph-theoretic notations and terminology. Of course, some terminology in graph theory will always inevitably vary among authors. For this reason, to avoid confusion, we take the time here to carefully go through and explain the technical language we employ, most of which will evade many of the fine and highly-specific technicalities that are commonly found in graph-theoretic literature, as we will simply not be needing them.

Definition 2.1.6 *Let $\mathcal{M}$ be a (possibly infinite) set. For an integer $1 \leq n \leq |\mathcal{M}|$, we define a cycle of length n in $\mathcal{M}$ (or an n-cycle in $\mathcal{M}$, for short) to be an $(n+1)$-tuple of the form $p = (p_0, p_1, \ldots, p_n) \in \mathcal{M}^{n+1}$, where each of the p_i are distinct except for $p_0 = p_n$. We will refer to the p_i as the vertices of the cycle p.*

Definition 2.1.7 *Let $\mathcal{M}$ be a (possibly infinite) set and fix an integer $1 \leq n \leq |\mathcal{M}|$. Given an n-cycle $p = (p_0, p_1, \ldots, p_n)$ in $\mathcal{M}$, we denote by $p' \coloneqq (p_n, p_{n-1}, \ldots, p_0)$ the cycle p in reverse.*

Definition 2.1.8 *Let $\mathcal{M}$ be a (possibly infinite) set and fix an integer $1 \leq n \leq |\mathcal{M}|$. In this paper, we will occasionally speak of undirected n-cycles in $\mathcal{M}$; by this we will mean n-cycles in $\mathcal{M}$ up to reversion.*

As a simple example of the previous definition, consider the case when $\mathcal{M} = \{1, 2, 3, 4\}$: there are exactly *six* 4-cycles in $\mathcal{M}$, namely $(1, 2, 3, 4, 1)$, $(1, 4, 3, 2, 1)$, $(1, 2, 4, 3, 1)$, $(1, 3, 4, 2, 1)$, $(1, 3, 2, 4, 1)$, $(1, 4, 2, 3)$; but there are only *three* undirected 4-cycles in $\mathcal{M}$. The below figure makes this clear.



Definition 2.1.9 *Let Λ be a (possibly infinite) set and let $\mathbb{F}$ be a field. Fix an integer $1 \le n \le |\Lambda|$. For a two-variable function $h : \Lambda^2 \to \mathbb{F}$ and an n-cycle, $p = (p_0, \dots, p_n)$, in Λ, we denote by $h[p]$ the product*

$$h[p] \coloneqq \prod_{i=1}^{n} h(p_{i-1}, p_i);$$

and by $h'[p]$ the analogous product with respect to the reverse cycle p':

$$h'[p] \coloneqq \prod_{i=1}^{n} h(p_i, p_{i-1}) \ \ (= h[p']).$$

Remark 2.1.10 *Let us define the function $\tau : \{0, 1, \dots, n\} \to \{0, 1, \dots, n-1\}$ to be the following shift operator:*

$$\tau(k) \equiv k + 1 \pmod{n}, \qquad k \in \{0, 1 \dots, n\}.$$

Though this is a trivial clarification, it is worth pointing out that for an n-cycle $p = (p_i)_{i=0}^n$ and any $j \in \mathbb{N}$, the n-cycle $(p_{\tau^j(i)})_{i=0}^n$, where τ^j denotes the j-th iteration of the shift operator τ, is indistinguishable from p in the sense that it describes the exact same graph-theoretic object; and obviously we have $h[p] = h[(p_{\tau^j(i)})_{i=0}^n]$ for any two-variable function h.

Definition 2.1.11 *Let Λ be a (possibly infinite) set and let $\mathbb{F}$ be a field. Fix an integer $1 \le n \le |\Lambda|$. We say that a two-variable function $c : \Lambda^2 \to \mathbb{F}$ satisfies the cocycle property for n-cycles if for every n-cycle p in Λ, $c[p] = 1$.*

If c satisfies the cocycle property for cycles in Λ of every length n, where $1 \le n \le |\Lambda|$, we say that c is a (full) cocycle function. Equivalently, c is a cocycle function if for all $z, w \in \Lambda$, $c(z, z) = 1$ and $c(z, w)c(w, z) = 1$, and for all integers $r > 2$ and every r-tuple $(z_1, \dots, z_r) \in \Lambda^r$,

$$c(z_1, z_2)c(z_2, z_3) \cdots c(z_{r-1}, z_r)c(z_r, z_1) = 1. \tag{2.1}$$

2.2 Self-Similar Markov Processes and Markov Additive Processes

Unless otherwise stated, we shall assume throughout the thesis that E is a locally compact and separable Banach space over the field of reals. We let $\mathcal{E}$ be a sigma-algebra on E, and ∂ a point not in E which acts as a cemetery state. We define $(E_\partial, \mathcal{E}_\partial) \coloneqq (E \cup \{\partial\}, \sigma(E \cup \{\partial\}))$ to be the measurable space $(E, \mathcal{E})$ *augmented by* ∂; cf., [8]. A standard practice (which we implicitly employ throughout the thesis actually) is to extend an $\mathcal{E}$-measurable function $f : E \to \mathbb{R}$ to an $\mathcal{E}_\partial$-measurable function by setting $f(\partial) = 0$.

We also note that throughout the thesis, all Markov processes are assumed to be càdlàg and defined on a filtered probability space satisfying the usual conditions.

2.2.1 Self-Similar Markov Processes and Stable Processes

Definition 2.2.1 *We say that two E-valued stochastic processes $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ living in probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{Q})$, respectively, have the same law – commonly abbreviated as $(\{X_t, t \geq 0\}, \mathbb{P}) \overset{d}{=} (\{Y_t, t \geq 0\}, \mathbb{Q})$ – if for every $n > 0$, $0 \leq t_1 < t_2 < \ldots < t_n$, bounded measurable functions $F : E^n \to \mathbb{R}$,*

$$\mathbb{E}_{\mathbb{P}}[F(X_{t_1}, X_{t_2}, \ldots, X_{t_n})] = \mathbb{E}_{\mathbb{Q}}[F(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n})],$$

where $\mathbb{E}_{\mathbb{P}}$ denotes the expectation with respect to the measure $\mathbb{P}$, and $\mathbb{E}_{\mathbb{Q}}$ the expectation with respect to the measure $\mathbb{Q}$. The above equation is usually abbreviated thus: for every $t \geq 0$,

$$\mathbb{E}_{\mathbb{P}}\Big[F\big(\{X_s : s \leq t\}\big)\Big] = \mathbb{E}_{\mathbb{Q}}\Big[F\big(\{Y_s : s \leq t\}\big)\Big].$$

We then that the law of X under $\mathbb{P}$ is equal to the law of Y under $\mathbb{Q}$.

Definition 2.2.2 *An E-valued càdlàg, quasi-left-continuous, strong Markov process $Z = (Z_t)_{t\geq 0}$ defined on some (filtered) probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, with probabilities $(\mathbb{P}_x, x \in E)$, is said to be a self-similar Markov process if it satisfies the self-similarity property: there exists an $\alpha > 0$ such that for every $c > 0$ and $x \in E$,*

$$(\{cZ_{c^{-\alpha}t}, t \geq 0\}, \mathbb{P}_x) \overset{d}{=} (\{Z_t, t \geq 0\}, \mathbb{P}_{cx}). \tag{2.2}$$

We then say that Z is a ssMp with index of self-similarity α.

We now go through the definition of an important example of a self-similar Markov process which plays a central role in the thesis.

Definition 2.2.3 *Fix $\alpha \in (0,2)$. By a one-dimensional α-stable process we mean a Lévy process (a careful definition of these processes will be presented in the next section) with jump measure Π taking the form*

$$\Pi(dx) = |x|^{-(1+\alpha)}(c_1 \mathbb{1}_{\{x>0\}} + c_2 \mathbb{1}_{\{x<0\}})dx, \qquad x \in \mathbb{R} \setminus \{0\}, \tag{2.3}$$

where

$$c_1 := \frac{\Gamma(1+\alpha)\sin(\pi\alpha\rho)}{\pi} \text{ and } c_2 := \frac{\Gamma(1+\alpha)\sin(\pi\alpha(1-\rho))}{\pi} \tag{2.4}$$

Moreover, its characteristic exponent satisfies

$$\Psi(z) = |z|^\alpha \left(\mathrm{e}^{\pi i \alpha(\frac{1}{2}-\rho)} \mathbf{1}_{(z>0)} + \mathrm{e}^{-\pi i \alpha(\frac{1}{2}-\rho)} \mathbf{1}_{(z<0)} \right) \tag{2.5}$$

$$= \int_{\mathbb{R}\setminus\{0\}} (1 - e^{-izx} + izx \mathbb{1}_{\{|x|<1\}})\Pi(dx), \qquad z \in \mathbb{R}. \tag{2.6}$$

The constant ρ from above is referred to as the *positivity parameter*. The setting of positive and negative jumps conforms to the parameter regime $\rho \in (0,1)$. Symmetric α-stable processes correspond to the setting $\rho = 1/2$. The spectrally positive setting (i.e., only positive jumps) conforms to the parameter regime that $\alpha \in (1,2)$ and $\alpha(1-\rho) = 1$.

2.2.2 Killed Lévy Processes and Background Results

Here we largely follow the classic book [36] on the subject.

Definition 2.2.4 *A real-valued process $X = (X_t)_{t\geq 0}$ defined on the (filtered) probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, with probabilities $(\mathbb{P}_x : x \in \mathbb{R})$, where we use the convention $\mathbb{P} \equiv \mathbb{P}_0$, is said to be a Lévy process if it possesses the following properties:*

(i) the paths of X are $\mathbb{P}_x$-almost surely càdlàg;

(ii) (stationary increments) for $0 \leq s \leq t$ and $x \in \mathbb{R}$,

$$(X_t - X_s, \mathbb{P}_x) \stackrel{d}{=} (X_{t-s}, \mathbb{P}) \tag{2.7}$$

(iii) (independent increments) for $0 \leq s \leq t$, $X_t - X_s$ is independent of $\mathcal{F}_s$.

One implication of stationary and independent increments is that a process having these two properties is fully characterized by just the distribution of its position at some arbitrarily chosen time $t \geq 0$. In fact, it is sufficient to consider $t = 1$:

Definition 2.2.5 *Let X be the Lévy process from Definition 2.2.4. Then, the function $\Psi : \mathbb{R} \to \mathbb{R}$ satisfying for every $\theta \in \mathbb{R}$,*

$$\mathbb{E}[e^{i\theta X_t}] = e^{-\Psi(\theta)t}, \qquad t \geq 0,$$

is called the characteristic exponent of X.

We now give the definition of an exponentially-killed Lévy process:

Definition 2.2.6 *Let $\tilde{\xi} = (\tilde{\xi}_t)_{t\geq 0}$ be a Lévy process with infinite lifetime, and let $\mathbb{e}_q$ be an independent exponentially-distributed random variable with parameter $q \geq 0$; where we understand $\mathbb{e}_0$ as infinity. We say that the process $\xi = (\xi_t)_{t\geq 0}$ defined by*

$$\xi_t = \begin{cases} \tilde{\xi}_t, & \text{if } t < \mathbb{e}_q \\ -\infty, & \text{if } t \geq \mathbb{e}_q, \end{cases}$$

is an exponentially-killed Lévy process with killing rate q and cemetery state $-\infty$.

The case of $q = 0$ refers to the case where there is no killing (and hence $\xi = \tilde{\xi}$).

The process ξ defined above, *while alive*, preserves the stationarity and independence of increments of the (standard) Lévy process $\tilde{\xi}$ (from the same definition). We formally define what we mean by stationary and independent increments *while alive* below:

Definition 2.2.7 *We say that a killed Lévy process $\xi = (\xi_t)_{t\geq 0}$ with lifetime ζ has stationary increments while alive if for every $t, s \geq 0$, the distribution of $(\xi_{t+s} - \xi_t)\mathbb{1}_{\{t+s<\zeta\}}$, conditioned on $\{\zeta > t\}$, only depends on s.*

Definition 2.2.8 *We say that a killed Lévy process $\xi = (\xi_t)_{t\geq 0}$ with lifetime ζ has independent increments while alive if for every $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \ldots < t_n$, conditioned on $\zeta > t_0$, the random variables*

$$(\xi_{t_1} - \xi_{t_0})\mathbb{1}_{\{t_1<\zeta\}}, (\xi_{t_2} - \xi_{t_1})\mathbb{1}_{\{t_2<\zeta\}}, \ldots, (\xi_{t_n} - \xi_{t_{n-1}})\mathbb{1}_{\{t_n<\zeta\}}$$

are mutually independent.

Lemma 2.2.9 *The process ξ from Definition 2.2.6, while alive, possesses stationary and independent increments.*

We claim that the following proposition is sufficient for establishing Lemma 2.2.9:

Proposition 2.2.10 *Let ξ and $\tilde{\xi}$ be the processes from Definition 2.2.6. Then, for every $n \geq 0$, $s, t \geq 0$, $0 \leq t_1 < \ldots < t_n \leq t$ and $\lambda, \lambda_1, \ldots, \lambda_n \in \mathbb{R}$,*

$$\mathbb{E}\left[\Big(\prod_{j=1}^{n} e^{i\lambda_j \xi_{t_j}}\Big) e^{i\lambda(\xi_{t+s}-\xi_t)}; t+s<\zeta\right] = \mathbb{E}\left[\prod_{j=1}^{n} e^{i\lambda_j \xi_{t_j}}; t<\zeta\right] \cdot e^{-s(q+\tilde{\Psi}(\lambda))},$$

where $\tilde{\Psi}$ denotes the characteristic exponent of $\tilde{\xi}$.

Before proving the above proposition, let us first see how it can be used to prove Lemma 2.2.9:

Proof (of Lemma 2.2.9) For stationary increments while alive, fix $t, s \geq 0$ and $\lambda \in \mathbb{R}$. It suffices to show that the quantity

$$\mathbb{E}\Big[e^{i\lambda(\xi_{t+s}-\xi_t)} \mathbb{1}_{\{t+s<\zeta\}} \Big| t<\zeta\Big]$$

only depends on s. Notice how Proposition 2.2.10 with $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$ yields

$$\mathbb{E}\Big[e^{i\lambda(\xi_{t+s}-\xi_t)} \mathbb{1}_{\{t+s<\zeta\}}\Big] = \mathbb{P}(\zeta > t) \cdot e^{-s(q+\tilde{\Psi}(\lambda))}.$$

Dividing both sides by $\mathbb{P}(\zeta > t)$ then yields

$$\mathbb{E}\Big[e^{i\lambda(\xi_{t+s}-\xi_t)} \mathbb{1}_{\{t+s<\zeta\}} \Big| t<\zeta\Big] = e^{-s(q+\tilde{\Psi}(\lambda))}, \tag{2.8}$$

which only depends on s, as required. Moreover, since the lifetime, ζ, of ξ is exponentially distributed with parameter q, and is also independent of $\tilde{\xi}$,

$$\begin{aligned}
\mathbb{E}\Big[e^{i\lambda(\xi_{t+s}-\xi_t)} \mathbb{1}_{\{t+s<\zeta\}} \Big| t<\zeta\Big] &= e^{-s(q+\tilde{\Psi}(\lambda))} \\
&= \mathbb{P}(\zeta > s) \cdot \mathbb{E}(e^{i\lambda\tilde{\xi}_s}) \\
&= \mathbb{E}(e^{i\lambda\tilde{\xi}_s} \mathbb{1}_{\{\zeta>s\}}) \qquad \text{(by independence of } \tilde{\xi} \text{ and } \zeta) \\
&= \mathbb{E}(e^{i\lambda\xi_s} \mathbb{1}_{\{s<\zeta\}}).
\end{aligned}$$

Now, for independent increments while alive, fix $0 \leq t_0 < \ldots < t_n$ and $\lambda_j \in \mathbb{R}$. It is enough to show that

$$\mathbb{E}\Bigg[\prod_{j=1}^{n} e^{i\lambda_j(\xi_{t_j}-\xi_{t_{j-1}})}\mathbb{1}_{\{t_n<\zeta\}}\Bigg|t_0<\zeta\Bigg] = \prod_{j=1}^{n}\mathbb{E}\Big[e^{i\lambda_j(\xi_{t_j}-\xi_{t_{j-1}})}\mathbb{1}_{\{t_j<\zeta\}}\Big|t_{j-1}<\zeta\Big].$$

By setting $t = t_{n-1}$ and $s = t_n - t_{n-1}$ in the statement of Proposition 2.2.10, as well as redefining the λ_j and λ therein appropriately, we obtain

$$\begin{aligned}
&\mathbb{E}\Bigg[\prod_{j=1}^{n} e^{i\lambda_j(\xi_{t_j}-\xi_{t_{j-1}})}; t_n<\zeta\Bigg] \\
&= \mathbb{E}\Bigg[\prod_{j=1}^{n-1} e^{i\lambda_j(\xi_{t_j}-\xi_{t_{j-1}})}; t_{n-1}<\zeta\Bigg]\cdot e^{-(t_n-t_{n-1})(q+\tilde{\Psi}(\lambda_n))} \\
&= \mathbb{E}\Bigg[\prod_{j=1}^{n-1} e^{i\lambda_j(\xi_{t_j}-\xi_{t_{j-1}})}; t_{n-1}<\zeta\Bigg]\cdot \mathbb{E}\Big[e^{i\lambda_n(\xi_{t_n}-\xi_{t_{n-1}})}\mathbb{1}_{\{t_n<\zeta\}}\Big|t_{n-1}<\zeta\Big] && \text{(by (2.8))} \\
&= \mathbb{E}\Big[e^{i\lambda_1(\xi_{t_1}-\xi_{t_0})}\mathbb{1}_{\{t_1<\zeta\}}\Big]\cdot\prod_{j=2}^{n}\mathbb{E}\Big[e^{i\lambda_j(\xi_{t_j}-\xi_{t_{j-1}})}\mathbb{1}_{\{t_j<\zeta\}}\Big|t_{j-1}<\zeta\Big] && \text{(by iterating)} \\
&= \mathbb{P}(\zeta>t_0)\cdot\prod_{j=1}^{n}\mathbb{E}\Big[e^{i\lambda_j(\xi_{t_j}-\xi_{t_{j-1}})}\mathbb{1}_{\{t_j<\zeta\}}\Big|t_{j-1}<\zeta\Big].
\end{aligned}$$

Dividing both sides by $\mathbb{P}(\zeta > t_0)$ then yields the result we are after. ∎

Let us now prove Proposition 2.2.10.

Proof (of Proposition 2.2.10) We let ζ denote the lifetime of ξ. Then,

$$\begin{aligned}
&\mathbb{E}\Bigg[\Big(\prod_{j=1}^{n} e^{i\lambda_j\xi_{t_j}}\Big)e^{i\lambda(\xi_{t+s}-\xi_t)}; t+s<\zeta\Bigg] \\
&= \mathbb{E}\Bigg[\Big(\prod_{j=1}^{n} e^{i\lambda_j\tilde{\xi}_{t_j}}\Big)e^{i\lambda(\tilde{\xi}_{t+s}-\tilde{\xi}_t)}\mathbb{1}_{\{t+s<\zeta\}}\Bigg] \\
&= \mathbb{E}\Bigg[\Big(\prod_{j=1}^{n} e^{i\lambda_j\tilde{\xi}_{t_j}}\Big)e^{i\lambda(\tilde{\xi}_{t+s}-\tilde{\xi}_t)}\Bigg]\mathbb{P}(t+s<\zeta) && \text{(independence of } \zeta \text{ and } \tilde{\xi}) \\
&= \mathbb{E}\Bigg[\prod_{j=1}^{n} e^{i\lambda_j\tilde{\xi}_{t_j}}\Bigg]\cdot\mathbb{E}\Big[e^{i\lambda(\tilde{\xi}_{t+s}-\tilde{\xi}_t)}\Big]\cdot\mathbb{P}(t+s<\zeta) && \text{(independent increments of } \tilde{\xi}) \\
&= \mathbb{P}(\zeta>t)\cdot\mathbb{E}\Bigg[\prod_{j=1}^{n} e^{i\lambda_j\tilde{\xi}_{t_j}}\Bigg]\cdot\mathbb{P}(\zeta>s)\cdot\mathbb{E}\Big(e^{i\lambda(\tilde{\xi}_{t+s}-\tilde{\xi}_t)}\Big) && \text{(memorylesness of } \zeta)
\end{aligned}$$

$$= e^{-qt} \cdot \mathbb{E}\left[\prod_{j=1}^{n} e^{i\lambda_j \tilde{\xi}_{t_j}}\right] \cdot e^{-qs} \cdot e^{-s\tilde{\Psi}(\lambda)} \qquad \text{(stationary increments of } \tilde{\xi}\text{)}.$$

Setting $\lambda_j = 0$ for all j and $t = 0$ in the above computation yields

$$\mathbb{E}\Big[e^{-i\lambda\xi_s}\mathbb{1}_{\{s<\zeta\}}\Big] = e^{-s\left(q+\tilde{\Psi}(\lambda)\right)}. \tag{2.9}$$

Also, setting $s = 0$ in that same computation yields

$$\mathbb{E}\left[\Big(\prod_{j=1}^{n} e^{i\lambda_j\xi_{t_j}}\Big); t < \zeta\right] = e^{-qt} \cdot \mathbb{E}\left[\prod_{j=1}^{n} e^{i\lambda_j\tilde{\xi}_{t_j}}\right]. \tag{2.10}$$

The result of the proposition then follows by plugging in equations (2.9) and (2.10) in the last equality of the previous computation. ∎

2.2.3 Markov Additive Processes and Background Results

Definition 2.2.11 *An $\mathbb{R} \times E$-valued càdlàg, quasi-left-continuous, strong Markov process $(\xi, \Xi) = (\xi_t, \Xi_t)_{t\geq 0}$, living in some (filtered) probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \mathbb{P})$, with probabilities $\big(\mathbb{P}_{(x,\theta)}, (x,\theta) \in \mathbb{R} \times E\big)$, and cemetery state $(-\infty, \partial)$, where ∂ is a point not in E, is called a Markov additive process (MAP) if $\Xi = (\Xi_t)_{t\geq 0}$ is a regular strong Markov process on E with cemetery state ∂ such that, for every bounded measurable function $f : \mathbb{R} \times E \to [0, \infty)$, $t, s \geq 0$ and $(y, \theta) \in \mathbb{R} \times E$,*

$$\mathbb{E}_{(y,\theta)}[f(\xi_{t+s} - \xi_t, \Xi_{t+s})\mathbb{1}_{\{t+s<\zeta\}}|\mathcal{G}_t] = \mathbb{E}_{(0,\theta')}[f(\xi_s, \Xi_s)\mathbb{1}_{\{s<\zeta\}}]\Big|_{\theta'=\Xi_t}, \quad \mathbb{P}_{(y,\theta)} - a.s., \tag{2.11}$$

where as usual $\zeta := \inf\{t > 0 : \Xi_t = \partial\}$ denotes the lifetime of (ξ, Ξ). We call the process $\Xi = (\Xi_t)_{t\geq 0}$ the modulator of the MAP, and $\xi = (\xi_t)_{t\geq 0}$ the ordinate.

The pioneering papers of Cinlar: [14], [15], [17], [16] and [18] are one of the main sources of information about general MAPs.

Notice how the defining property (2.11) of MAPs is reminiscent of the stationary and independent increments property of Lévy processes. To illustrate this resemblance we now derive the so-called *translation invariance property* of the ordinate of the MAP (which plays a crucial role in the proof of the norm-dependent connection of MAPs with ssMps):

Lemma 2.2.12 *Fix $a, x \in \mathbb{R}$ and $\theta \in E$. If (ξ, Ξ) is a MAP, then*

$$(\{(a + \xi_t, \Xi_t), t \geq 0\}, \mathbb{P}_{(x,\theta)}) \overset{d}{=} (\{(\xi_t, \Xi_t), t \geq 0\}, \mathbb{P}_{(a+x,\theta)}).$$

Proof We need to show for every bounded measurable function g and $t \geq 0$,

$$\mathbb{E}_{(a+x,\theta)}\Big[g\big(\{(\xi_s, \Xi_s) : s \leq t\}\big)\Big] = \mathbb{E}_{(x,\theta)}\Big[g\big(\{(a+\xi_s, \Xi_s) : s \leq t\}\big)\Big]. \tag{2.12}$$

To this end, we fix a bounded measurable function $g : (\mathbb{R} \times E)^n \to \mathbb{R}$, $n \geq 1$, and we proceed to prove equation (2.12) by induction on n. We start with the base case:

$$\begin{aligned}
&\mathbb{E}_{(x,\theta)}\Big[g\big(a+\xi_t, \Xi_t\big)\Big] \\
&= \mathbb{E}_{(x,\theta)}\Big[g\big((a+x)+(\xi_t-\xi_0), \Xi_t\big)\Big] && (\mathbb{P}_{(x,\theta)}((\xi_0,\Xi_0)=(x,\theta))=1) \\
&= \mathbb{E}_{(x,\theta)}\Big[\mathbb{E}_{(x,\theta)}\Big[g\big((a+x)+(\xi_t-\xi_0), \Xi_t\big)\Big|\mathcal{G}_0\Big]\Big] && \text{(tower property)} \\
&= \mathbb{E}_{(x,\theta)}\Big[\mathbb{E}_{(0,\theta')}\Big[g\big((a+x)+\xi_t, \Xi_t\big)\Big]\Big|_{\theta'=\Xi_0}\Big] && \text{(by equation (2.11))} \\
&= \mathbb{E}_{(x,\theta)}\Big[\mathbb{E}_{(0,\theta')}\Big[g\big((a+x)+\xi_t, \Xi_t\big)\Big]\Big|_{\theta'=\Xi_0}\mathbb{1}_{\{(\xi_0,\Xi_0)=(x,\theta)\}}\Big] && (\mathbb{P}_{(x,\theta)}((\xi_0,\Xi_0)=(x,\theta))=1) \\
&= \mathbb{E}_{(x,\theta)}\Big[\mathbb{E}_{(0,\theta')}\Big[g\big((a+x)+\xi_t, \Xi_t\big)\Big]\Big|_{\theta'=\theta}\mathbb{1}_{\{(\xi_0,\Xi_0)=(x,\theta)\}}\Big] \\
&= \mathbb{E}_{(x,\theta)}\Big[\mathbb{E}_{(0,\theta)}\Big[g\big((a+x)+\xi_t, \Xi_t\big)\Big]\mathbb{1}_{\{(\xi_0,\Xi_0)=(x,\theta)\}}\Big] \\
&= \mathbb{E}_{(0,\theta)}\Big[g\big((a+x)+\xi_t, \Xi_t\big)\Big]\cdot\mathbb{E}_{(x,\theta)}\Big[\mathbb{1}_{\{(\xi_0,\Xi_0)=(x,\theta)\}}\Big] \\
&= \mathbb{E}_{(0,\theta)}\Big[g\big((a+x)+\xi_t, \Xi_t\big)\Big]\cdot\mathbb{P}_{(x,\theta)}\Big((\xi_0,\Xi_0)=(x,\theta)\Big) \\
&= \mathbb{E}_{(0,\theta)}\Big[g\big((a+x)+\xi_t, \Xi_t\big)\Big] && (\mathbb{P}_{(x,\theta)}((\xi_0,\Xi_0)=(x,\theta))=1) \\
&= \mathbb{E}_{(a+x,\theta)}\Big[\mathbb{E}_{(0,\theta)}\Big[g((a+x)+\xi_t, \Xi_t)\Big]\mathbb{1}_{\{(\xi_0,\Xi_0)=(a+x,\theta)\}}\Big] \\
&= \mathbb{E}_{(a+x,\theta)}\Big[\mathbb{E}_{(0,\theta')}\Big[g\big((a+x)+\xi_t, \Xi_t\big)\Big]\Big|_{\theta'=\Xi_0}\mathbb{1}_{\{(\xi_0,\Xi_0)=(a+x,\theta)\}}\Big] \\
&= \mathbb{E}_{(a+x,\theta)}\Big[\mathbb{E}_{(a+x,\theta)}\Big[g\big((a+x)+(\xi_t-\xi_0), \Xi_t\big)\Big|\mathcal{G}_0\Big]\Big] && \text{(by equation (2.11))} \\
&= \mathbb{E}_{(a+x,\theta)}\Big[g\big((a+x)+(\xi_t-\xi_0), \Xi_t\big)\Big] && \text{(tower property)} \\
&= \mathbb{E}_{(a+x,\theta)}\Big[g\big(\xi_t, \Xi_t\big)\Big],
\end{aligned}$$

as required. For the inductive step, we let $g : (\mathbb{R} \times E)^n \to \mathbb{R}$, for $n > 1$, be bounded and measurable, and proceed to prove that (2.12) holds. Fix $0 \leq t_1 < t_2 < \ldots, t_n \leq t$. Then,

$$\begin{aligned}
&\mathbb{E}_{(a+x,\theta)}\Big[g\big(\xi_{t_1}, \Xi_{t_1}, \ldots, \xi_{t_n}, \Xi_{t_n}\big)\Big] \\
&= \mathbb{E}_{(a+x,\theta)}\Big[\mathbb{E}_{(a+x,\theta)}\Big[g\big(\xi_{t_1}, \Xi_{t_1}, \ldots, \xi_{t_n}, \Xi_{t_n}\big)\Big|\mathcal{G}_{t_{n-1}}\Big]\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_{(a+x,\theta)}\Big[\mathbb{E}_{(a+x,\theta)}\Big[g\big(\xi_{t_1},\Xi_{t_1},\ldots,(\xi_{t_n}-\xi_{t_{n-1}})+\xi_{t_{n-1}},\Xi_{t_n}\big)\Big|\mathcal{G}_{t_{n-1}}\Big]\Big] \\
&= \mathbb{E}_{(a+x,\theta)}\Big[\mathbb{E}_{(0,\theta')}\Big[g\big(y_1,x_1,\ldots,\xi_{t_n-t_{n-1}}+y_{n-1},\Xi_{t_n-t_{n-1}}\big)\Big]\Big|_{\substack{\theta'=\Xi_{t_{n-1}} \\ (y_1,x_1)=(\xi_{t_1},\Xi_{t_1}) \\ \vdots \\ (y_{n-1},x_{n-1})=(\xi_{t_{n-1}},\Xi_{t_{n-1}})}}\Big].
\end{aligned}
$$

where on the last equality we have used the fact that the (ξ_{t_i},Ξ_{t_i}), $i<n$, are all $\mathcal{G}_{t_{n-1}}$-measurable, and have applied equation (2.11) (with $t=t_{n-1}$ and $s=t_n-t_{n-1}$ therein).

Observe how $G(y_1,x_1,\ldots,y_{n-1},x_{n-1})=\mathbb{E}_{(0,\theta')}\Big[g\big(y_1,x_1,\ldots,\xi_{t_n-t_{n-1}}+y_{n-1},\Xi_{t_n-t_{n-1}}\big)\Big]\Big|_{\theta'=x_{n-1}}$ defines a bounded measurable function on $(\mathbb{R}\times E)^{n-1}$. It therefore follows, by the inductive hypothesis, that

$$
\begin{aligned}
&\mathbb{E}_{(a+x,\theta)}\Big[g\big(\xi_{t_1},\Xi_{t_1},\ldots,\xi_{t_n},\Xi_{t_n}\big)\Big] \\
&= \mathbb{E}_{(a+x,\theta)}\Big[G\big(\xi_{t_1},\Xi_{t_1},\ldots,\xi_{t_{n-1}},\Xi_{t_{n-1}}\big)\Big] \\
&= \mathbb{E}_{(x,\theta)}\Big[G\big(a+\xi_{t_1},\Xi_{t_1},\ldots,a+\xi_{t_{n-1}},\Xi_{t_{n-1}}\big)\Big] \\
&= \mathbb{E}_{(x,\theta)}\Big[\mathbb{E}_{(0,\theta')}\Big[g\big(a+\xi_{t_1},\Xi_{t_1},\ldots,\xi_{t_n-t_{n-1}}+(a+\xi_{t_{n-1}}),\Xi_{t_n-t_{n-1}}\big)\Big]\Big|_{\theta'=\Xi_{t_{n-1}}}\Big] \\
&= \mathbb{E}_{(x,\theta)}\Big[\mathbb{E}_{(x,\theta)}\Big[g\big(a+\xi_{t_1},\Xi_{t_1},\ldots,(\xi_{t_n}-\xi_{t_{n-1}})+(a+\xi_{t_{n-1}}),\Xi_{t_n}\big)|\mathcal{G}_{t_{n-1}}\Big]\Big] \\
&= \mathbb{E}_{(x,\theta)}\Big[g\big(a+\xi_{t_1},\Xi_{t_1},\ldots,a+\xi_{t_n},\Xi_{t_n}\big)\Big],
\end{aligned}
$$

as required, where on the second-last equality we have again used equation (2.11) in the same way we had done earlier; and on the last equality we have simply applied the tower property of conditional expectations. ∎

Remark 2.2.13 *From the proof of Lemma 2.2.12, we see that the additive property (2.11), first and foremost, yields*

$$
(\{(x+\xi_t,\Xi_t),t\geq 0\},\mathbb{P}_{(0,\theta)}) \stackrel{d}{=} (\{(\xi_t,\Xi_t),t\geq 0\},\mathbb{P}_{(x,\theta)}), \qquad (x,\theta)\in\mathbb{R}\times E, \tag{2.13}
$$

which is then exploited to derive the more general equation

$$
(\{(a+\xi_t,\Xi_t),t\geq 0\},\mathbb{P}_{(x,\theta)}) \stackrel{d}{=} (\{(\xi_t,\Xi_t),t\geq 0\},\mathbb{P}_{(a+x,\theta)}), \qquad (x,\theta)\in\mathbb{R}\times E,\ \ a\in\mathbb{R}. \tag{2.14}
$$

It is now clear that equation (2.14) provides the main intuition behind the common labelling of equation (2.11) as the "additive property" of a MAP. To see how this is the analogous "stationary independent increments property" of Lévy processes, let $Y=(Y_t)_{t\geq 0}$ be a (1-dimensional) Lévy *process defined on the filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *with the family of probabilities* $(\mathbb{P}_x : x \in \mathbb{R})$, *where we use the convention* $\mathbb{P} \equiv \mathbb{P}_0$. *Then, an equation comparable to (2.14) can be brought about for* Y *deriving out of its stationary independent increments, namely, the equation*

$$(\{x + Y_t, t \geq 0\}, \mathbb{P}) \stackrel{d}{=} (\{Y_t, t \geq 0\}, \mathbb{P}_x), \quad x \in \mathbb{R}. \tag{2.15}$$

One can use induction in the same way that was done in the proof of Lemma 2.2.12. We only provide the proof of the (analogous) base case: fix a bounded measurable function $f : \mathbb{R} \to \mathbb{R}$. *Then,*

$$\begin{aligned}
\mathbb{E}_x[f(Y_t)] &= \mathbb{E}_x\Big[f\big(x + (Y_t - Y_0)\big)\mathbb{1}_{\{Y_0 = x\}}\Big] && (Y_0 = x\ \mathbb{P}_x\text{-a.s.}) \\
&= \mathbb{E}_x\Big[\mathbb{E}_x\Big[f\big(x + (Y_t - Y_0)\big)\mathbb{1}_{\{Y_0 = x\}}|\mathcal{F}_0\Big]\Big] && \text{(tower property)} \\
&= \mathbb{E}_x\Big[\mathbb{1}_{\{Y_0 = x\}}\mathbb{E}_x\Big[f\big(x + (Y_t - Y_0)\big)|\mathcal{F}_0\Big]\Big] && \text{(pull-out property)} \\
&= \mathbb{E}_x\Big[\mathbb{1}_{\{Y_0 = x\}}\mathbb{E}_x\Big[f\big(x + (Y_t - Y_0)\big)\Big]\Big] && \text{(independent increments of } Y\text{)} \\
&= \mathbb{E}_x\Big[f\big(x + (Y_t - Y_0)\big)\Big] \cdot \mathbb{P}_x(Y_0 = x) \\
&= \mathbb{E}_x\Big[f\big(x + (Y_t - Y_0)\big)\Big] && (Y_0 = x\ \mathbb{P}_x\text{-a.s.}) \\
&= \mathbb{E}_0\Big[f\big(x + Y_t\big)\Big] && \text{(stationary increments of } Y\text{)},
\end{aligned}$$

as required.

In the previous section we carefully introduced and defined the killing rate of a general exponentially-killed Lévy process. We now go through the analogous discussion for MAPs and highlight the main differences. Firstly, from Definition 2.2.11, we see that the lifetime of the MAP (ξ, Ξ) is precisely the lifetime of its modulator Ξ. It follows that the killing rate will only depend on Ξ. So, unlike the class of exponentially-killed Lévy processes, the killing rate of a MAP is going to depend on the position of the associated modulator (and is therefore going to be a function of the modulator's state space). Let us first define exactly what we mean by the killing rate of a process

Definition 2.2.14 *Let* $X = (X_t)_{t\geq 0}$ *be a (killed) stochastic process defined on some filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *with lifetime* ζ. *We define the killing rate of* X *at time* $t \geq 0$ *by*

$$q_t := \lim_{h \downarrow 0} \frac{\mathbb{P}(\zeta \in (t, t+h)|\mathcal{F}_t)}{h}, \qquad \text{on } \{\zeta > t\},$$

provided that the limit exists.

Note that for each time $t \geq 0$, the killing rate is an $\mathcal{F}_t$-measurable random variable, and therefore depends on the path of X up to time t. If we additionally assume that X is Markovian, then it can be shown (and we do this below) that this dependence will only be on the position of X at exactly time t. To this end, suppose that X from the previous definition additionally is a Markov process with state space E and let $\mathbb{P}_x$, $x \in E$, denote the law of X initiated from state x, and denote the respective expectation by $\mathbb{E}_x$. We claim that for every $t \geq 0$,

$$q_t = \mathbb{1}_{\{\zeta > t\}} q(X_t), \tag{2.16}$$

where $q : E \to [0, \infty)$, $q(x) = \lim_{h \downarrow 0} \frac{\mathbb{P}_x(\zeta < h)}{h}$, $x \in E$, whenever the limit exists.

Indeed, by the Markov property, for every $t \geq 0$ and $x \in E$,

$$\mathbb{P}_x(\zeta \in (t, t+h) | \mathcal{F}_t) = \mathbb{1}_{\{\zeta > t\}} \mathbb{P}_{X_t}(\zeta < h),$$

from which (2.16) immediately follows by dividing by h and taking limits.

Going back to MAPs, we recall from Definition 2.2.11 that the lifetime of the MAP (ξ, Ξ) is precisely the lifetime of its modulator Ξ. Let's denote the latter by ζ. This means that the killing rate of the MAP must also solely depend on the path of the modulator. Moreover, by the Markov property of MAPs and equation (2.16), we have that the killing rate (function) of the MAP, $q : E \to [0, \infty)$, satisfies

$$q(x) = \lim_{h \downarrow 0} \frac{\mathbb{P}_{(0,x)}(\zeta < h)}{h}, \qquad x \in E;$$

and that the killing rate at time $t \geq 0$ is precisely $q_t = q(\Xi_t)$.

Let us now consider the closest analogue of an exponentially-killed Lévy process in the MAP universe: let $(\xi, \Xi) = (\xi_t, \Xi_t)_{t \geq 0}$ be a MAP with lifetime ζ satisfying

$$\mathbb{P}_{(0,\theta)}(\zeta > t | \Xi) = \exp\left\{ - \int_0^t q(\Xi_s) ds \right\}, \qquad \theta \in E, \;\; t \geq 0,$$

for some continuous function $q : E \to [0, \infty)$. One may notice that the above (conditional on Ξ) probability is an $\mathcal{F}_t$-measurable random quantity, where $(\mathcal{F}_t)_{t \geq 0}$ here denotes the natural filtration of the MAP. More precisely, said probability is dependent on the path of Ξ up to time t. In contrast, for exponentially-killed Lévy processes this was not the case: there was no dependence on the path of the process, and hence the corresponding probability was deterministic.

It is then not difficult to verify that the continuous function q is precisely the killing rate function of the particular MAP: for an initial state $\theta \in E$,

$$\mathbb{P}_{(0,\theta)}(\zeta > t) = \mathbb{E}_{(0,\theta)} \left[\mathbb{P}_{(0,\theta)}(\zeta > t | \Xi) \right] = \mathbb{E}_{(0,\theta)} \left[\exp\left\{ - \int_0^t q(\Xi_s) ds \right\} \right].$$

Thus, for $h > 0$,

$$\mathbb{P}_{(0,\theta)}(\zeta \le h) = 1 - \mathbb{E}_{(0,\theta)}\Big[\exp\Big\{-\int_0^h q(\Xi_s)ds\Big\}\Big].$$

By then Taylor-expanding and using right-continuity of Ξ to approximate the integral via the almost-sure initial value, θ, of Ξ, we get for small $h > 0$,

$$\mathbb{P}_{(0,\theta)}(\zeta \le h) = hq(\theta) + o(h),$$

from which the claim readily follows.

2.2.4 The Lamperti Transform

The classical Lamperti transform, as the name suggests, is due to mathematician John Lamperti. It was first written in 1972 in Lamperti's original paper [41]. Its proof can be found in the lecture notes [12], as well as in Chapter 13.3 of [36] (in a more modern and complete presentation). Said theorem establishes a one-to-one correspondence between positive self-similar Markov processes (take $E = (0, \infty)$ in Definition 2.2.2) and Lévy processes. Such a bijection has proved to be instrumental both in the understanding of the path and distributional properties of the former class, and in pushing the boundaries of the knowledge around the celebrated Wiener-Hopf factorization for Lévy processes. The precise formulation of the result is the following.

Theorem 2.2.15 *(Lamperti's transform, [41]) Fix $\alpha > 0$ and $x > 0$. Let $X = (X_t)_{t\ge 0}$ be a pssMp with index of self-similarity α, initiated from the state $x > 0$. Then, there exists a process $\xi^* = (\xi^*_t)_{t\ge 0}$ from the class of exponentially-killed Lévy processes, with killing rate $q^* \ge 0$, such that*

$$X_t \mathbb{1}_{\{t<\zeta\}} = x \exp\Big\{\xi^*_{\phi(x^{-\alpha}t)}\Big\}, \qquad t \ge 0,$$

where $\phi(t) := \int_0^t X_s^{-\alpha} ds$ and $\zeta := \inf\{s > 0 : X_s = 0\}$. The process ξ^* is usually referred to as the underlying Lévy process of the pssMp X.

Conversely, let $\xi = (\xi_t)_{t\ge 0}$ be an exponentially-killed Lévy process. Then, for each $x > 0$, the process $X = (X_t)_{t\ge 0}$ defined by

$$X_t = x \exp\Big\{\xi_{\tilde{\phi}(x^{-\alpha}t)}\Big\}\mathbb{1}_{\{t<x^\alpha I_\infty\}}, \qquad t \ge 0,$$

where $\tilde{\phi}(t) = \inf\{s > 0 : \int_0^s e^{\alpha\xi^*_s} ds > t\}$ and $I_\infty = \int_0^\infty e^{\alpha\xi_s} ds$, defines a pssMp initiated (almost surely) from state x.

Essentially, what the proof of Theorem 2.2.15 all comes down to is showing that stationary independent increments translate to self-similarity, and vice versa, through a space-time transform of the path of the process.

Several extensions of this result have been developed, including real-valued and multidimensional versions (cf., [13], [1]) and they involve MAPs instead of Lévy processes.

2.2.5 Generators and Lévy Systems

For pure-jump Markov processes, the jump structure is encoded by a Lévy system. We adopt the standard formulation from, e.g., [5], [40], etc. For reflected processes and processes with boundary behaviour, generators with boundary conditions provide a natural characterisation. Background material on reflecting diffusions and Skorokhod problems can be found in, e.g., [60], [61], [48], [47], etc.

All of the Markov processes that are studied in the thesis are characterized via their infinitesimal generator. For pure-jump Markov processes, that is, Markov processes whose generator consists purely of an integral (nonlocal) operator, their generator is wholly determined by their jump structure, and this is probabilistically encoded by the associated Lévy system/jump kernel. Because it is in the setting of MAPs that these types of processes will arise in the thesis, we provide directly the definition for the Lévy system of a general MAP.

Definition 2.2.16 *Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be two measurable spaces. We say that a function $L : E \times \mathcal{F} \to [0, \infty)$ is a transition kernel from $(E, \mathcal{E})$ to $(F, \mathcal{F})$ if:*

(i) for fixed $A \in \mathcal{F}$, the function $E \ni \boldsymbol{\phi} \mapsto L(\boldsymbol{\phi}, A)$ is $\mathcal{E}$-measurable;

(ii) for fixed $\boldsymbol{\phi} \in E$, the function $\mathcal{F} \ni A \mapsto L(\boldsymbol{\phi}, A)$ defines a measure on $(F, \mathcal{F})$.

Definition 2.2.17 *Let $(\xi, \Theta) = (\xi_t, \Theta_t)_{t\geq 0}$ be a $\mathbb{R} \times E$-valued MAP. By a Lévy system for (ξ, Θ) we mean a couple (H, L), where H is a predictable additive functional $t \mapsto H_t$, and L is a kernel from $(E, \mathcal{E})$ to $(E \times \mathbb{R}, \mathcal{E} \bigotimes \mathcal{B}(\mathbb{R}))$, such that for every bounded measurable function $f : \mathbb{R}^2 \times E^2 \to [0, \infty)$, $\boldsymbol{\theta} \in E$ and $t \geq 0$,*

$$\begin{aligned} &\mathbb{E}_{(0,\boldsymbol{\theta})}\left[\sum_{s\leq t} f(\xi_{s-}, \Delta\xi_s, \Theta_{s-}, \Theta_s)\mathbb{1}_{\{\Delta\xi_s\neq 0\}}\right] \\ &= \mathbb{E}_{(0,\boldsymbol{\theta})}\left[\int_0^t \int_{\mathbb{R}\times E} f(\xi_s, y, \Theta_s, \boldsymbol{\phi})L(\Theta_s, d\boldsymbol{\phi}, dy)dH_s\right]. \end{aligned} \tag{2.17}$$

We refer to the transition kernel L as the jump kernel of the MAP.

As remarked in [40], in most examples of MAPs appearing in the literature, the additive functional H takes the form $H_t = t \wedge \zeta$, where ζ denotes the lifetime of the MAP. In fact, this also happens to be the case with all the MAPs we characterize in the thesis as well. As such, we will always omit writing about the Lévy system of the respective MAP, and solely refer to its jump kernel (hence it will be understood that H is of the above form).

For reflected Markov processes and those with boundary behaviour, generators with boundary conditions provide a natural characterisation. In particular, in the thesis, for such a Markov process X, with state space E, we derive operators A such that for functions f in a suitable class, the process

$$f(X_t) - f(X_0) - \int_0^t Af(X_s)ds, \qquad t \geq 0, \tag{2.18}$$

is a martingale. This is what is commonly referred to as the *martingale problem.* The class of test functions f that we will use in the thesis consists of bounded twice continuously differentiable functions (commonly denoted by $C_b^2(E)$) that satisfy appropriate boundary conditions. The $C_b^2(E)$ class is quite standard and natural for applying Itô's formula and deriving explicit expressions for the generator. We emphasize that the processes considered in this thesis are constructed independently, and the martingale relations of the previous form are verified for them a posteriori. In other words, we will not be defining the processes via the martingale problem − the martingale relation (2.18) will rather be a *property of the process.* This being the case, the well-posedness of the corresponding martingale problems (i.e., whether the class of test functions uniquely characterizes the law of the process) is not required for our arguments nor is it addressed any further in the thesis. We refer the reader to Chapter 4 of [23] for a detailed discussion on martingale problems and their role in characterizing Markov processes.

In line with the literature, in the thesis we make consistent use of the following notations with regard to the generator A of the Markov process X:

$$A : \mathcal{D}(A) \subset C_b(E) \to C_b(E),$$

where $\mathcal{D}(A)$ denotes the domain of A.

We emphasize, however, that in this thesis we do not attempt to characterise the full domain $\mathcal{D}(A)$. Instead, our analysis is carried out on a class of test functions. Specifically, in the thesis, we perform computations in the class $C_b^2(E)$ mentioned earlier, augmented with the appropriate boundary conditions for the process under consideration, and regard this class as a subset of the respective $\mathcal{D}(A)$.

We also note that, unlike the Feller semigroup framework (see, for example, [33]), in the thesis we do not require functions in $\mathcal{D}(A)$ to vanish at infinity. As explained earlier, our choice of the $C_b^2(E)$ class is motivated by the applicability of Itô's formula and the related verification of the martingale property (2.18), rather than by semigroup considerations.

2.2.6 Stochastic Differential Equations With Boundary Conditions

For reflected processes and processes with boundary behaviour, generators with boundary conditions provide a natural characterisation. We now go through the required background material for the thesis on reflecting diffusions and Skorokhod problems.

The classical Skorokhod problem concerns the construction of a process constrained to remain in a domain by means of reflection at the boundary. In its original formulation, introduced by Skorokhod (cf., [56], [57]), reflection occurs in the inward normal direction for the half-space $\mathbb{R}^{d-1} \times [0, \infty)$. This framework has since been substantially generalised to allow reflection in more complex domains and, in particular, to permit oblique (i.e. non-normal) reflection directions along the boundary.

In the thesis, we encounter reflected processes whose boundary behaviour does not fall within the scope of the classical normal-reflection setting. It is therefore necessary to adopt a more general formulation of the Skorokhod problem that allows for prescribed reflection fields along the boundary. For this purpose, we follow the abstract treatment given in Chapter 2.2 of [47], which provides a flexible framework suitable for the reflected stochastic differential equations considered later in the thesis:

Let $D \subseteq \mathbb{R}^d$ be an open set with a boundary ∂D. Denote by $\bar{D}$ the closure of D. Assume that for any $x \in \partial D$, a non-empty set of reflecting directions $K_x \subseteq \mathbb{R}^d$ is given. Assume further that $|v| \neq 0$ for any $v \in K_x$. Let $a : \bar{D} \to \mathbb{R}$ and $b_k : \bar{D} \to \mathbb{R}$, $1 \leq k \leq m$ be measurable functions.

Definition 2.2.18 *A pair $(X_t, l_t)_{t \geq 0}$ of continuous processes is a solution of the reflecting SDE*

$$dX_t = a(X_t)dt + \sum_{k=1}^{m} b_k(X_t)dW_k(t) + v(X_t)dl_t, \qquad t \geq 0, \tag{2.19}$$

with initial condition $X_0 = 0$, where the $W_k = (W_k(t))_{t \geq 0}$, $1 \leq k \leq m$, are independent Wiener processes, if

$$X_t \in \bar{D}, \qquad t \geq 0;$$

l is non-decreasing, $l_0 = 0$,

$$\int_0^t \big|v(X_s)\big| dl_s < \infty, \qquad \int_0^t \mathbb{1}_{X_s \notin \partial D} dl_s = 0, \qquad t \geq 0,$$

where $v(X_s) \in K_{X_s}$ if $X_s \in \partial D$.

The following theorem gives sufficient conditions on the functions a and b_k, $1 \leq k \leq m$, that guarantee a unique weak solution to (2.19).

Theorem 2.2.19 *Assume that the functions a and b_k, $1 \leq k \leq m$, satisfy the*

1. *global Lipschitz condition:*

$$\exists L > 0 \ \forall t \geq 0 \ \forall x_1, x_2 \in \bar{D} : |a(x_1) - a(x_2)| + \sum_{k=1}^{m} |b(x_1) - b(x_2)| \leq L|x_1 - x_2|;$$

2. *linear growth condition:*

$$\exists L > 0 \ \forall t \geq 0 \ \forall x \in \bar{D} : |a(x)| + \sum_{k=1}^{m} |b_k(x)| \leq C(1 + |x|).$$

Then there exists a unique weak solution to (2.19).

Finally, we describe how a solution to the martingale problem associated with a given generator can be represented as a reflected stochastic differential equation. Once again, we follow [47] closely, and more precisely, Chapter 3.2:

Fix $a_i : \bar{D} \to \mathbb{R}$, $\gamma_i : \bar{D} \to \mathbb{R}$, $1 \leq i \leq d$, and $b_{ki} : \bar{D} \to \mathbb{R}$, $1 \leq k \leq m$, $1 \leq i \leq d$ some bounded measurable functions.

Let $J : C_b^2(\bar{D}) \to C_b^2(\bar{D})$ be an operator supported on the boundary ∂D that takes the form

$$Jf(x) = \sum_{i=1}^{d} \gamma_i(x) \frac{\partial f}{\partial x_i}(x), \qquad x \in \partial D,$$

for functions $f \in C_b^2(\bar{D})$ twice continuously differentiable with bounded derivatives up to the order two and domain D.

Let $L : C_b^2(\bar{D}) \to C_b^2(\bar{D})$ be an operator of the form

$$Lf(x) = \sum_{i=1}^{d} a_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{d} \sigma_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x), \qquad x \in \bar{D},$$

where $\sigma_{ij}(x) = \sum_{k=1}^{m} b_{ki}(x) b_{kj}(x)$.

Suppose that $X = (X_t)_{t \geq 0}$ is a Markov process with values in $\bar{D}$ such that the process

$$f(X_t) - \int_0^t Lf(X_s) ds, \qquad t \geq 0,$$

is a martingale for every $f \in C_b^2(D)$ satisfying

$$Jf(x) = 0, \qquad x \in \partial D.$$

Then there exists a filtered probability space carrying an m-dimensional Brownian motion $(W_1, \ldots, W_m)^T = (W_1(t), \ldots, W_m(t))_{t \geq 0}^T$ and a process $l = (l_t)_{t \geq 0}$ as described in Definition 2.2.18 (with v therein replaced by $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_d)^T$) such that (X, l) is a weak solution to the following reflecting SDE.

$$dX_t = \boldsymbol{a}(X_t) dt + \sum_{k=1}^{m} \boldsymbol{b}_k(X_t) dW_k(t) + \boldsymbol{\gamma}(X_t) dl_t, \qquad t \geq 0,$$

where $\boldsymbol{a} = (a_1, \ldots, a_d)^T$, $\boldsymbol{b_k} = (b_{k1}, \ldots, b_{kd})^T$.

Chapter 3

Determinantally Equivalent Functions

3.1 Introduction and Main Result

In [44], Loewy investigates the relation between diagonal similarity and the concept of equal corresponding principal minors of two matrices. Diagonal similarity of matrices has long been studied in linear algebra; c.f., [4], [20], [24], [21]; as well as its connections with graph theory; cf., [22], [54]. The allusion that it may have a connection with the concept of equal corresponding principal minors stems from the observation that if two $n \times n$ matrices A and B are diagonally similar (up to a transposition), i.e., $B = D^{-1}AD$ or $B = D^{-1}A^TD$ for some non-singular diagonal matrix D, then all their corresponding principal minors agree. It is then the converse statement and the extent of its validity that would naturally trigger one's curiosity. It was discovered by Loewy in the aforementioned paper that the following two conditions (stated below informally) are enough for this converse to hold:

- A is irreducible;
- some particular sub-matrices of A have rank at least 2.

Although purely linear-algebraic, the same exact problem can be found expressed, albeit in random matrix theory jargon, in probabilistic literature as well; and more specifically, in the theory of discrete determinantal point process (DPP). Said stochastic processes, in layman's terms, serve as models of random sets of finitely-many points. Their key feature is their ability to effectively model repulsion/diversity of points. This, and many other desirable features, has resulted in their ever-increasing popularity in the machine-learning community; cf. [34], [63], [11]. Their connection with Loewy's linear-algebraic problem stems from the fact that a DPP which models the configuration of, say n, random points which we label for simplicity $\{1, \ldots, n\}$, has a probability distribution that is completely characterized by a (deterministic) $n \times n$ positive definite matrix, say $K = (K_{ij})_{i,j=1}^n$, called the *kernel* of the DPP, whereby the probability of a particular arrangement of points is given by the corresponding principal minor of the kernel; for example, the probability of observing the points 2 and 3 together is given by the principal minor $\begin{vmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{vmatrix}$. As such, it should be clear to the reader that the kernel of a (discrete) DPP is not unique. In particular, any $n \times n$ matrix Q with equal corresponding principal minors (to those of K) is also a kernel of the DPP, and is commonly referred to in the literature as an *equivalent kernel*. Thus, the formulation of Loewy's problem in this DPP language is as follows: given equivalent kernels K and Q, to what extent is it true that they are diagonally similar?

Although (discrete) DPPs with non-symmetric kernels have been studied, e.g., [25], [2], [9]; they have not received nearly the amount of attention their symmetric counterparts have, e.g., [7], [52], [3]. That being the case, it was of primary interest to answer the above question restricting to the case of symmetric (or Hermitian) equivalent kernels that is symmetric (or Hermitian) matrices with equal corresponding principal minors. It was discovered in [35] that in the (real or complex) symmetric setting, equivalent kernels K and Q will be diagonally similar with no additional conditions on the kernels/matrices. In [42], the investigation was taken a bit further by restricting to the (more general) case of (complex) Hermitian matrices. The respective result follows from a specialization of [44].

After this brief review of the origins and history of the original linear-algebraic problem, we now describe the functional extension we are concerned with. Let $K : \Lambda^2 \to \mathbb{F}$ and $Q : \Lambda^2 \to \mathbb{F}$ be two functions, where Λ is some abstract set (of arbitrary cardinality) and $\mathbb{F}$ is an arbitrary field. Suppose that K and Q are *determinantally equivalent* in the sense that

$$\det(Q(x_i, x_j))_{i,j=1}^n = \det(K(x_i, x_j))_{i,j=1}^n \qquad \forall x_1, \ldots, x_n \in \Lambda \;\; \forall n \in \mathbb{N}. \tag{3.1}$$

What are, then, all the possible transformations that transform Q into K? In the paper that first formulated this problem, [59], the following transformations were conjectured in Conjecture 1.4 therein:

- *conjugation transformations.* There exists a nowhere-zero function $g : \Lambda \to \mathbb{F}$ such that for every $x, y \in \Lambda$,
$$Q(x, y) = g(x)g(y)^{-1}K(x, y). \tag{3.2}$$
We refer to g as the *conjugation function* of the transformation.
- *transposition transformations.* $Q(x, y) = K(y, x)$ for every $x, y \in \Lambda$.

More precisely, it was conjectured that

Conjecture 1 (Conjecture 1.4, [59]) *If K and Q are equivalent kernels, then they can be transformed into one another by transposition and conjugation transformations.*

It was then proved in the same paper, via elementary techniques involving cycles, that restricting to symmetric functions K and Q, the conjecture holds. It is also worth pointing out that this exact result, in the case when the underlying 'set Λ' is finite, is precisely the relevant result from [35] involving symmetric matrices which we had discussed earlier. Both of the aforementioned results use very similar proof techniques. The problem of relaxing these symmetry constraints and solving this conjecture in that general setting was subsequently left open, and is what this chapter is dedicated to. Specifically, we obtain the following result.

Theorem 3.1.1 *Let Λ be a set, $\mathbb{F}$ a field, and let $K, Q : \Lambda^2 \to \mathbb{F}$ be two (not necessarily symmetric) nowhere-zero functions, except possibly on the set $\{(x, x) : x \in \Lambda\}$. Suppose further that for every pairwise distinct $x, y, z, w \in \Lambda$,*

$$\begin{vmatrix} Q(x,y) & Q(x,w) \\ Q(z,y) & Q(z,w) \end{vmatrix} \neq 0. \tag{3.3}$$

If K and Q are determinantally equivalent, then Q can be transformed into K through only conjugation and transposition transformations.

We note that, in the particular case of $4 < |\Lambda| < \infty$, condition (3.3) on its own implies both of Loewy's conditions (from the two earlier bullet points): for the second bullet point, by looking up in [44] the more precise formulation of the condition, one immediately sees this implication; for the first, suppose for a contradiction that A is reducible and pick a row or column with three zeros (possible for $n \geq 5$); then at least two of these zeros with two other entries of A show that (3.3) does not hold. With that said, we should clarify that while our main result is related to a particular case of Loewy's result, the approach taken here is substantially different. In particular, our proof relies only on elementary arguments and avoids the more technical linear-algebraic machinery used by Loewy. This simplification is made possible by the stronger hypothesis assumed here regarding nowhere-zero functions, which is essential for applying the key tools developed in Section 3.4. Without this assumption, certain steps — specifically involving division by zero — would break down, and our method would not be viable. We believe that this elementary approach offers a more accessible perspective on the result and may be adaptable to other contexts.

For completeness, we make a few additional remarks about DPPs serving to illustrate the chapter's connection with probability theory. We emphasize that the technical definitions and discussions below are not required for any of the arguments presented in the chapter and are included only as supplementary background of independent interest.

Besides discrete, there are also DPPs defined in the continuum, which have also been extensively studied; cf., [32], [58]. In that setting, there are, not a finite number of points which the DPP models, but uncountably-many. As such, it is clear that there can no longer be any square matrix of finite size that can characterize the DPP. Instead, it is now a measurable function, known as the *correlation function*, that characterizes the DPP, which is of a special "determinantal form".

Definition 3.1.2 *Let Λ be a locally compact Polish space and μ a Radon measure on Λ. Let $K : \Lambda^2 \to \mathbb{C}$ be a measurable function. A point process $\mathcal{X}$ on Λ is said to be a determinantal point process with correlation function K if it is simple (i.e., it almost surely assigns at most measure* 1 *to singletons) and for $k \geq 1$ and any family of mutually disjoint subsets $D_1, D_2, \ldots, D_k$ of Λ,*

$$\mathbb{E}\Big[\prod_{i=1}^{k} \mathcal{X}(D_i)\Big] = \int_{D_1 \times \ldots D_k} \det(K(x_i, x_j))_{1 \leq i,j \leq k} d\mu(x_1) \ldots d\mu(x_k).$$

One immediately sees from the above definition that there are various additional measure-theoretic considerations that arise; the main one being that the correlation function of a DPP is defined μ-almost everywhere as opposed to just 'everywhere'. As such, by neglecting the measure space structure of Λ in the above definition and just considering Λ as a set, our Theorem 3.1.1 gives rise to the following corollary, concerning the classification of transformations of equivalent correlation functions of DPPs:

Corollary 3.1.3 *Let Λ be a set and $\mathcal{X}$ a DPP on Λ with a correlation function $K : \Lambda^2 \to \mathbb{C}$ that is nowhere-zero except possibly on the set $\{(x, x) : x \in \Lambda\}$. Suppose Q is another correlation function for $\mathcal{X}$ satisfying* (3.3)*, then Q can be transformed into K through only conjugation and transposition transformations.*

The rest of the chapter is structured as follows. We begin, in Section 3.2, by discussing why the conjecture of [59] cannot hold in the general setting, and we then provide some insight into the hypotheses of our Theorem 3.1.1. In Section 3.3 we provide a short proof of the main result of [59] with the additional assumption that the functions at hand, besides symmetric, are also nowhere-zero and the characteristic of the underlying field is not 2. We do this by utilising "a shortcut" given by Proposition 3.3.4. Doing so gives us the opportunity to illustrate some of the techniques of said paper which we also deploy in proving our Theorem 3.1.1, and which were a source of inspiration for the conception of the novel techniques of the present paper. In Section 3.4 we describe three simple algebraic identities using graphs, which our proof of the main result is essentially built upon. Lastly, in Section 3.5, by using the identities and graph-theoretic setting from Section 3.4 we discover an underlying combinatorial structure in the problem; we derive various relations between determinantally equivalent functions, and we then use them to prove our theorem.

3.2 Counterexamples to the conjecture

For reasons that will be explained later on in the section, our analysis is restricted to functions $K, Q : \Lambda \to \mathbb{F}$ that are nowhere-zero except possibly on the set $\{(x,x) : x \in \Lambda\}$. A simple and canonical counterexample to the conjecture of [59] is the following (cf., [29]): Let Λ be a set such that $|\Lambda| = 4$ – and so then the problem is actually none other than the linear-algebraic one between two matrices described in the introductory section. For simplicity, let us assume $\Lambda = \{1,2,3,4\}$. Consider the functions/matrices $K : \Lambda^2 \to \mathbb{F}$ and $Q : \Lambda^2 \to \mathbb{F}$ defined by

$$(K(x,y))_{1\le x,y\le 4} = \begin{pmatrix} a & b & 1 & 1 \\ c & d & 1 & 1 \\ 1 & 1 & e & f \\ 1 & 1 & g & h \end{pmatrix}, \qquad (Q(x,y))_{1\le x,y\le 4} = \begin{pmatrix} a & c & 1 & 1 \\ b & d & 1 & 1 \\ 1 & 1 & e & f \\ 1 & 1 & g & h \end{pmatrix}, \tag{3.4}$$

for some $b, c, f, g \in \mathbb{F} \setminus \{0\}$ and $a, d, e, h \in \mathbb{F}$ such that $c \neq b$ and $f \neq g$. In this case, one finds, through a straightforward application of the formula for determinants of block matrices, that K is indeed determinantally equivalent to Q. However, one will also find that K cannot be obtained from Q by applying transposition and conjugation transformations to Q. Thus, the conjectured classification of transformations does not hold: we have just identified one additional type of transformation from the two conjectured which yields determinantally equivalent functions; perhaps a suitable name for it would be a "*partial transposition transformation*", since only *part* of the matrix is being transposed. Said in linear-algebraic language, while the above two matrices have equal corresponding principal minors, they are not diagonally similar (up to a transposition), but only "partially".

Our first step is to seek for a condition to impose on an arbitrary pair of determinantally equivalent functions that rules out the pair K and Q of the previously-described block forms. The sub-matrices of ones in the upper-right and lower-left blocks of the above two block matrices are the ones that bring about the block structure that enables this "partial transposition transformation" discussed previously to be a viable transformation that transforms the two matrices with equal corresponding principal minors into one another. And so by far the most intuitive condition to impose on our pair of matrices that we hope would be enough to make this type of transformation non-permissible would be to require non-zero determinants in the upper-right and lower-left 2×2 sub-matrix regions, that is, to require

$$\begin{vmatrix} Q(1,3) & Q(1,4) \\ Q(2,3) & Q(2,4) \end{vmatrix} \neq 0, \quad \begin{vmatrix} Q(3,1) & Q(3,2) \\ Q(4,1) & Q(4,2) \end{vmatrix} \neq 0,$$

which, in our current case of $|\Lambda| = 4$, is, up to simultaneous permutations of rows and columns, the same as requiring for every pairwise distinct $x, y, z, w \in \Lambda$,

$$\begin{vmatrix} Q(x,y) & Q(x,w) \\ Q(z,y) & Q(z,w) \end{vmatrix} \neq 0.$$

We show that this condition, in conjunction with the nowhere-zero condition stated earlier, in fact, are sufficient in solving the conjecture for a general abstract set Λ.

It remains to explain the reason for our 'nowhere-zero functions' requirement. Plainly, it all comes down to the fact that two of the three algebraic identities involving 3-cycles and 4-cycles (cf., Section 3.4), which our proof of Theorem 3.1.1 in Section 3.5 is essentially built upon, involve divisors that are of the form $h(x, y)$, where $x \neq y$, with $h = K$ or $h = Q$; in which case, the condition $h(x, y) \neq 0$ is, of course, necessary.

After the statement of Theorem 3.1.1 we had remarked that in the case of $4 < |\Lambda| < \infty$, condition (3.3) on its own implies both of Loewy's sufficient conditions from [44]. So, although the proof we give here certainly requires the 'nowhere-zero functions' condition, it remains an open question whether the proof can be adapted to require weaker assumptions.

3.3 Proof of the symmetric case

In addition to our own, in later sections, we have employed and taken inspiration from some of the tools and techniques from [59]. Thus, we believe it is worth starting off by giving a short proof of said paper's main result under the further assumption that the functions at hand are nowhere-zero (except perhaps on the set $\{(x, x) : x \in \Lambda\}$). In this way, we hope the reader gets a clear illustration of the main concepts in the simpler symmetric setting before we move on to extend them to the more involved general setting. We emphasize that the techniques in this section (in particular, the proof of Proposition 3.3.4) are fairly standard.

We assume throughout this section that the characteristic of the field $\mathbb{F}$ is *not* two. The result is stated as follows.

Theorem 3.3.1 (Modified Theorem 1.5 from [59]) *Let Λ be a set, $\mathbb{F}$ a field, and let $K, Q : \Lambda^2 \to \mathbb{F}$ be symmetric functions that are nowhere-zero except perhaps on the set $\{(x,x) : x \in \Lambda\}$. If K and Q are determinantally equivalent (i.e., equation (3.1) holds), then it must be the case that K and Q are conjugation transformations of one another.*

Definition 3.3.2 *If $Q : \Lambda^2 \to \mathbb{F}$ and $K : \Lambda^2 \to \mathbb{F}$ are functions that satisfy*

$$Q(x,y) = c(x,y)K(x,y) \qquad \forall x, y \in \Lambda$$

for some cocycle function $c : \Lambda^2 \to \mathbb{F}$, we say that Q is a cocycle transformation of K.

As remarked in [59], we make the following key observation.

Proposition 3.3.3 *Let Λ be a set, $\mathbb{F}$ a field, and let $K : \Lambda^2 \to \mathbb{F}$ and $Q : \Lambda^2 \to \mathbb{F}$ be two (not necessarily nowhere-zero) functions of two variables. Then, Q is a conjugation transformation of K if and only if Q is a cocycle transformation of K.*

Proof Suppose there exists a nowhere-zero function $g : \Lambda \to \mathbb{F}$ such that

$$Q(x,y) = g(x)g(y)^{-1}K(x,y) \quad \forall x, y \in \Lambda.$$

Consider the function $c : \Lambda^2 \to \mathbb{F}$, given by

$$c(x,y) = g(x)g(y)^{-1}, \qquad x, y \in \Lambda.$$

It is then not difficult to see that c defined in this way is a cocycle function, and hence, that Q is a cocycle transformation of K with respective cocycle function c.

Conversely, suppose there exists a cocycle function $c : \Lambda^2 \to \mathbb{F}$ such that

$$Q(x,y) = c(x,y)K(x,y) \quad \forall x, y \in \Lambda.$$

Let $x, y \in \Lambda$ and fix an arbitrary $x_0 \in \Lambda$. Thanks to the cocycle property,

$$c(x,y)c(y,x_0)c(x_0,x) = 1,$$

and thus

$$c(x,y) = \frac{c(x_0,x)^{-1}}{c(y,x_0)}.$$

But also, again thanks to the cocycle property, we have

$$c(x, x_0) = c(x_0, x)^{-1}.$$

Therefore,

$$c(x, y) = \frac{c(x, x_0)}{c(y, x_0)}.$$

Consider the nowhere-zero function $g : \Lambda \to \mathbb{F}$, given by

$$g(z) = c(z, x_0), \qquad z \in \Lambda.$$

It is then clear that Q is a conjugation transformation of K with conjugation function g. ∎

We now state and prove the "shortcut result" we had mentioned earlier in the introduction.

Proposition 3.3.4 *Let Λ be a set and $\mathbb{F}$ a field. If a two-variable function $c : \Lambda^2 \to \mathbb{F}$ satisfies*

(i) $c(x, x) = 1$ *for every* $x \in \Lambda$*;*

(ii) $c(x, y)c(y, x) = 1$ *for every* $x, y \in \Lambda$*;*

(iii) $c(x, y)c(y, z)c(z, x) = 1$ *for every* $x, y, z \in \Lambda$*,*

then c is a cocycle function.

In other words, c satisfying the cocycle property for cycles of lengths 1, 2 and 3 is both a necessary and sufficient condition for c to be a (full) cocycle function, that is, for it to satisfy the cocycle property for cycles of any length.

Proof By (i) and (ii) of the hypothesis, we already have for all $z, w \in \Lambda$, $c(z, z) = 1$ and $c(z, w)c(w, z) = 1$. We proceed by induction on $r \geq 3$ in (2.1). The base case $r = 3$ is already satisfied by c thanks to (iii) of the hypothesis. We now move on to the inductive step. To this end, let $(z_1, \ldots, z_{r+1}) \in \Lambda^{r+1}$ be an $(r + 1)$-tuple. Then,

$$\begin{aligned}
&c(z_1, z_2)c(z_2, z_3) \cdots c(z_r, z_{r+1})c(z_{r+1}, z_1) \\
&= c(z_1, z_2)c(z_2, z_3)c(z_3, z_1)c(z_3, z_1)^{-1}\Big(\prod_{i=4}^{r+1} c(z_{i-1}, z_i)\Big)c(z_{r+1}, z_1) \\
&= c(z_1, z_3)\Big(\prod_{i=4}^{r+1} c(z_{i-1}, z_i)\Big)c(z_{r+1}, z_1) && \text{(by (iii) and (ii))}
\end{aligned}$$

$= 1,$

where the last equality is due to the inductive hypothesis, since $(z_1, z_3, z_4, \dots, z_{r+1}) \in \Lambda^r$ is an r-tuple. ∎

We are now in the position to provide a short proof of the (modified) main result of [59]:

Proof of Theorem 3.3.1 Consider the function $S : \Lambda^2 \to \mathbb{F}$, given by

$$S(x,y) = \begin{cases} \frac{Q(x,y)}{K(x,y)}, & \text{if } x \neq y \\ 1, & \text{if } x = y \end{cases}, \qquad x, y \in \Lambda.$$

Since $Q(x,y) = S(x,y)K(x,y)$ for every $x, y \in \Lambda$, if we can show that S is a cocycle function, the result will follow immediately from Proposition 3.3.3.

By equation (3.1) with $n \in \{1, 2\}$, S satisfies conditions (i) and (ii) of Proposition 3.3.4. To see that property (iii) of the proposition is also satisfied, we make use of equation (3.1) with $n = 3$ and the Leibniz formula for determinants: let $x_1, x_2, x_3 \in \Lambda$ be pairwise distinct, then

$$\sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \prod_{i=1}^{3} K(x_i, x_{\sigma(i)}) = \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \prod_{i=1}^{3} Q(x_i, x_{\sigma(i)}). \tag{3.5}$$

If a permutation $\sigma \in S_3$ fixes a point, then since S satisfies properties (i) and (ii) of Proposition 3.3.4 we have

$$\prod_{i=1}^{3} K(x_i, x_{\sigma(i)}) = \prod_{i=1}^{3} Q(x_i, x_{\sigma(i)}). \tag{3.6}$$

We can then subtract these terms from (3.5) and be left with the terms that come from the permutations $\sigma = (123)$ and $\sigma' = (321)$; in which case, (3.5) simplifies to

$$\begin{gathered} K(x_1, x_2)K(x_2, x_3)K(x_3, x_1) + K(x_3, x_2)K(x_2, x_1)K(x_1, x_3) \\ = \\ Q(x_1, x_2)Q(x_2, x_3)Q(x_3, x_1) + Q(x_3, x_2)Q(x_2, x_1)Q(x_1, x_3), \end{gathered} \tag{3.7}$$

which, by the symmetry assumptions on both K and Q, further simplifies to

$$2K(x_1, x_2)K(x_2, x_3)K(x_3, x_1) = 2Q(x_1, x_2)Q(x_2, x_3)Q(x_3, x_1). \tag{3.8}$$

Hence, property (iii) of Proposition 3.3.4 is also satisfied by S. ∎

3.4 Cycles on 4 vertices

As alluded to in Section 3.3, cycles will also play a key role in deriving the main result of this paper. In essence, as we shall see in the final section, the proof of our Theorem 3.1.1 comes down to establishing the cocycle property for cycles of length 3 in Λ of one of two particular functions of two variables. Foreshadowing the proof of Theorem 3.1.1 some more, in addition to equation (3.1) with $n = 3$, we will also be making use of equation (3.1) with $n = 4$. This, along with the usual application of the Leibniz formula for determinants, will mean that cycles of length 4 will also enter the scene. Now, the key feature of cycles is their elementary nature, and by developing suitable connections between 3-cycles and 4-cycles, we will arrive at a proof of Theorem 3.1.1 that is both conceptually simple and intuitive. These connections take the form, as we will see in this section, of three remarkably simple yet ingenious algebraic identities, which are then applied in Section 3.5 to uncover a hidden structure in the problem and thereby yield the promised elementary proof.

For the remainder of this section, we set $\mathcal{M} := \{1, 2, 3, 4\}$. We had explained in Section 2.1.3 that there are exactly six distinct directed 4-cycles in the set $\mathcal{M}$. It is not difficult to list them, for there are only three distinct undirected 4-cycles in $\mathcal{M}$, and each has exactly two possible directions for its trail.

We will use the following labelling for the 4-cycles in $\mathcal{M}$ henceforth:

$$q^{[1]} := (1, 2, 3, 4, 1), \qquad q^{[2]} := (1, 2, 4, 3, 1), \qquad q^{[3]} := (1, 3, 2, 4, 1). \tag{3.9}$$

In the same way, we also know that there are exactly eight distinct directed 3-cycles in $\mathcal{M}$; we will use the following labelling henceforth:

$$p^{(1)} := (1, 2, 3, 1), \qquad p^{(2)} := (1, 2, 4, 1), \qquad p^{(3)} := (1, 3, 4, 1), \qquad p^{(4)} := (2, 3, 4, 2). \tag{3.10}$$

Lemma 3.4.1 *Let $\mathbb{F}$ be a field, and let $h : \mathcal{M}^2 \to \mathbb{F}$ be a function of two variables. Then,*

$$h[q^{[2]}]h[q^{[3]}] = h(1, 3)h(3, 1) \cdot h[p^{(2)}]h'[p^{(4)}]; \tag{3.11}$$

and

$$h[q^{[2]}]h'[q^{[3]}] = h(2, 4)h(4, 2) \cdot h[p^{(1)}]h'[p^{(3)}]. \tag{3.12}$$

Proof It is not difficult to verify the two equations. ∎

Lemma 3.4.2 *Let $\mathbb{F}$ be a field, and let $h : \mathcal{M}^2 \to \mathbb{F}$ be a function of two variables that is nowhere-*

zero except possibly on the set $\{(x,x) : x \in \mathcal{M}\}$. *Then,*

$$h[q^{[1]}] = \frac{h[p^{(1)}]h[p^{(3)}]}{h(1,3)h(3,1)} = \frac{h[p^{(2)}]h[p^{(4)}]}{h(2,4)h(4,2)}. \tag{3.13}$$

Proof In the below figure we provide a pictorial proof of the lemma:

Figure 3.1: In the graph on the left, in red is the 3-cycle $p^{(1)}$, and in blue is the 3-cycle $p^{(3)}$. In the graph on the right, in red is the 3-cycle $p^{(2)}$, and in blue is the 3-cycle $p^{(4)}$. $\blacksquare$

Lemma 3.4.3 *Let* $\mathbb{F}$ *be a field, and let* $h : \mathcal{M}^2 \to \mathbb{F}$ *be a function of two variables that is nowhere-zero except possibly on the set* $\{(x,x) : x \in \mathcal{M}\}$. *Then,*

$$h[p^{(1)}] = \frac{h[p^{(2)}]h'[p^{(3)}]h[p^{(4)}]}{h(4,1)h(1,4) \cdot h(3,4)h(4,3) \cdot h(2,4)h(4,2)}.$$

Proof In the below figure we provide a pictorial proof of the lemma.

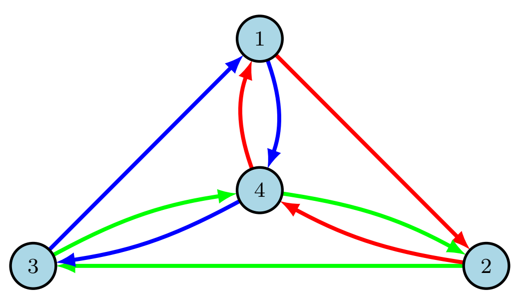


Figure 3.2: In red is the 3-cycle $p^{(2)}$, in green is the 3-cycle $p^{(4)}$, and in blue is the 3-cycle $p^{(3)}$ in reverse. $\blacksquare$

3.5 Proof of the Main Result

The backbone of the proof of our main theorem is the following elementary fact.

Lemma 3.5.1 *Let $a, b, a', b' \in \mathbb{F}$ satisfy*

$$a + b = a' + b' \tag{3.14}$$

and

$$ab = a'b'. \tag{3.15}$$

Then, we either have (i) $a = a'$ and $b = b'$; or (ii) $a = b'$ and $b = a'$.

Proof Consider the two quadratic equations

$$x^2 - (a + b)x + ab = 0$$

and

$$x^2 - (a' + b')x + a'b' = 0.$$

The first equation has roots a and b, and the second equation has roots a' and b'. Thanks to equations (3.14) and (3.15) the two quadratic equations are equal, and so must be the roots. ∎

For the same reasons as in Section 3.3, to prove Theorem 3.1.1 it will suffice to prove that, for the pair of determinantally equivalent functions K and Q from the statement, either the function $S : \Lambda^2 \to \mathbb{F}$, given by

$$S(x, y) = \begin{cases} \frac{Q(x,y)}{K(x,y)}, & \text{if } x \neq y \\ 1, & \text{if } x = y \end{cases}, \qquad x, y \in \Lambda,$$

or the function $\tilde{S} : \Lambda^2 \to \mathbb{F}$, given by

$$\tilde{S}(x, y) = \begin{cases} \frac{Q(x,y)}{K(y,x)}, & \text{if } x \neq y \\ 1, & \text{if } x = y \end{cases}, \qquad x, y \in \Lambda,$$

is a cocycle function.

Utilizing equation (3.1) with $n \in \{1, 2\}$, it is easy to check that both S and $\tilde{S}$ satisfy the cocycle property for cycles in Λ of lengths 1 and 2, and that $Q(x, y) = S(x, y)K(x, y) = \tilde{S}(x, y)K(y, x)$ for every $x, y \in \Lambda$. Indeed, for every $x, y \in \Lambda$ distinct, equation (3.1) with $n \in \{1, 2\}$ yields

$$K(x, x) = Q(x, x)$$

and

$$K(x, y)K(y, x) = Q(x, y)Q(y, x). \tag{3.16}$$

Thus, by Proposition 3.3.4 it suffices to show that either S or $\tilde{S}$ satisfies the cocycle property for cycles of length 3 in Λ. Expressing this in our shorthand notation, we need to prove that either

$$S[p] = 1 \text{ for every 3-cycle } p \coloneqq (p_i)_{i=0}^{3} \text{ in } \Lambda; \tag{3.17}$$

or

$$\tilde{S}[p] = 1 \text{ for every 3-cycle } p \coloneqq (p_i)_{i=0}^{3} \text{ in } \Lambda. \tag{3.18}$$

Definition 3.5.2 *Let Λ be a set, $\mathbb{F}$ a field, and let $K, Q : \Lambda^2 \to \mathbb{F}$ be two functions of two-variables. We say that a 3-cycle, $p = (p_i)_{i=0}^{3}$, in Λ belongs to Case 1 or Case 2 if*

$$\textbf{\textit{Case 1:}}\ K[p] = Q[p] \ \textit{and}\ K'[p] = Q'[p];$$

or

$$\textbf{\textit{Case 2:}}\ K[p] = Q'[p] \ \textit{and}\ K'[p] = Q[p],$$

respectively.

Remark 3.5.3 *Notice how if K and Q in the above definition are determinantally equivalent and nowhere-zero except possibly on the set $\{(x, x) : x \in \Lambda\}$, then in both Case 1 and Case 2 the first equality implies the second, and vice versa. For example, if $K[p] = Q[p]$, then*

$$K'[p] = \frac{K[p] \cdot K'[p]}{K[p]} = \frac{Q[p] \cdot Q'[p]}{Q[p]} = Q'[p],$$

where the second equality is an immediate consequence of (3.16). The remaining implications can be verified using the same argument. This fact is implicitly used in much of the remaining chapter.

We make an important observation:

Lemma 3.5.4 *Let Λ be a set, and $\mathbb{F}$ a field. If functions $K : \Lambda^2 \to \mathbb{F}$ and $Q : \Lambda^2 \to \mathbb{F}$ are determinantally equivalent, then every 3-cycle, $p = (p_i)_{i=0}^{3}$, in Λ belongs to at least one of Case 1 or Case 2.*

Proof As we had seen in Section 3.3 (recall equation (3.5)), equation (3.1) with $n = 3$ in conjunction with the Leibniz formula for determinants yields

$$\sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \prod_{i=1}^{3} K(p_i, p_{\sigma(i)}) = \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \prod_{i=1}^{3} Q(p_i, p_{\sigma(i)}).$$

By recalling the arguments from the proof of Theorem 3.3.1, we can conclude that

$$K[p] + K'[p] = Q[p] + Q'[p]. \tag{3.19}$$

But, thanks to equation (3.16), we also have

$$K[p]K'[p] = Q[p]Q'[p]. \tag{3.20}$$

Equations (3.19) and (3.20) then allow us to apply Lemma 3.5.1 to complete the proof. ∎

Remark 3.5.5 *The conclusion of the above lemma holds true even when the determinantally equivalent functions K and Q are not necessarily nowhere-zero.*

Notice now how, thanks to Lemma 3.5.4, proving that either (3.17) or (3.18) is satisfied is equivalent to showing that 3-cycles in Λ belong to the same Case. It turns out that it suffices to prove this statement for every subset $\mathcal{M} \subseteq \Lambda$ with $|\mathcal{M}| = 4$. More specifically, it suffices to prove the following proposition:

Proposition 3.5.6 *Let Λ be a set and $\mathbb{F}$ be a field. Suppose that functions $K : \Lambda^2 \to \mathbb{F}$ and $Q : \Lambda^2 \to \mathbb{F}$ are determinantally equivalent and nowhere-zero except possibly on the set $\{(x, x) : x \in \Lambda\}$. Suppose further that condition (3.3) is satisfied. Let $\mathcal{M} \subseteq \Lambda$ be a set such that $|\mathcal{M}| = 4$. Then, it is either the case that*

every 3-cycle in $\mathcal{M}$ belongs to Case 1;

or

every 3-cycle in $\mathcal{M}$ belongs to Case 2.

Let us first see how, after one has verified the above proposition, one can then easily derive Theorem 3.1.1.

Proof of Theorem 3.1.1 Pick any two 3-cycles $p := (p_i)_{i=0}^3$ and $q := (q_i)_{i=0}^3$ in Λ. We need to show that p and q belong to the same Case. We have the following scenarios:

1. *The cycles p and q share the same vertices.* In this scenario, p and q are obviously the same 3-cycle up to a possible reversion, in which case p and q belong to the same Case trivially by definition.

2. *The cycles p and q share exactly two vertices.* Define $\mathcal{M}$ to be the set of vertices of p and q. It is clear that $|\mathcal{M}| = 4$ and that p and q are 3-cycles in $\mathcal{M}$, and so we are done by Proposition 3.5.6.

3. *The cycles p and q share exactly one vertex.* In this case there is a unique index i such that q_i is a vertex of p. Denote by q_{l_1} and q_{l_2} the other two distinct vertices of q; and by p_{m_1} and p_{m_2} the other two distinct vertices of p. Note how the 3-cycles p and $(q_i, q_{l_1}, p_{m_1}, q_i)$ share exactly two vertices; namely, q_i and p_{m_1}. By the conclusion of the above scenario, we must then have that the 3-cycles p and $(q_i, q_{l_1}, p_{m_1}, q_i)$ belong to the same Case. Furthermore, the 3-cycles q and $(q_i, q_{l_1}, p_{m_1}, q_i)$ also share exactly two vertices; namely, q_i and q_{l_1}. Hence, for the same reason, it must be the case that the 3-cycles q and $(q_i, q_{l_1}, p_{m_1}, q_i)$ belong to the same Case as well. We are done by transitivity.

4. *The cycles p and q have completely different vertices.* Since the 3-cycles p and (q_0, q_1, p_1, q_0) share exactly one vertex, it follows, thanks to the conclusion of the above scenario, that the 3-cycles p and (q_0, q_1, p_1, q_0) belong to the same Case. Additionally, the 3-cycles q and (q_0, q_1, p_1, q_0) share exactly two vertices; and so by the conclusion of the second scenario, we then have that the 3-cycles q and (q_0, q_1, p_1, q_0) belong to the same Case. We are done by transitivity.

We have exhausted all possible scenarios for the two cycles p and q and have successfully proved in each one of them that the 3-cycles p and q belong to the same Case, as required. ∎

It remains to prove Proposition 3.5.6. We recall that said proposition establishes a relationship between a pair of determinantally equivalent functions, K and Q, restricted on sets of cardinality four. As such, to simplify notations, we will assume throughout this section, without loss of generality, that K and Q have domain $\mathcal{M}^2$, where $\mathcal{M} := \{1, 2, 3, 4\}$; and we will be consistent with the labelling of the 4-cycles and 3-cycles in $\mathcal{M}$ that we had given in (3.9) and (3.10) from our previous section.

From the pigeonhole principle, we know that there will always be two 3-cycles $p^{(i)}$ (out of the four from (3.10)) belonging to the same Case. Since the action of the symmetric group S_4 is transitive on pairs of distinct undirected 3-cycles on four vertices, we can actually assume, without loss of generality, which specific two 3-cycles in $\mathcal{M}$ belong to Case 1:

Running assumptions for the remainder of Section 3.5: $p^{(2)}$ and $p^{(4)}$ belong to Case 1.

The remainder of this section is therefore concerned in proving that $p^{(1)}$ and $p^{(3)}$ also belong to Case 1.

We need a couple of lemmas; the first lemma we present below states that if we have knowledge that one of the 3-cycles $p^{(1)}$ or $p^{(3)}$ belongs to Case 1, then the other necessarily follows suit.

Lemma 3.5.7 *Let $\mathbb{F}$ be a field, and suppose that functions $K : \mathcal{M}^2 \to \mathbb{F}$ and $Q : \mathcal{M}^2 \to \mathbb{F}$ are determinantally equivalent and nowhere-zero except possibly on the set $\{(x,x) : x \in \mathcal{M}\}$. Suppose that one of the 3-cycles $p^{(1)}$ or $p^{(3)}$ belongs to Case 1, then so must the other.*

Proof Let us suppose without loss of generality that it is the 3-cycle $p^{(3)}$ that belongs to Case 1. By Lemma 3.4.3, equation (3.16) and the fact that the 3-cycles $p^{(2)}$, $p^{(3)}$ and $p^{(4)}$ are assumed to belong to Case 1, we have

$$\begin{aligned} K[p^{(1)}] &= \frac{K[p^{(2)}]K'[p^{(3)}]K[p^{(4)}]}{K(4,1)K(1,4) \cdot K(3,4)K(4,3) \cdot K(2,4)K(4,2)} \\ &= \frac{Q[p^{(2)}]Q'[p^{(3)}]Q[p^{(4)}]}{Q(4,1)Q(1,4) \cdot Q(3,4)Q(4,3) \cdot Q(2,4)Q(4,2)} \\ &= Q[p^{(1)}]. \end{aligned}$$

This proves that the remaining 3-cycle $p^{(1)}$ also belongs to Case 1. ∎

Remark 3.5.8 *The above lemma does not require condition (3.3) from our theorem's hypothesis. It does however require $K(x,y) \neq 0$ for $x \neq y$, for otherwise we would divide by zero.*

Remark 3.5.9 *Modulo a relabelling of the vertices of each of the $p^{(i)}$, the above lemma in essence states that if three of the four 3-cycles, $p^{(1)}, \ldots, p^{(4)}$, in $\mathcal{M}$ belong to the same Case, then so must the remaining one.*

The above result implies that if we have knowledge that either of the two cycles $p^{(1)}$ and $p^{(3)}$ belong to Case 1, then the proof of Proposition 3.5.6 is complete; so let us suppose henceforth that $p^{(1)}$ and $p^{(3)}$ *both* belong to Case 2.

Lemma 3.5.10 *Let $\mathbb{F}$ be a field, and suppose that functions $K : \mathcal{M}^2 \to \mathbb{F}$ and $Q : \mathcal{M}^2 \to \mathbb{F}$ are determinantally equivalent and nowhere-zero except possibly on the set $\{(x,x) : x \in \mathcal{M}\}$. Then,*

$$K[q^{[1]}] + K'[q^{[1]}] = Q[q^{[1]}] + Q'[q^{[1]}]. \tag{3.21}$$

Proof It follows from Lemma 3.4.2, equation (3.16), as well as our assumption that the 3-cycles $p^{(1)}$ and $p^{(3)}$ belong to Case 2, that

$$K[q^{[1]}] = \frac{K[p^{(1)}]K[p^{(3)}]}{K(1,3)K(3,1)} = \frac{Q'[p^{(1)}]Q'[p^{(3)}]}{Q(1,3)Q(3,1)} = Q'[q^{[1]}],$$

in which case, $K'[q^{[1]}] = Q[q^{[1]}]$. The result now immediately follows. ∎

Lemma 3.5.11 *Let $\mathbb{F}$ be a field, and suppose that functions $K : \mathcal{M}^2 \to \mathbb{F}$ and $Q : \mathcal{M}^2 \to \mathbb{F}$ are determinantally equivalent. Then,*

$$\sum_{i=1}^{3} \Big(K[q^{[i]}] + K'[q^{[i]}]\Big) = \sum_{i=1}^{3} \Big(Q[q^{[i]}] + Q'[q^{[i]}]\Big). \tag{3.22}$$

Moreover, if K and Q are nowhere-zero except possibly on the set $\{(x,x) : x \in \mathcal{M}\}$, then

$$\Big(K[q^{[2]}] + K'[q^{[2]}]\Big) + \Big(K[q^{[3]}] + K'[q^{[3]}]\Big) = \Big(Q[q^{[2]}] + Q'[q^{[2]}]\Big) + \Big(Q[q^{[3]}] + Q'[q^{[3]}]\Big). \tag{3.23}$$

Proof Since $|\mathcal{M}| = 4$, we can use equation (3.1) with $n = 4$ in conjunction with the Leibniz formula for determinants to obtain, for $x_1, x_2, x_3, x_4 \in \mathcal{M}$ distinct,

$$\sum_{\sigma \in S_4} \operatorname{sgn}(\sigma) \prod_{i=1}^{4} K(x_i, x_{\sigma(i)}) = \sum_{\sigma \in S_4} \operatorname{sgn}(\sigma) \prod_{i=1}^{4} Q(x_i, x_{\sigma(i)}). \tag{3.24}$$

It is clear that for a permutation $\sigma \in S_4$ not a 3- or 4-cycle, the summands on both sides of (3.24) agree. Therefore, as we had done when we were analyzing equation (3.5), we can subtract such terms from both sides of (3.24) and be left with

$$\begin{gathered}\sum_{i=1}^{4} K(p_*^{(i)}, p_*^{(i)})\Big(K[p^{(i)}] + K'[p^{(i)}]\Big) - \sum_{i=1}^{3} \Big(K[q^{[i]}] + K'[q^{[i]}]\Big) \\ = \\ \sum_{i=1}^{4} Q(p_*^{(i)}, p_*^{(i)})\Big(Q[p^{(i)}] + Q'[p^{(i)}]\Big) - \sum_{i=1}^{3} \Big(Q[q^{[i]}] + Q'[q^{[i]}]\Big),\end{gathered} \tag{3.25}$$

where, for $i \in \{1,2,3,4\}$, $p_*^{(i)} \in \mathcal{M}$ denotes the unique vertex not in the cycle $p^{(i)}$. Now, we know from equation (3.1) with $n = 1$ that for every $i \in \{1,2,3,4\}$,

$$K(p_*^{(i)}, p_*^{(i)}) = Q(p_*^{(i)}, p_*^{(i)}).$$

Since $p^{(i)}$, by Lemma 3.5.4, belongs to either Case 1 or 2, we have

$$\{K[p^{(i)}], K'[p^{(i)}]\} = \{Q[p^{(i)}], Q'[p^{(i)}]\} \quad \forall\ i \in \{1,2,3,4\}.$$

Therefore, we can subtract the first sum of both the left hand side and right hand side of equation (3.25) and we are done.

The second claim of the lemma follows immediately from Lemma 3.5.10 by subtracting (3.21) from the established equation (3.22). ∎

Lemma 3.5.12 *Let $\mathbb{F}$ be a field, and suppose that functions $K : \mathcal{M}^2 \to \mathbb{F}$ and $Q : \mathcal{M}^2 \to \mathbb{F}$ are determinantally equivalent and nowhere-zero except possibly on the set $\{(x,x) : x \in \mathcal{M}\}$. Then,*

$$\Big(K[q^{[2]}] + K'[q^{[2]}]\Big) \cdot \Big(K[q^{[3]}] + K'[q^{[3]}]\Big) = \Big(Q[q^{[2]}] + Q'[q^{[2]}]\Big) \cdot \Big(Q[q^{[3]}] + Q'[q^{[3]}]\Big).$$

Proof It follows from Lemma 3.4.1, equation (3.16), our assumption that the 3-cycles $p^{(2)}$ and $p^{(4)}$ belong to Case 1, as well as our assumption that the 3-cycles $p^{(1)}$ and $p^{(3)}$ belong to Case 2, that

$$K[q^{[2]}]K[q^{[3]}] = K(1,3)K(3,1) \cdot K[p^{(2)}]K'[p^{(4)}] = Q(1,3)Q(3,1) \cdot Q[p^{(2)}]Q'[p^{(4)}] = Q[q^{[2]}]Q[q^{[3]}], \tag{3.26}$$

and

$$K[q^{[2]}]K'[q^{[3]}] = K(2,4)K(4,2) \cdot K[p^{(1)}]K'[p^{(3)}] = Q(2,4)Q(4,2) \cdot Q'[p^{(1)}]Q[p^{(3)}] = Q'[q^{[2]}]Q[q^{[3]}]; \tag{3.27}$$

in which case,

$$K'[q^{[2]}]K[q^{[3]}] = Q[q^{[2]}]Q'[q^{[3]}] \tag{3.28}$$

and

$$K'[q^{[2]}]K'[q^{[3]}] = Q'[q^{[2]}]Q'[q^{[3]}]. \tag{3.29}$$

The result now follows immediately. ∎

Corollary 3.5.13 *Let $\mathbb{F}$ be a field, and suppose that functions $K : \mathcal{M}^2 \to \mathbb{F}$ and $Q : \mathcal{M}^2 \to \mathbb{F}$ are determinantally equivalent and nowhere-zero except possibly on the set $\{(x,x) : x \in \mathcal{M}\}$. Then, one of the following two statements holds:*

(i) $K[q^{[2]}] = Q[q^{[2]}]$ *and* $K'[q^{[2]}] = Q'[q^{[2]}]$ *or* $K[q^{[2]}] = Q'[q^{[2]}]$ *and* $K'[q^{[2]}] = Q[q^{[2]}]$,

and

$K[q^{[3]}] = Q[q^{[3]}]$ *and* $K'[q^{[3]}] = Q'[q^{[3]}]$ *or* $K[q^{[3]}] = Q'[q^{[3]}]$ *and* $K'[q^{[3]}] = Q[q^{[3]}]$;

(ii) $K[q^{[2]}] + K'[q^{[2]}] = Q[q^{[3]}] + Q'[q^{[3]}]$ *and* $K[q^{[3]}] + K'[q^{[3]}] = Q[q^{[2]}] + Q'[q^{[2]}]$.

Proof By Lemma 3.5.11 (more specifically, equation (3.23) therein) and Lemma 3.5.12, we have, by a simple application of Lemma 3.5.1, that either statement (ii) holds; or the equations

$$K[q^{[2]}] + K'[q^{[2]}] = Q[q^{[2]}] + Q'[q^{[2]}] \text{ and } K[q^{[3]}] + K'[q^{[3]}] = Q[q^{[3]}] + Q'[q^{[3]}]$$

hold. We note that, thanks to equation (3.16), we also have the equations

$$K[q^{[2]}] \cdot K'[q^{[2]}] = Q[q^{[2]}] \cdot Q'[q^{[2]}] \text{ and } K[q^{[3]}] \cdot K'[q^{[3]}] = Q[q^{[3]}] \cdot Q'[q^{[3]}]$$

in our possession. An application of Lemma 3.5.1 then brings up statement (i). ∎

With this corollary in our disposal, we are now in the position to place the last piece of the puzzle on our board, that is, to prove that, in fact, the 3-cycles $p^{(1)}$ and $p^{(3)}$ actually belong to Case 1:

First suppose there is some index $i \in \{1, 2, 3, 4\}$ such that $K[p^{(i)}] = K'[p^{(i)}]$. Then, by Lemma 3.5.4,

$$Q[p^{(i)}] = K[p^{(i)}] = K'[p^{(i)}] = Q'[p^{(i)}],$$

that is, $p^{(i)}$ belongs to Case 1 and 2 simultaneously. Consequently, three of $p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}$ belong to the same Case; and so then we are done by Lemma 3.5.7 (cf., the second remark after the lemma). So let us suppose henceforth that

$$K[p^{(i)}] \neq K'[p^{(i)}] \quad \forall i \in \{1, 2, 3, 4\}. \tag{3.30}$$

In other words, we are supposing that each of the 3-cycles $p^{(i)}$ belong to exactly one Case. Specifically, we are supposing that $p^{(2)}$ and $p^{(4)}$ *only* belong to Case 1, and that $p^{(1)}$ and $p^{(3)}$ *only* belong to Case 2. We seek for a contradiction. We have two scenarios to examine corresponding to the cases in Corollary 3.5.13; let us start with case (i).

It follows from Lemma 3.4.2 (through a simple relabelling of vertices) that for every function $h : \mathcal{M}^2 \to \mathbb{F}$ that is nowhere-zero, except perhaps on the set $\{(x, x) : x \in \mathcal{M}\}$, that

$$h[q^{[2]}] = \frac{h[p^{(1)}]h'[p^{(4)}]}{h(2,3)h(3,2)} = \frac{h'[p^{(3)}]h[p^{(2)}]}{h(1,4)h(4,1)}. \tag{3.31}$$

Suppose first that the instance $K[q^{[2]}] = Q[q^{[2]}]$ from (i) of Corollary 3.5.13 holds. Then, (3.31) in conjunction with equation (3.16) yields

$$\frac{K[p^{(1)}]K'[p^{(4)}]}{K(2,3)K(3,2)} = K[q^{[2]}] = Q[q^{[2]}] = \frac{Q[p^{(1)}]Q'[p^{(4)}]}{Q(2,3)Q(3,2)} = \frac{Q[p^{(1)}]Q'[p^{(4)}]}{K(2,3)K(3,2)}.$$

Thus,

$$K[p^{(1)}]K'[p^{(4)}] = Q[p^{(1)}]Q'[p^{(4)}].$$

Since we have assumed $p^{(4)}$ to be a 3-cycle belonging to Case 1, the above equation implies that $p^{(1)}$ also belongs to Case 1; contradicting our earlier assumption that $p^{(1)}$ *only* belongs to Case 2.

Let us now suppose that the instance $K[q^{[2]}] = Q'[q^{[2]}]$ from (i) of Corollary 3.5.13 holds. Then, (3.31) in conjunction with equation (3.16) yields

$$\frac{K[p^{(1)}]K'[p^{(4)}]}{K(2,3)K(3,2)} = K[q^{[2]}] = Q'[q^{[2]}] = \frac{Q'[p^{(1)}]Q[p^{(4)}]}{Q(2,3)Q(3,2)} = \frac{Q'[p^{(1)}]Q[p^{(4)}]}{K(2,3)K(3,2)}.$$

Thus,

$$K[p^{(1)}]K'[p^{(4)}] = Q'[p^{(1)}]Q[p^{(4)}].$$

Since we have assumed $p^{(1)}$ to be a 3-cycle belonging to Case 2, the above equation implies that $p^{(4)}$ also belongs to Case 2; contradicting our earlier assumption that $p^{(4)}$ *only* belongs to Case 1.

We conclude that (i) of Corollary 3.5.13 cannot possibly hold; so let us now assume that (ii) of Corollary 3.5.13 holds. In the proof of Lemma 3.5.12 we had established equations (3.26) – (3.29). By dividing (3.26) by (3.28), we get

$$K[q^{[2]}]Q'[q^{[3]}] = K'[q^{[2]}]Q[q^{[3]}]. \tag{3.32}$$

Now, by our usual application of Lemma 3.5.1, the first equation from (ii) of Corollary 3.5.13 together with equation (3.32) brings about the following two possibilities:

(I) $K[q^{[2]}] + K'[q^{[2]}] = 0$ and $Q[q^{[3]}] + Q'[q^{[3]}] = 0$;

(II) $K[q^{[2]}] = Q[q^{[3]}]$ and $K'[q^{[2]}] = Q'[q^{[3]}]$.

Let us first assume that (I) holds. By dividing (3.27) by (3.26), we get

$$\begin{aligned}
\frac{Q'[q^{[2]}]}{Q[q^{[2]}]} = \frac{K'[q^{[3]}]}{K[q^{[3]}]} &= -\frac{K'[q^{[3]}]}{K[q^{[3]}]}\frac{K[q^{[2]}]}{K'[q^{[2]}]} && \text{(by assumption (I), } K[q^{[2]}] = -K'[q^{[2]}]) \\
&= -\frac{K(2,4)K(4,2)\cdot K[p^{(1)}]K'[p^{(3)}]}{K(2,4)K(4,2)\cdot K'[p^{(1)}]K[p^{(3)}]} && \text{(by Lemma 3.4.1, equation (3.12))} \\
&= -1,
\end{aligned}$$

where the last line follows from our assumption that the 3-cycles $p^{(1)}$ and $p^{(3)}$ belong to Case 2. Therefore,

$$K[q^{[2]}] + K'[q^{[2]}] = 0 = Q[q^{[2]}] + Q'[q^{[2]}].$$

By equation (3.16), we also have $K[q^{[2]}] \cdot K'[q^{[2]}] = Q[q^{[2]}] \cdot Q'[q^{[2]}]$; and so, by the usual application of Lemma 3.5.1, we have

$$K[q^{[2]}] = Q[q^{[2]}] \text{ and } K'[q^{[2]}] = Q'[q^{[2]}] \quad \text{or} \quad K[q^{[2]}] = Q'[q^{[2]}] \text{ and } K'[q^{[2]}] = Q[q^{[2]}].$$

But this is precisely the first part of (i) of Corollary 3.5.13, which we had investigated earlier and had obtained a contradiction.

The remaining scenario to check is when (II) holds; so let us now assume that (II) holds. Thanks to (3.31), equation (3.16), and the fact that the 3-cycles $p^{(1)}$ and $p^{(4)}$ belong to Cases 2 and 1, respectively, we have

$$K[q^{[2]}] = \frac{K[p^{(1)}]K'[p^{(4)}]}{K(2,3)K(3,2)} = \frac{Q'[p^{(1)}]Q'[p^{(4)}]}{Q(2,3)Q(3,2)}.$$

This equation in conjunction with the assumed $K[q^{[2]}] = Q[q^{[3]}]$ of (II) yields

$$\cancel{Q(1,3)}\cancel{Q(3,2)}\cancel{Q(2,4)}Q(4,1) = \frac{\cancel{Q(1,3)}\cancel{Q(3,2)}Q(2,1) \cdot \cancel{Q(2,4)}Q(4,3)\cancel{Q(3,2)}}{Q(2,3)\cancel{Q(3,2)}},$$

which simplifies to

$$Q(2,3)Q(4,1) = Q(2,1)Q(4,3),$$

that is,

$$\begin{vmatrix} Q(2,1) & Q(2,3) \\ Q(4,1) & Q(4,3) \end{vmatrix} = 0;$$

a contradiction to our theorem's hypothesis. Since Proposition 3.5.6 is now proven, we can formally conclude Theorem 3.1.1.

Chapter 4

Markov Additive Processes and Self-Similar Markov Processes

4.1 Background

Lamperti's transform for self-similar Markov processes establishes a bijection between positive self-similar Markov processes (pssMps) and Lévy processes; cf. [41]. Its proof can be found in the lecture notes [12], as well as in Chapter 13.3 of [36] (in a more modern and complete presentation). Such a bijection has proved to be instrumental both in the understanding of the path and distributional properties of the former class, and in pushing the boundaries of the knowledge around the celebrated Wiener-Hopf factorization for Lévy processes. Particularly, it has given rise to the construction of new classes of Lévy processes for which explicit identities can be obtained. The recent book by Kyprianou and Pardo, [37], and the references therein present a thorough account of this. One work that had sparked this chain of events is that of Caballero and Chaumont, [10], where the authors obtained a precise description of the Lévy processes underlying a *stable process killed at its first passage time below zero*, a *stable process conditioned to stay positive* and *one conditioned to hit zero continuously.*

Later, Chaumont, Panti and Rivero in [13] extended Lamperti's result from the positive half-line to the real line, and then subsequently, Alili, Chaumont, Graczyk and Zak in [1] to $\mathbb{R}^d$, by proving that the class of $\mathbb{R}^d$-valued ssMps is in bijection with Markov additive processes (MAPs). This bijection has been very helpful in the understanding of processes related to stable processes and to lay the foundations of a general fluctuation theory for MAPs, extending the one for Markov additive processes where the modulator has a discrete state space. The latter have played a prominent role in, e.g., classical applied probability models for queues and dams.

This thesis chapter concerns the proof of an extension of the above-described bijections to cover the case of ssMps valued in a general Banach space with a given norm. Specifically, in Section 4.2, we state and prove the promised norm-dependent connection between ssMps and MAPs. We remark that a version of this very result, in a different context and applied in a different way from that which we do in the thesis, had been published by Siri-Jégousse and Wences in [55] a bit after our (then unpublished) derivation of said result.

4.2 Norm-Dependent "Lamperti-Type" Transform Between MAPs and ssMps

In order to state the earlier-mentioned generalization of Lamperti's transform for a general Banach space E over the field of reals, equipped with a given norm $\|\cdot\|$, we recall that any element $x \in E$ possesses a (unique) $\|\cdot\|$-*polar decomposition*: x can be uniquely identified with the vector

$$(\|x\|, \arg(x)) \in [0,\infty) \times \mathcal{S}_{\|\cdot\|}, \tag{4.1}$$

where $\arg(x) := \frac{1}{\|x\|}x$, and $\mathcal{S}_{\|\cdot\|} := \{z \in E : \|z\| = 1\}$ denotes the unit sphere of E with respect to $\|\cdot\|$. As a convention, we take $(0, \partial)$ to be the $\|\cdot\|$-polar representation of the vector $0 \in E$.

We make some further enhancements on our notation for the normed unit sphere to deal with the particular case of $E = \mathbb{R}^d$, for $d \geq 2$, along with the L^p norm, $\|\cdot\|_p$, for $p \geq 1$, on said space. We will need these notations later on in the thesis when we start exploring, through examples of ssMps that live in the positive orthant of $\mathbb{R}^d$, the norm-dependence of the impending generalization of Lamperti's transform.

$$\mathcal{S}^d_{\|\cdot\|} := \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| = 1\};$$

$$\mathcal{S}^d_p := \mathcal{S}^d_{\|\cdot\|_{L^p}} := \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\|_{L^p} = 1\}, \qquad p \geq 1;$$

$$\mathcal{S}^{d,+}_{\|\cdot\|} := \mathcal{S}^d_{\|\cdot\|} \cap [0,\infty)^d; \qquad \mathcal{S}^{d,+}_p := \mathcal{S}^d_p \cap [0,\infty)^d, \qquad p \geq 1.$$

In this section we prove that space-time-transforming a given (possibly killed) MAP in an appropriate way yields the $\|\cdot\|$-polar decomposition of some E-valued (possibly killed) ssMp. Conversely, we show that by applying the same space-time transform on a suitable MAP we recover the $\|\cdot\|$-polar decomposition of a given E-valued ssMp killed on its first hitting time of $0 \in E$. We call this MAP the *underlying MAP* of the associated ssMp (always with respect to the chosen norm $\|\cdot\|$ on E). Precisely, we prove the following theorem.

Theorem 4.2.1 *Let E be a locally compact and separable Banach space over the field of reals, equipped with a norm $\|\cdot\|$. Fix $\alpha > 0$. Let $(\xi, \Xi) = (\xi_t, \Xi_t)_{t\geq 0}$ be an $\mathbb{R} \times \mathcal{S}_{\|\cdot\|}$-valued MAP with cemetery state $(-\infty, \partial)$ and lifetime ζ. Set $\phi(t) := \inf\{s > 0 : \int_0^s \mathrm{e}^{\alpha\xi_r} dr > t\}$, $t \geq 0$. Then, the E-valued stochastic process that has the $\|\cdot\|$-polar decomposition*

$$(\mathrm{e}^{\xi_{\phi(t)}}, \Xi_{\phi(t)})_{t\geq 0}, \tag{4.2}$$

where we understand $\mathrm{e}^{-\infty} = 0$, is a ssMp with index of self-similarity α, cemetery state $0 \in E$ and lifetime $K = \int_0^\zeta \mathrm{e}^{\alpha\xi_s} ds$.

Conversely, let $Z = (Z_t)_{t\geq 0}$ be an E-valued ssMp with index of self-similarity α, cemetery state 0 and lifetime $K := \inf\{t > 0 : \|Z_t\| = 0\}$. The $\mathbb{R} \times \mathcal{S}_{\|\cdot\|}$-valued process $(\xi, \Xi) = (\xi_t, \Xi_t)_{t\geq 0}$, defined by the transformation

$$(\xi_t, \Xi_t) = \begin{cases} (\log(\|Z_{I_t}\|), \frac{1}{\|Z_{I_t}\|} Z_{I_t}), & \text{if } t < \zeta \\ (-\infty, \partial), & \text{if } t \geq \zeta; \end{cases} \tag{4.3}$$

where $I_t := \inf\{s > 0 : \int_0^s \|Z_r\|^{-\alpha} dr > t\}$, $t \geq 0$; $\zeta := \int_0^K \|Z_r\|^{-\alpha} dr$, is a MAP with cemetery state $(-\infty, \partial)$ and lifetime ζ. We call (ξ, Ξ) the underlying MAP of the ssMp Z (with respect to the norm $\|\cdot\|$).

Proof We begin by proving the first part of the theorem. To this end, fix $c > 0$ and $x \in E$ and denote the probabilities of (ξ, Ξ) by $\mathbb{P}_{(\rho,\theta)}$, where $(\rho, \theta) \in \mathbb{R} \times \mathcal{S}_{\|\cdot\|}$. Denote by $Z = (Z_t)_{t\geq 0}$ the process with the $\|\cdot\|$-polar decomposition from (4.2). Since ξ is càdlàg, the mapping $s \mapsto \int_0^s \mathrm{e}^{\alpha\xi_r} dr$ is continuous and non-decreasing. Hence, its inverse ϕ is well-defined, continuous, and constitutes a family of stopping times $(\phi(t))_{t\geq 0}$. The assertions regarding Z's cemetery state and lifetime then follow. Moreover, since (ξ, Ξ) is càdlàg and quasi-left-continuous and the time-change ϕ is continuous and non-decreasing, the process $(\xi_{\phi(t)}, \Xi_{\phi(t)})_{t\geq 0}$ is also càdlàg and quasi-left-continuous, and hence so is Z.

All that is really needed to prove is the self-similarity and Markov properties of Z. We start by proving the former. For this, we first need to define a family of probabilities, $(P_x, x \in \mathbb{R})$, for Z in terms of the given family of probabilities of (ξ, Ξ). We do this just below:

$$\mathbb{P}_{(x,\theta)} = P_{\mathrm{e}^x\theta}, \qquad (x, \theta) \in \mathbb{R} \times \mathcal{S}_{\|\cdot\|}, . \tag{4.4}$$

It is then easy to see that proving the self-similarity property of Z, i.e., proving for every $c > 0$ and $x \in E$ that

$$(\{Z_t, t \geq 0\}, P_{cx}) \overset{\mathrm{d}}{=} (\{cZ_{c^{-\alpha}t}, t \geq 0\}, P_x),$$

is equivalent to proving, for every $c > 0$ and $t \geq 0$, that

$$\left(\Big\{(\exp\{\xi_{\phi(t)}\}, \Xi_{\phi(t)}), t \geq 0\Big\}, \mathbb{P}_{(\eta',\theta)}\right) \stackrel{\mathrm{d}}{=} \left(\Big\{(c\exp\{\xi_{\phi(c^{-\alpha}t)}\}, \Xi_{\phi(c^{-\alpha}t)}), t \geq 0\Big\}, \mathbb{P}_{(\eta,\theta)}\right), \tag{4.5}$$

where $(\eta, \theta) = \big(\log\|x\|, \arg(x)\big)$ and $\eta' = \log\|x\| + \log(c)$.

If we can show that $(\{\phi(t), t \geq 0\}, \mathbb{P}_{(\eta',\theta)}) \stackrel{\mathrm{d}}{=} (\{\phi(c^{-\alpha}t), t \geq 0\}, \mathbb{P}_{(\eta,\theta)})$, then (4.5) will follow immediately from Lemma 2.2.12. Fix a bounded measurable function F and $t \geq 0$. Then,

$$\begin{aligned}
&\mathbb{E}_{(\eta,\theta)}\Big[F\big(\phi(c^{-\alpha}s) : s \leq t\big)\Big] \\
&= \mathbb{E}_{(\eta,\theta)}\left[F\left(\inf\{r > 0 : \int_0^r \mathrm{e}^{\alpha\xi_u} du > \mathrm{e}^{-\alpha(\log c)}s\} : s \leq t\right)\right] \\
&= \mathbb{E}_{(\eta',\theta)}\left[F\left(\{\inf\{r > 0 : \int_0^r \mathrm{e}^{\alpha\xi_l} dl > s\} : s \leq t\}\right)\right] \qquad \text{(by Lemma 2.2.12)} \\
&= \mathbb{E}_{(\eta',\theta)}[F(\phi(s) : s \leq t)],
\end{aligned}$$

as required.

It remains to prove the strong Markov property of Z. Denote by $(\mathcal{G}_t)_{t\geq 0}$ the natural filtration of the underlying MAP (ξ, Ξ). This means that $(\mathcal{H}_t)_{t\geq 0}$, where $\mathcal{H}_t := \mathcal{G}_{\phi(t)}$, is the natural filtration of the process from (4.2). It suffices to show that for any $(\mathcal{H}_t)_{t\geq 0}$-stopping time τ and for every bounded measurable function $g : [0, \infty] \times \mathcal{S}_{\|\cdot\|} \to \mathbb{R}$, on $\{\tau < \zeta\}$,

$$\mathbb{E}_{(\eta,\theta)}\Big[g(\exp\{\xi_{\phi(\tau+s)}\}, \Xi_{\phi(\tau+s)})\mathbb{1}_{\{\tau+s<\zeta\}}\big|\mathcal{H}_\tau\Big] = \mathbb{E}_{(\eta',\theta')}\Big[g(\exp\{\xi_{\phi(s)}\}, \Xi_{\phi(s)})\Big]\Big|_{\substack{\eta'=\log\|X_\tau\|,\\ \theta'=\arg(X_\tau)}}, \tag{4.6}$$

where $(\eta, \theta) = (\log\|x\|, \arg(x))$.

Of course, for equation (4.6) to even make sense, the stopped σ-algebra $\mathcal{H}_\tau = \mathcal{G}_{\phi(\tau)}$ needs to be well-defined. To show that this is indeed the case we need to show that $\phi(\tau)$ is a $(\mathcal{G}_t)_{t\geq 0}$-stopping time. First notice how, for a deterministic $u > 0$, $\phi(u)$ is a $(\mathcal{G}_t)_{t\geq 0}$-stopping time: $\{\phi(u) < s\} = \{\int_0^s e^{\alpha\xi_r} dr > u\} \in \mathcal{G}_s$, since this event depends only on the information of the process (ξ, Ξ) up to time s and no later.

Now, to prove $\phi(\tau)$ is a $(\mathcal{G}_t)_{t\geq 0}$-stopping time, fix a deterministic $s > 0$. Then, note

$$\{\phi(\tau) < s\} = \{\int_0^s e^{\alpha\xi_r} dr > \tau\}$$

$$= \bigcup_{u \in \mathbb{Q} \cap (0,\infty)} (\{\tau < u\} \cap \{u < \int_0^s e^{\alpha \xi_r} dr\})$$
$$= \bigcup_{u \in \mathbb{Q} \cap (0,\infty)} (\{\tau < u\} \cap \{\phi(u) < s\}).$$

Since τ is an $(\mathcal{H}_t)_{t\geq 0}$-stopping time, we have $\{\tau < u\} \in \mathcal{H}_u$. But $\mathcal{H}_u$ is equal to $\mathcal{G}_{\phi(u)}$ by definition. We showed above that $\phi(u)$, for deterministic u, is a $(\mathcal{G}_t)_{t\geq 0}$-stopping time. Thus, $\mathcal{H}_u$ is the stopped σ-algebra of $(\mathcal{G}_t)_{t\geq 0}$ with respect to the $(\mathcal{G}_t)_{t\geq 0}$-stopping time $\phi(u)$. Now, $\{\tau < u\}$ being an element of $\mathcal{H}_u$ by definition means that it satisfies $\{\tau < u\} \cap \{\phi(u) < r\} \in \mathcal{G}_r$ for every $r \geq 0$. In particular, we have that for every $u \in \mathbb{Q} \cap (0, \infty)$, $\{\tau < u\} \cap \{\phi(u) < s\} \in \mathcal{G}_s$. Thus, by closure under countable unions of $\mathcal{G}_s$, $\{\phi(\tau) < s\} \in \mathcal{G}_s$; which proves that $\phi(\tau)$ is indeed a $(\mathcal{G}_t)_{t\geq 0}$-stopping time. Alternatively, one could simply invoke a standard stopping-time result: since ϕ is adapted continuous and non-decreasing, and τ is an $(\mathcal{H}_t)_{t\geq 0}$-stopping time, then $\phi(\tau)$ is a stopping time.

To conclude the proof of the Markov property, we first note that

$$\begin{aligned}
\phi(\tau + s) &= \inf\{r > 0 : \int_0^r e^{\alpha \xi_u} du > \tau + s\} \\
&= \inf\{r > \phi(\tau) : \int_0^r e^{\alpha \xi_u} du - \int_0^{\phi(\tau)} e^{\alpha \xi_u} du > s\} \\
&= \inf\{r > \phi(\tau) : \int_0^{r-\phi(\tau)} e^{\alpha \xi_u} du > s\} \\
&= \phi(\tau) + \phi(s) \circ \theta_{\phi(\tau)}.
\end{aligned}$$

So, on $\{\tau < \zeta\}$,

$$\begin{aligned}
&\mathbb{E}_{(\eta,\theta)}\Big[g\big(\exp\{\xi_{\phi(\tau+s)}\}, \Xi_{\phi(\tau+s)}\big)\mathbb{1}_{\{\tau+s<\zeta\}}\big|\mathcal{H}_\tau\Big] \\
&= \mathbb{E}_{(\eta,\theta)}\Big[g\big(\exp\{\xi_{\phi(\tau)+\phi(s)\circ\theta_{\phi(\tau)}}\}, \Xi_{\phi(\tau)+\phi(s)\circ\theta_{\phi(\tau)}}\big)\mathbb{1}_{\{s<\zeta\circ\theta_{\phi(\tau)}\}}\big|\mathcal{H}_\tau\Big] \\
&= \mathbb{E}_{(\eta,\theta)}\Big[g\big(\exp\{\xi_{\phi(s)}\}, \Xi_{\phi(s)}\big)\mathbb{1}_{\{s<\zeta\}} \circ \theta_{\phi(\tau)}|\mathcal{G}_{\phi(\tau)}\Big] \\
&= \mathbb{E}_{(\eta',\theta')}\Big[g\big(\exp\{\xi_{\phi(s)}\}, \Xi_{\phi(s)}\big)\mathbb{1}_{\{s<\zeta\}}\Big]\Big|_{\substack{\eta'=\xi_{\phi(\tau)} \\ \theta'=\Xi_{\phi(\tau)}}} \\
&= \mathbb{E}_{(\eta',\theta')}\Big[g\big(\exp\{\xi_{\phi(s)}\}, \Xi_{\phi(s)}\big)\Big]\Big|_{\substack{\eta'=\log\|X_\tau\|, \\ \theta'=\arg(X_\tau)}}
\end{aligned}$$

where the third equality follows from càdlàg paths and the strong Markov property of the MAP (ξ, Ξ), as well as of course the fact that $\phi(\tau)$ is a stopping time, as required.

We now deal with the second part of the theorem. We need to prove that the bivariate process (ξ, Ξ) from (4.3) is indeed a MAP as described in the statement of the second part of the theorem.

Our first claim is that

$$\tilde{\phi}(t) := \int_0^t \|Z_s\|^{-\alpha} ds = \inf\{s > 0 : \int_0^s \mathrm{e}^{\alpha\xi_r} dr > t\} = \phi(t), \qquad t \geq 0.$$

By taking the derivative with respect to t of both sides and applying the chain rule we get $\tilde{\phi}'(I_t) dI_t/dt = 1$, and hence $dI_t/dt = (\tilde{\phi}'(I_t))^{-1}$. By observing that $\tilde{\phi}'(t) = \|Z_t\|^{-\alpha}$, we obtain $dI_t/dt = \|Z_{I_t}\|^{\alpha} = \mathrm{e}^{\alpha\xi_t}$, and by finally integrating both sides of this equation with respect to t, we are done.

Suppose that the ssMp at hand, Z, has filtration $(\mathcal{F}_t)_{t\geq 0}$ and probabilities $(P_x : x \in E)$. It is then evident that $(\mathcal{G}_t)_{t\geq 0}$, defined by $\mathcal{G}_t = \mathcal{F}_{I_t}$, is the natural filtration of (ξ, Ξ). Strictly speaking, for the aforementioned filtration to even make sense, one would need to check that I_t is an $(\mathcal{F}_t)_{t\geq 0}$-stopping time, but this follows directly from the fact that Z is càdlàg and $\tilde{\phi}$ is non-decreasing and continuous.

To show that (ξ, Ξ) is a MAP, we first need to equip (ξ, Ξ) with a family of probabilities, $\mathbb{P}_{(x,\theta)}$, $(x, \theta) \in \mathbb{R} \times \mathcal{S}_{\|\cdot\|}$, in terms of the given family of probabilities of Z. By noting that $(\xi_0, \Xi_0) = (x, \theta)$ if and only if $Z_0 = \mathrm{e}^x\theta$, we see that we can simply define $\mathbb{P}_{(x,\theta)}$ as in (4.4).

The first step is then to confirm that the lifetime, ζ, of said process is indeed only dependent on Ξ. Since ζ satisfies $\zeta = \int_0^K \|Z_r\|^{-\alpha} dr$, proving that for every $x \in \mathbb{R}$,

$$(\zeta, \mathbb{P}_{(x,\theta)}) \stackrel{\mathrm{d}}{=} (\zeta, \mathbb{P}_{(0,\theta)}), \qquad \theta \in \mathcal{S}_{\|\cdot\|},$$

is equivalent to proving that for every $x \in \mathbb{R}$,

$$\left(\int_0^K \|Z_r\|^{-\alpha} dr, P_{\mathrm{e}^x\theta} \right) \stackrel{\mathrm{d}}{=} \left(\int_0^K \|Z_r\|^{-\alpha} dr, P_\theta \right), \qquad \theta \in \mathcal{S}_{\|\cdot\|}.$$

Thanks to self-similarity of Z,

$$(K, P_{\mathrm{e}^x\theta}) \stackrel{\mathrm{d}}{=} (\mathrm{e}^{\alpha x} K, P_\theta), \qquad (x, \theta) \in \mathbb{R} \times \mathcal{S}_{\|\cdot\|},$$

and

$$\Big(\{Z_r, r \geq 0\}, P_{\mathrm{e}^x\theta}\Big) \stackrel{\mathrm{d}}{=} \Big(\{\mathrm{e}^x Z_{\mathrm{e}^{-\alpha x} r}, r \geq 0\}, P_\theta\Big), \qquad (x, \theta) \in \mathbb{R} \times \mathcal{S}_{\|\cdot\|}.$$

From these two equations we get the desired

$$\left(\int_0^K \|Z_r\|^{-\alpha} dr, P_{\mathrm{e}^x\theta} \right) \stackrel{\mathrm{d}}{=} \left(\int_0^{\mathrm{e}^{\alpha x} K} \|Z_{\mathrm{e}^{-\alpha x} r}\|^{-\alpha} \mathrm{e}^{-\alpha x} dr, P_\theta \right) = \left(\int_0^K \|Z_r\|^{-\alpha} dr, P_\theta \right),$$

for $(x,\theta) \in \mathbb{R} \times \mathcal{S}_{\|\cdot\|}$, where the last equality follows from a simple change-of-variables.

We now prove that (ξ, Ξ) is a strong Markov process possessing the additive property (2.11). We start off with the strong Markov property: let τ be a $(\mathcal{G}_t)_{t\geq 0}$-stopping time and $g : \mathbb{R} \times \mathcal{S}_{\|\cdot\|} \to \mathbb{R}$ a bounded measurable function, which is extended to take value zero on $(-\infty, \partial)$. It is not difficult to derive on $\{\tau < \zeta\}$,

$$I_{\tau+s} = I_\tau + I_s \circ \theta_{I_\tau}, \qquad s \geq 0. \tag{4.7}$$

Of course, one needs to then check that I_τ too is an $(\mathcal{F}_t)_{t\geq 0}$-stopping time. This is a standard exercise and we omit the details, preferring instead to proceed with the proof of the strong Markov property. We remark that in the forthcoming calculations we have denoted the expectation with respect to the measure P_z, $z \in E$, by E_z. Let $(x,\theta) \in \mathbb{R} \times \mathcal{S}_{\|\cdot\|}$, then, on $\{\tau < \zeta\}$,

$$\begin{aligned}
&\mathbb{E}_{(x,\theta)}[g(\xi_{\tau+s}, \Xi_{\tau+s})|\mathcal{G}_\tau] \\
&= E_{\mathrm{e}^x\theta}\left[g\left(\log\big(\|Z_{I_{\tau+s}}\|\big), \frac{1}{\|Z_{I_{\tau+s}}\|} Z_{I_{\tau+s}} \right) \middle| \mathcal{F}_{I_t} \right] \\
&= E_{\mathrm{e}^x\theta}\left[g\left(\log(\|Z_{I_s}\|), \frac{1}{\|Z_{I_s}\|} Z_{I_s} \right) \circ \theta_{I_\tau} \middle| \mathcal{F}_{I_t} \right] && \text{(by (4.7))} \\
&= E_{Z_{I_t}}\left[g\left(\log(\|Z_{I_s}\|), \frac{1}{\|Z_{I_s}\|} Z_{I_s} \right) \right] && \text{(Z is a strong Markov process)} \\
&= E_{\exp\{\log(\|Z_{I_\tau}\|)\}\frac{1}{\|Z_{I_\tau}\|} Z_{I_\tau}}\left[g\left(\log(\|Z_{I_s}\|), \frac{1}{\|Z_{I_s}\|} Z_{I_s} \right) \right] \\
&= \mathbb{E}_{(\log(\|Z_{I_\tau}\|), \frac{1}{\|Z_{I_\tau}\|} Z_{I_\tau})}[g(\xi_s), \Xi_s)] \\
&= \mathbb{E}_{(\xi_\tau, \Xi_\tau)}[g(\xi_s, \Xi_s)],
\end{aligned}$$

which proves the strong Markov property for (ξ, Ξ).

We now prove the additive property. For this we need to make two important observations by exploiting the self-similarity property of Z. Our first claim is that for any $x \in \mathbb{R}$ and $\theta \in \mathcal{S}_{\|\cdot\|}$,

$$(I_t, P_{\mathrm{e}^x\theta}) \stackrel{\mathrm{d}}{=} (\mathrm{e}^{\alpha x} I_t, P_\theta), \qquad t \geq 0. \tag{4.8}$$

Indeed, for a bounded measurable function F and $t \geq 0$,

$$\begin{aligned}
E_{\mathrm{e}^x\theta}[F(I_t)] &= E_{\mathrm{e}^x\theta}\left[F\left(\inf\{s > 0 : \int_0^s \|Z_r\|^{-\alpha} dr > t\} \right) \right] \\
&= E_\theta\left[F\left(\inf\{s > 0 : \int_0^s \|\mathrm{e}^x Z_{\mathrm{e}^{-\alpha x} r}\|^{-\alpha} dr > t\} \right) \right] && \text{(by self-similarity of Z)}
\end{aligned}$$

$$= E_\theta\left[F\left(\mathrm{e}^{\alpha x}\cdot\inf\{s>0:\int_0^s \|Z_r\|^{-\alpha}dr>t\}\right)\right]$$
$$= E_\theta[F(\mathrm{e}^{\alpha x}I_t)].$$

Our second claim is an immediate consequence of the self-similarity property of Z: for any $x\in\mathbb{R}$ and $\theta\in\mathcal{S}_{\|\cdot\|}$,

$$(Z_t, P_{\mathrm{e}^x\theta}) \overset{\mathrm{d}}{=} (\mathrm{e}^x Z_{\mathrm{e}^{-\alpha x}t}, P_\theta), \qquad t\geq 0. \tag{4.9}$$

Having established the above two claims we proceed with the proof of the additive property for (ξ,Ξ). Fix a bounded measurable function $g:\mathbb{R}\times\mathcal{S}_{\|\cdot\|}\to\mathbb{R}$, $s,t\geq 0$ and $(x,\theta)\in E\times\mathcal{S}_{\|\cdot\|}$. Then, on $\{t<\zeta\}$,

$$\begin{aligned}
&\mathbb{E}_{(x,\theta)}[g(\xi_{t+s}-\xi_t,\Xi_{t+s})\mathbb{1}_{\{t+s<\zeta\}}|\mathcal{G}_t]\\
&= E_{\mathrm{e}^x\theta}\left[g\left(\log\left(\frac{\|Z_{I_{t+s}}\|}{\|Z_{I_t}\|}\right),\frac{1}{\|Z_{I_{t+s}}\|}Z_{I_{t+s}}\right)\mathbb{1}_{\{t+s<\tilde{\phi}(K)\}}\middle|\mathcal{F}_{I_t}\right]\\
&= E_{\mathrm{e}^x\theta}\left[g\left(\log\left(\frac{\|Z_{I_t+I_s\circ\theta_{I_t}}\|}{\|Z_{I_0\circ\theta_{I_t}}\|}\right),\frac{1}{\|Z_{I_t+I_s\circ\theta_{I_t}}\|}Z_{I_t+I_s\circ\theta_{I_t}}\right)\mathbb{1}_{\{I_t+I_s\circ\theta_{I_t}<K\}}\middle|\mathcal{F}_{I_t}\right]\\
&= E_{\mathrm{e}^x\theta}\left[g\left(\log\left(\frac{\|Z_{I_t+I_s\circ\theta_{I_t}}\|}{\|Z_{I_0\circ\theta_{I_t}}\|}\right),\frac{1}{\|Z_{I_t+I_s\circ\theta_{I_t}}\|}Z_{I_t+I_s\circ\theta_{I_t}}\right)\mathbb{1}_{\{I_s\circ\theta_{I_t}<K\circ\theta_{I_\tau}\}}\middle|\mathcal{F}_{I_t}\right]\\
&= E_{\mathrm{e}^x\theta}\left[\left(g\left(\log\left(\frac{\|Z_{I_s}\|}{\|Z_0\|}\right),\frac{1}{\|Z_{I_s}\|}Z_{I_s}\right)\mathbb{1}_{\{I_s<K\}}\right)\circ\theta_{I_\tau}\middle|\mathcal{F}_{I_t}\right]\\
&= E_{Z_{I_t}}\left[g\left(\log\left(\frac{\|Z_{I_s}\|}{\|Z_0\|}\right),\frac{1}{\|Z_{I_s}\|}Z_{I_s}\right)\mathbb{1}_{\{I_s<K\}}\right],
\end{aligned}$$

where the last line is an immediate consequence of the strong Markov property of Z. We rewrite the last expression in the following conducive way on $\{t<\zeta\}$,

$$E_{\mathrm{e}^{x'}.\theta'}\left[g\left(\log\left(\frac{\|Z_{I_s}\|}{\|Z_0\|}\right),\frac{1}{\|Z_{I_s}\|}Z_{I_s}\right)\mathbb{1}_{\{I_s<K\}}\right]_{\substack{x'=\log(\|Z_{I_t}\|)\\ \theta'=\frac{1}{\|Z_{I_t}\|}Z_{I_t}}}.$$

Carrying on our previous calculation with the aid of (4.8) and (4.9), we obtain on $\{t<\zeta\}$,

$$\begin{aligned}
&\mathbb{E}_{(x,\theta)}[g(\xi_{t+s}-\xi_t,\Xi_{t+s})\mathbb{1}_{\{t+s<\zeta\}}|\mathcal{G}_t]\\
&= E_{\mathrm{e}^{x'}.\theta'}\left[g\left(\log\left(\frac{\|Z_{I_s}\|}{\|Z_0\|}\right),\frac{1}{\|Z_{I_s}\|}Z_{I_s}\right)\mathbb{1}_{\{I_s<K\}}\right]_{\substack{x'=\log(\|Z_{I_t}\|)\\ \theta'=\frac{1}{\|Z_{I_t}\|}Z_{I_t}}}
\end{aligned}$$

$$
\begin{aligned}
&= E_{\mathrm{e}^{x'}\cdot\theta'}\left[g\left(\log(\|Z_{I_s}\|) - x', \frac{1}{\|Z_{I_s}\|}Z_{I_s}\right)\mathbb{1}_{\{I_s<K\}}\right]_{\substack{x'=\log\left(\|Z_{I_t}\|\right)\\ \theta'=\frac{1}{\|Z_{I_t}\|}Z_{I_t}}} \\
&= E_{\theta'}\left[g\left(\log\left(\|\mathrm{e}^{x'}Z_{\mathrm{e}^{-\alpha x'}(\mathrm{e}^{\alpha x'}I_s)}\|\right) - x', \frac{1}{\|Z_{I_s}\|}Z_{I_s}\right)\mathbb{1}_{\{I_s<K\}}\right]_{\substack{x'=\log\left(\|Z_{I_t}\|\right)\\ \theta'=\frac{1}{\|Z_{I_t}\|}Z_{I_t}}} \\
&= E_{\theta'}\left[g\left(\log(\|Z_{I_s}\|), \frac{1}{\|Z_{I_s}\|}Z_{I_s}\right)\mathbb{1}_{\{I_s<K\}}\right]_{\theta'=\frac{1}{\|Z_{I_t}\|}Z_{I_t}} \\
&= \mathbb{E}_{(0,\theta')}[g(\xi_s,\Xi_s)\mathbb{1}_{\{s<\zeta\}}]_{\theta'=\frac{1}{\|Z_{I_t}\|}Z_{I_t}} \\
&= \mathbb{E}_{(0,\Xi_t)}[g(\xi_s,\Xi_s)\mathbb{1}_{\{s<\zeta\}}],
\end{aligned}
$$

which is precisely the desired additive property.

Lastly, that (ξ, Ξ) is càdlàg follows from the fact that it is the time-change of a càdlàg process by a continuous and non-decreasing family of stopping times. ∎

Remark 4.2.2 *In the proof of the second part of Theorem 4.2.1, the time-change*

$$I_t \coloneqq \inf\{s > 0 : \int_0^s \|Z_r\|^{-\alpha} dr > t\}, \qquad t < \zeta,$$

implicitly required sufficient path regularity of the ssMp Z to ensure that it is well-behaved and does not blow up – this is taken care of by the standard assumption of càdlàg and quasi-left-continuous paths. In particular, for the additive functional

$$\tilde{\phi}(s) = \int_0^s \|Z_r\|^{-\alpha} dr, \qquad s < K,$$

to be continuous and non-decreasing on $[0, K)$, and hence for its right-continuous inverse I_t to be well-defined and finite for every $t < \zeta$, both right-continuous paths of Z and quasi-left-continuity are at play – the latter property specifically guaranteeing that there are no sudden predictable drops of $\|Z\|$ to zero.

The bijection established in Theorem 4.2.1 should be compared with two principal generalisations of Lamperti's classical representation: the $\mathbb{R}^d$ extension of Alili, Chaumont, Graczyk and Zak [1], and the infinite-dimensional framework developed by Siri-Jégousse and Wences [55].

As we had mentioned earlier, Alili et al. in [1] establish a bijection between $\mathbb{R}^d$-valued self-similar Markov processes and Markov additive processes whose modulator lives in the *Euclidean unit sphere*.

Their setting is therefore finite-dimensional and Euclidean in nature. By contrast, Siri-Jégousse and Wences in [55] work in a general locally compact and separable metric space, allowing in particular for infinite-dimensional state spaces. Their framework is formulated in terms of standard processes in the sense of Blumenthal and Getoor (cf., [8]). Theorem 4.2.1 in the thesis operates on a locally compact and separable Banach space E, which in particular implies a finite-dimensional setting – just like in [1] – but differs from the latter in that the modulator evolves on the *unit sphere with respect to an arbitrarily-chosen norm* $\|\cdot\|$. It is precisely this norm-dependence that is the main conceptual extension of Theorem 4.2.1 relative to the version established by Alili et al.: varying the norm alters the geometry of the unit sphere and hence produces different modulators and therefore different underlying MAPs for the same ssMp.

In regards to the regularity of the processes, it should also be remarked that, in contrast to the cited works – which assume standard processes – we do not explicitly impose quasi-left-continuity on the paths of the processes. Instead we proceed directly by explicit construction of the space-time change, verification of the self-similarity property, verification of the strong Markov property via stopping-time arguments, and direct verification of the defining property of MAPs, that is their additive property (2.11). So while both the proofs presented in [55] and in our thesis follow the heuristics in the proof in [1], we believe that the one we present here is more elementary and constructive in nature and exhibits more clearly the resemblance with the original arguments due to Lamperti in [41] connecting Lévy processes and $(0,\infty)$-valued ssMps but now in the setting of there being a Markov modulating component process present. We should emphasize, however, that the result obtained in [55] is formulated (and works) within a more general abstract topological setting.

In conclusion, while the work by Siri-Jégousse and Wences and the current part of the thesis both concern "norm-dependent" Lamperti-type transforms, the norm-induced geometry on the underlying modulator that we had discussed earlier – which is central to the applications developed in the subsequent chapters of this thesis – is not the main focus of [55]. In particular, for the ssMps that we will be studying in the following chapters of the thesis, we will quickly find out that the L^1 norm is the "most appropriate" choice of norm.

Chapter 5

Stable Processes Killed Upon Exiting a Cone in the Positive Orthant of $\mathbb{R}^d$

5.1 Introduction

In this thesis chapter, we consider a concrete example of an ssMp: a stable process with index of self-similarity $\alpha \in (0,2)$ killed on its first exit from a cone in $\mathbb{R}^d$, $d \geq 1$, for which we perform practical calculations to attain valuable information about its underlying Lévy or Markov Additive process. We start off in Section 5.2 by providing all the notations and formal definitions we will consistently be using in this chapter, as well as subsequent chapters of the thesis, in regards to stable processes. We also use the aforementioned section to derive and present a couple of basic preliminary results in regards to their jump structure that will be used in our proofs later on.

In dimension $d = 1$, the cone will be the positive real line $(0, \infty)$, and the stable process will be a standard one-dimensional stable process with both upward and downward jumps, initiated from some positive state. As such, this particular stable process killed on its first exit from $(0, \infty)$ is a pssMp. Thus, by Theorem 2.2.15, it possesses an underlying Lévy process, $\xi^* = (\xi^*_t)_{t \geq 0}$, killed at an independent (of its path) rate $q^* > 0$. This setting has already been extensively explored (cf., [37]) and all of the results presented in the thesis on this subject are well-established. In particular, a derivation of q^* can be found in Section 5.3 of [37] and it does not utilize Lévy systems/compensation formula. Because in later sections we will be extensively applying higher-dimensional versions of the compensation formula to tackle our novel problems, we choose to dedicate Section 5.3.1 to the derivation of this (already known) q^* using the classic (one-dimensional) compensation formula (cf., Theorem 4.4 of [36]). We emphasize that the techniques used to this end are quite standard.

It is in dimension $d > 1$ where all the novel problems that we tackle in this chapter lie. In this setting, the cone will be the positive orthant of $\mathbb{R}^d$, and the (d-dimensional) stable process of study will be the d-dimensional (random) vector with each of its coordinates being an independent (one-dimensional) α-stable process with both upward and downward jumps, initiated from a positive state. As such, this particular multi-dimensional stable process killed on its first exit from the first orthant is an example of an $\mathbb{R}^d$-valued ssMp. Thus, by Theorem 4.2.1, there exists an underlying (norm-dependent) MAP, $(\xi, \Xi) = (\xi_t, \Xi_t)_{t\geq 0} \in \mathbb{R} \times \mathcal{S}_{\|\cdot\|}$. Unlike the one-dimensional setting described in the previous paragraph, the killing rate of this underlying MAP will, in fact, very much depend on the position/path of the MAP. Indeed, one would intuitively expect said killing rate to be larger the nearer the process is to the boundary of the cone. Moreover, it goes without saying that said killing rate (function) is very much dependent on the choice of norm, $\|\cdot\|$, on $\mathbb{R}^d$. This makes up the contents of Section 5.4.1. For the characterization of the MAP to be complete, we need to also describe the MAP's jump structure (in virtue of the fact that it is pure-jump); which we do by deriving its jump kernel in Section 5.4.2.

Additionally, we will also study a special case of the stable process described in the above paragraph, namely the case of $d = 2$ and $\|\cdot\| = \|\cdot\|_2$, where we will be considering a more "general" cone than the positive orthant ("quadrant" in this case) of $\mathbb{R}^2$. This cone will be given via polar coordinates in the way one would naturally describe an, albeit two-dimensional, cone inside the first quadrant; namely, as

$$\{(r, \theta) \in [0, \infty) \times [0, 2\pi) : \theta \in (\phi_1, \phi_2)\}, \qquad 0 \leq \phi_1 < \phi_2 \leq \frac{\pi}{2}.$$

Notice how the above set when $\phi_1 = 0$ and $\phi_2 = \frac{\pi}{2}$ describes precisely the positive orthant (quadrant) of $\mathbb{R}^2$. A natural question to ask is how exactly will the killing rate of the underlying MAP of this particular 2-dimensional stable process depend on the angles ϕ_1 and ϕ_2. Will there exist a neat closed-form formula for the killing rate? This makes up the contents of Section 5.4.3. Engaging in some more involved trigonometry, in Section 5.4.4, we tackle the analogous problem in the third-dimension using spherical coordinates.

5.2 High-Dimensional Stable Processes

We let $X = (X_t)_{t\geq 0}$ be the d-dimensional α-stable process with index of self-similarity $\alpha \in (0, 2)$ given by

$$X_t = (X_t^{(1)}, \ldots, X_t^{(d)}) \in \mathbb{R}^d, \qquad t \geq 0, \tag{5.1}$$

where $X^{(1)} = (X_t^{(1)})_{t\geq 0}, \ldots, X^{(d)} = (X_t^{(d)})_{t\geq 0}$ is a collection of independent (one-dimensional) α-stable processes with positivity parameters $\rho_1, \ldots, \rho_d$, respectively. For this thesis chapter we will assume henceforth that $\rho_1, \ldots, \rho_d \in (0, 1)$. Let us denote the probabilities of X of this section by

$(P_{\boldsymbol{x}}, \boldsymbol{x} \in \mathbb{R}^d)$, and the expectation with respect to $P_{\boldsymbol{x}}$ by $E_{\boldsymbol{x}}$. By definition, each $X^{(i)}$, $1 \leq i \leq d$, is self-similar, and hence so is X. It follows that the process $Z = (Z_t)_{t\geq 0}$ defined by

$$Z_t = X_t \mathbb{1}_{\{t<\tau^D\}}, \qquad t \geq 0, \tag{5.2}$$

where $\tau^D := \inf\{t > 0 : X_t \notin D\}$, where $D = [0, \infty)^d$ denotes the positive orthant of $\mathbb{R}^d$, is an example of an ssMp absorbed at the zero vector, $\mathbf{0}_d \in \mathbb{R}^d$, and thus admits a (norm-dependent) underlying MAP through Theorem 4.2.1. Indeed,

Lemma 5.2.1 *The process $Z = (Z_t)_{t\geq 0}$ defined by*

$$Z_t = X_t \mathbb{1}_{\{t<\tau^D\}}, \qquad t \geq 0,$$

where $D = [0, \infty)^d$ and $\tau^D := \inf\{t > 0 : X_t \notin D\}$, is a (killed) ssMp with index of self-similarity α, lifetime τ^D and cemetery state $\mathbf{0}_d \in \mathbb{R}^d$.

Proof One can follow analogous steps to those taken in Chapter 5 of [37]. In particular, by iteratively applying Lemma 5.8 therein (which is the one-dimensional version of the statement of the current lemma) and by noticing how the process Z can be written as

$$Z_t = X_t \mathbb{1}_{\bigcap_{i=1}^{d}\{\underline{X}_t^{(i)}\geq 0\}}, \qquad t \geq 0,$$

where, for $1 \leq i \leq d$, $\underline{X}^{(i)} = (\underline{X}_t^{(i)})_{t\geq 0}$, denotes the running infimum of $X^{(i)}$, $\underline{X}_t^{(i)} := \inf_{s\leq t} X_s^{(i)}$, $t \geq 0$, one will obtain the result. ∎

Throughout the entire thesis, we shall be consistent in denoting the underlying MAP of the ssMp Z by $(\xi, \Xi) = (\xi_t, \Xi_t)_{t\geq 0}$, and its respective lifetime by ζ. Let us also denote the probabilities of (ξ, Ξ) by $\mathbb{P}_{(x,\boldsymbol{\theta})}$, $(x, \boldsymbol{\theta}) \in \mathbb{R} \times \mathcal{S}^{d,+}_{\|\cdot\|}$; which we know are obtained from those of Z through the description given in the second part of Theorem 4.2.1. We shall denote the expectation with respect to $\mathbb{P}_{(x,\boldsymbol{\theta})}$ by $\mathbb{E}_{(x,\boldsymbol{\theta})}$.

Since the stable processes $X^{(j)}$ defined above admit both positive and negative jumps, their positivity parameters satisfy $\rho_j \in (0, 1)$, and hence, consistent with (2.3)–(2.5), their respective Lévy measures, $\Pi^{(j)}$, and respective characteristic exponents, $\Psi^{(j)}$, are given by

$$\Pi^{(j)}(dx) = |x|^{-(1+\alpha)}(c_1^{(j)}\mathbb{1}_{\{x>0\}} + c_2^{(j)}\mathbb{1}_{\{x<0\}})dx, \quad x \in \mathbb{R} \setminus \{0\}, \tag{5.3}$$

and

$$\Psi^{(j)}(\lambda) = \int_{\mathbb{R}\setminus\{0\}} (1 - e^{-i\lambda x} + i\lambda x \mathbb{1}_{\{|x|<1\}})\Pi^{(j)}(dx), \qquad \lambda \in \mathbb{R}, \tag{5.4}$$

where
$$c_1^{(j)} := \frac{\Gamma(1+\alpha)\sin(\pi\alpha\rho_j)}{\pi} \text{ and } c_2^{(j)} := \frac{\Gamma(1+\alpha)\sin(\pi\alpha(1-\rho_j))}{\pi}. \tag{5.5}$$

Henceforth, we shall denote the Lévy measure of X from (5.1) by Π. From the fact that the $X^{(i)}$ are independent, and hence cannot jump simultaneously, we derive the following explicit form of Π in terms of the $\Pi^{(i)}$:

Lemma 5.2.2 *X from (5.1) has an infinitely divisible distribution with generating triplet $(\mathbf{0}_{d\times d}, \Pi, \mathbf{0}_d)$ (as per terminology from Theorem 8.1 of [33]), where $\mathbf{0}_{d\times d}$ denotes the $d\times d$ null-matrix. The measure Π is given by*

$$\Pi(d\boldsymbol{x}) = \sum_{j=1}^{d}(\bigotimes_{k\neq j}\delta_0(dx_k))\otimes\Pi^{(j)}(dx_j), \qquad \boldsymbol{x}=(x_1,\dots,x_d)\in\mathbb{R}^d\setminus\{\mathbf{0}_d\}, \tag{5.6}$$

where δ denotes the classic Dirac measure. Moreover, X is an α-stable process.

Proof To prove the first statement of the lemma regarding the generating triplet of X, fix $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_d)\in\mathbb{R}^d$, $\boldsymbol{x}=(x_1,\dots,x_d)\in\mathbb{R}^d$, denote by $\langle\,,\rangle$ and $|\cdot|$ the Euclidean inner product and norm, respectively. Denote by Ψ the characteristic exponential of X. Then, by independence of the $X^{(i)}$,
$$\Psi(\boldsymbol{\lambda}) = \sum_{i=1}^{n}\Psi^{(i)}(\lambda_i). \tag{5.7}$$

By (5.4), for every $1\le j\le d$,

$$\begin{aligned}\Psi^{(j)}(\lambda_j) &= \int_{\mathbb{R}\setminus\{0\}}(1-e^{-i\lambda_j x_j}+i\lambda_j x_j\mathbb{1}_{\{|x_j|<1\}})\Pi^{(j)}(dx_j)\\ &= \int_{\mathbb{R}^d\setminus\{\mathbf{0}_d\}}(1-e^{-i\langle\boldsymbol{\lambda},\boldsymbol{x}\rangle}+i\langle\boldsymbol{\lambda},\boldsymbol{x}\rangle\mathbb{1}_{\{|\boldsymbol{x}|<1\}})(\prod_{\substack{1\le k\le d\\ k\neq j}}\delta_0(dx_k))\Pi^{(j)}(dx_j).\end{aligned}$$

By summing up the above over $1\le j\le d$ and applying (5.7), we get the first statement of the lemma. Lastly, it follows that X is an α-stable (d-dimensional) process because its generating triplet satisfies that given in Theorem 14.3 of [33]. ∎

Remark 5.2.3 *We know that the Lévy measure of a Lévy process is what determines its jump structure. The intrinsic nature of the Lévy measure of X, given in (5.6), is such that the only jumps X can experience are through one and only one of its coordinates, that is, if t is a jump time* *of X, and we denote by $\Delta X_t := X_t - X_{t-}$ the jump, then necessarily*

$$\Delta X_t := X_t - X_{t-} \in \bigsqcup_{j=1}^{d} J_j,$$

where $J_j := \{0\}^{j-1} \times (\mathbb{R} \setminus \{0\}) \times \{0\}^{d-j}$, $1 \le j \le d$; and we understand $J_1 := (\mathbb{R} \setminus \{0\}) \times \{0\}^{d-1}$ and $J_d := \{0\}^{d-1} \times (\mathbb{R} \setminus \{0\})$.

We now present and prove a useful property that Π from above possesses, which we will be utilizing extensively in the sequel.

Lemma 5.2.4 *Let Π be the Lévy measure of X (given in Lemma 5.2.2). For every $\lambda > 0$ and $\boldsymbol{x} \in \mathbb{R}^d \setminus \{\mathbf{0}_d\}$, $\Pi(\lambda d\boldsymbol{x}) = \lambda^{-\alpha}\Pi(d\boldsymbol{x})$.*

Proof We carry out the proof for $n = 2$ inasmuch as it contains all the essential features of the general case. It is not difficult to see, using Lemma 5.2.2, that in this setting, our Π satisfies

$$\Pi(d\boldsymbol{x}) = \delta_0(dx_2)\Pi^{(1)}(dx_1) + \delta_0(dx_1)\Pi^{(2)}(dx_2), \qquad \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 \setminus \{\mathbf{0}_2\}.$$

To determine $\Pi(\lambda B)$ for a general Borel set $B \in \mathcal{B}(\mathbb{R}^2 \setminus \{\mathbf{0}_2\})$, it suffices to determine $\Pi(\lambda B)$ for B in some π-system of $\mathbb{R}^2 \setminus \{\mathbf{0}_2\}$ that generates the Borel sigma-algebra $\mathcal{B}(\mathbb{R}^2 \setminus \{\mathbf{0}_2\})$. The most convenient such π-system is the collection of left-open right-closed cylinder sets. So, fix $B = (b_1, d_1] \times (b_2, d_2] \subseteq \mathbb{R}^2 \setminus \{\mathbf{0}_2\}$. If $\mathbf{0}_2 \notin B$, then the statement of the lemma is immediate, since $\Pi(B) = 0$ in that case. Suppose without loss of generality that $(b_1, d_1]$ in an interval in the positive real line and $(b_2, d_2]$ an interval in the negative real line (0 inclusive), then

$$\Pi(B) = \Pi^{(1)}((b_1, d_1]) = c_1^{(1)} \int_{b_1}^{d_1} \frac{1}{x^{1+\alpha}} dx$$

and

$$\Pi(\lambda B) = \Pi^{(1)}(\lambda(b_1, d_1]) = c_1^{(1)} \int_{\lambda b_1}^{\lambda d_1} \frac{1}{x^{1+\alpha}} dx = \lambda^{-\alpha}\Pi^{(1)}((b_1, d_1]) = \lambda^{-\alpha}\Pi(B),$$

by a simple change of variables. ∎

5.3 Positive Stable Process Killed Upon Exiting $(0, \infty)$

5.3.1 Killing Rate of the Underlying Lévy Process

We emphasize once again that none of the results in this small section are novel, and that the techniques employed are quite standard.

Let $X = (X_t)_{t\geq 0}$ be a (one-dimensional) α-stable process, with probabilities $(\mathbb{P}_x : x \in \mathbb{R})$, that experiences both positive and negative jumps, almost surely; in which case it possesses the jump measure Π from (2.3) with associated constants c_1 and c_2 as given in (2.4) with $\alpha \in (0, 2)$ and $\rho \in (0, 1)$. We then define the process $Z = (Z_t)_{t\geq 0}$ by setting $Z_t = X_t \mathbb{1}_{\{\underline{X}_t \geq 0\}}$. It is formally proved in the beginning of Section 5.3 of [37] that Z, equipped with the probabilities $(\mathbb{P}_x : x > 0)$, is a pssMp with cemetery state 0. Clearly, the lifetime of this pssMp is $\tau_0^- := \inf\{t > 0 : X_t < 0\}$. Moreover, by Table 3.1 from [37], this lifetime is almost-surely finite, and $X_{\tau_0^-} < 0$ almost surely, that is, X does not creep downwards (see Exercise 7.4 of [36]). Therefore, Z is a pssMp which falls into category 3 from the trichotomy of pssMps given in Lemma 13.2 of [36]. As such, the Lamperti-transform guarantees that the underlying (exponentially-killed) Lévy process, $\xi^* = (\xi_t^*)_{t\geq 0}$, will have an almost-surely finite killing rate q^* (with cemetery state $-\infty$, recall). A complete characterization of ξ^* through its characteristic exponent can be found in [10], and a detailed summary in Chapter 5.3 of [37].

We proceed with the promised computation of q^* through the use of Lévy systems. We emphasize once again that the explicit expression of q^* is well known, and that the techniques used in this small subsection are standard. As discussed in the introduction, we revisit this computation in detail here primarily to illustrate the use of the classical compensation formula in the simpler one-dimensional setting before applying higher-dimensional versions of this formula to our new problems in dimensions $d > 1$ in the subsequent sections of this chapter.

The first step is to derive the law of τ_0^-. We do this with the help of the compensation formula: fix a bounded measurable function $f : [0, \infty) \to [0, \infty)$, let N be the Poisson random measure associated with the jumps of X and let Π be the Lévy measure of X (given in (2.3)). Then, for $x > 0$,

$$
\begin{aligned}
\mathbb{E}_x[f(\tau_0^-)] &= \mathbb{E}_x\Big[\sum_{t\geq 0} f(t)\mathbb{1}_{\{X_t<0\}}\mathbb{1}_{\{X_{t-}\geq 0\}}\mathbb{1}_{\{t<\tau_0^-\}}\Big] \\
&= \mathbb{E}_x\Big[\sum_{t\geq 0} f(t)\mathbb{1}_{\{\Delta X_t + X_{t-}<0\}}\mathbb{1}_{\{X_{t-}\geq 0\}}\mathbb{1}_{\{t<\tau_0^-\}}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}_x\left[\int_0^\infty \int_{-\infty}^0 f(t)\mathbb{1}_{\{\theta+X_{t-}<0\}}\mathbb{1}_{\{X_{t-}\geq 0\}}\mathbb{1}_{\{t<\tau_0^-\}}N(dt\times d\theta)\right] \\
&= \mathbb{E}_x\left[\int_0^\infty \int_{-\infty}^0 f(t)\mathbb{1}_{\{\theta+X_{t-}<0\}}\mathbb{1}_{\{X_{t-}\geq 0\}}\mathbb{1}_{\{t<\tau_0^-\}}\Pi(d\theta)dt\right] && \text{(compensation formula)} \\
&= \mathbb{E}_x\left[\int_0^\infty \int_{-\infty}^0 f(t)\mathbb{1}_{\{\theta+X_{t}<0\}}\mathbb{1}_{\{X_{t}\geq 0\}}\mathbb{1}_{\{t<\tau_0^-\}}\Pi(d\theta)dt\right] && \text{(right-continuity of } X\text{)} \\
&= \int_0^\infty f(t)\int_{-\infty}^0 \Big(\int_0^{-\theta} \mathbb{P}_x(X_t\in dz, t<\tau_0^-)\Big)\Pi(d\theta)dt && \text{(Tonelli's theorem)} \\
&= \int_0^\infty f(t)\int_0^\infty \Big(\int_{-\infty}^{-z} \Pi(d\theta)\Big)\mathbb{P}_x(X_t\in dz, t<\tau_0^-)dt.
\end{aligned}
$$

By introducing the shorthand notation $\bar{\Pi}^-(z) = \Pi((-\infty,-z))$, for $z>0$, we get

$$
\mathbb{E}_x[f(\tau_0^-)] = \int_0^\infty f(t)\mathbb{E}_x\Big[\bar{\Pi}^-(X_t); t<\tau_0^-\Big]dt, \tag{5.8}
$$

which implies

$$
\mathbb{E}_x\left[\int_0^{\tau_0^-} \bar{\Pi}^-(X_s)ds\right] = 1. \tag{5.9}
$$

We claim that $\int_0^{\tau_0^-} \bar{\Pi}^-(X_s)ds$ is an exponential random variable with rate 1. To prove this, it suffices to show that for every $n\in\mathbb{N}$,

$$
\mathbb{E}_x\left[\Big(\int_0^{\tau_0^-} \bar{\Pi}^-(X_s)ds\Big)^n\right] = n!. \tag{5.10}
$$

Indeed, the random variable $\int_0^{\tau_0^-} \bar{\Pi}^-(X_s)ds$ is moment-determinate:

$$
\int_0^{\tau_0^-} \bar{\Pi}^-(X_s)ds = \frac{c_2}{\alpha}\int_0^{\tau_0^-} X_s^{-\alpha}ds,
$$

and since X is bounded away from zero up to the time of the jump that causes the exit τ_0^-, and since this time is almost surely finite, the quantity $\int_0^{\tau_0^-} X_s^{-\alpha}ds$ is finite almost surely. More importantly, one shows through standard estimates for stable processes that for sufficiently small $\lambda>0$,

$$
\mathbb{E}_x\Big[\exp\big\{\lambda\int_0^{\tau_0^-} X_s^{-\alpha}ds\big\}\Big] < \infty,
$$

which is sufficient for moment-determinacy.

We carry out the proof for $n = 2$ in (5.10) inasmuch as it contains all the essential features of the general case. We denote the natural filtration of X by $(\mathcal{F}_t)_{t\geq 0}$.

$$
\begin{aligned}
&\mathbb{E}_x\Bigg[\Big(\int_0^{\tau_0^-} \bar{\Pi}^-(X_s)ds\Big)^2\Bigg] \\
&= 2\mathbb{E}_x\Bigg[\int_0^{\tau_0^-} \bar{\Pi}^-(X_{s_2})\int_0^{s_2} \bar{\Pi}^-(X_{s_1})ds_1 ds_2\Bigg] \\
&= 2\mathbb{E}_x\Bigg[\int_0^{\tau_0^-} \bar{\Pi}^-(X_{s_1})\Big(\int_{s_1}^{\tau_0^-} \bar{\Pi}^-(X_{s_2})ds_2\Big)ds_1\Bigg] \\
&= 2\mathbb{E}_x\Bigg[\int_0^{\infty} \mathbb{1}_{\{\tau_0^->s_1\}}\bar{\Pi}^-(X_{s_1})\Big(\int_{s_1}^{\tau_0^-} \bar{\Pi}^-(X_{s_2})ds_2\Big)ds_1\Bigg] \\
&= 2\int_0^{\infty} \mathbb{E}_x\Bigg[\mathbb{1}_{\{\tau_0^->s_1\}}\bar{\Pi}^-(X_{s_1})\Big(\int_{s_1}^{\tau_0^-} \bar{\Pi}^-(X_{s_2})ds_2\Big)\Bigg]ds_1 && \text{(by Tonelli's theorem)} \\
&= 2\int_0^{\infty} \mathbb{E}_x\Bigg[\mathbb{1}_{\{\tau_0^->s_1\}}\bar{\Pi}^-(X_{s_1})\mathbb{E}_x\Bigg[\int_{s_1}^{\tau_0^-} \bar{\Pi}^-(X_{s_2})ds_2\Big|\mathcal{F}_{s_1}\Bigg]\Bigg]ds_1 && \text{(tower property)} \\
&= 2\int_0^{\infty} \mathbb{E}_x\Bigg[\mathbb{1}_{\{\tau_0^->s_1\}}\bar{\Pi}^-(X_{s_1})\mathbb{E}_y\Bigg[\int_0^{\tau_0^-} \bar{\Pi}^-(X_{s_2})ds_2\Bigg]\Bigg|_{y=X_{s_1}}\Bigg]ds_1 && \text{(Markov property of } X\text{)} \\
&= 2\int_0^{\infty} \mathbb{E}_x\Big[\mathbb{1}_{\{\tau_0^->s_1\}}\bar{\Pi}^-(X_{s_1})\Big]ds_1 && \text{(by (5.9))} \\
&= 2\mathbb{E}_x\Bigg[\int_0^{\tau_0^-} \bar{\Pi}^-(X_{s_1})ds_1\Bigg] && \text{(by Tonelli's theorem)} \\
&= 2 && \text{(by (5.9))},
\end{aligned}
$$

as required.

From Lamperti's theorem, we know that the lifetime of the underlying Lévy process ξ^* is equal to $\int_0^{\tau_0^-} X_s^{-\alpha}ds$ and is exponentially-distributed. It is easy to check that for $s < \tau_0^-$,

$$X_s^{-\alpha} = \frac{\bar{\Pi}^-(X_s)}{\bar{\Pi}^-(1)}.$$

Therefore, the lifetime of ξ^* is equal to

$$\int_0^{\tau_0^-} X_s^{-\alpha}ds = \frac{1}{\bar{\Pi}^-(1)}\int_0^{\tau_0^-} \bar{\Pi}^-(X_s)ds = \frac{\alpha}{c_2}\mathbb{e}_1,$$

where $\mathbb{e}_1$ is an exponential random variable of parameter 1. It is not difficult to check that, for

$\mu > 0$, the random variable μW, where W is some exponential random variable with parameter $\lambda > 0$, is also an exponential random variable with parameter $\frac{\lambda}{\mu}$. Using this simple fact, we deduce that the lifetime of ξ^* is exponentially-distributed with parameter $\frac{c_2}{\alpha}$; said another way, ξ^* is killed at rate

$$q^* = \frac{c_2}{\alpha}. \tag{5.11}$$

5.4 High-Dimensional Stable Process Killed Upon Exiting the Positive Orthant Of $\mathbb{R}^d$

5.4.1 Killing Rate Function of the Underlying MAP

We now derive the killing rate of the MAP (ξ, Ξ). To this end, we will be using higher-dimensional versions of the classical compensation formula which we just saw in action in Section 5.3.1.

Lemma 5.4.1 *Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$. Let $(\xi, \Xi) = (\xi_t, \Xi_t)_{t\geq 0}$ be the underlying MAP of Z from (5.2) with respect to norm $\|\cdot\|$. The killing rate of (ξ, Ξ) is given by*

$$q(\boldsymbol{x}) = \frac{1}{\alpha}\sum_{k=1}^{d}\frac{\Gamma(1+\alpha)\sin(\pi\alpha(1-\rho_k))}{\pi}(\frac{x_k}{\|\boldsymbol{x}\|})^{-\alpha}, \qquad \boldsymbol{x} = (x_1, \ldots, x_d) \in [0, \infty^d) \setminus \{\boldsymbol{0}_d\}.$$

Proof Going back to the proof of Theorem 4.2.1, we see from (4.3) that the killing time, ζ, of (ξ, Ξ) equals

$$\zeta = \int_0^{\tau^D} \|X_s\|_1^{-\alpha} ds, \qquad \tau^D := \inf\{s > 0 : X_s \notin [0, \infty)^d\}.$$

Fix $\boldsymbol{\theta} = (\theta_1, \ldots, \theta^d) \in \mathcal{S}^{d,+}_{\|\cdot\|}$ and a positive bounded measurable function f on $[0, \infty)$. Now, recalling that stable processes in this subsection don't creep, and hence, that τ^D is a jump time of X, we get

$$\begin{aligned}
&\mathbb{E}_{(0,\boldsymbol{\theta})}[f(\zeta)] \\
&= E_{\boldsymbol{\theta}}\left[\sum_{t\geq 0} f\left(\int_0^t \|X_s\|_1^{-\alpha} ds\right) \mathbb{1}_{\{X_{t-}\in(0,\infty)^d, X_t \notin (0,\infty)^d\}} \mathbb{1}_{\{t<\tau^D\}}\right] \\
&= E_{\boldsymbol{\theta}}\left[\sum_{t\geq 0} f\left(\int_0^t \|X_s\|_1^{-\alpha} ds\right) \mathbb{1}_{\{X_{t-}\in(0,\infty)^d, X_{t-}+\Delta X_t \notin (0,\infty)^d\}} \mathbb{1}_{\{t<\tau^D\}}\right] \\
&= E_{\boldsymbol{\theta}}\left[\int_{(0,\infty)\times\mathbb{R}^d\setminus\{\boldsymbol{0}_d\}} f\left(\int_0^t \|X_s\|_1^{-\alpha} ds\right) \mathbb{1}_{\{X_{t-}\in(0,\infty)^d, X_{t-}+\boldsymbol{z} \notin (0,\infty)^d\}} \mathbb{1}_{\{t<\tau^D\}} N(d\boldsymbol{z}\times dt)\right]
\end{aligned}$$

$$= E_{\boldsymbol{\theta}}\left[\int_{(0,\infty)\times\mathcal{M}_{t-}} f\left(\int_0^t \|X_s\|_1^{-\alpha}ds\right)\mathbb{1}_{\{t<\tau^D\}}N(d\boldsymbol{z}\times dt)\right],$$

where N denotes the Poisson random measure associated with the jumps of X, and

$$\mathcal{M}_t \coloneqq \bigcup_{j=1}^d \left(\{0\}^{j-1}\times(-\infty,-X_t^{(j)})\times\{0\}^{d-j}\right), \qquad t<\tau^D.$$

Thanks to the compensation formula and right-continuous paths of X, the latter expectation takes the form

$$\mathbb{E}_{(0,\boldsymbol{\theta})}[f(\zeta)] = E_{\boldsymbol{\theta}}\left[\int_0^{\tau^D}\int_{\mathcal{M}_t} f\left(\int_0^t \|X_s\|_1^{-\alpha}ds\right)\Pi(d\boldsymbol{z})dt\right],$$

By then performing the change-of-variables $\boldsymbol{w} = \|X_t\|^{-1}\boldsymbol{z}$, and denoting the k-th coordinate of the vector $\arg(X_t)$ by $\arg(X_t)^{(k)}$, one obtains thanks to Lemma 5.2.4,

$$\begin{aligned}\mathbb{E}_{(0,\boldsymbol{\theta})}[f(\zeta)] &= E_{\boldsymbol{\theta}}\left[\int_0^{\tau^D}\int_{\mathcal{M}_t'} f\left(\int_0^t \|X_s\|_1^{-\alpha}ds\right)\Pi\big(\|X_t\|_1 d\boldsymbol{w}\big)dt\right]\\ &= E_{\boldsymbol{\theta}}\left[\int_0^{\tau^D} f\left(\int_0^t \|X_s\|_1^{-\alpha}ds\right)\left(\int_{\mathcal{M}_t'}\Pi(d\boldsymbol{w})\right)\|X_t\|_1^{-\alpha}dt\right],\end{aligned}$$

where $\mathcal{M}_t'$ is simply the set $\mathcal{M}_t$ with $X_t^{(j)}$ therein replaced by $\arg(X_t)^{(j)}$ for every j. By performing the change of variables

$$t = I_v \coloneqq \inf\{s>0 : \int_0^s \|X_r\|_1^{-\alpha}dr > v\}, \qquad dv = \|X_t\|_1^{-\alpha}dt,$$

and recalling the definitions of (ξ,Ξ) and Π from Lemma 5.2.4, we get

$$\begin{aligned}\mathbb{E}_{(0,\boldsymbol{\theta})}[f(\zeta)] &= \mathbb{E}_{(0,\boldsymbol{\theta})}\left[\int_0^{\zeta} f(v)\Big(\sum_{j=1}^d \Pi^{(j)}\big((-\infty,-\Xi_t^{(j)})\big)\Big)dv\right]\\ &= \frac{1}{\alpha}\sum_{j=1}^d c_2^{(j)}\mathbb{E}_{(0,\boldsymbol{\theta})}\left[\int_0^{\zeta}\big(\Xi_v^{(j)}\big)^{-\alpha}f(v)dv\right].\end{aligned}$$

We define the function $q : \mathcal{S}_{\|\cdot\|}^{d,+}\to[0,\infty)$,

$$q(\boldsymbol{\theta}) = \frac{1}{\alpha}\sum_{j=1}^d c_2^{(j)}\theta_j^{-\alpha}, \qquad \boldsymbol{\theta} = (\theta_1,\ldots,\theta_d)\in\mathcal{S}_{\|\cdot\|}^{d,+}.$$

The previous expectation can then be written as

$$\mathbb{E}_{(0,\boldsymbol{\theta})}[f(\zeta)] = \mathbb{E}_{(0,\boldsymbol{\theta})}\left[\int_0^\zeta q(\Xi_v) f(v)dv\right].$$

By setting $f(v) = \mathbb{1}_{(0,T)}(v)$, $T > 0$, above and computing the derivative with respect to T evaluated at 0 one obtains the desired killing rate function. ∎

5.4.2 Jump Structure of the Underlying MAP

In the previous section we derived the killing rate function of (ξ, Ξ); it remains to describe the latter by means of its infinitesimal generator. We observe that (ξ, Ξ) is a pure-jump process, and hence possesses a generator with no local part; it is therefore totally determined by its jump structure, which in turn is determined by its respective Lévy system (recall Definition 2.2.17). We work out the latter below, again using the compensation formula.

Theorem 5.4.2 *Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$. Let $(\xi, \Xi) = (\xi_t, \Xi_t)_{t\geq 0}$ be the underlying MAP of Z from (5.2) with respect to norm $\|\cdot\|$. The jump kernel of (ξ, Ξ) is the pushforward measure of*

$$L(\boldsymbol{\theta}, d\boldsymbol{\phi}, dy) = \sum_{j=1}^d (c_1^{(j)} \mathbb{1}_{(0,\infty)}(y) + c_2^{(j)} \mathbb{1}_{(-\theta^{(j)},0)}(y))\delta_{\boldsymbol{e}_j}(d\boldsymbol{\phi})|y|^{-(1+\alpha)}dy, \qquad (\boldsymbol{\phi}, y) \in \mathcal{S}^{d,+}_{\|\cdot\|} \times \mathbb{R},$$

where $\boldsymbol{\theta} = (\theta^{(1)}, \ldots, \theta^{(d)}) \in \mathcal{S}^{d,+}_{\|\cdot\|}$, under the transformation

$$\mathcal{S}^{d,+}_{\|\cdot\|} \times \mathbb{R} \ni (\boldsymbol{\phi}, y) \mapsto (\log\|\boldsymbol{\theta} + y\boldsymbol{\phi}\|, \arg(\boldsymbol{\theta} + y\boldsymbol{\phi})), \qquad \boldsymbol{\theta} \in \mathcal{S}^{d,+}_{\|\cdot\|}.$$

Proof We will follow familiar steps to those from the proof of Lemma 5.4.1. Firstly, from the very construction of (ξ, Ξ) (recall (4.3)), we have, for any $t > 0$ and positive bounded measurable function f on $\mathbb{R}^2 \times \mathcal{S}^{d,+}_{\|\cdot\|} \times \mathcal{S}^{d,+}_{\|\cdot\|}$,

$$\begin{aligned}
&\mathbb{E}_{(0,\boldsymbol{\theta})}\Big[\sum_{s\leq t} f\big(\xi_{s-}, \Delta\xi_s, \Xi_{s-}, \Xi_s\big)\mathbb{1}_{\{\Delta\xi_s\neq 0\}}\Big] \\
&= E_{\boldsymbol{\theta}}\left[\sum_{s\leq t} f\Big(\log\|X_{I_s-}\|, \log\frac{\|X_{I_s}\|}{\|X_{I_s-}\|}, \arg(X_{I_s-}), \arg(X_{I_s})\Big)\mathbb{1}_{\{\Delta X_{I_s}\neq 0\}}\right] \\
&= E_{\boldsymbol{\theta}}\left[\sum_{s\leq I_t} f\Big(\log\|X_{s-}\|, \log\frac{\|X_s\|}{\|X_{s-}\|}, \arg(X_{s-}), \arg(X_s)\Big)\mathbb{1}_{\{\Delta X_s\neq 0\}}\right].
\end{aligned}$$

Using the compensation formula in a similar way to how we had done in the proof of Lemma 5.4.1, we can rewrite the above as

$$
\begin{aligned}
&E_{\boldsymbol{\theta}}\left[\sum_{s\le I_t} f\Big(\log\|X_{s-}\|, \log\frac{\|X_s\|}{\|X_{s-}\|}, \arg(X_{s-}), \arg(X_s)\Big)\mathbb{1}_{\{\Delta X_s\neq 0\}}\right] \\
&= E_{\boldsymbol{\theta}}\left[\int_0^{I_t} ds \int \Pi(d\boldsymbol{x})\mathbb{1}_{\{X_s+\boldsymbol{x}\in(0,\infty)^d\}} f\Big(\log\|X_s\|, \log\frac{\|X_s+\boldsymbol{x}\|}{\|X_s\|}, \arg(X_s), \arg(X_s+\boldsymbol{x})\Big)\right].
\end{aligned}
$$

Now, by making the change of variables $s = I_v$, and then invoking Lemma 5.2.4, the above becomes

$$
\begin{aligned}
&= E_{\boldsymbol{\theta}}\left[\int_0^{t} dv \|X_{I_v}\|^{\alpha} \int \Pi(d\boldsymbol{x})\mathbb{1}_{\{X_{I_v}+\boldsymbol{x}\in D\}} f\Big(\log\|X_{I_v}\|, \log\frac{\|X_{I_v}+\boldsymbol{x}\|}{\|X_{I_v}\|}, \arg(X_{I_v}), \arg(X_{I_v}+\boldsymbol{x})\Big)\right] \\
&= E_{\boldsymbol{\theta}}\Bigg[\int_0^{t} dv \int \Pi(d\boldsymbol{x})\mathbb{1}_{\{X_{I_v}+\|X_{I_v}\|\boldsymbol{x}\in D\}} \\
&\qquad\qquad \times f\Big(\log\|X_{I_v}\|, \log\frac{\|X_{I_v}+\|X_{I_v}\|\boldsymbol{x}\|}{\|X_{I_v}\|}, \arg(X_{I_v}), \arg(X_{I_v}+\|X_{I_v}\|\boldsymbol{x})\Big)\Bigg].
\end{aligned}
$$

Appealing again to (4.3) and Lemma 5.2.2, the above expression becomes

$$
\begin{aligned}
&\mathbb{E}_{(0,\boldsymbol{\theta})}\Big[\sum_{s\le t} f(\xi_{s-}, \Delta\xi_s, \Xi_{s-}, \Xi_s)\mathbb{1}_{\{\Delta\xi_s\neq 0\}}\Big] \\
&= \mathbb{E}_{(0,\boldsymbol{\theta})}\Bigg[\int_0^{t\wedge\zeta}\int_{\mathbb{R}}\Bigg\{\sum_{j=1}^{d}\Big(c_2^{(j)}\mathbb{1}_{(-\Xi_v^{(j)},0)}(w) + c_1^{(j)}\mathbb{1}_{(0,\infty)}(w)\Big)\cdot \\
&\qquad\qquad \times f\Big(\xi_v, \log\|\Xi_v + w\boldsymbol{e}_j\|_1, \Xi_v, \arg(\Xi_v + w\boldsymbol{e}_j)\Big)|w|^{-(1+\alpha)}\Bigg\}dwdv\Bigg].
\end{aligned}
$$

It remains to bring the above resulting expression in the desired form displayed in the RHS of (2.17) in terms of an appropriate transition kernel L. One will then indeed find that the said transition kernel is precisely that which is given in the statement of the theorem. ∎

We remark that the jump kernel L from Theorem 5.4.2 is as close to a closed form as we can ever hope to get when dealing with an abstract norm, $\|\cdot\|$, on $\mathbb{R}^d$, since various terms in our calculations involving the norm cannot possibly be further evaluated unless we have some further information on the norm. The L^p norm, for $p \ge 1$, is a particularly nice norm to this end. In order to state the result, we introduce the following notation for a particular vector in $\mathbb{R}^d$ that makes a frequent appearance in our thesis: for $p \ge 1$, $y \in \mathbb{R}$, $1 \le j \le d$, and $\boldsymbol{\theta} = (\theta^{(1)}, \ldots, \theta^{(d)}) \in \mathcal{S}_p^{d,+}$, we define the vector

$$
\boldsymbol{v}_{j,y}^{p,\boldsymbol{\theta}} := e^{-y}((\theta^{(j)})^p - 1 + e^{py})^{1/p}\boldsymbol{e}_j + \sum_{k\neq j} e^{-y}\theta^{(k)}\boldsymbol{e}_k, \tag{5.12}
$$

where as usual, $\{\boldsymbol{e}_j : 1 \le j \le d\}$ denotes the standard orthonormal basis of $\mathbb{R}^d$.

Remark 5.4.3 *For $\boldsymbol{\theta} = (\theta^{(1)}, \theta^{(2)}) \in \mathcal{S}_1^{2,+}$ and $y \in \mathbb{R}$, using the fact that $\theta^{(1)} + \theta^{(2)} = 1$, one gets*

$$\boldsymbol{v}_{1,y}^{1,\boldsymbol{\theta}} = (1-(1-\theta^{(1)})e^{-y}, e^{-y}\theta^{(2)})^T = (1-e^{-y}\theta^{(2)}, e^{-y}\theta^{(2)})^T = (\theta^{(1)} + (1-e^{-y})\theta^{(2)}, e^{-y}\theta^{(2)})^T;$$

$$\boldsymbol{v}_{2,y}^{1,\boldsymbol{\theta}} = (e^{-y}\theta^{(1)}, 1-(1-\theta^{(2)})e^{-y})^T = (e^{-y}\theta^{(1)}, 1-e^{-y}\theta^{(1)})^T = (e^{-y}\theta^{(1)}, \theta^{(2)} + (1-e^{-y})\theta^{(1)})^T,$$

We now state the result of Theorem 5.4.2 in the case of $\|\cdot\| = \|\cdot\|_p$, for $p \geq 1$:

Corollary 5.4.4 *Fix $p \geq 1$. Let $(\xi, \Xi) = (\xi_t, \Xi_t)_{t\geq 0}$ be the underlying MAP of Z from (5.2) with respect to $\|\cdot\|_p$. The jump kernel of (ξ, Ξ) is given, for $(\boldsymbol{\phi}, y) \in \mathcal{S}_p^{d,+} \times \mathbb{R}$ and $\boldsymbol{\theta} = (\theta^{(1)}, \dots, \theta^{(d)}) \in \mathcal{S}_p^{d,+}$, by*

$$\begin{aligned} &L(\boldsymbol{\theta}, d\boldsymbol{\phi}, dy) \\ &= \sum_{j=1}^{d} (c_1^{(j)} \mathbb{1}_{(0,\infty)}(y) + c_2^{(j)} \mathbb{1}_{(\frac{1}{p}\log(1-(\theta^{(j)})^p),0)}(y)) \delta_{\boldsymbol{v}_{j,y}^{p,\boldsymbol{\theta}}}(d\boldsymbol{\phi}) \frac{e^{py}(e^{py}-1+(\theta^{(j)})^p)^{\frac{1-p}{p}}}{|(e^{py}-1+(\theta^{(j)})^p)^{1/p} - \theta^{(j)}|^{1+\alpha}} dy. \end{aligned}$$

Proof One can rewrite the statement of Theorem 5.4.2 (with $\|\cdot\| = \|\cdot\|_p$) thus: for every $\boldsymbol{\theta} \in \mathcal{S}_p^{d,+}$, $t \geq 0$ and positive bounded measurable function f on $\mathbb{R} \times \mathbb{R} \times \mathcal{S}_p^{d,+} \times \mathcal{S}_p^{d,+}$,

$$\begin{aligned} &\mathbb{E}_{(0,\boldsymbol{\theta})}\Big[\sum_{s\leq t} f(\xi_{s-}, \Delta\xi_s, \Xi_{s-}, \Xi_s)\mathbb{1}_{\{\Delta\xi_s \neq 0\}}\Big] \\ &= \mathbb{E}_{(0,\boldsymbol{\theta})}\Bigg[\int_0^{t\wedge\zeta} \Bigg\{ \sum_{j=1}^{d} c_1^{(j)} \int_0^\infty f(\xi_s, \log\|\Xi_s + w\boldsymbol{e}_j\|_p, \Xi_s, \arg(\Xi_s + w\boldsymbol{e}_j)) \cdot w^{-(1+\alpha)} dw \\ &\qquad + \sum_{j=1}^{d} c_2^{(j)} \int_{-\Xi_s^{(j)}}^0 f(s, \xi_s, \log\|\Xi_s + w\boldsymbol{e}_j\|_p, \Xi_s, \arg(\Xi_s + w\boldsymbol{e}_j)) \cdot (-w)^{-(1+\alpha)} dw \Bigg\} ds \Bigg]. \end{aligned}$$

We note that $s < \zeta$ and $w > -\Xi_s^{(j)}$ imply that $\Xi_s^{(k)} > 0$, for every $1 \leq k \leq d$, and $\Xi_s^{(j)} + w > 0$; and so in particular,

$$\|\Xi_s + w\boldsymbol{e}_j\|_p = (|\Xi_s^{(j)} + w|^p + \sum_{k\neq j} |\Xi_s^{(k)}|^p)^{1/p} = ((\Xi_s^{(j)} + w)^p + \sum_{k\neq j} (\Xi_s^{(k)})^p)^{1/p} = ((\Xi_s^{(j)} + w)^p + 1 - (\Xi_s^{(j)})^p)^{1/p},$$

where, in the last equality, we have used the fact that Ξ_s lies in the L^p unit-sphere. By plugging this expression into the previous expectation and then performing the following change-of-variables:

$$y = \frac{1}{p}\log\Big((\Xi_s^{(j)} + w)^p + 1 - (\Xi_s^{(j)})^p\Big), \;\; w = (e^{py} - 1 + (\Xi_s^{(j)})^p)^{1/p} - \Xi_s^{(j)}, \;\; \frac{dw}{dy} = e^{py}(e^{py} - 1 + (\Xi_s^{(j)})^p)^{\frac{1-p}{p}},$$

the result follows. ∎

Taking $p = 1$ significantly cleans up the expression for the above jump kernel:

Corollary 5.4.5 *Let $(\xi, \Xi) = (\xi_t, \Xi_t)_{t\geq 0}$ be the underlying MAP of Z from (5.2) with respect to $\|\cdot\|_1$. The jump kernel of (ξ, Ξ) is given, for $\boldsymbol{\theta} = (\theta^{(1)}, \dots, \theta^{(d)}) \in \mathcal{S}_1^{d,+}$, by*

$$L(\boldsymbol{\theta}, d\boldsymbol{\phi}, dy) = \sum_{j=1}^{d} (c_1^{(j)} \mathbb{1}_{(0,\infty)}(y) + c_2^{(j)} \mathbb{1}_{(\log(1-\theta^{(j)}),0)}(y)) \delta_{\boldsymbol{v}^{1,\boldsymbol{\theta}}_{j,y}}(d\boldsymbol{\phi}) \frac{e^y}{|e^y - 1|^{1+\alpha}} dy, \quad (\boldsymbol{\phi}, y) \in \mathcal{S}_1^{d,+} \times \mathbb{R}.$$

The dimension $d = 2$ version of the above result has a crispier formula for the jump kernel:

Lemma 5.4.6 *Let $(\xi, \Xi) = (\xi_t, \Xi_t)_{t\geq 0}$ be the underlying MAP of Z from (5.2) with respect to $\|\cdot\|_1$ in dimension $d = 2$. The jump kernel of (ξ, Ξ) is given, for $(\boldsymbol{\phi}, y) \in \mathcal{S}_1^{2,+} \times \mathbb{R}$, by*

$$\begin{aligned} L(\boldsymbol{\theta}, d\boldsymbol{\phi}, dy) = \Big(&(c_1^{(1)} \mathbb{1}_{(0,\infty)}(y) + c_2^{(1)} \mathbb{1}_{(\log\theta^{(2)},0)}(y)) \delta_{\boldsymbol{v}_1(y,\boldsymbol{\theta})}(d\boldsymbol{\phi}) \\ &+ (c_1^{(2)} \mathbb{1}_{(0,\infty)}(y) + c_2^{(2)} \mathbb{1}_{(\log\theta^{(1)},0)}(y)) \delta_{\boldsymbol{v}_2(y,\boldsymbol{\theta})}(d\boldsymbol{\phi}) \Big) \frac{\mathrm{e}^y}{|\mathrm{e}^y - 1|^{1+\alpha}} dy, \end{aligned}$$

where $\boldsymbol{\theta} = (\theta^{(1)}, \theta^{(2)})^T \in \mathcal{S}_1^{2,+}$, $\boldsymbol{v}_1(y, \boldsymbol{\theta}) := (1 - \mathrm{e}^{-y}\theta^{(2)}, \mathrm{e}^{-y}\theta^{(2)})^T$, $\boldsymbol{v}_2(y, \boldsymbol{\theta}) := (\mathrm{e}^{-y}\theta^{(1)}, 1 - \mathrm{e}^{-y}\theta^{(1)})^T$, and $\delta_{\boldsymbol{x}}$, for any $\boldsymbol{x} \in \mathbb{R}^2$, denotes the classic Dirac measure on $\mathbb{R}^2$.

5.4.3 Killing Rate Function of the Underlying MAP of a 2-dimensional Stable Process Killed Upon Exiting a Cone in the Positive Quadrant of $\mathbb{R}^2$

As the title of this subsection suggests, we will be working in dimension $d = 2$. We would like to investigate how the killing rate of the MAP (ξ, Ξ) from last section is related to the norm even further by studying some more specific and interesting instances of our multi-dimensional stable process Z from (5.2) by varying $D = [0, \infty)^2$ therein, as well as the norm for which (ξ, Ξ) is with respect to. In particular, we are interested in a more "general" cone in $\mathbb{R}^2$ than the first quadrant we had explored in our last section. We observe that the first quadrant of $\mathbb{R}^2$, $D = [0, \infty)^2$, takes the following form in polar coordinates:

$$D = \{(r, \theta) \in [0, \infty) \times [0, 2\pi) : 0 \leq r < \infty, \theta \in [0, \frac{\pi}{2}]\}.$$

And so a natural way of formulating a more general cone D in $\mathbb{R}^2$ is by allowing for θ to range between two arbitrary angles ϕ_1 and ϕ_2 in $[0, \frac{\pi}{2}]$. More precisely, this cone will be given by the

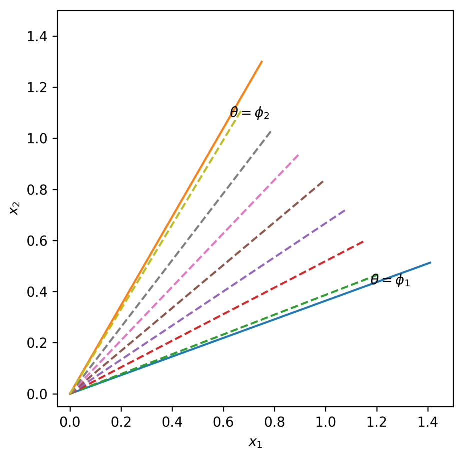


Figure 5.1: A general "polar cone" in the first orthant of $\mathbb{R}^2$.

following set written in polar coordinates (illustrated in Figure 5.1):

$$D = \{(r, \theta) \in [0, \infty) \times [0, 2\pi) : 0 \leq r < \infty, \theta \in [\phi_1, \phi_2]\}, \quad 0 \leq \phi_1 < \phi_2 \leq \frac{\pi}{2}. \tag{5.13}$$

We should note that it is thanks to the *"conic"* nature of the above-defined set D that the self-similarity property of X from (5.1) is preserved by the process $Z = (Z_t)_{t \geq 0}$,

$$Z_t = X_t \mathbb{1}_{\{t < \tau^D\}}, \qquad t \geq 0,$$

where $\tau^D := \inf\{s > 0 : X_s \notin D\}$, with D given in (5.13). By the "conic" nature of D we mean the fact that D satisfies

$$\{c\boldsymbol{x} : \boldsymbol{x} \in D\} =: cD = D.$$

We then seek, like in the previous subsection, to derive the killing rate function, $q : \mathcal{S}_2^{2,+} \to [0, \infty)$, of the underlying MAP (ξ, Ξ) (with respect to the L^2 norm, $\|\cdot\|_2$, on $\mathbb{R}^2$) of the ssMp Z (in dimension $d = 2$) from (5.2), with D now being the set described in polar coordinates in (5.13). The reason for the particular choice of the L^2 norm should now become more apparent: it is by far the most natural given the nature of the problem because by viewing the position of the process Z in the plane being composed by (a) the angle it makes with the positive axis and (b) its length from the origin, and bearing in mind that we are studying the first time the argument process of Z, $\arg(Z) = (\arg(Z_t))_{t \geq 0}$, falls outside a given range of angles, imposing its underlying modulator process, Ξ, to live in the (Euclidean) unit circle (which can only be achieved if Ξ is with respect to the L^2 norm) is the most suitable to this end not least because its position can conveniently be expressed in polar coordinates. What is more, this choice of norm, as we will see, brings about a very explicit and clean killing rate function.

There is a major difference in the setup of this problem and that of Section 5.4.1. This difference stems from the fact that at time τ^D, with D now being the more "general" cone described in (5.13), X can jump in four different possible directions: up, down, left, right; whereas in Section 5.4.1, when D was the entire first quadrant, at time τ^D, X had only two possible directions for its exit jump: down and left. As one may predict, this difference introduces several trigonometric problems.

We begin by performing similar calculations to those from Section 5.4.1 involving the compensation formula to obtain, for a non-negative measurable function H and $\boldsymbol{\theta} \in \mathcal{S}_2^{2,+} \cap D$,

$$\mathbb{E}_{(0,\boldsymbol{\theta})}[H(\zeta)] = \mathbb{E}_{(0,\boldsymbol{\theta})}\left[\int_0^\infty \int_{\mathbb{R}^2\setminus\{\mathbf{0}_2\}} H(v)\mathbb{1}_{\{\Xi_v\in D,\boldsymbol{z}+\Xi_v\notin D\}}\mathbb{1}_{\{v<\zeta\}}\Pi(d\boldsymbol{z})dv\right], \tag{5.14}$$

where we recall that $\zeta := \int_0^{\tau^D} \|X_r\|_2^{-\alpha} dr$ is the lifetime of the underlying MAP, (ξ, Ξ), of this subsection, and Π is the Lévy measure of X (from Lemma 5.2.2).

To proceed we consider the position in the plane of the vector Ξ_v, $v \geq 0$, in polar coordinates (as opposed to Cartesian ones). We thus perform the following change of variables:

$$(\Xi_v^{(1)}, \Xi_v^{(2)}) = (r_v \cos\theta_v, r_v \sin\theta_v), \qquad v \geq 0, \tag{5.15}$$

where r_v denotes the (Euclidean) distance of Ξ_v from the origin, and θ_v denotes the angle Ξ_v makes with the positive axis. Of course, since Ξ_v lies in the unit circle, we have $r_v \equiv 1$ for every $v \geq 0$. Therefore, (5.15) becomes

$$(\Xi_v^{(1)}, \Xi_v^{(2)}) = (\cos\theta_v, \sin\theta_v), \qquad v \geq 0. \tag{5.16}$$

On the event $\{\Xi_v \in D\}$, the only measurable subsects, C_i, $i \in \{1,2,3,4\}$, of $\mathbb{R}^2 \setminus \{\mathbf{0}_2\}$ that are not Π-null, such that $\boldsymbol{z} + \Xi_v \notin D$ and $\boldsymbol{z} \in C_i$, are of the form

$$C_1 = \{0\} \times (-\infty, \kappa_1),\ \ C_2 = \{0\} \times (\kappa_2, \infty),\ \ C_3(-\infty, \kappa_3) \times \{0\},\ \ C_4 = (\kappa_4, \infty) \times \{0\}, \tag{5.17}$$

where the κ_i are (random) variables to be determined that depend on the process Ξ (and hence the process $\theta = (\theta_v)_{v\geq 0}$ from (5.15)) as well as the angles ϕ_1 and ϕ_2 from (5.13).

The above makes mathematically precise the statement made previously regarding the only four possible directions X can jump in to make its first exit from D. Determining these four different κ_i are the four trigonometric problems we need to solve in order to work out the killing rate, q, of this subsection.

To find the appropriate κ_1 so that on the event $\{\Xi_v \in D\}$ we have $\boldsymbol{z}+\Xi_v \notin D$ for $\boldsymbol{z} \in \{0\}\times(-\infty, \kappa_1)$, one needs to determine the vertical distance from the point (in polar coordinates) $(1, \theta_v)$, where $\theta_v \in [\phi_1, \phi_2]$, to the line (again in polar coordinates) $\theta = \phi_1$. We are therefore analysing the case when X first exits the cone by jumping vertically down. One can check that the solution, κ_1, to this trigonometric problem is

$$\kappa_1 = -\frac{\sin(\theta_v - \phi_1)}{\cos\phi_1}.$$

We now analyze the case when X's first exit from D happens with a vertical jump upwards. The solution, κ_2, to this analogous trigonometric problem is

$$\kappa_2 = \frac{\sin(\phi_2 - \theta_v)}{\cos\phi_2}.$$

The analogous problem for the case when X's first exit from D happens by an horizontal jump to the left has solution, κ_3, given by

$$\kappa_3 = -\frac{\sin(\phi_2 - \theta_v)}{\sin\phi_2}.$$

Finally, the analogous problem for the case when X's first exit from D happens by an horizontal jump to the right has solution, κ_4, given by

$$\kappa_4 = \frac{\sin(\theta_v - \phi_1)}{\sin\phi_1}.$$

Going back to our calculation from (5.14) we obtain

$$\begin{aligned}
\mathbb{E}_{(0,\boldsymbol{\theta})}[H(\zeta)] = {} & \mathbb{E}_{(0,\boldsymbol{\theta})}\left[\int_0^\zeta \int_{\{0\}\times(-\infty, -\frac{\sin(\theta_v-\phi_1)}{\cos\phi_1})} H(v)\Pi(d\boldsymbol{z})dv\right] \\
& + \mathbb{E}_{(0,\boldsymbol{\theta})}\left[\int_0^\zeta \int_{\{0\}\times(\frac{\sin(\phi_2-\theta_v)}{\cos\phi_2}, \infty)} H(v)\Pi(d\boldsymbol{z})dv\right] \\
& + \mathbb{E}_{(0,\boldsymbol{\theta})}\left[\int_0^\zeta \int_{(-\infty, -\frac{\sin(\phi_2-\theta_v)}{\sin\phi_2})\times\{0\}} H(v)\Pi(d\boldsymbol{z})dv\right] \\
& + \mathbb{E}_{(0,\boldsymbol{\theta})}\left[\int_0^\zeta \int_{(\frac{\sin(\theta_v-\phi_1)}{\sin\phi_1}, \infty)\times\{0\}} H(v)\Pi(d\boldsymbol{z})dv\right].
\end{aligned}$$

Then, since Π takes the form

$$\Pi(d\boldsymbol{z}) = \Pi(d(z_1, z_2)) = \delta_0(dz_2)\Pi^{(1)}(dz_1) + \delta_0(dz_1)\Pi^{(2)}(dz_2), \qquad \boldsymbol{z} = (z_1, z_2) \in \mathbb{R}^2 \setminus \{\mathbf{0}_d\},$$

we can further evaluate the above expectation and obtain

$$\mathbb{E}_{(0,\boldsymbol{\theta})}[H(\zeta)] = \mathbb{E}_{(0,\boldsymbol{\theta})}\left[\int_0^\zeta H(v)\Pi^{(2)}\Big((-\infty, -\frac{\sin(\theta_v - \phi_1)}{\cos\phi_1})\Big)dv\right]$$

$$
\begin{aligned}
&+\mathbb{E}_{(0,\boldsymbol{\theta})}\Big[\int_0^{\zeta} H(v)\Pi^{(2)}\Big(\big(\frac{\sin(\phi_2-\theta_v)}{\cos\phi_2},\infty\big)\Big)dv\Big]\\
&+\mathbb{E}_{(0,\boldsymbol{\theta})}\Big[\int_0^{\zeta} H(v)\Pi^{(1)}\Big(\big(-\infty,-\frac{\sin(\phi_2-\theta_v)}{\sin\phi_2}\big)\Big)dv\Big]\\
&+\mathbb{E}_{(0,\boldsymbol{\theta})}\Big[\int_0^{\zeta} H(v)\Pi^{(1)}\Big(\big(\frac{\sin(\theta_v-\phi_1)}{\sin\phi_1},\infty\big)\Big)dv\Big].
\end{aligned}
$$

By some basic properties of Lévy measures of 1-dimensional stable processes, for $z>0$ and $i=1,2$,

$$
\Pi^{(i)}((z,\infty))=\frac{c_1^{(i)}}{\alpha}z^{-\alpha}\quad\text{and}\quad\Pi^{(i)}((-\infty,-z))=\frac{c_2^{(i)}}{\alpha}z^{-\alpha},
$$

where $c_1^{(i)}=\frac{\Gamma(1+\alpha)\sin(\pi\alpha\rho_i)}{\pi}$ and $c_2^{(i)}=\frac{\Gamma(1+\alpha)\sin(\pi\alpha(1-\rho_i))}{\pi}$. Thus,

$$
\begin{aligned}
\mathbb{E}_{(0,\boldsymbol{\theta})}[H(\zeta)]=&\frac{c_2^{(2)}}{\alpha}\mathbb{E}_{(0,\boldsymbol{\theta})}\Big[\int_0^{\zeta} H(v)\Big(\frac{\sin(\theta_v-\phi_1)}{\cos\phi_1}\Big)^{-\alpha}dv\Big]\\
&+\frac{c_1^{(2)}}{\alpha}\mathbb{E}_{(0,\boldsymbol{\theta})}\Big[\int_0^{\zeta} H(v)\Big(\frac{\sin(\phi_2-\theta_v)}{\cos\phi_2}\Big)^{-\alpha}dv\Big]\\
&+\frac{c_2^{(1)}}{\alpha}\mathbb{E}_{(0,\boldsymbol{\theta})}\Big[\int_0^{\zeta} H(v)\Big(\frac{\sin(\phi_2-\theta_v)}{\sin\phi_2}\Big)^{-\alpha}dv\Big]\\
&+\frac{c_1^{(1)}}{\alpha}\mathbb{E}_{(0,\boldsymbol{\theta})}\Big[\int_0^{\zeta} H(v)\Big(\frac{\sin(\theta_v-\phi_1)}{\sin\phi_1}\Big)^{-\alpha}dv\Big].
\end{aligned}
$$

We can further evaluate the above by exploiting the following three known trigonometric identities

$$
\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta; \tag{5.18}
$$

$$
\cos(\arcsin x)=\sqrt{1-x^2}; \tag{5.19}
$$

$$
\sin(\arccos x)=\sqrt{1-x^2}, \tag{5.20}
$$

as follows:

$$
\begin{aligned}
\frac{\sin(\theta_v-\phi_1)}{\cos\phi_1}&=\frac{\sin\theta_v\cos\phi_1-\cos\theta_v\sin\phi_1}{\cos\phi_1} && \text{(by (5.18))}\\
&=\sin\theta_v-\cos\theta_v\tan\phi_1\\
&=\Xi_v^{(2)}-\cos\Big(\arcsin(\Xi_v^{(2)})\Big)\tan\phi_1 && \text{(by (5.16))}\\
&=\Xi_v^{(2)}-\tan\phi_1\sqrt{1-(\Xi_v^{(2)})^2} && \text{(by (5.19))}\\
&=\Xi_v^{(2)}-\Xi_v^{(1)}\tan\phi_1 && (\Xi_v\text{ lives in the unit circle});
\end{aligned}
$$

and in like fashion,

$$\frac{\sin(\phi_2-\theta_v)}{\cos\phi_2} = \Xi_v^{(1)}\tan\phi_2 - \Xi_v^{(2)};$$

$$\frac{\sin(\phi_2-\theta_v)}{\sin\phi_2} = \Xi_v^{(1)} - \Xi_v^{(2)}\cot\phi_2;\ \text{and}$$

$$\frac{\sin(\theta_v-\phi_1)}{\sin\phi_1} = \Xi_v^{(2)}\cot\phi_1 - \Xi_v^{(1)}.$$

Thus, by taking $H(v) = \mathbb{1}_{(0,T)}(v)$ with $T > 0$ arbitrarily small, we get

$$\begin{aligned}\mathbb{P}_{(0,\boldsymbol{\theta})}(\zeta < T) = &\frac{c_2^{(2)}}{\alpha}\int_0^T \mathbb{E}_{(0,\boldsymbol{\theta})}\left[(\Xi_v^{(2)} - \Xi_v^{(1)}\tan\phi_1)^{-\alpha}\right]dv \\ &+ \frac{c_1^{(2)}}{\alpha}\int_0^T \mathbb{E}_{(0,\boldsymbol{\theta})}\left[(\Xi_v^{(1)}\tan\phi_2 - \Xi_v^{(2)})^{-\alpha}\right]dv \\ &+ \frac{c_2^{(1)}}{\alpha}\int_0^T \mathbb{E}_{(0,\boldsymbol{\theta})}\left[(\Xi_v^{(1)} - \Xi_v^{(2)}\cot\phi_2)^{-\alpha}\right]dv \\ &+ \frac{c_1^{(1)}}{\alpha}\int_0^T \mathbb{E}_{(0,\boldsymbol{\theta})}\left[(\Xi_v^{(2)}\cot\phi_1 - \Xi_v^{(1)})^{-\alpha}\right]dv.\end{aligned}$$

Finally, by computing $\lim_{T\to 0}\frac{\mathbb{P}_{(0,\boldsymbol{\theta})}(\zeta<T)}{T}$ – which is nothing other than the derivative of the function $\mathbb{P}_{(0,\boldsymbol{\theta})}(\zeta < T)$ with respect to T evaluated at 0 – one obtains the desired killing rate:

Lemma 5.4.7 *The killing rate, $q : \mathcal{S}_2^{2,+} \to [0,\infty)$ of the underlying MAP $(\xi,\Xi) = (\xi_t,\Xi_t)_{t\geq 0}$ of the ssMp Z (with $d = 2$ and D given in (5.13)) from (5.2) with respect to the L^2 norm is precisely*

$$\begin{aligned}q(\boldsymbol{x}) = \mathbb{1}_D(\boldsymbol{x})\Big[&\frac{c_2^{(2)}}{\alpha}(x_2 - x_1\tan\phi_1)^{-\alpha} + \frac{c_1^{(2)}}{\alpha}(x_1\tan\phi_2 - x_2)^{-\alpha} \\ &+ \frac{c_2^{(1)}}{\alpha}(x_1 - x_2\cot\phi_2)^{-\alpha} + \frac{c_1^{(1)}}{\alpha}(x_2\cot\phi_1 - x_1)^{-\alpha}\Big], \quad \boldsymbol{x} = (x_1,x_2) \in \mathcal{S}_2^{2,+},\end{aligned}$$

where $c_1^{(i)} = \frac{\Gamma(1+\alpha)\sin(\pi\alpha\rho_i)}{\pi}$ and $c_2^{(i)} = \frac{\Gamma(1+\alpha)\sin(\pi\alpha(1-\rho_i))}{\pi}$.

Remark 5.4.8 *Note how Lemma 5.4.1 with $d = 2$ and $\|\cdot\| = \|\cdot\|_2$ yields, for $\boldsymbol{x} = (x_1,x_2) \in \mathcal{S}_2^{2,+}$,*

$$q(\boldsymbol{x}) = \frac{c_2^{(1)}}{\alpha}x_1^{-\alpha} + \frac{c_2^{(2)}}{\alpha}x_2^{-\alpha}. \tag{5.21}$$

We recall that the above killing rate is for the case when D is the entire first orthant (i.e., the first quadrant of $\mathbb{R}^2$), and therefore, for when $\phi_1 = 0$ and $\phi_2 = \frac{\pi}{2}$ in (5.13). This agrees (as it should) with the killing rate displayed in Lemma 5.4.7, since $\tan 0 = 0$, $\cot 0 = \infty$, $\tan\frac{\pi}{2} = \infty$ *and* $\cot\frac{\pi}{2} = 0$.

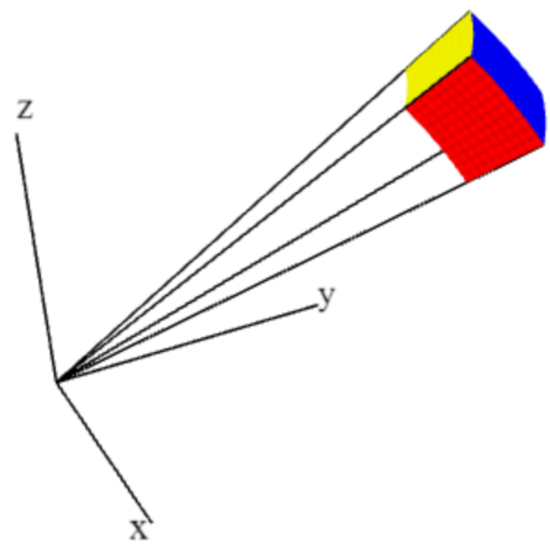


Figure 5.2: A general "spherical cone" in the first orthant of $\mathbb{R}^3$.

5.4.4 Killing Rate Function of the Underlying MAP of a 3-dimensional Stable Process Killed Upon Exiting a Cone in the Positive Orthant of $\mathbb{R}^3$

As one may guess, the methods from the previous subsection can be carried over to the three-dimensional setting where we can now consider, not a "polar cone" (as described in (5.13)), but a natural three-dimensional analogue that is the "spherical" cone D which is described in the classical spherical coordinate system as

$$D = \{(r, \theta, \phi) \in [0, \infty) \times [0, 2\pi) \times [0, \pi] : \theta \in (\theta_1, \theta_2), \phi \in (\phi_1, \phi_2)\}, \tag{5.22}$$

for some $0 \le \theta_1 < \theta_2 \le \frac{\pi}{2}$ and $0 \le \phi_1 < \phi_2 \le \frac{\pi}{2}$.

Note how the first orthant of $\mathbb{R}^3$ can be described in spherical coordinates by setting $\theta_1 = 0 = \phi_1$ and $\theta_2 = \frac{\pi}{2} = \phi_2$ in (5.22). In this section we are interested in deriving the killing rate function of the underlying MAP, $(\xi, \Xi) = (\xi_t, \Xi_t)_{t \ge 0}$, of the ssMp Z (in dimension $d = 3$ and D as described in (5.22) above) from (5.2) with respect to the L^2 norm on $\mathbb{R}^3$.

The nature of the problem of this section and the fact that the modulator Ξ is with respect to the L^2 norm means that it is both possible and conducive to express the position of Ξ in spherical coordinates:

$$(\Xi_v^{(1)}, \Xi_v^{(2)}, \Xi_v^{(3)}) = (r_v \cos\theta_v \sin\phi_v, r_v \sin\theta_v \sin\phi_v, r_v \cos\phi_v) = (\cos\theta_v \sin\phi_v, \sin\theta_v \sin\phi_v, \cos\phi_v), \tag{5.23}$$

for $v \geq 0$, where θ_v denotes the angle Ξ_v makes with the positive x-axis, ϕ_v denotes the angle Ξ_v makes with the positive z-axis, and r_v denotes the (Euclidean) distance of Ξ_v from the origin $\mathbf{0}_3$, which, since Ξ_v lives in the Euclidean unit sphere, is always equal to 1.

Following analogous steps to those from Section 5.4.3, we find that the only measurable subsets, C_i, $i \in \{1, \ldots, 6\}$, of $\mathbb{R}^3 \setminus \{\mathbf{0}_3\}$ that are not Π-null (recall that Π is the Lévy measure of X from Lemma 5.2.2) with the property that on the event $\{\Xi_v \in D\}$ we have $\boldsymbol{z} + \Xi_v \notin D$ for $z \in C$ are of the form

$$C_1 = \{0\}^2 \times (-\infty, \kappa_1), \;\; C_2 = \{0\}^2 \times (\kappa_2, \infty), \;\; C_3 = \{0\} \times (-\infty, \kappa_3) \times \{0\},$$

and

$$C_4 = \{0\} \times (\kappa_4, \infty) \times \{0\}, \;\; C_5 = (-\infty, \kappa_5) \times \{0\}^2, \;\; C_6 = (\kappa_6, \infty) \times \{0\}^2,$$

where the κ_i are (random) variables to be determined that depend on the process Ξ (and hence the processes $\theta = (\theta_v)_{v \geq 0}$ and $\phi = (\phi_v)_{v \geq 0}$ from (5.23)) as well as the angles θ_1, θ_2, ϕ_1 and ϕ_2 from (5.22).

The increased number of (analogous) random variables ρ_i, compared to the four from last section (recall (5.17)), reflects the increased number of ways in which Ξ_v can jump out of D on its first exit, as a result of Ξ_v now living in three-dimensional space. This means that we must now solve six (slightly more involved) trigonometric problems in order to work out the killing rate of this subsection.

To work out the appropriate κ_1 is to determine the vertical distance from the point – in spherical coordinates – $(1, \theta_v, \phi_v)$, where $\theta_v \in (\theta_1, \theta_2)$ and $\phi_v \in (\phi_1, \phi_2)$, to the bottom surface of the "cone" D. This is the scenario whereby X makes its first exit from D by a vertical jump downwards. By drawing appropriate sketches one will find that

$$\kappa_1 = \cot \phi_2 \sin \phi_v - \cos \phi_v.$$

The trigonometric problem of finding κ_2 concerns the scenario where X first exits D by a vertical jump upwards. One can check that

$$\kappa_2 = \cot \phi_1 \sin \phi_v - \cos \phi_v.$$

The trigonometric problem of finding κ_3 concerns the scenario where X first exits D by a jump in the direction of the negative y-axis. One can check that

$$\kappa_3 = \tan\phi_1 \cos\phi_v - \sin\theta_v \sin\phi_v.$$

The trigonometric problem of finding κ_4 concerns the scenario where X first exits D by a jump in the direction of the positive y-axis. One can check that

$$\kappa_4 = \tan\phi_2 \cos\phi_v - \sin\theta_v \sin\phi_v.$$

The trigonometric problem of finding κ_5 concerns the scenario where X first exits D by a jump in the direction of the negative x-axis. One can check that

$$\kappa_5 = \cot\theta_2 \sin\theta_v \sin\phi_v - \cos\theta_v \sin\phi_v.$$

The trigonometric problem of finding κ_6 concerns the scenario where X first exits D by a jump in the direction of the positive x-axis. One can check that

$$\kappa_6 = \cot\theta_1 \sin\theta_v \sin\phi_v - \cos\theta_v \sin\phi_v.$$

Using (5.23), it is not difficult to see that

$$\begin{aligned}
\kappa_1 &= \cot\phi_2 \sqrt{(\Xi_v^{(1)})^2 + (\Xi_v^{(2)})^2} - \Xi_v^{(3)};\\
\kappa_2 &= \cot\phi_1 \sqrt{(\Xi_v^{(1)})^2 + (\Xi_v^{(2)})^2} - \Xi_v^{(3)};\\
\kappa_3 &= \Xi_v^{(3)} \tan\phi_1 - \Xi_v^{(2)};\\
\kappa_4 &= \Xi_v^{(3)} \tan\phi_2 - \Xi_v^{(2)};\\
\kappa_5 &= \Xi_v^{(2)} \cot\theta_2 - \Xi_v^{(1)};\\
\kappa_6 &= \Xi_v^{(2)} \cot\theta_1 - \Xi_v^{(1)}.
\end{aligned}$$

Following analogous steps to those from the previous section, we obtain

Lemma 5.4.9 *The killing rate, $q : \mathcal{S}_2^{3,+} \to [0,\infty)$ of the underlying MAP $(\xi,\Xi) = (\xi_t,\Xi_t)_{t\geq 0}$ of* *the ssMp Z (with $d = 3$ and D given in (5.22)) from (5.2) with respect to the L^2 norm is precisely*

$$q(\boldsymbol{x}) = \mathbb{1}_D(\boldsymbol{x})\Big[\frac{c_2^{(3)}}{\alpha}(x_3 - \cot\phi_2\sqrt{x_1^2 + x_2^2})^{-\alpha} + \frac{c_1^{(3)}}{\alpha}(\cot\phi_1\sqrt{x_1^2 + x_2^2} - x_3)^{-\alpha}$$
$$+ \frac{c_2^{(2)}}{\alpha}(x_2 - x_3\tan\phi_1)^{-\alpha} + \frac{c_1^{(2)}}{\alpha}(x_3\tan\phi_2 - x_2)^{-\alpha}$$
$$+ \frac{c_2^{(1)}}{\alpha}(x_1 - x_2\cot\theta_2)^{-\alpha} + \frac{c_1^{(1)}}{\alpha}(x_2\cot\theta_1 - x_1)^{-\alpha}\Big], \quad \boldsymbol{x} = (x_1, x_2, x_3) \in \mathcal{S}_2^{3,+},$$

where $c_1^{(i)} = \frac{\Gamma(1+\alpha)\sin(\pi\alpha\rho_i)}{\pi}$ *and* $c_2^{(i)} = \frac{\Gamma(1+\alpha)\sin(\pi\alpha(1-\rho_i))}{\pi}$.

Remark 5.4.10 *Note how Lemma 5.4.1 with $d = 3$ and $\|\cdot\| = \|\cdot\|_2$ yields, for $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathcal{S}_2^{3,+}$,*

$$q(\boldsymbol{x}) = \frac{c_2^{(1)}}{\alpha}x_1^{-\alpha} + \frac{c_2^{(2)}}{\alpha}x_2^{-\alpha} + \frac{c_2^{(3)}}{\alpha}x_3^{-\alpha}. \tag{5.24}$$

We recall that the above killing rate is for the case when D is the entire first orthant of $\mathbb{R}^3$, and therefore, for when $\phi_1 = 0 = \theta_1$ and $\phi_2 = \frac{\pi}{2} = \theta_2$ in (5.22). This agrees (as it should) with the killing rate displayed in Lemma 5.4.9, since $\tan 0 = 0$, $\cot 0 = \infty$, $\tan\frac{\pi}{2} = \infty$ *and* $\cot\frac{\pi}{2} = 0$.

Chapter 6

Reflected Stable Processes in the Positive Orthant of $\mathbb{R}^d$

6.1 Introduction

Let $X = (X_t)_{t\geq 0}$ be the d-dimensional stable process from (5.1), where we take the $(X^{(i)})$, $1 \leq i \leq d$, therein to be iid (one-dimensional) α-stable processes, each with positivity parameter ρ, that is, $\rho_i = \rho \in (0,1)$ for every $1 \leq i \leq d$; and we abide to the notation of our Section 5.2. In this thesis chapter we study two types of reflected ssMps:

- for $\alpha \in (0,2)$ and $\rho = \frac{1}{2}$, the d-dimensional process $\hat{X} = (\hat{X}_t)_{t\geq 0}$ defined by

$$\hat{X}_t = (|X_t^{(1)}|, \ldots, |X_t^{(d)}|), t \geq 0, \tag{6.1}$$

  where $|\cdot|$ denotes the usual absolute value in $\mathbb{R}$. We assume that the process $\hat{X}$ is killed and absorbed at its first hitting time of $\mathbf{0}_d \in \mathbb{R}^d$. This process is the subject of study in Section 6.2. More specifically, the dimension $d = 1$ is dealt with in Section 6.2.1, and is essentially a review/summary of some of the findings in [39]. Our novel dimension $d > 1$ setting is dealt with in Section 6.2.2.

- for $\alpha \in (1,2)$ and $\rho \geq 0$ satisfying $\alpha(1-\rho) = 1$, the d-dimensional process $R = (R_{t\geq 0})_{t\geq 0}$ defined by

$$R_t = (X_t^{(1)} - (0 \wedge \underline{X}_t^{(1)}), \ldots, X_t^{(d)} - (0 \wedge \underline{X}_t^{(d)})), \qquad t \geq 0, \tag{6.2}$$

  where $\underline{X}_t^{(i)} \coloneqq \inf_{s\leq t} X_s^{(i)}$ denotes the running infimum of $X^{(i)}$. We assume that the process R

is killed and absorbed at its first hitting time of $\mathbf{0}_d \in \mathbb{R}^d$. This process is the subject of study in Section 6.3.

It can be shown that $\hat{X}$ from the first bullet point above is an ssMp. It therefore possesses a (norm-dependent) underlying MAP, $(\hat{\xi}, \hat{\Xi}) = (\hat{\xi}_t, \hat{\Xi}_t)_{t\geq 0}$. Like in the one-dimensional version of this particular example (cf., [39]) relating to underlying Lévy processes via the classic Lamperti transform (of which we provide a brief summary in Section 6.2.1), it would be natural to also expect that the underlying MAP $(\hat{\xi}, \hat{\Xi}) = (\hat{\xi}_t, \hat{\Xi}_t)_{t\geq 0}$ of $\hat{X}$ is interconnected with (ξ, Ξ), the underlying MAP of the process $Z = (Z_t)_{t\geq 0}$ from last section. Indeed, it is clear that up to, but not including, the killing time, ζ, of (ξ, Ξ), the paths of these two MAPs coincide, but then at time ζ, instead of being sent to the cemetery state, $(\hat{\xi}, \hat{\Xi})$ has to make a "corrective" jump to stay in the orthant. By the strong Markov property, this procedure is then repeated ad infinitum.

It can also be shown that R from the second bullet point above is an ssMp. As such, up to its first hitting time of $\mathbf{0}_d \in \mathbb{R}^d$, it possesses a (norm-dependent) underlying MAP, $(\xi^R, \Xi^R) = (\xi_t^R, \Xi_t^R)_{t\geq 0}$. We note that, besides the notion of reflection being completely different, a major difference between R and $\hat{X}$ is that the former's reflection occurs in a smooth/continuous fashion – and we have already explained in the previous paragraph that $\hat{X}$, instead, always makes a jump at its reflection time. This difference stems from that of our choice of parameters α and ρ for the two processes. Now, the derivation of the generator of (ξ^R, Ξ^R) is done in a couple of steps: in Section 6.3.1, we first derive the generator of the process R from (6.2) in dimension $d = 1$; in Section 6.3.2, we extend this (type of) result to dimension $d > 1$; and finally in Section 6.3.3, through basic calculus and the Volkonskii formula, we "convert" the generator obtained in the previously mentioned section to the one that the underlying MAP (ξ^R, Ξ^R) must possess.

We take the time in this introductory section to also briefly discuss the long-time behaviour of the ssMps $\hat{X}$ and R. In dimension $d = 1$ definitive statements regarding their absorption at the origin can readily be made. More specifically in dimension one the question of whether or not $\hat{X}$ gets absorbed at zero in finite time is the same as asking whether a symmetric one-dimensional α-stable process hits zero in finite time. The answer to the latter is well-known: if $\alpha > 1$, then the stable process hits points almost surely, and therefore hits zero in a finite time almost surely; if $\alpha \leq 1$, then it almost surely never hits points, and therefore almost surely never hits zero. As for whether the process R in dimension $d = 1$ gets absorbed at zero in finite time, it is a classical result that such reflected processes almost surely return to the origin infinitely often. Hence the hitting time of zero is almost surely finite.

The above discussion in dimension $d > 1$ is more involved and less definitive statements can be made. The question of whether $\hat{X}$ gets absorbed at $\mathbf{0}_d$ in finite time is then the question of whether d independent symmetric one-dimensional α-stable processes all hit zero simultaneously in finite time. As we discussed above, in the case when $\alpha \le 1$, this almost surely never happens, that is, $\hat{X}$ almost surely never gets absorbed at $\mathbf{0}_d$. As for the case when $\alpha > 1$, we saw above that each of the coordinates will almost surely hit zero in finite time; whether or not they will do so simultaneously is then the question. One possible method for investigating this question – and the analogous one for the process R in dimension $d > 1$ – is through the theory of independent regenerative sets / Markov random sets and their intersection; see, for example, [6], [31], [30].

6.2 Reflected Symmetric Stable Processes

The choice of parameters α and ρ for this section (given in the introduction) ensure that each $X^{(i)}$, and therefore X, is symmetric. Before we go ahead and study the underlying MAP of $\hat{X}$, which we have denoted, and will be consistently denoting, by $(\hat{\xi}, \hat{\Xi})$, we first succinctly explore in Section 6.2.1 the process $\hat{X}$ in dimension $d = 1$ (which is a pssMp) and its underlying Lévy process via Lamperti's transform. We do this as a way of gently introducing and providing the heuristics for the main novel problem we tackle in Section 6.2.2 regarding $\hat{X}$ in dimension $d > 1$.

We introduce some preliminary notations for the entire Section 6.2: we denote the law of $(\hat{\xi}, \hat{\Xi})$ initiated from $(\rho, \theta) \in \mathbb{R} \times \mathcal{S}^{d,+}_{\|\cdot\|}$ by $\mathbb{P}_{(\rho,\theta)}$. We observe that the law of X issued from $X_0 = \boldsymbol{x} \in [0,\infty)^d$, which we shall denote by $P_{\boldsymbol{x}}$, is the image measure of $\mathbb{P}_{(\|\boldsymbol{x}\|, \arg(\boldsymbol{x}))}$ under the (analogous) transformation given in (4.2). Additionally, as is customary, we shall be denoting the respective expectation operators of the two aforementioned probability measures by $\mathbb{E}_{(\rho,\theta)}$ and $E_{\boldsymbol{x}}$, respectively.

6.2.1 Underlying Lévy Process of a Reflected 1-dimensional Symmetric Stable Process

To simplify notation, in this subsection we shall write X instead of $X^{(1)}$. Thus, we define $\hat{X} = (\hat{X}_t)_{t\ge 0}$ from (6.1) with $d = 1$ by $\hat{X}_t = |X_t|$, $t \ge 0$. And, just for this subsection, we define the process $Z = (Z_t)_{t\ge 0}$ by $Z_t = X_t \mathbb{1}_{\{t<\tau_0^-\}}$, $t \ge 0$, where $\tau_0^- := \inf\{t > 0 : X_t < 0\}$. We shall denote by $\hat{\xi} = (\hat{\xi}_t)_{t\ge 0}$ the underlying Lévy process of $\hat{X}$.

The process Z defined in the above paragraph and its underlying Lévy process ξ^* were explored in Section 5.3.1. In that section we had derived the explicit killing rate of ξ^*, denoted by q^*, via the compensation formula. Let us denote the lifetime of ξ^* by ζ^*; in which case, $\zeta^* \overset{\mathrm{d}}{=} \mathrm{Exp}(q^*)$.

A connection between the Lévy processes ξ^* and $\hat{\xi}$ is naturally expected. Indeed, such is the case, and this makes up the content of Proposition 5.20 in Chapter 5.7 of [37]. We summarize the main idea in the remainder of this subsection.

Up to, but not including, the killing time ζ^* it is clear that $\hat{\xi}$ has the same dynamics as ξ^*. At time ζ^*, ξ^* gets sent to its cemetery state $-\infty$; and correspondingly, at time τ_0^-, Z gets absorbed at 0 and X makes a jump below 0. It is at this time instance when $\hat{X}$ first reflects X, via a jump, back up in the positive y-axis. By the strong Markov property of $\hat{\xi}$, the restarted process $(\hat{\xi}_{\zeta^*+t})_{t\geq 0}$ will also experience the corresponding jump at an independent and exponentially-distributed (with parameter q^*) time. Repeated application of the strong Markov property in this way leads us to conclude that a portion of the jumps that $\hat{\xi}$ makes occur at the jump times of a Poisson process with intensity q^*. Indeed, our previous argument reveals independent and q^*-exponentially distributed waiting times for the jumps of $\hat{\xi}$ that correspond, albeit through a transform (of time), to the reflection times of $\hat{X}$. This establishes the existence of a process $\xi^{C_2} = (\xi^{C_2})_{t\geq 0}$ in the sum-decomposition of $\hat{\xi}$ that jumps at the jump times of a Poisson process of intensity q^*, or said another way, which jumps at the rate which a Poisson process does, but whose jump sizes may depend on the position of $\hat{\xi}$ just prior to the jumps.

As explained above, we have $\hat{\xi}_t \overset{\mathrm{d}}{=} \xi^*_t$ for every $t < \zeta^*$. Moreover, $\xi_t^{C_2} = 0$ for $t < \zeta^*$, since ξ^{C_2} makes its first jump at time ζ^*, by construction. This leads us to conclude that in its sum-decomposition $\hat{\xi}$ must additionally consist of a process ξ^L coming directly from ξ^*; but recognise how this ξ^L could not possibly be ξ^* itself, for the latter gets sent to its cemetery state at time ζ^*, which would then imply that $\hat{\xi}$ does as well; which cannot be, since it is the underlying Lévy process of $\hat{X}$. Therefore, by the strong Markov property of $\hat{\xi}$, the process ξ^L is necessarily the process ξ^* but with the effect of killing removed. The process ξ^L is thus a Lévy process with characteristic exponent $\Psi^* - q^*$, where Ψ^* denotes the characteristic exponent of ξ^*. We therefore have the following path sum-decomposition of $\hat{\xi}$:

$$\hat{\xi} = \xi^L + \xi^{C_2}. \tag{6.3}$$

Going back to the process ξ^{C_2}; we had established that its jump times were those of a Poisson process of intensity q^*. Let's denote the latter process by $N = (N_t)_{t\geq 0}$. We further claim that ξ^{C_2} is a compound Poisson process. More precisely, we claim that ξ^{C_2} takes the form

$$\xi_t^{C_2} = \sum_{j=1}^{N_t} \delta_j,$$

where the δ_j are iid random variables independent of N. Of course, the δ_j are precisely the jump sizes of ξ^{C_2}, which are, by the Markov property, equal in distribution to $\Delta\xi_{\zeta^*}^{C_2}$. To prove the above claim, note how the fact that q^* is independent of the path of ξ^*, implies that the waiting times of N are independent of the jump sizes (the δ_j) of ξ^{C_2} – the latter all being independent of each other by recursive application of the strong Markov property of $\hat{\xi}$. So, ξ^{C_2} is indeed a compound

Poisson process with jump rate q^* and jump distribution being the law of $\Delta\xi^{C_2}_{\zeta^*}$.

Our final claim is that ξ^L and ξ^{C_2} are independent. We know from Lamperti's theorem that the jump rate of ξ^{C_2}, q^*, is independent of ξ^L. So all that remains to show is that the jump distribution of ξ^{C_2} is also independent of ξ^L. It suffices, by the Markov property, to show that $\Delta\xi^{C_2}_{\zeta^*}$ is independent of $(\xi^L_t : t < \zeta^*)$. Since the jump times of ξ^{C_2} are independent of those of ξ^L, it follows that ξ^{C_2} and ξ^L cannot jump simultaneously. Utilizing solely this fact we can compute

$$\begin{aligned}\Delta\hat{X}_{\tau_0^-} &\overset{\mathrm{d}}{=} \exp\Big\{\xi^L_{\zeta^*} + \xi^{C_2}_{\zeta^*}\Big\} - \exp\Big\{\hat{\xi}_{\zeta^*-}\Big\} \\ &\overset{\mathrm{d}}{=} \exp\Big\{\xi^L_{\zeta^*-} + \Delta\xi^{C_2}_{\zeta^*} + \xi^{C_2}_{\zeta^*-}\Big\} - \exp\big\{\xi^*_{\zeta^*-}\big\},\end{aligned}$$

where the last line follows from the fact that ζ^* is a jump time of ξ^{C_2} – and so not a jump time of ξ^L – and the fact that $\hat{\xi}_t \overset{\mathrm{d}}{=} \xi^*_t$ for $t < \zeta^*$. Note further how $\xi^*_{\zeta^*-} \overset{\mathrm{d}}{=} \hat{\xi}_{\zeta^*-} = \xi^L_{\zeta^*-} + \xi^{C_2}_{\zeta^*-}$ and $\exp\Big\{\xi^*_{\zeta^*-}\Big\} \overset{\mathrm{d}}{=} Z_{\tau_0^- -} \overset{\mathrm{d}}{=} X_{\tau_0^- -}$. Thus,

$$\Delta\hat{X}_{\tau_0^-} \overset{\mathrm{d}}{=} X_{\tau_0^- -}(e^{\Delta\xi^{C_2}_{\zeta^*}} - 1).$$

This, in conjunction with the fact that $\Delta\hat{X}_{\tau_0^-} = \hat{X}_{\tau_0^-} - \hat{X}_{\tau_0^- -} \overset{\mathrm{d}}{=} -X_{\tau_0^-} - X_{\tau_0^- -}$ implies

$$e^{\Delta\xi^{C_2}_{\zeta^*}} \overset{\mathrm{d}}{=} 1 - \frac{X_{\tau_0^-} + X_{\tau_0^- -}}{X_{\tau_0^- -}} = -\frac{X_{\tau_0^-}}{X_{\tau_0^- -}}.$$

And so the problem has now reduced to proving that $-\frac{X_{\tau_0^-}}{X_{\tau_0^- -}}$ is independent of $(X_t : t < \tau_0^-)$, since $(\xi^L_t : t < \zeta^*)$ is the underlying Lévy process of the latter. This goes far beyond the focus of the thesis and we thus omit it; a proof can be found in Chapter 5.7 of [37].

6.2.2 Underlying MAP of a Reflected d-dimensional Symmetric Stable Process

In this subsection we tackle the novel problem we briefly stated in the introductory section of this thesis chapter. We make it more precise as follows:

We consider a multidimensional version of the subject of the previous Section 6.2.1. More precisely, we study the d-dimensional ssMp $\hat{X} = (\hat{X}_t)_{t\geq 0}$ defined by

$$\hat{X}_t = (|X^{(1)}_t|, \ldots, |X^{(d)}_t|), \qquad t \geq 0, \tag{6.4}$$

where $|\cdot|$ denotes the usual absolute value in $\mathbb{R}$, and the $X^{(i)}$ are the 1-dimensional stable processes from (5.1) with $\alpha \in (0,2)$ and $\rho_i = \frac{1}{2}$. We assume this process dies and gets absorbed at its first hitting time of $\mathbf{0}_d \in \mathbb{R}^d$.

We first introduce some notation: since the $X^{(i)}$ of this section are chosen to have the same positivity parameters $\rho_i = \frac{1}{2}$, it follows that their respective Lévy measures all agree and are equal to the measure Π from (2.3) with

$$c := \pi^{-1}\Gamma(1+\alpha)\sin(\pi\alpha/2) = c_1 = c_2. \tag{6.5}$$

Our first step is to formally prove that $\hat{X}$ is in fact an ssMp. Self-similarity follows immediately from that of each of the $X^{(i)}$. The proof of the Markov property requires a bit more work, but notice how a vector consisting of d independent Markov processes will, again, be a Markov process. So, to prove that $\hat{X}$ is a Markov process, it will suffice to show that for a (one-dimensional) symmetric α-stable process $Y = (Y_t)_{t\geq 0}$, the process $\hat{Y} = (|Y_t|)_{t\geq 0}$ preserves Markovianity:

Proposition 6.2.1 *Fix $\alpha \in (0,2)$. Let $Y = (Y_t)_{t\geq 0}$ be a symmetric one-dimensional α-stable Lévy process. Then, the process $\hat{Y} = (|Y_t|)_{t\geq 0}$ is a Markov process with respect to its natural filtration $\mathcal{G} = (\mathcal{G}_t)_{t\geq 0}$, where $\mathcal{G}_t := \sigma(|Y_s| : s \leq t)$.*

Proof Let $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ denote the natural filtration of Y. Let $g : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function and let $x > 0$. Then, since $\mathcal{G}_t \subset \mathcal{F}_t$ for all $t \geq 0$, by the tower property we have

$$\mathbb{E}_x\Big[g(|Y_{t+s}|)|\mathcal{G}_t\Big] = \mathbb{E}_x\Big[\mathbb{E}_x\Big[g(|Y_{t+s}|)|\mathcal{F}_t\Big]\Big|\mathcal{G}_t\Big].$$

It's easy to see that we can write $g(|Y_{t+s}|)$ in the form $h(Y_{t+s})$, for some suitable bounded measurable function h. In this way we can apply the Markov property of Y on the inner-most expectation above to obtain

$$\mathbb{E}_x\Big[g(|Y_{t+s}|)|\mathcal{G}_t\Big] = \mathbb{E}_x\Big[\mathbb{E}_x\Big[h(Y_{t+s})\Big|\mathcal{F}_t\Big]\Big|\mathcal{G}_t\Big] = \mathbb{E}_x\Big[\mathbb{E}_w\Big[h(Y_s)\Big]\Big|_{w=Y_t}\Big|\mathcal{G}_t\Big] = \mathbb{E}_x\Big[\mathbb{E}_w\Big[g(|Y_s|)\Big]\Big|_{w=Y_t}\Big|\mathcal{G}_t\Big].$$

To proceed, note how, thanks to Y's symmetry, we have for every $z \in \mathbb{R}$,

$$(|Y|, \mathbb{P}_z) \overset{\mathrm{d}}{=} (|Y|, \mathbb{P}_{|z|}). \tag{6.6}$$

Indeed, for $z \geq 0$, (6.6) is trivially true; so let's assume $z < 0$, in which case $|z| = -z$. We then have

$$(|Y|, \mathbb{P}_z) \overset{\mathrm{d}}{=} (|z+Y|, \mathbb{P}) \overset{\mathrm{d}}{=} (|z-Y|, \mathbb{P}) \overset{\mathrm{d}}{=} (|-z+Y|, \mathbb{P}) \overset{\mathrm{d}}{=} (||z|+Y|, \mathbb{P}) \overset{\mathrm{d}}{=} (|Y|, \mathbb{P}_{|z|}),$$

where the first and final equalities are due to stationary and independent increments of Y, and the second equality due to the symmetry of Y. By applying the now-established (6.6) we get

$$\begin{aligned}
\mathbb{E}_x\Big[g(|Y_{t+s}|)|\mathcal{G}_t\Big] &= \mathbb{E}_x\Big[\mathbb{E}_w\Big[g(|Y_s|)\Big]\Big|_{w=Y_t}|\mathcal{G}_t\Big] \\
&= \mathbb{E}_x\Big[\mathbb{E}_w\Big[g(|Y_s|)\Big]\Big|_{w=|Y_t|}|\mathcal{G}_t\Big] && \text{(by (6.6))} \\
&= \mathbb{E}_{|Y_t|}(g(|Y_s|)),
\end{aligned}$$

where the last equality is due to the pull-out property of conditional expectations, thanks to the $\mathcal{G}_t$-measurability of $\mathbb{E}_w(g(|Y_s|))\Big|_{w=|Y_t|}$. ∎

The aim of this section is to characterize the underlying MAP $(\hat{\xi}, \hat{\Xi})$ of the d-dimensional ssMp $\hat{X}$ of this section. To proceed with our analysis, we need to define the following *"negation" and "reflection" operators* on $\mathbb{R}^d$:

Definition 6.2.2 *For $j \in \{1, \ldots, d\}$, define the operator $N^{(j)} : \mathbb{R}^d \to \mathbb{R}^d$ as follows:*

$$\begin{aligned}
N^{(j)}(\boldsymbol{x}) &= (x_1, x_2, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_d), \qquad \boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d \\
&= -x_j \boldsymbol{e}_j + \sum_{i \neq j} x_i \boldsymbol{e}_i,
\end{aligned}$$

where as usual, $\{\boldsymbol{e}_j : 1 \le j \le d\}$ denotes the standard orthonormal basis of $\mathbb{R}^d$. The reflection operator $R : \mathbb{R}^d \to \mathbb{R}^d$ is defined by

$$R(\boldsymbol{x}) = \sum_{j=1}^{d} \mathbb{1}_{\{x_j < 0\}}(\boldsymbol{x}) N^{(j)}(\boldsymbol{x}), \qquad \boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

We will be applying the above operators solely to the process $X = (X^{(1)}, \ldots, X^{(d)})$ at the times when it jumps out of the first orthant. We know that such jumps can only occur in exactly one of d possible directions, which is to say that at each of X's exit times from the orthant, one and only one of its coordinates $X^{(i)}$ becomes negative. As such, what the above-defined reflection operator R does to X at each of these jump times is to identify which of the d coordinates has become negative and then multiply that coordinate by -1 to make it positive/reflect it.

In this section of the thesis we will require a particular, yet natural, condition on the norms on $\mathbb{R}^d$ for which $(\hat{\xi}, \hat{\Xi})$ will be with respect to.

Definition 6.2.3 *We say that a norm $\|\cdot\|$ on $\mathbb{R}^d$ satisfies property $\mathcal{N}$ if for every $j \in \{1, \ldots, d\}$,*

$$\|\boldsymbol{x}\| = \|N^{(j)}(\boldsymbol{x})\|.$$

Remark 6.2.4 *Not every norm on $\mathbb{R}^d$ satisfies property $\mathcal{N}$: Consider the norm $\|\cdot\|$ on $\mathbb{R}^2$ defined by*

$$\|(x_1, x_2)\| = |x_1 + x_2| + |x_2|, \qquad (x_1, x_2) \in \mathbb{R}^2.$$

Take the vector $\boldsymbol{x} = (1, 1)$, then

$$\|N^{(1)}(\boldsymbol{x})\| = \|(-1, 1)\| = |-1 + 1| + |1| = 1,$$

but $\|\boldsymbol{x}\| = 3$.

Remark 6.2.5 *Many commonly used norms do satisfy property $\mathcal{N}$, however. In particular, the L^p norm, for $p \geq 1$, obviously satisfies property $\mathcal{N}$. It is also clear that any abstract norm $\|\cdot\|$ on $\mathbb{R}^d$ satisfying property $\mathcal{N}$ also satisfies $\|\hat{X}_t\| = \|X_t\|$ for every $t \geq 0$.*

The following recursive construction of $\hat{X}$ will be useful. For $k \in \mathbb{N}$, define the sequence of processes $\hat{X}^{(k)} = (\hat{X}_t^{(k)})_{t\geq 0}$ by

$$\hat{X}_t^{(k)} = \begin{cases} \hat{X}_t^{(k-1)}, & \text{if } t < \tau_{k-1}^D \\ R(\hat{X}_{\tau_{k-1}^D}^{(k-1)}) + \tilde{X}_t^{(k)}, & \text{if } t \geq \tau_{k-1}^D \end{cases},$$

where $(\hat{X}_t^0)_{t\geq 0} = (X_t)_{t\geq 0}$, and $\tau_0^D = \tau^D$, $\tilde{X}_t^{(k)} := \hat{X}_t^{(k-1)} - \hat{X}_{\tau_{k-1}^D}^{(k-1)}$ (for $t \geq \tau_{k-1}^D$), $\tau_k^D := \inf\{t > \tau_{k-1}^D : \hat{X}_t^{(k)} \notin D\}$. Then for all $k \in \mathbb{N}$, $\hat{X}$ is compatible with $\hat{X}^{(k)}$ on the time interval $0 \leq t < \tau_k^D$.

Because $\hat{X}$ is confined to the positive orthant and is a self-similar Markov process, we can identify its MAP $(\hat{\xi}, \hat{\Xi})$ simply by the general transformation (4.3).

Lemma 6.2.6 *Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$ satisfying property $\mathcal{N}$. Let $(\hat{\xi}, \hat{\Xi}) = (\hat{\xi}_t, \hat{\Xi}_t)_{t\geq 0}$ be the underlying MAP of $\hat{X}$ from (6.4) with respect to $\|\cdot\|$. Denote by L the time of the first "corrective" jump of $(\hat{\xi}, \hat{\Xi})$ due to reflection. Let τ^D denote the first exit time from the orthant of the d-dimensional stable process X, as defined in (5.1) and (5.2). The distribution of the "corrective"* *jumps of the MAP* $(\hat{\xi}, \hat{\Xi})$ *is determined by the first exit of* X *from the orthant as follows*

$$E_{\boldsymbol{\theta}}\left[G\left(\log\left(\frac{\|X_{\tau^D}\|}{\|X_{\tau^D-}\|}\right), \left(\frac{|X^{(i)}_{\tau^D}|}{\|X_{\tau^D}\|}, i \in \{1, \dots, d\}\right)\right) \middle| X_{\tau^D-}\right]$$
$$= \mathbb{E}_{(0,\boldsymbol{\theta})}\left[G\left(\Delta\hat{\xi}_L, \hat{\Xi}_L\right) \middle| \hat{\Xi}_{L-} = X_{\tau^D-}/\|X_{\tau^D-}\|\right], \tag{6.7}$$

for any bounded measurable function $G : \mathbb{R} \times \mathcal{S}^{d,+}_{\|\cdot\|} \to \mathbb{R}$ *and any* $\boldsymbol{\theta} \in \mathcal{S}^{d,+}_{\|\cdot\|}$*. The latter can be computed further via*

$$\mathbb{E}_{(0,\boldsymbol{\theta})}[G(\Delta\hat{\xi}_L, \hat{\Xi}_L)|\hat{\Xi}_{L-}] = H(\hat{\Xi}_{L-}) \tag{6.8}$$

where, for a deterministic vector $\Xi = (\Xi^{(1)}, \dots, \Xi^{(d)}) \in \mathcal{S}^{d,+}_{\|\cdot\|}$*,*

$$H(\Xi) = q(\Xi)^{-1} \cdot \sum_{j=1}^{d} \int_{\Xi^{(j)}}^{\infty} G(\log\|\Xi - x\boldsymbol{e}_j\|, \frac{1}{\|\Xi - x\boldsymbol{e}_j\|} N^{(j)}(\Xi - x\boldsymbol{e}_j)) x^{-(1+\alpha)} dx, \tag{6.9}$$

with q *being the function from Lemma 5.4.1 by taking* $\rho_k = 1/2$ *for all* k*.*

Proof By the very definition of conditional expectation, we must show that for every bounded measurable function $h : \mathcal{S}^{d,+}_{\|\cdot\|} \to \mathbb{R}$ and $\boldsymbol{\theta} \in \mathcal{S}^{d,+}_{\|\cdot\|}$,

$$\mathbb{E}_{(0,\boldsymbol{\theta})}[H(\hat{\Xi}_{L-})h(\hat{\Xi}_{L-})] = \mathbb{E}_{(0,\boldsymbol{\theta})}[G(\Delta\hat{\xi}_L, \hat{\Xi}_L)h(\hat{\Xi}_{L-})]. \tag{6.10}$$

We begin by computing the LHS utilizing the compensation formula in effectively the same way we had done in our proof of Lemma 5.4.1 (and for this reason we skip many intermediate steps):

$$\begin{aligned}
&\mathbb{E}_{(0,\boldsymbol{\theta})}[H(\hat{\Xi}_{L-})h(\hat{\Xi}_{L-})] \\
&= E_{\boldsymbol{\theta}}\Big[H\big(\arg(X_{\tau^D-})\big)h\big(\arg(X_{\tau^D-})\big)\Big] \\
&= c\int_0^\infty \mathbb{E}_{(0,\boldsymbol{\theta})}\Big[\sum_{m=1}^{d}\int_{-\infty}^{-\hat{\Xi}^{(m)}_v} H(\hat{\Xi}_v)h(\hat{\Xi}_v)|l|^{-(1+\alpha)}dl; v < L\Big]dv \\
&= c\int_0^\infty \mathbb{E}_{(0,\boldsymbol{\theta})}\left[H(\hat{\Xi}_v)h(\hat{\Xi}_v)\Big(\sum_{m=1}^{d}\int_{\hat{\Xi}^{(m)}_v}^{\infty} l^{-(1+\alpha)}dl\Big); v < L\right]dv \\
&= \int_0^\infty \mathbb{E}_{(0,\boldsymbol{\theta})}\Big[H(\hat{\Xi}_v)h(\hat{\Xi}_v)\Big(\frac{c}{\alpha}\sum_{m=1}^{d}(\hat{\Xi}^{(j)}_v)^{-\alpha}\Big); v < L\Big]dv \\
&= \int_0^\infty \mathbb{E}_{(0,\boldsymbol{\theta})}\Big[H(\hat{\Xi}_v)h(\hat{\Xi}_v)q(\hat{\Xi}_v); v < L\Big]dv.
\end{aligned}$$

where we recall Lemma 5.4.1 with $\rho_1 = \dots = \rho_d = 1/2$ for the last equality. By the way we had defined H, the $q(\hat{\Xi}_v)$ in the above integrand cancels and we get

$$\begin{aligned}
&\mathbb{E}_{(0,\boldsymbol{\theta})}[H(\hat{\Xi}_{L-})h(\hat{\Xi}_{L-})] \\
&= \mathbb{E}_{(0,\boldsymbol{\theta})}\left[\int_0^L \left\{h(\hat{\Xi}_v)\sum_{j=1}^d \int_{\hat{\Xi}_v^{(j)}}^\infty G(\log\|\hat{\Xi}_v - x\boldsymbol{e}_j\|, \frac{1}{\|\hat{\Xi}_v - x\boldsymbol{e}_j\|}N^{(j)}(\hat{\Xi}_v - x\boldsymbol{e}_j))x^{-(1+\alpha)}dx\right\}dv\right].
\end{aligned}$$

We will now work out the RHS of (6.10) and show that it equals the above expectation. Firstly,

$$\Delta\hat{\xi}_L = \log\left(\frac{\|\hat{X}_{\tau_D}\|_1}{\|\hat{X}_{\tau_D-}\|_1}\right) = \log\left(\frac{\|R(X_{\tau_D})\|_1}{\|X_{\tau_D-}\|_1}\right) = \log\left(\frac{\|X_{\tau_D}\|_1}{\|X_{\tau_D-}\|_1}\right). \tag{6.11}$$

Secondly, we also need to derive an expression of $\hat{\Xi}_L$. Using only the independence of all the $X^{(i)}$, we observe that on the event $\{\tau^D = (\tau_0^-)^{(j)}\}$, where $(\tau_0^-)^{(j)} := \inf\{t > 0 : X_t^{(j)} < 0\}$, i.e., on the event that the first time X exits the orthant is through a jump in the j-th direction, we have

$$\hat{\Xi}_L = N^{(j)}\big(\arg(X_{\tau_D})\big) \tag{6.12}$$

Therefore, by using the very definition of the MAP $(\hat{\xi}, \hat{\Xi})$ and applying the usual compensation formula method that brought about Lemma 5.4.1 and Theorem 5.4.2 (for which we skip many of the familiar intermediate steps and algebraic manipulations),

$$\begin{aligned}
&\mathbb{E}_{(0,\boldsymbol{\theta})}\Big[G(\Delta\hat{\xi}_L, \hat{\Xi}_L)h(\hat{\Xi}_{L-})\Big] \\
&= \sum_{j=1}^d E_{\boldsymbol{\theta}}\left[G\Big(\log\frac{\|X_{\tau_D}\|}{\|X_{\tau_D-}\|}, N^{(j)}\big(\arg(X_{\tau_D})\big)\Big)h\big(\arg(X_{\tau_D-})\big)\mathbb{1}_{\{\tau_D=(\tau_0^-)^{(j)}\}}\right] \\
&= \mathbb{E}_{(0,\boldsymbol{\theta})}\left[\int_0^L \left\{h(\hat{\Xi}_v)\sum_{j=1}^d \int_{-\infty}^{-\hat{\Xi}_v^{(j)}} G\Big(\log\|\hat{\Xi}_v + \ell\boldsymbol{e}_j\|, \frac{1}{\|\hat{\Xi}_v + \ell\boldsymbol{e}_j\|}N^{(j)}(\hat{\Xi}_v + \ell\boldsymbol{e}_j)\Big)c|\ell|^{-(1+\alpha)}d\ell\right\}dv\right],
\end{aligned}$$

as required. ∎

In view of the many similarities between MAPs and Lévy processes, it would be natural to suspect that $(\hat{\xi}, \hat{\Xi})$ admits a path-decomposition in terms of (ξ, Ξ) that is analogous to the one depicted in (6.3) concerning underlying Lévy processes in dimension $d = 1$. We conjecture the following:

Conjecture 2 *Assume the setup of Lemma 6.2.6. Let (ξ^*, Ξ^*) be a copy of the MAP (ξ, Ξ) with the effect of killing removed. Moreover, let $L^* \geq 0$ be a random variable dependent on Ξ^*, whose conditional survival probability satisfies* $\Pr(L^* > t \,|\, \Xi^*) = \exp\Big(-\int_0^t q(\Xi_s^*)ds\Big)$, *where q is the function from Lemma 5.4.1 with $\rho_1 = \cdots = \rho_d = \frac{1}{2}$.*

(i) Recalling L is the time of the first corrective jump of $(\hat{\xi}, \hat{\Xi})$, then, for each $x \in \mathbb{R}$ and $\boldsymbol{\theta} \in \mathcal{S}^{d,+}_{\|\cdot\|}$, under $\mathbb{P}_{(x,\boldsymbol{\theta})}$, $(\hat{\xi}_t, \hat{\Xi}_t)_{0\le t<L}$ is equal in law to $(\xi^*, \Xi^*)_{0\le t<L^*}$ with $(\xi^*_0, \Xi^*_0) = (x, \boldsymbol{\theta})$. Moreover, the joint law of $(\Delta\hat{\xi}_L, \hat{\Xi}_L)$ is given by Lemma 6.2.6. Given $(\hat{\xi}_L, \hat{\Xi}_L)$, thanks to the Strong Markov property, the process $(\hat{\xi}, \hat{\Xi})$ can be constructed iteratively until the next corrective jump and so on.

(ii) As such, conditional on the sequence of pairs $(L_n, \hat{\Xi}_{L_n-})$, $n \in \mathbb{N}_0$, with L_n ($L_0 = 0$ and $L_1 = L$) denoting the corrective jump times of $\hat{\Xi}$, we have

$$\hat{\xi}_t \overset{d}{=} \xi^*_t + \sum_{k=0}^{N_t} F_k,$$

where, $N_t = \sup\{k \in \mathbb{N}_0 : L_k \le t\}$, and $(F_0 = 0), F_1, F_2, \ldots$ are independent such that, for real Borel sets A, $\Pr\Big(F_k \in A \,|\, \hat{\Xi}_{L_k-}\Big)$ is given by (6.8) with $G(x, \boldsymbol{\theta}) = \mathbf{1}_A(x)$ and all occurrences of $\hat{\Xi}_{L-}$ therein replaced by $\hat{\Xi}_{L_k-}$.

Further investigation is required to determine the validity of this result under the stated assumptions. To maintain full transparency and academic rigour in the thesis, and to emphasize the incomplete nature of the current argument, we have chosen to present it as a conjecture as opposed to an established theorem. A definitive resolution thus remains an open problem for future research.

We can get an even more explicit formula for $\mathbb{E}_{(0,\boldsymbol{\theta})}[G(\Delta\hat{\xi}_L, \hat{\Xi}_L)|\hat{\Xi}_{L-}]$ from (6.9) by taking $\|\cdot\| = \|\cdot\|_p$, for $p \ge 1$:

Corollary 6.2.7 *Fix $p \ge 1$ and assume the setup/notations of Theorem 2 with $\|\cdot\| = \|\cdot\|_p$. Let $(\hat{\xi}, \hat{\Xi})$ be the underlying MAP of $\hat{X}$ from (6.4) with respect to $\|\cdot\|_p$. Then, for every $\boldsymbol{\theta} \in \mathcal{S}^{d,+}_p$ and bounded measurable function $G : \mathbb{R} \times \mathcal{S}^{d,+}_p \to \mathbb{R}$,*

$$\begin{aligned}&\mathbb{E}_{(0,\boldsymbol{\theta})}[G(\Delta\hat{\xi}_L, \hat{\Xi}_L)|\hat{\Xi}_{L-}]\\ &= q(\hat{\Xi}_{L-})^{-1} \sum_{j=1}^{d} \int_{\frac{1}{p}\log\left(1-(\hat{\Xi}^{(j)}_{L-})^p\right)}^{\infty} G(x, \boldsymbol{v}^{p,\hat{\Xi}_{L-}}_{j,x}) \frac{e^{px}(e^{px} - 1 + (\hat{\Xi}^{(j)}_{L-})^p)^{\frac{1-p}{p}}}{\left(\hat{\Xi}^{(j)}_{L-} + (e^{px} - 1 + (\hat{\Xi}^{(j)}_{L-})^p)^{1/p}\right)^{1+\alpha}} dx,\end{aligned}$$

where $\boldsymbol{v}^{p,\boldsymbol{\theta}}_{j,y}$ denotes the vector from (5.12), and q is the function from Lemma 5.4.1 with $\|\cdot\| = \|\cdot\|_p$ and $\rho_1 = \cdots = \rho_d = 1/2$.

Proof For the usual reasons, for $j \in \{1, \ldots, d\}$, $x > \hat{\Xi}^{(j)}_{L-}$ implies

$$\|\hat{\Xi}_{L-} - x\boldsymbol{e}_j\|_p = ((x - \hat{\Xi}^{(j)}_{L-})^p + 1 - (\hat{\Xi}^{(j)}_{L-})^p)^{1/p}.$$

By plugging this into (6.9) (with $\|\cdot\| = \|\cdot\|_p$ and $\Xi = \hat{\Xi}_{L-}$) and performing the following change-of-variables:

$$z = \frac{1}{p}\log\Big((x - \hat{\Xi}^{(j)}_{L-})^p + 1 - (\hat{\Xi}^{(j)}_{L-})^p\Big),\ \ x = (e^{pz} - 1 + (\hat{\Xi}^{(j)}_{L-})^p)^{1/p} + \hat{\Xi}^{(j)}_{L-},$$
$$\frac{dx}{dz} = e^{pz}(e^{pz} - 1 + (\hat{\Xi}^{(j)}_{L-})^p)^{\frac{1-p}{p}},$$

one obtains the expression

$$q(\hat{\Xi}_{L-})^{-1}\sum_{j=1}^{d}\int_{\frac{1}{p}\log\left(1-(\hat{\Xi}^{(j)}_{L-})^p\right)}^{\infty} G\left(x, \frac{N^{(j)}\Big(\hat{\Xi}_{L-} - \Big(\hat{\Xi}^{(j)}_{L-} + (e^{pz} - 1 + (\hat{\Xi}^{(j)}_{L-})^p)^{1/p}\Big)\boldsymbol{e}_j\Big)}{e^z}\right)$$
$$\times \frac{e^{pz}(e^{pz} - 1 + (\hat{\Xi}^{(j)}_{L-})^p)^{\frac{1-p}{p}}}{\Big(\hat{\Xi}^{(j)}_{L-} + \big(e^{pz} - 1 + (\hat{\Xi}^{(j)}_{L-})^p\big)^{1/p}\Big)^{1+\alpha}}dz.$$

It remains to observe that for $z > \frac{1}{p}\log\Big(1 - (\hat{\Xi}^{(j)}_{L-})^p\Big)$,

$$N^{(j)}\Big(\hat{\Xi}_{L-} - \Big(\hat{\Xi}^{(j)}_{L-} + \big(e^{pz} - 1 + (\hat{\Xi}^{(j)}_{L-})^p\big)^{1/p}\Big)\boldsymbol{e}_j\Big) = \boldsymbol{v}^{p,\hat{\Xi}_{L-}}_{j,x}. \quad \blacksquare$$

And, as usual, in the case of the L^1 norm, the above result cleans up significantly:

Corollary 6.2.8 *Assume the setup/notations of Theorem 2 with $\|\cdot\| = \|\cdot\|_1$. Let $(\hat{\xi}, \hat{\Xi})$ be the underlying MAP of $\hat{X}$ from (6.4) with respect to $\|\cdot\|_1$. Then, for every $\boldsymbol{\theta} \in \mathcal{S}^{d,+}_1$ and bounded measurable function $G : \mathbb{R} \times \mathcal{S}^{d,+}_1 \to \mathbb{R}$,*

$$\begin{aligned}&\mathbb{E}_{(0,\boldsymbol{\theta})}[G(\Delta\hat{\xi}_L, \hat{\Xi}_L)|\hat{\Xi}_{L-}]\\ &= q(\hat{\Xi}_{L-})^{-1}\cdot\sum_{j=1}^{d}\int_{\log\left(1-\hat{\Xi}^{(j)}_{L-}\right)}^{\infty} G\Big(x, \frac{1}{e^x}\big(\hat{\Xi}_{L-} + (e^x - 1)\boldsymbol{e}_j\big)\Big)\cdot(e^x + 2\hat{\Xi}^{(j)}_{L-} - 1)^{-(1+\alpha)}\cdot e^x dx,\end{aligned}$$

where q is the function from Lemma 5.4.1 with $\|\cdot\| = \|\cdot\|_1$ and $\rho_1 = \cdots = \rho_d = 1/2$.

The random variable $\Delta\hat{\xi}_L$ gives the size of the jump the ordinate of the underlying MAP of $\hat{X}$ makes as a result of reflection. It follows that this is a real-valued random variable which is positive/negative if there is an increase/decrease in the norm of X at the time of this jump. As we had explained previously, the value of $\Delta\hat{\xi}_L$ is very much dependent on the position of the underlying modulator just before the jump, that is the position of the (random) vector $\hat{\Xi}_{L-}$. It is then natural to want to find out explicitly how the distribution of $\Delta\hat{\xi}_L$ depends on $\hat{\Xi}_{L-}$, that is to compute the conditional probability

$$\mathbb{P}_{(0,\boldsymbol{\theta})}(\Delta\hat{\xi}_L < J|\hat{\Xi}_{L-}).$$

The above quantity is readily available from Lemma 6.2.6 by setting $G(x,\boldsymbol{\phi}) = \mathbb{1}_{(0,J)}(x),\ x \in \mathbb{R}, \boldsymbol{\phi} \in \mathcal{S}^{d,+}_{\|\cdot\|}$. By considering the specific case of $\|\cdot\| = \|\cdot\|_1$, however, one gets an even crispier formula:

Corollary 6.2.9 *Assume the setup/notations of Theorem 2 with $\|\cdot\| = \|\cdot\|_1$. Let $(\hat{\xi},\hat{\Xi})$ be the underlying MAP of $\hat{X}$ from (6.4) with respect to $\|\cdot\|_1$. The "corrective" jump of the ordinate, $\hat{\xi}$, has distribution function given by*

$$\mathbb{P}_{(0,\boldsymbol{\theta})}(\Delta\hat{\xi}_L < J|\hat{\Xi}_{L-}) = \begin{cases} \frac{1}{c} - \frac{1}{\alpha} q(\hat{\Xi}_{L-})^{-1}\sum_{j=1}^{d}(e^J + 2\hat{\Xi}^{(j)}_{L-} - 1)^{-\alpha}, & \text{if } J > 0 \\ q(\hat{\Xi}_{L-})^{-1}\cdot\dfrac{1}{\alpha}\displaystyle\sum_{j=1}^{d}\Big((\hat{\Xi}^{(j)}_{L-})^{-\alpha} \\ \qquad\qquad - (e^J + 2\hat{\Xi}^{(j)}_{L-} - 1)^{-\alpha}\Big)\mathbb{1}_{\{\hat{\Xi}^{(j)}_{L-} > 1 - e^J\}}(\hat{\Xi}_{L-}), & \text{if } J < 0 \end{cases},$$

where $\boldsymbol{\theta} \in \mathcal{S}^{d,+}_1$ and q is the function from Lemma 5.4.1 with $\|\cdot\| = \|\cdot\|_1$ and $\rho_1 = \cdots = \rho_d = 1/2$.

Proof It follows immediately from Corollary 6.2.8 that for every bounded measurable function G,

$$\mathbb{E}_{(0,\boldsymbol{\theta})}\big[G(\Delta\hat{\xi}_L)|\hat{\Xi}_{L-}\big] = q(\hat{\Xi}_{L-})^{-1}\sum_{j=1}^{d}\int_{\log\left(1-\hat{\Xi}^{(j)}_{L-}\right)}^{\infty} G(l)\cdot(e^l + 2\hat{\Xi}^{(j)}_{L-} - 1)^{-(1+\alpha)}e^l dl. \tag{6.13}$$

Setting $G(l) = \mathbb{1}_{(-\infty,J)}(l)$ with $J > 0$ yields

$$\begin{aligned}\mathbb{P}_{(0,\boldsymbol{\theta})}(\Delta\hat{\xi}_L < J|\hat{\Xi}_{L-}) &= q(\hat{\Xi}_{L-})^{-1}\sum_{j=1}^{d}\int_{\log\left(1-\hat{\Xi}^{(j)}_{L-}\right)}^{J}(e^l + 2\hat{\Xi}^{(j)}_{L-} - 1)^{-(1+\alpha)}e^l dl \\ &= q(\hat{\Xi}_{L-})^{-1}\sum_{j=1}^{d}\int_{\hat{\Xi}^{(j)}_{L-}}^{2\hat{\Xi}^{(j)}_{L-}+e^J-1} l^{-(1+\alpha)}dl,\end{aligned}$$

from which the equality from the statement of the corollary regarding the case when $J > 0$ follows from the very definition of the function q.

Finally, we examine the case when $G(l) = \mathbb{1}_{(-\infty,J)}(l)$ with $J < 0$. Note then, how for a j such that $\hat{\Xi}^{(j)}_{L-} < 1 - e^J$, the j-th integral appearing on the RHS of (6.13) will be zero. Taking this into account and following analogous steps we get the other equality from the statement of the corollary regarding the case when $J < 0$. ∎

And in dimension $d = 2$, the above result becomes

Corollary 6.2.10 *Assume the setup/notations of Theorem 2 with $\|\cdot\| = \|\cdot\|_1$ and $d = 2$. Let $(\hat{\xi}, \hat{\Xi})$ be the underlying MAP of $\hat{X}$ (in dimension $d = 2$) from (6.4) with respect to $\|\cdot\|_1$. The "corrective" jump of the ordinate, $\hat{\xi}$, has distribution function given by*

$$\mathbb{P}_{(0,\boldsymbol{\theta})}(\Delta\hat{\xi}_L < J|\hat{\Xi}_{L-}) = \begin{cases} 1 - \dfrac{1}{\alpha} q(\hat{\Xi}_{L-})^{-1}\Big[(e^J + \hat{\Xi}^{(1)}_{L-} - \hat{\Xi}^{(2)}_{L-})^{-\alpha} \\ \qquad\qquad + (e^J + \hat{\Xi}^{(2)}_{L-} - \hat{\Xi}^{(1)}_{L-})^{-\alpha}\Big], & \text{if } J > 0 \\ \dfrac{1}{\alpha} q(\hat{\Xi}_{L-})^{-1}\Bigg[\Big((\hat{\Xi}^{(1)}_{L-})^{-\alpha} \\ \qquad\qquad - (e^J + \hat{\Xi}^{(1)}_{L-} - \hat{\Xi}^{(2)}_{L-})^{-\alpha}\Big)\mathbb{1}_{\{\hat{\Xi}^{(1)}_{L-} > 1 - e^J\}} & \text{if } J < 0 \\ \qquad + \Big((\hat{\Xi}^{(2)}_{L-})^{-\alpha} \\ \qquad\qquad - (e^J + \hat{\Xi}^{(2)}_{L-} - \hat{\Xi}^{(1)}_{L-})^{-\alpha}\Big)\mathbb{1}_{\{\hat{\Xi}^{(2)}_{L-} > 1 - e^J\}}\Bigg], \end{cases},$$

where $\boldsymbol{\theta} \in \mathcal{S}_1^{2,+}$ and q is the function from Lemma 5.4.1 with $d = 2$, $\|\cdot\| = \|\cdot\|_1$ and $\rho_1 = \rho_2 = 1/2$.

6.3 Skorokhod-Reflected Spectrally-Positive Lévy Processes

In this section we take $\alpha \in (1, 2)$ and each $X^{(i)} = (X^{(i)}_t)_{t\geq 0}$, $1 \leq i \leq d$, an iid α-stable process with positivity parameter $\rho \geq 0$ satisfying $\alpha(1 - \rho) = 1$. Said conditions ensure that each $X^{(i)}$ is a spectrally-positive stable process that will hit 0 at a finite time, and do so continuously. It is clear that the positivity parameter ρ above cannot possibly equal $\frac{1}{2}$. This means that each $X^{(i)}$ is non-symmetric; so we cannot conclude that $|X^{(i)}|$ is a Markov process. Consequently, the process $\hat{X}$ from Section 6.2 is no longer Markovian under the current sections' parametrization, and so does not possess an underlying MAP. We must then conceive another natural notion of reflection that preserves the Markovianity of the process X from (5.1) with the parametrizations of this section. Skorokhod in papers [56] and [57] does just that; but only for continuous Markov processes. Later, his notion of reflection was extended to càdlàg processes (see, for example, [62], [27], [19]); and came to be known as Skorokhod-reflection. Namely, the *Skorokhod-reflection* of some càdlàg function $f : [0, \infty) \to \mathbb{R}$ is precisely the càdlàg function $z : [0, \infty) \to \mathbb{R}$ defined by

$$z(t) = f(t) - (0 \wedge \inf_{s\leq t} f(s)), \quad t \geq 0.$$

Accordingly, we define the (Skorokhod-)reflection process of X as the process $\tilde{R} = (\tilde{R}_t)_{t\geq 0}$ defined by

$$\tilde{R}_t = (X_t^{(1)} - (0 \wedge \underline{X}_t^{(1)}), \ldots, X_t^{(d)} - (0 \wedge \underline{X}_t^{(d)})), \qquad t \geq 0, \tag{6.14}$$

where $\underline{X}_t^{(i)} := \inf_{s\leq t} X_s^{(i)}$, $t \geq 0$. As mentioned at the start, this type of reflection preserves the Markovianity of X:

Lemma 6.3.1 *Set $\alpha \in (0,2)$, and let $Y = (Y_t)_{t\geq 0}$ be a (possibly non-symmetric) α-stable process. Then, the process $Z = (Z_t)_{t\geq 0}$ defined by $Z_t = Y_t - (0 \wedge \underline{Y}_t)$, $t \geq 0$, is an ssMp.*

Proof Fix $t \geq 0$. Define the process $\tilde{Y} = (\tilde{Y}_s)_{s\geq 0}$ by $\tilde{Y}_s = \tilde{Y}_{t+s} - \tilde{Y}_t$, $s \geq 0$. First, notice how for $s \geq 0$,

$$\begin{aligned} Z_{t+s} = Y_{t+s} - (0 \wedge \underline{Y}_{t+s}) &= \tilde{Y}_s + ((Y_t - (0 \wedge \underline{Y}_t)) \vee (Y_t - \inf_{t\leq u\leq t+s} Y_u)) \\ &= \tilde{Y}_s - (-Z_t \wedge (\inf_{t\leq u\leq t+s} (Y_u - Y_t))) \\ &= \tilde{Y}_s - (-Z_t \wedge \inf_{u\leq s} \tilde{Y}_u). \end{aligned}$$

Then, by denoting the natural filtration of Y by $(\mathcal{F}_t)_{t\geq 0}$ and its probabilities by $\mathbb{P}_z$, $z \geq 0$, with the usual convention that $\mathbb{P}_0 \equiv \mathbb{P}$; and observing how Y is initiated from $z > 0$ if and only if Z is initiated from z, we find for a bounded measurable function g,

$$\begin{aligned} \mathbb{E}_z[g(Z_{t+s})|\mathcal{F}_t] &= \mathbb{E}_z\Big[\tilde{Y}_s - (-Z_t \wedge \inf_{u\leq s} \tilde{Y}_u)\big|\mathcal{F}_t\Big] \\ &= \mathbb{E}\Big[g\big(Y_s - (w \wedge \underline{Y}_s)\big)\Big]\Big|_{w=-Z_t} \\ &= \mathbb{E}\Big[g\big(Y_s - (0 \wedge (\underline{X}_s - w) + w)\big)\Big]\Big|_{w=-Z_t} \\ &= \mathbb{E}\left[g\left((-w) + Y_s - \Big(0 \wedge \big((-w) + \underline{X}_s\big)\Big)\right)\right]\Bigg|_{w=-Z_t} \\ &= \mathbb{E}_w\Big[g\big(Y_s - (0 \wedge \underline{Y}_s)\big)\Big]\Big|_{w=Z_t} \\ &= \mathbb{E}_w[g(Z_s)]\Big|_{w=Z_t}, \end{aligned}$$

where the second and second-last equalities follow from independent and stationary increments of Y and adaptability of Z. This proves the Markov property.

The self-similarity property is a trivial check which follows immediately from the self-similarity of Y. ∎

Before moving on to the objective of this section, let us succinctly describe the type of reflection Z describes of Y in the above lemma: the path of Z is in agreement with that of Y up to the first time Y goes below zero. Thereafter, if Y is *not* at a time when it's attaining a new infimum, in which case Z is a constant 0, the path of Z will precisely be that of the excursion Y makes away from its previously-attained infimum.

It follows by Lemma 6.3.1 that the process $R = (R_t)_{t\geq 0}$,

$$R_t = \tilde{R}_t \mathbb{1}_{\{t<v\}}, \qquad t \geq 0, \tag{6.15}$$

where $v = \inf\{s > 0 : \|\tilde{R}_s\| = 0\}$, is an example of an ssMp in the positive orthant of $\mathbb{R}^d$ that gets absorbed at the zero vector $\mathbf{0}_d \in \mathbb{R}^d$; and hence possesses a (norm-dependent) underlying MAP, which to be consistent with the introductory section, we denote by $(\xi^R, \Xi^R) = (\xi^R_t, \Xi^R_t)_{t\geq 0}$. Our objective in this section is to characterize the latter. We should note that it is now no longer enough to solely describe its jump structure. Indeed, the nature of the kind of reflection we are studying in this section (happening in a continuous fashion), raises the necessity of a mathematical object that says something about (ξ^R, Ξ^R)'s behaviour near the boundary of $D = [0, \infty)^d$ as well, and in particular, the rate at which it "bounces" as it hits the boundary and gets reflected. To this end, the generator of (ξ^R, Ξ^R), as well as all the (reflecting) boundary conditions satisfied by functions in its domain, are the most appropriate in fully characterizing it.

6.3.1 Infinitesimal Generator of a Skorokhod-Reflected 1-dimensional Spectrally-Positive Lévy Process

The derivation of the generator of (ξ^R, Ξ^R) will be split into several steps. The first step will be to derive the generator of the Skorokhod-reflection of a (general) one-dimensional spectrally-positive Lévy process; and will be done in this subsection. We will use the same stochastic calculus strategy that brought about the corresponding mathematical objects for the sticky one-dimensional (spectrally-positive) Lévy process in Proposition 5 of [49]. More specifically, we will utilize Itô's formula to obtain a suitable martingale that reveals the desired generator (recall the discussion in our preliminary Section 2.2.5). For some further martingales associated with the Skorokhod-reflection of a Lévy process in one-dimension we refer the reader to [45], where there is an additional emphasis on their applications in fluctuation theory.

We will be using the following Lévy-Itô decomposition of a (standard) one-dimensional spectrally-positive Lévy process $X = (X_t)_{t\geq 0}$:

$$X = X^{(1)} + X^{(2)} + X^{(3)},$$

where $X^{(1)} = (X_t^{(1)})_{t\geq 0}$ is the process defined by $X_t^{(1)} := bt + \sigma B_t$, $t \geq 0$, with $b \in \mathbb{R}$, $\sigma > 0$ and $B = (B_t)_{t\geq 0}$ a (one-dimensional) standard Brownian motion; the process $X^{(2)} = (X_t^{(2)})_{t\geq 0}$, with

$$X_t^{(2)} = \int_{(0,t]\times[1,\infty)} xN(ds, dx), \qquad t \geq 0,$$

where N denotes the Poisson random measure associated with the jumps of X, is the compound Poisson process responsible for the jumps of X that are of size bigger than 1 – independent of $X^{(1)}$; and $X^{(3)} = (X_t^{(3)})_{t\geq 0}$ is the square-integrable martingale responsible for the jumps of X that are of size smaller than 1 – independent of both $X^{(1)}$ and $X^{(2)}$.

Theorem 6.3.2 *Let X be a standard one-dimensional spectrally-positive Lévy process, initiated from some positive state x, with a Lévy-Itô decomposition as given above. Denote the Skorokhod-reflection of X by $R := X - (0 \wedge \underline{X})$, where as usual, $\underline{X}$ denotes the running infimum process of X. Then, for $g \in C_b^2([0,\infty))$ (i.e., a real-valued function g with bounded and continuous partial derivatives up to the order 2 with domain $[0,\infty)$) satisfying the boundary condition $g'(0) = 0$, we have that the process*

$$g(R_t) - g(R_0) - \int_0^t \mathcal{L}g(R_s)ds, \qquad t \geq 0,$$

is a martingale, where the operator $\mathcal{L}$ is given on this class of functions by

$$\mathcal{L}F(z) = bF'(z) + \frac{\sigma^2}{2}F''(z) + \int_0^\infty (F(z+x) - F(z) - F'(z)x\mathbb{1}_{(0,1)}(x))\Pi(dx),$$

with Π being the Lévy measure of X, is the extended generator of X (as given in Chapter VII of [50]).

Proof By Theorem 9.1, Chapter II of [48], X is a semimartingale. Since X does not make any downward jumps and is monotone-decreasing, $\underline{\mathrm{X}}$ is an adapted process with continuous paths that are of finite variation; hence, by Theorem 7, Chapter II of [48], it is a semimartingale. Since both X and $\underline{\mathrm{X}}$ are semimartingales, $f(X, Z)$, where $Z = 0 \wedge \underline{\mathrm{X}}$, is a semimartingale for $f \in C_b^2(\mathbb{R}^2)$, and Itô's formula (Theorem 33, Chapter II of [48]) is applicable.

Fix $g \in C_b^2([0,\infty))$. Then, define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = g(x - y), \quad y \leq x.$$

Note how

$$\frac{\partial f}{\partial x}(x, y) = g'(x - y), \quad \frac{\partial f}{\partial y}(x, y) = -g'(x - y), \quad \frac{\partial^2 f}{\partial x \partial y}(x, y) = -g''(x - y)$$

and

$$\frac{\partial^2 f}{\partial x^2}(x,y) = g''(x-y), \quad \frac{\partial^2 f}{\partial y^2}(x,y) = g''(x-y)$$

are all continuous and bounded (thanks to g being in C_b^2). The aforementioned Itô's formula then tells us that for $t \geq 0$,

$$\begin{aligned}
&f(X_t, Z_t) - f(X_0, Z_0) \\
&= \int_0^t \frac{\partial f}{\partial x}(X_{s-}, Z_{s-}) dX_s + \int_0^t \frac{\partial f}{\partial y}(X_{s-}, Z_{s-}) d\underline{\mathrm{X}}_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(X_{s-}, Z_{s-}) d[X,X]_s^c \\
&+ \int_0^t \frac{\partial^2 f}{\partial x \partial y}(X_{s-}, Z_{s-}) d[X,Z]_s^c + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial y^2}(X_{s-}, Z_{s-}) d[Z,Z]_s^c \\
&+ \sum_{0<s\leq t} \left(\Delta f(X_s, Z_s) - \frac{\partial f}{\partial x}(X_{s-}, Z_{s-}) \cdot \Delta X_s - \frac{\partial f}{\partial y}(X_{s-}, Z_{s-}) \cdot \Delta Z_s\right),
\end{aligned}$$

where $[\cdot,\cdot]^c$ denotes the path-by-path continuous part of the quadratic (co-)variation process $[\cdot,\cdot]$.

Since $\underline{\mathrm{X}}$ is a semimartingale with continuous paths of finite variation, by Corollary 22.1, Chapter II of [48] we have $[Z,Z] = 0$. Because $\underline{\mathrm{X}}$ is continuous (and therefore makes no jumps), $[Z,Z] = [Z,Z]^c$, by definition. Thus,

$$[Z,Z]^c = 0. \tag{6.16}$$

Consequently, by Theorem 28, Chapter II of [48] we have $[X,Z] = 0$. Moreover, since $\underline{\mathrm{X}}$ makes no jumps, $\Delta[X,Z]$ – which we know from Theorem 23 (i), Chapter II of [48] is nothing other than $\Delta X_s \cdot \Delta Z_s$ – is equal to 0. These two facts yield

$$[Z,X]^c = 0 \tag{6.17}$$

as well.

Now, thanks to the independence of the $X^{(i)}$ and arguments similar to those we've already seen above, note that if we set $Y_t := bt + X_t^{(2)} + X_t^{(3)}$, for $t \geq 0$ – a known pure-jump semimartingale (i.e., $[Y,Y]^c = 0$), as was explained in Chapter 2.6 of [48] – then,

$$\begin{aligned}
[X,X]_t &= \sigma^2[B,B]_t + [Y,Y]_t \\
&= \sigma^2 t + [Y,Y]_t^c + \sum_{s\leq t}(\Delta Y_s)^2 \\
&= \sigma^2 t + \sum_{s\leq t}(\Delta X_s^{(2)})^2 + \sum_{s\leq t}(\Delta X_s^{(3)})^2 + 2\sum_{s\leq t}\Delta X_s^{(2)} \cdot \Delta X_s^{(3)} \\
&= \sigma^2 t + \sum_{s\leq t}(\Delta X_s)^2 \mathbb{1}_{\{|\Delta X_s|>1\}}(s) + \sum_{s\leq t}(\Delta X_s)^2 \mathbb{1}_{\{|\Delta X_s|\leq 1\}}(s) \qquad (\text{since } X^{(2)} \perp X^{(3)})
\end{aligned}$$

$$= \sigma^2 t + \sum_{s\le t} (\Delta X_s)^2,$$

where we have used the classic result $[B,B]_t = t$, whose proof can be found in, e.g., [46]; and the result $\Delta[Y,Y] = (\Delta Y)^2$ from Theorem 22, Chapter II of [48]. Thus,

$$[X,X]^c_t = [X,X]_t - \sum_{s\le t} (\Delta X_s)^2 = \sigma^2 t, \qquad t \ge 0, \tag{6.18}$$

Having established (6.16), (6.17) and (6.18), by writing our specific f in terms of g, and using the fact that $M_t := \int_0^t g'(R_{s-}) dX^{(3)}_s$, $t \ge 0$, is a local martingale thanks to the fact that stochastic integration preserves local martingales (cf., Theorem 20, Chapter II of [48]); and that, by Corollary 3.2.6, $V_t := \int_0^t g'(R_{s-}) dB_s$, $t \ge 0$, is a martingale, we can now further evaluate our earlier formula for $f(X_t, Z_t) - f(X_0, Z_0)$ thus:

$$\begin{aligned}
&g(R_t) - g(R_0) \\
&= b\int_0^t g'(R_s)ds + \sigma\int_0^t g'(R_{s-})dB_s + \int_0^t g'(R_{s-})dX^{(2)}_s + \int_0^t g'(R_{s-})dX^{(3)}_s - \int_0^t g'(R_{s-})dZ_s \\
&\quad + \frac{\sigma^2}{2}\int_0^t g''(R_{s-})ds + \sum_{0<s\le t} \left(g(R_s) - g(R_{s-}) - g'(R_{s-})\cdot \Delta X_s\right) \\
&= b\int_0^t g'(R_s)ds + \sigma\int_0^t g'(R_{s-})dB_s + \int_0^t g'(R_{s-})dX^{(2)}_s + \int_0^t g'(R_{s-})dX^{(3)}_s \\
&\quad - \int_0^t g'(R_{s-})\mathbb{1}_{(\tau^{\{0\}},t)}(s)d\underline{X}_s + \frac{\sigma^2}{2}\int_0^t g''(R_{s-})ds + \sum_{0<s\le t} \left(g(R_s) - g(R_{s-}) - g'(R_{s-})\cdot \Delta X_s\right) \\
&= b\int_0^t g'(R_s)ds + \sigma V_t + \sum_{s\le t} g'(R_{s-})\cdot \Delta X_s \mathbb{1}_{\{|\Delta X_s|>1\}} + M_t + \frac{\sigma^2}{2}\int_0^t g''(R_s)ds \\
&\quad - g'(0)\int_0^t \mathbb{1}_{(\tau^{\{0\}},t)}(s)\mathbb{1}_{\{s:\underline{X}_{s-}=X_{s-}\}}(s)d\underline{X}_s \\
&\quad + \int_{(0,t]\times(0,\infty)} (g(R_{s-}+x) - g(R_{s-}) - g'(R_{s-})x)N(ds\times dx),
\end{aligned}$$

where the second equality follows from the fact that Z is constant zero up to the first time that X reaches the origin, denoted by $\tau^{\{0\}}$, and thereafter its path is exactly that of $\underline{X}$; and the third equality follows from the fact that the support of the Stieltjes measure $d\underline{X}$ coincides with that of the Stieltjes measure dL, where $L := x - \underline{X}$ is local time at the minimum of X – and we know that the latter's support is the closure of the (random) set of times $\{t \ge 0 : X_t = \underline{X}_t\}$, which, since X makes no jumps during these times (due to spectral-positivity), is identical to the set of times

$\{s \geq 0 : R_{s-} = 0\}$. Additionally, by Corollary 4.6 of [36],

$$\tilde{M}_t := \int_{(0,t]\times(0,\infty)} (g(R_{s-}+x) - g(R_{s-}) - g'(R_{s-})x) N(ds \times dx) \\ - \int_0^t \int_0^\infty (g(R_s + x) - g(R_s) - g'(R_s)x)\Pi(dx)ds, \qquad t \geq 0,$$

where Π is the Lévy measure of X, is a known martingale as well. Thus,

$$g(R_t) - g(R_0) = \sigma V_t + M_t + \tilde{M}_t + \bar{M}_t + \int_0^t \mathcal{L}g(R_s)ds \\ - g'(0) \int_0^t \mathbb{1}_{(\tau^{\{0\}},t)}(s) \mathbb{1}_{\{s:\underline{X}_{s-}=X_{s-}\}} d\underline{X}_s, \qquad t \geq 0,$$

where

$$\bar{M}_t := \int_{(0,t]\times(1,\infty)} g'(R_{s-})x\big(N(ds \times dx) - \Pi(dx)ds\big), \qquad t \geq 0,$$

is a martingale, by Corollary 4.6 of [36]. It now follows that for $g \in C_b^2([0,\infty))$ with $g'(0) = 0$,

$$g(R_t) - g(R_0) - \int_0^t \mathcal{L}g(R_s)ds, \qquad t \geq 0,$$

is a local martingale. To establish that the above is an actual martingale it remains to show that M is not just a local, but an actual, martingale. Thanks to Theorem 51, Chapter II of [48], it suffices to prove that for every $t \geq 0$,

$$\mathbb{E}[\sup_{s\leq t} |M_s|] < \infty.$$

We first note that, since g has bounded derivatives, the inequality

$$|M_s| \leq \|g'\|_\infty \cdot |X_t^{(3)}|, \qquad s \leq t,$$

holds; and so

$$\mathbb{E}[\sup_{s\leq t} |M_s|] \leq \|g'\|_\infty \cdot \mathbb{E}[\sup_{s\leq t} |X_s^{(3)}|].$$

Now,

$$\mathbb{E}[\sup_{s\leq t} |X_s^{(3)}|] = \mathbb{E}\Big[\sup_{s\leq t} |X_s^{(3)}| \mathbb{1}_{\{\sup_{s\leq t}|X_s^{(3)}|\leq 1\}}\Big] + \mathbb{E}\Big[\sup_{s\leq t} |X_s^{(3)}| \mathbb{1}_{\{\sup_{s\leq t}|X_s^{(3)}|> 1\}}\Big] \\ \leq 1 + \mathbb{E}\Big[(\sup_{s\leq t} |X_s^{(3)}|)^2\Big] = 1 + \mathbb{E}\Big[\sup_{s\leq t} |X_s^{(3)}|^2\Big]$$

$$\leq 1 + 4\mathbb{E}[|X_t^{(3)}|^2] < \infty,$$

where we have used in the second line above the fact that for any sequence with non-negative terms the supremum of the terms squared is equal to the square of their supremum; and on the final line we've used Doob's maximal inequality for square-integrable martingales. ∎

6.3.2 Infinitesimal generator of a Skorokhod-reflected d-dimensional Spectrally-Positive Lévy Process

Our second step is to obtain the corresponding mathematical objects of the previous Section 6.3.1 in the higher-dimensional setting, now for the Skorokhod-reflection of the process $X = (X^{(1)}, \dots, X^{(d)})$, where the $X^{(i)} = (X_t^{(i)})_{t\geq 0}$ are independent one-dimensional spectrally-positive Lévy processes – the method is analogous and so we many intermediate steps. To this end, let us use the following Lévy-Itô decomposition of $X^{(i)}$:

$$X^{(i)} = X^{(i,1)} + X^{(i,2)} + X^{(i,3)},$$

where $X^{(i,1)} = (X^{(i,1)})_{t\geq 0}$ is the process defined by $X_t^{(i,1)} := b_i t + \sigma_i B_t^{(i)}$, $t \geq 0$, with $b_i \in \mathbb{R}$, $\sigma_i > 0$ and $B^{(i)} = (B_t^{(i)})_{t\geq 0}$ a (one-dimensional) standard Brownian motion; the process $X^{(i,2)} = (X_t^{(i,2)})_{t\geq 0}$ with

$$X_t^{(i,2)} = \int_{(0,t]\times[1,\infty)} x N^{(i)}(ds, dx), \qquad t \geq 0,$$

where $N^{(i)}$ denotes the Poisson random measure associated with the jumps of $X^{(i)}$, is the compound Poisson process responsible for the jumps of $X^{(i)}$ that are of size bigger than 1 – independent of $X^{(i,1)}$; and $X^{(i,3)} = (X_t^{(i,3)})_{t\geq 0}$ is the square-integrable martingale responsible for the jumps of $X^{(i)}$ that are of size smaller than 1 – independent of both $X^{(i,1)}$ and $X^{(i,2)}$.

Theorem 6.3.3 *Let $X = (X_t)_{t\geq 0}$, with $X_t = (X_t^{(1)}, \dots, X_t^{(d)})$, where the $X^{(i)}$ are independent one-dimensional spectrally-positive Lévy processes, each initiated from some positive state $x^{(i)}$, $1 \leq i \leq d$, with Lévy-Itô decomposition as given above. Denote the Skorokhod-reflection of X by the process $R = (R^{(1)}, \dots, R^{(d)})$, where $R^{(i)} := X^{(i)} - (0 \wedge \underline{X}^{(i)})$ for $1 \leq i \leq d$, and as usual $\underline{X}^{(i)}$ denotes the running infimum process of $X^{(i)}$. Then, for $g \in C_b^2([0,\infty)^d)$ satisfying the boundary conditions*

$$\frac{\partial g}{\partial z_i}(z_1, \dots, z_{i-1}, 0, z_{i+1}, \dots, z_d) = 0 \ \ \forall z_k > 0, \qquad i = 1, \dots, d,$$

the process

$$g(R_t) - g(R_0) - \int_0^t \mathcal{L} g(R_s) ds, \qquad t \geq 0,$$

is a martingale, where the operator $\mathcal{L}$ is given on this class of functions by

$$\mathcal{L}F(\boldsymbol{w}) = \sum_{i=1}^{d} \Big(b_i \frac{\partial F}{\partial z_i}(\boldsymbol{w}) + \frac{\sigma_i^2}{2} \frac{\partial^2 F}{\partial z_i^2}(\boldsymbol{w}) + \int_0^\infty (F(\boldsymbol{w} + x\boldsymbol{e}_i) - F(\boldsymbol{w}) - \frac{\partial F}{\partial z_i}(\boldsymbol{w}) x \mathbb{1}_{(0,1)}(x)) \Pi^{(i)}(dx) \Big),$$

where $\Pi^{(i)}$ denotes the Lévy measure of $X^{(i)}$.

Proof Without loss of generality, and in order to simplify notations, we choose $x^{(i)} = 0$ so that $R^{(i)} = X^{(i)} - \underline{X}^{(i)}$ for $1 \leq i \leq d$ (we had already gone through in great detail in our proof of Theorem 6.3.2 how the proof can be adapted accordingly for when the initial states are positive). Fix a function $g \in C_b^2([0,\infty)^d)$ and consider the $C_b^2(\mathbb{R}^{2d})$ function f given by

$$f(x_1, y_1, x_2, y_2, \ldots, x_d, y_d) = g(x_1 - y_1, x_2 - y_2, \ldots, x_d - y_d), \quad y_i \leq x_i$$

By Theorem 33, Chapter 2 of [48],

$$\begin{aligned}
& f(X_t^{(1)}, \underline{X}_t^{(1)}, X_t^{(2)}, \underline{X}_t^{(2)}, \ldots, X_t^{(d)}, \underline{X}_t^{(d)}) - f(X_0^{(1)}, \underline{X}_0^{(1)}, X_0^{(2)}, \underline{X}_0^{(2)}, \ldots, X_0^{(d)}, \underline{X}_0^{(d)}) \\
&= \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_i}(X_{s-}^{(1)}, \underline{X}_{s-}^{(1)}, \ldots, X_{s-}^{(d)}, \underline{X}_{s-}^{(d)}) dX_s^{(i)} + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial y_i}(X_{s-}^{(1)}, \underline{X}_{s-}^{(1)}, \ldots, X_{s-}^{(d)}, \underline{X}_{s-}^{(d)}) d\underline{X}_s^{(i)} \\
&\quad + \frac{1}{2} \sum_{i=1}^{d} \int_0^t \frac{\partial^2 f}{\partial x_i^2}(X_{s-}^{(1)}, \underline{X}_{s-}^{(1)}, \ldots, X_{s-}^{(d)}, \underline{X}_{s-}^{(d)}) d[X^{(i)}, X^{(i)}]_s^c \\
&\quad + \frac{1}{2} \sum_{i=1}^{d} \int_0^t \frac{\partial^2 f}{\partial y_i^2}(X_{s-}^{(1)}, \underline{X}_{s-}^{(1)}, \ldots, X_{s-}^{(d)}, \underline{X}_{s-}^{(d)}) d[\underline{X}^{(i)}, \underline{X}^{(i)}]_s^c \\
&\quad + \sum_{1 \leq i < j \leq d} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s-}^{(1)}, \underline{X}_{s-}^{(1)}, \ldots, X_{s-}^{(d)}, \underline{X}_{s-}^{(d)}) d[X^{(i)}, X^{(j)}]_s^c \\
&\quad + \sum_{1 \leq i < j \leq d} \int_0^t \frac{\partial^2 f}{\partial y_i \partial y_j}(X_{s-}^{(1)}, \underline{X}_{s-}^{(1)}, \ldots, X_{s-}^{(d)}, \underline{X}_{s-}^{(d)}) d[\underline{X}^{(i)}, \underline{X}^{(j)}]_s^c \\
&\quad + \sum_{i=1}^{d} \int_0^t \frac{\partial^2 f}{\partial x_i \partial y_i}(X_{s-}^{(1)}, \underline{X}_{s-}^{(1)}, \ldots, X_{s-}^{(d)}, \underline{X}_{s-}^{(d)}) d[X^{(i)}, \underline{X}^{(i)}]_s^c \\
&\quad + \sum_{1 \leq i < j \leq d} \int_0^t \frac{\partial^2 f}{\partial x_i \partial y_j}(X_{s-}^{(1)}, \underline{X}_{s-}^{(1)}, \ldots, X_{s-}^{(d)}, \underline{X}_{s-}^{(d)}) d[X^{(i)}, \underline{X}^{(j)}]_s^c \\
&\quad + \sum_{s \leq t} (\Delta f(X_s^{(1)}, \underline{X}_s^{(1)}, \ldots, X_s^{(d)}, \underline{X}_s^{(d)}) - \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(X_{s-}^{(1)}, \underline{X}_{s-}^{(1)}, \ldots, X_{s-}^{(d)}, \underline{X}_{s-}^{(d)}) \Delta X_s^{(i)} \\
&\qquad\qquad - \sum_{i=1}^{d} \frac{\partial f}{\partial y_i}(X_{s-}^{(1)}, \underline{X}_{s-}^{(1)}, \ldots, X_{s-}^{(d)}, \underline{X}_{s-}^{(d)}) \Delta \underline{X}_s^{(i)}).
\end{aligned}$$

By the exact same arguments that brought about equations (6.16)–(6.18) in the proof of Theorem 6.3.2, we also have $[\underline{X}^{(i)}, \underline{X}^{(i)}]^c = 0$, $[\underline{X}^{(i)}, X^{(i)}]^c = 0$ and $[X^{(i)}, X^{(i)}]^c = \sigma_i^2 t$ for every i. Additionally, by the independence of the $X^{(i)}$, $[X^{(i)}, X^{(j)}]^c = [\underline{X}^{(i)}, \underline{X}^{(j)}]^c = [\underline{X}^{(i)}, X^{(j)}]^c = 0$ for every $i \neq j$. Using also the fact that $\underline{X}^{(i)}$ makes no jumps due to the fact that $X^{(i)}$ is spectrally-positive, the previous equation reduces to

$$
\begin{aligned}
&f(X_t^{(1)}, \underline{X}_t^{(1)}, X_t^{(2)}, \underline{X}_t^{(2)}, \ldots, X_t^{(d)}, \underline{X}_t^{(d)}) - f(X_0^{(1)}, \underline{X}_0^{(1)}, X_0^{(2)}, \underline{X}_0^{(2)}, \ldots, X_0^{(d)}, \underline{X}_0^{(d)}) \\
&= \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(X_{s-}^{(1)}, \underline{X}_{s-}^{(1)}, \ldots, X_{s-}^{(d)}, \underline{X}_{s-}^{(d)}) dX_s^{(i)} + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial y_i}(X_{s-}^{(1)}, \underline{X}_{s-}^{(1)}, \ldots, X_{s-}^{(d)}, \underline{X}_{s-}^{(d)}) d\underline{X}_s^{(i)} \\
&\quad + \sum_{i=1}^d \frac{\sigma_i^2}{2} \int_0^t \frac{\partial^2 f}{\partial x_i^2}(X_{s-}^{(1)}, \underline{X}_{s-}^{(1)}, \ldots, X_{s-}^{(d)}, \underline{X}_{s-}^{(d)}) ds \\
&\quad + \sum_{s \le t} (\Delta f(X_s^{(1)}, \underline{X}_s^{(1)}, \ldots, X_s^{(d)}, \underline{X}_s^{(d)}) - \sum_{i=1}^d \frac{\partial f}{\partial x_i}(X_{s-}^{(1)}, \underline{X}_{s-}^{(1)}, \ldots, X_{s-}^{(d)}, \underline{X}_{s-}^{(d)}) \Delta X_s^{(i)}).
\end{aligned}
$$

By employing the definition of our function f in terms of the fixed function g and making use of our Lévy-Itô decomposition of $X^{(i)}$ in the same way we did in our proof of Theorem 6.3.2, it is not difficult to see that the above equation is equivalent to the following equation:

$$
\begin{aligned}
&g(R_t) - g(R_0) \\
&= \sum_{i=1}^d \int_0^d \frac{\partial g}{\partial z_i}(R_{s-}) dX_s^{(i)} - \sum_{i=1}^d \int_0^t \frac{\partial g}{\partial z_i}(R_{s-}) d\underline{X}_s^{(i)} + \sum_{i=1}^d \frac{\sigma_i^2}{2} \int_0^t \frac{\partial^2 g}{\partial dz_i^2}(R_s) ds \\
&\quad + \sum_{s \le t} (g(R_s) - g(R_{s-}) - \sum_{i=1}^d \frac{\partial g}{\partial z_i}(R_{s-}) \cdot \Delta X_s^{(i)}) \\
&= \sum_{i=1}^d b_i \int_0^t \frac{\partial g}{\partial z_i}(R_s) ds + \sum_{i=1}^d (\sigma_i V_t^{(i)} + M_t^{(i)}) + \sum_{i=1}^d \sum_{s \le t} \frac{\partial g}{\partial z_i}(R_{s-}) \cdot \Delta X_s^{(i)} \cdot \mathbb{1}_{\{|\Delta X_s^{(i)}| > 1\}} \\
&\quad - \sum_{i=1}^d \int_0^t \frac{\partial g}{\partial z_i}(R_{s-}) \cdot \mathbb{1}_{\{s : X_s^{(i)} = \underline{X}_s^{(i)}\}}(s) d\underline{X}_s^{(i)} + \sum_{i=1}^d \frac{\sigma_i^2}{2} \int_0^t \frac{\partial^2 g}{\partial dz_i^2}(R_s) ds \\
&\quad + \sum_{s \le t} (g(R_s) - g(R_{s-}) - \sum_{i=1}^d \frac{\partial g}{\partial z_i}(R_{s-}) \cdot \Delta X_s^{(i)}) \\
&= \sum_{i=1}^d b_i \int_0^t \frac{\partial g}{\partial z_i}(R_s) ds + \int_{(0,t] \times \mathbb{R}^d \setminus \{\mathbf{0}_d\}} (\sum_{i=1}^d \frac{\partial g}{\partial z_i}(R_{s-}) \cdot x_i \cdot \mathbb{1}_{(1,\infty)}(x_i)) N(ds, d\boldsymbol{x}) \\
&\quad + \sum_{i=1}^d (\sigma_i V_t^{(i)} + M_t^{(i)}) - \sum_{i=1}^d \int_0^t \frac{\partial g}{\partial z_i}(R_{s-}^{(1)}, \ldots, R_{s-}^{(i-1)}, 0, R_{s-}^{(i+1)}, \ldots, R_{s-}^{(d)}) \cdot \mathbb{1}_{\{s : X_s^{(i)} = \underline{X}_s^{(i)}\}}(s) d\underline{X}_s^{(i)}
\end{aligned}
$$

$$+\sum_{i=1}^d \frac{\sigma_i^2}{2}\int_0^t \frac{\partial^2 g}{\partial dz_i^2}(R_s)ds + \int_{(0,t]\times\mathbb{R}^d\setminus\{\mathbf{0}_d\}} (g(R_{s-}+\boldsymbol{x}) - g(R_{s-}) - \sum_{i=1}^d \frac{\partial g}{\partial z_i}(R_{s-})\cdot x_i)N(ds,d\boldsymbol{x}),$$

where N is the Poisson random measure associated with the jumps of X, $V_t^{(i)} := \int_0^t \frac{\partial g}{\partial z_i}(R_{s-})dB_s^{(i)}$, $t \geq 0$, is a known martingale, and $M_t^{(i)} := \int_0^t \frac{\partial g}{\partial z_i}(R_{s-})dX_s^{(i,3)}$, $t \geq 0$ – as we had also seen in our proof of Theorem 6.3.2 – is also an (actual) martingale.

By using the fact that only one coordinate, $X^{(i)}$, can jump at a time (by independence), and the fact that $\underline{\mathrm{X}}^{(i)}$ has continuous paths, we get

$$\begin{aligned}
&g(R_t) - g(R_0)\\
&= \sum_{i=1}^d b_i \int_0^t \frac{\partial g}{\partial z_i}(R_s)ds + \sum_{i=1}^d \frac{\sigma_i^2}{2}\int_0^t \frac{\partial^2 g}{\partial dz_i^2}(R_s)ds + \int_{(0,t]\times(1,\infty)} (\sum_{i=1}^d \frac{\partial g}{\partial z_i}(R_{s-})\cdot x)N^{(i)}(ds,dx)\\
&\quad + \sum_{i=1}^d(\sigma_i V_t^{(i)} + M_t^{(i)}) - \sum_{i=1}^d \int_0^t \frac{\partial g}{\partial z_i}(R_{s-}^{(1)},\ldots,R_{s-}^{(i-1)},0,R_{s-}^{(i+1)},\ldots,R_{s-}^{(d)})\cdot \mathbb{1}_{\{s: X_s^{(i)} = \underline{\mathrm{X}}_s^{(i)}\}}(s)d\underline{\mathrm{X}}_s^{(i)}\\
&\quad + \sum_{i=1}^d \int_{(0,t]\times(0,\infty)} (g(R_{s-}+x\boldsymbol{e}_i) - g(R_{s-}) - \frac{\partial g}{\partial z_i}(R_{s-})\cdot x)N^{(i)}(ds,dx)\\
&= \sum_{i=1}^d b_i \int_0^t \frac{\partial g}{\partial z_i}(R_s)ds + \sum_{i=1}^d(\sigma_i V_t^{(i)} + M_t^{(i)} + \bar{M}_t^{(i)} + \tilde{M}_t^{(i)}) + \sum_{i=1}^d \frac{\sigma_i^2}{2}\int_0^t \frac{\partial^2 g}{\partial dz_i^2}(R_s)ds\\
&\quad - \sum_{i=1}^d \int_0^t \frac{\partial g}{\partial z_i}(R_{s-}^{(1)},\ldots,R_{s-}^{(i-1)},0,R_{s-}^{(i+1)},\ldots,R_{s-}^{(d)})\cdot \mathbb{1}_{\{s: X_s^{(i)} = \underline{\mathrm{X}}_s^{(i)}\}}(s)d\underline{\mathrm{X}}_s^{(i)}\\
&\quad + \sum_{i=1}^d \int_0^t \int_0^\infty (g(R_s + x\boldsymbol{e}_i) - g(R_s) - \frac{\partial g}{\partial z_i}(R_s)x\mathbb{1}_{(0,1)}(x))\Pi^{(i)}(dx)ds,
\end{aligned}$$

where $N^{(i)}$ is the Poisson random measure associated with the jumps of $X^{(i)}$ and $\Pi^{(i)}$ is the respective Lévy measure; and

$$\bar{M}_t^{(i)} := \int_{(0,t]\times(1,\infty)} \frac{\partial g}{\partial z_i}(R_{s-})\cdot x(N^{(i)}(ds,dx) - \Pi^{(i)}(dx)ds), \qquad t \geq 0,$$

and

$$\tilde{M}_t^{(i)} := \int_0^t \int_0^\infty (g(R_{s-}+x\boldsymbol{e}_i) - g(R_{s-}) - \frac{\partial g}{\partial z_i}(R_{s-})x)(N^{(i)}(ds,dx) - \Pi^{(i)}(dx)ds), \qquad t \geq 0,$$

are both martingales, by Corollary 4.6 of [36]. The result now follows immediately. ∎

6.3.3 Infinitesimal Generator of the Underlying MAP of a Skorokhod-Reflected d-dimensional Spectrally-Positive Stable Process

Before stating our results we need to introduce and explain some unconventional terminology and notation which we employ in the remainder of this section (as well as Chapter 7). Because the MAP we are concerned with takes values in the product space $\mathbb{R} \times \mathcal{S}^{d,+}_{\|\cdot\|}$, which is a subset of $\mathbb{R}^{d+1}$, we need to come up with a more convenient/natural way of expressing the $(d+1)$-dimensional vector

$$(y, \boldsymbol{\theta}) = (y, \theta^{(1)}, \ldots, \theta^{(d)}) \in \mathbb{R} \times \mathcal{S}^{d,+}_{\|\cdot\|},$$

through the orthonormal basis of $\mathbb{R}^{d+1}$. To this end, we shall call the first (traditional) coordinate thereof (containing the entry $y \in \mathbb{R}$) *the* 0*-th coordinate* of the vector; and, for $1 \leq j \leq d$, the $(j+1)$-th (traditional) coordinate thereof (containing the entry $\theta^{(j)}$) *the* j*-th coordinate* of the vector. This being the case, we define the $(d+1)$-dimensional vector $\boldsymbol{e}_0$ to be the vector

$$\boldsymbol{e}_0 \coloneqq (1, \underbrace{0, \ldots, 0}_{d\text{-many}});$$

and the $(d+1)$-dimensional vector $\boldsymbol{e}_j$, for $1 \leq j \leq d$, to be the vector

$$\boldsymbol{e}_j \coloneqq (0, \underbrace{0, \ldots, 0, 1, 0, \ldots, 0}_{d\text{-many coordinates}}),$$

where the "1"-entry is the (traditional) $(j+1)$-th entry of the above vector. The collection of vectors $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\}$ is then, actually, nothing other than the standard orthonormal basis of $\mathbb{R}^{d+1}$.

Definition 6.3.4 *We say that a function $f \in C^2_b(\mathbb{R} \times \mathcal{S}^{d,+}_{\|\cdot\|})$, i.e., a real-valued function with bounded and continuous partial derivatives up to the order* 2 *with domain $\mathbb{R} \times \mathcal{S}^{d,+}_{\|\cdot\|} \subseteq \mathbb{R}^{d+1}$, is of class $\mathcal{D}^d_{\|\cdot\|}$ if it satisfies for each $1 \leq i \leq d$ and $\boldsymbol{w} = (w_1, \ldots, w_d) \in [0, \infty)^d \setminus \{\mathbf{0}_d\}$ with $w_i = 0$, the condition*

$$\Big(\frac{\partial}{\partial z_i}(\|\boldsymbol{w}\|)\boldsymbol{e}_0 + \boldsymbol{e}_i - \frac{\partial}{\partial z_i}(\|\boldsymbol{w}\|) \sum_{\substack{1 \leq j \leq d \\ j \neq i}} \arg(\boldsymbol{w})^{(j)} \boldsymbol{e}_j \Big) \cdot \boldsymbol{\nabla} f\big(\log\|\boldsymbol{w}\|, \arg(\boldsymbol{w})\big) = 0, \tag{6.19}$$

where the set of vectors $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\}$ is just the standard orthonormal basis of $\mathbb{R}^{d+1}$, and as usual $\arg(\boldsymbol{w})^{(j)}$ denotes the j-th coordinate of the vector $\arg(\boldsymbol{w}) \coloneqq \frac{1}{\|\boldsymbol{w}\|}\boldsymbol{w}$.

We now state the main result of this section regarding the MAP (ξ^R, Ξ^R). In essence what we do is we make use of the classic Volkonskii formula (cf., (21.4) of Chapter III.3, [51]) and basic calculus to "convert" the result of Theorem 6.3.3 to the analogous one for (ξ^R, Ξ^R), since the latter is a particular time-changed transformation of R.

Theorem 6.3.5 *Let $\|\cdot\|$ be some norm on $\mathbb{R}^d$. Let $(\xi^R, \Xi^R) = (\xi^R_t, \Xi^R_t)_{t\geq 0}$ be the underlying MAP of R from (6.15) with respect to the norm $\|\cdot\|$. Then, for $f \in C_b^2(\mathbb{R} \times \mathcal{S}^{d,+}_{\|\cdot\|})$ of class $\mathcal{D}^d_{\|\cdot\|}$, the process*

$$f(\xi^R_t, \Xi^R_t) - f(\xi^R_0, \Xi^R_0) - \int_0^t Af(\xi^R_s, \Xi^R_s)ds, \qquad t \geq 0,$$

is a martingale, where the operator A is given on this class of functions by

$$\begin{aligned} &Af(x, \boldsymbol{\theta}) \\ &= c_1 e^{\alpha x} \int_0^\infty \Bigg\{ \sum_{i=1}^d \Bigg(f\big(\log \|e^x \boldsymbol{\theta} + y\boldsymbol{e}_i\|, \arg(e^x\boldsymbol{\theta} + y\boldsymbol{e}_i)\big) - f(x, \boldsymbol{\theta}) \\ &\quad - \Big[\Big(e^x \frac{\partial}{\partial z_i}(\|\boldsymbol{\theta}\|)\boldsymbol{e}_0 + (1 - e^x\theta^{(i)}\frac{\partial}{\partial z_i}(\|\boldsymbol{\theta}\|))\boldsymbol{e}_i - e^x \frac{\partial}{\partial z_i}(\|\boldsymbol{\theta}\|) \sum_{\substack{1\leq j\leq d \\ j\neq i}} \theta^{(j)}\boldsymbol{e}_j\Big) \cdot \boldsymbol{\nabla} f(x, \boldsymbol{\theta})\Big] \times \\ &\qquad \times y e^{-x} \mathbb{1}_{(0,1)}(y) \Bigg) \Bigg\} \frac{1}{y^{1+\alpha}} dy, \end{aligned}$$

where $x \in \mathbb{R}$, $\boldsymbol{\theta} = (\theta^{(1)}, \ldots, \theta^{(d)}) \in \mathcal{S}^{d,+}_{\|\cdot\|}$, and $c_1 = \pi^{-1}\Gamma(1 + \alpha)\sin(\pi\alpha)$.

Proof Because this theorem concerns $X^{(i)}$ being one-dimensional spectrally-positive α-stable processes – which we know do not possess a linear Brownian motion component in their Lévy-Itô decomposition – we take $b_i = \sigma_i = 0$ in the statement of Theorem 6.3.3. By Volkonskii's formula (cf., (21.4) of Chapter III.3, [51]), for functions $g \in C_b^2([0, \infty)^d)$ satisfying the boundary conditions from the statement of Theorem 6.3.3, the process

$$g(R_{I_t}) - g(R_{I_0}) - \int_0^t \mathcal{A}g(R_{I_s})ds, \qquad t \geq 0,$$

is a martingale, where the operator $\mathcal{A}$ is given on this class of functions by

$$\mathcal{A}g(\boldsymbol{w}) = \|\boldsymbol{w}\|^\alpha \mathcal{L}g(\boldsymbol{w}),$$

where $\mathcal{L}$ is the operator from the statement of Theorem 6.3.3 (with $b_i = \sigma_i = 0$, and $\Pi^{(i)}$ being the respective Lévy measure of the $X^{(i)}$ from the setting of this theorem), and $I_t \coloneqq \inf\{s > 0 : \int_0^s \|R_u\|^{-\alpha} du > t\}$, $t \geq 0$, is the analogous transform from the second part of Theorem 4.2.1 for the process R.

Let us now consider the $f \in C_b^2(\mathbb{R} \times \mathcal{S}^{d,+}_{\|\cdot\|})$ from our theorem hypothesis. We define the following $C_b^2([0,\infty)^d)$-function g:

$$g : [0,\infty)^d \setminus \{\mathbf{0}_d\} \to \mathbb{R}, \quad g(\boldsymbol{w}) = f(\log\|\boldsymbol{w}\|, \arg(\boldsymbol{w})). \tag{6.20}$$

We claim that g satisfies the boundary conditions from the statement of Theorem 6.3.3. Indeed, by writing out the function g explicitly,

$$g(\boldsymbol{w}) = f\big(\log\|\boldsymbol{w}\|, \arg(\boldsymbol{w})\big) = f(\log\|\boldsymbol{w}\|, \frac{w_1}{\|\boldsymbol{w}\|}, \frac{w_2}{\|\boldsymbol{w}\|}, \dots, \frac{w_d}{\|\boldsymbol{w}\|}),$$

and applying the chain rule, for $\boldsymbol{w} = (w_1, \dots, w_d) \in [0,\infty)^d \setminus \{\mathbf{0}_d\}$ and $i \in \{1, \dots, d\}$, one gets

$$\begin{aligned}
\frac{\partial g}{\partial z_i}(\boldsymbol{w}) &= \frac{\partial}{\partial z_i}(\log\|\boldsymbol{w}\|) f_{y_0}(\log\|\boldsymbol{w}\|, \arg(\boldsymbol{w})) + \sum_{j=1}^d \frac{\partial}{\partial z_i}(\arg(\boldsymbol{w})^{(j)}) f_{y_j}(\log\|\boldsymbol{w}\|, \arg(\boldsymbol{w})) \\
&= \|\boldsymbol{w}\|^{-1}\frac{\partial}{\partial z_i}(\|\boldsymbol{w}\|) f_{y_0}(\log\|\boldsymbol{w}\|, \arg(\boldsymbol{w})) - \frac{\partial}{\partial z_i}(\|\boldsymbol{w}\|)\sum_{j\neq i}\frac{w_j}{\|\boldsymbol{w}\|^2} f_{y_j}(\log\|\boldsymbol{w}\|, \arg(\boldsymbol{w})) \\
&\quad + \Big(\frac{1}{\|\boldsymbol{w}\|} - \frac{\partial}{\partial z_i}(\|\boldsymbol{w}\|)\frac{w_i}{\|\boldsymbol{w}\|^2}\Big) f_{y_i}(\log\|\boldsymbol{w}\|, \arg(\boldsymbol{w})) \\
&= \|\boldsymbol{w}\|^{-1}\Big(\frac{\partial}{\partial z_i}(\|\boldsymbol{w}\|) f_{y_0}(\log\|\boldsymbol{w}\|, \arg(\boldsymbol{w})) + f_{y_i}(\log\|\boldsymbol{w}\|, \arg(\boldsymbol{w})) \\
&\quad - \frac{\partial}{\partial z_i}(\|\boldsymbol{w}\|)\sum_{j=1}^d \arg(\boldsymbol{w})^{(j)} f_{y_j}(\log\|\boldsymbol{w}\|, \arg(\boldsymbol{w}))\Big).
\end{aligned}$$

It now follows immediately from the theorem's hypothesis on f that g satisfies said boundary conditions. Finally, by writing the above operator $\mathcal{A}$ in the coordinate system given by $(x, \boldsymbol{\theta}) = (\log\|\boldsymbol{w}\|, \arg(\boldsymbol{w}))$, for $\boldsymbol{w} \in [0,\infty)^d \setminus \{\mathbf{0}_d\}$, one obtains the desired operator A from the theorem statement. Indeed, first note that the quantities $\|\boldsymbol{w}\|$, $\frac{\partial}{\partial z_i}(\|\boldsymbol{w}\|)$ and $\boldsymbol{w}$, written in the aforementioned coordinate system, are precisely

$$\|\boldsymbol{w}\| = e^x, \qquad \boldsymbol{w} = e^x\boldsymbol{\theta}, \qquad \frac{\partial}{\partial z_i}(\|\boldsymbol{w}\|) = e^x\frac{\partial}{\partial\theta^{(i)}}(\|\boldsymbol{\theta}\|).$$

It is also immediate that in this new coordinate system, the function $g(\boldsymbol{w})$ from (6.20) is precisely $f(x, \boldsymbol{\theta})$. Therefore, $\frac{\partial g}{\partial z_i}(\boldsymbol{w})$ from above, written in this new coordinate system, is precisely:

$$\begin{aligned}
&e^{-x}\Big(e^x\frac{\partial}{\partial\theta^{(i)}}(\|\boldsymbol{\theta}\|)\cdot\frac{\partial f}{\partial x}(x,\boldsymbol{\theta}) + \big(1 - e^x\theta^{(i)}\frac{\partial}{\partial\theta^{(i)}}(\|\boldsymbol{\theta}\|)\big)\cdot\frac{\partial f}{\partial\theta^{(i)}}(x,\boldsymbol{\theta}) - e^x\frac{\partial}{\partial\theta^{(i)}}(\|\boldsymbol{\theta}\|)\sum_{j\neq i}\theta^{(j)}\frac{\partial f}{\partial\theta^{(j)}}(x,\boldsymbol{\theta})\Big) \\
&\quad = e^{-x}\Big[\Big(e^x\frac{\partial}{\partial\theta^{(i)}}(\|\boldsymbol{\theta}\|)\boldsymbol{e}_0 + \big(1 - e^x\theta^{(i)}\frac{\partial}{\partial\theta^{(i)}}(\|\boldsymbol{\theta}\|)\big)\boldsymbol{e}_i - e^x\theta^{(i)}\frac{\partial}{\partial\theta^{(i)}}\sum_{j\neq i}\theta^{(j)}\boldsymbol{e}_j\Big)\cdot\boldsymbol{\nabla} f(x,\boldsymbol{\theta})\Big].
\end{aligned}$$

Therefore, the action of the operator $\mathcal{A}$ from before on this very function g, expressed in the new coordinate system, is precisely

$$c_1 e^{\alpha x}\int_0^\infty \Bigg\{ \sum_{i=1}^d \Bigg(f\big(\log\|e^x\boldsymbol{\theta}+y\boldsymbol{e}_i\|, \arg(e^x\boldsymbol{\theta}+y\boldsymbol{e}_i)\big) - f(x,\boldsymbol{\theta}) \\ - \Big[\Big(e^x\frac{\partial}{\partial\theta^{(i)}}(\|\boldsymbol{\theta}\|)\boldsymbol{e}_0 + \big(1-e^x\theta^{(i)}\frac{\partial}{\partial\theta^{(i)}}(\|\boldsymbol{\theta}\|)\big)\boldsymbol{e}_i - e^x\theta^{(i)}\frac{\partial}{\partial\theta^{(i)}}\sum_{j\neq i}\theta^{(j)}\boldsymbol{e}_j\Big)\cdot\boldsymbol{\nabla} f(x,\boldsymbol{\theta})\Big]\times \\ \times\, y e^{-x}\mathbb{1}_{(0,1)}(y)\Bigg)\Bigg\}\frac{1}{y^{1+\alpha}}dy,$$

which is exactly the expression of $Af(x,\boldsymbol{\theta})$ given in the statement of the theorem. ∎

As usual, we can get a more explicit closed-form expression for A when it is with respect to the L^p norm, $p \geq 1$. The following identities concerning the L^p norm will prove useful in this endeavour (and will also be extensively used in Chapter 7 for analogous derivations in the setting of Brownian motion).

Lemma 6.3.6 *Let $p \geq 1$, and $\|\cdot\| = \|\cdot\|_p$ be the L^p norm on $\mathbb{R}^d$. Then, for $\boldsymbol{w} \in [0,\infty)^d \setminus \{\mathbf{0}_d\}$ and $i \in \{1,\ldots,d\}$,*

$$\frac{\partial}{\partial z_i}(\|\boldsymbol{w}\|) = (\arg(\boldsymbol{w})^{(i)})^{p-1};$$

$$\frac{\partial^2}{\partial z_i^2}(\|\boldsymbol{w}\|) = (p-1)\|\boldsymbol{w}\|^{-1}((\arg(\boldsymbol{w})^{(i)})^{p-2} - (\arg(\boldsymbol{w})^{(i)})^{2p-2});$$

$$\frac{\partial}{\partial z_i}((\arg(\boldsymbol{w})^{(i)})^p) = p\|\boldsymbol{w}\|^{-1}((\arg(\boldsymbol{w})^{(i)})^{p-1} - (\arg(\boldsymbol{w})^{(i)})^{2p-1}),$$

where $\arg(\boldsymbol{w}) = \frac{1}{\|\boldsymbol{w}\|}\boldsymbol{w} = (\arg(\boldsymbol{w})^{(1)},\ldots,\arg(\boldsymbol{w})^{(d)})$.

Proof The norm $\|\cdot\|$ is none other than the function that takes $\boldsymbol{w} \in \mathbb{R}^d$ and returns $(\sum_{j=1}^d |w_j|^p)^{\frac{1}{p}}$. If each coordinate of $\boldsymbol{w}$ is non-negative, then, by a simple application of the chain rule,

$$\frac{\partial}{\partial z_i}(\|\boldsymbol{w}\|) = w_i^{p-1}\cdot(\sum_{j=1}^d w_j^p)^{\frac{1-p}{p}} = (w_i^{-1}(\sum_{j=1}^d w_j^p)^{\frac{1}{p}})^{1-p} = (\arg(\boldsymbol{w})^{(i)})^{p-1}.$$

Similarly, by the chain and product rules from calculus,

$$\begin{aligned}\frac{\partial^2}{\partial z_i^2}(\|\boldsymbol{w}\|) &= \frac{\partial}{\partial z_i}((\arg(\boldsymbol{w})^{(i)})^{p-1}) \\ &= \frac{\partial}{\partial z_i}(\|\boldsymbol{w}\|^{-(p-1)}w_i^{p-1})\end{aligned}$$

$$= (p-1)w_i^{p-2}\|\boldsymbol{w}\|^{-(p-1)} - (p-1)\|\boldsymbol{w}\|^{-p}(\arg(\boldsymbol{w})^{(i)})^{p-1}w_i^{p-1}$$
$$= (p-1)\|\boldsymbol{w}\|^{-1}\|\boldsymbol{w}\|^{-(p-2)}w_i^{p-2} - (p-1)\|\boldsymbol{w}\|^{-1}\|\boldsymbol{w}\|^{-(p-1)}w_i^{p-1}(\arg(\boldsymbol{w})^{(i)})^{p-1}$$
$$= (p-1)\|\boldsymbol{w}\|^{-1}((\arg(\boldsymbol{w})^{(i)})^{p-2} - (\arg(\boldsymbol{w})^{(i)})^{p-1}(\arg(\boldsymbol{w})^{(i)})^{p-1}).$$

In the same way,

$$\begin{aligned}\frac{\partial}{\partial z_i}((\arg(\boldsymbol{w})^{(i)})^p) &= \frac{\partial}{\partial z_i}(\|\boldsymbol{w}\|^{-p}w_i^p)\\ &= p\|\boldsymbol{w}\|^{-p}w_i^{p-1} - p\|\boldsymbol{w}\|^{-(p+1)}(\arg(\boldsymbol{w})^{(i)})^{p-1}w_i^p\\ &= p\|\boldsymbol{w}\|^{-1}\|\boldsymbol{w}\|^{-(p-1)}w_i^{p-1} - p\|\boldsymbol{w}\|^{-1}\|\boldsymbol{w}\|^{-p}w_i^p(\arg(\boldsymbol{w})^{(i)})^{p-1}\\ &= p\|\boldsymbol{w}\|^{-1}(\|\boldsymbol{w}\|^{-(p-1)}w_i^{p-1} - (\arg(\boldsymbol{w})^{(i)})^{p-1}\|\boldsymbol{w}\|^{-p}w_i^p)\\ &= p\|\boldsymbol{w}\|^{-1}((\arg(\boldsymbol{w})^{(i)})^{p-1} - (\arg(\boldsymbol{w})^{(i)})^{p-1}(\arg(\boldsymbol{w})^{(i)})^p). \quad \blacksquare\end{aligned}$$

Corollary 6.3.7 *Fix $p \geq 1$. Let $(\xi^R, \Xi^R) = (\xi_t^R, \Xi_t^R)_{t\geq 0}$ be the underlying MAP of R from (6.15) with respect to the norm $\|\cdot\|_p$ on $\mathbb{R}^d$. Then, for $f \in C_b^2(\mathbb{R}\times\mathcal{S}_p^{d,+})$ of class $\mathcal{D}^d_{\|\cdot\|_p}$, the process*

$$f(\xi_t^R, \Xi_t^R) - f(\xi_0^R, \Xi_0^R) - \int_0^t Af(\xi_s^R, \Xi_s^R)ds, \qquad t \geq 0,$$

is a martingale, where the operator A is given on this class of functions by

$$\begin{aligned}&Af(x,\boldsymbol{\theta})\\ &= c_1\int_0^\infty \Bigg\{\sum_{i=1}^d \Bigg(f\big(x+y, \boldsymbol{v}_{i,y}^{p,\boldsymbol{\theta}}\big) - f(x,\boldsymbol{\theta})\\ &\quad - \Big[\Big((\theta^{(i)})^{p-1}\boldsymbol{e}_0 + (1-(\theta^{(i)})^p)\boldsymbol{e}_i - (\theta^{(i)})^{p-1}\sum_{\substack{1\leq j\leq d\\ j\neq i}}\theta^{(j)}\boldsymbol{e}_j\Big)\cdot\boldsymbol{\nabla}f(x,\boldsymbol{\theta})\Big]\times\\ &\qquad \times\Big(\Big(e^{py}-1+(\theta^{(i)})^p\Big)^{1/p} - \theta^{(i)}\Big)\mathbb{1}_{\left(0,\gamma_i(p,x)-x\right)}(y)\Bigg)\Bigg\}\frac{e^{py}\Big(e^{py}-1+(\theta^{(i)})^p\Big)^{\frac{1-p}{p}}}{\Big(\Big(e^{py}-1+(\theta^{(i)})^p\Big)^{1/p}-\theta^{(i)}\Big)^{1+\alpha}}dy,\end{aligned}$$

where $x \in \mathbb{R}$ and $\boldsymbol{\theta} = (\theta^{(1)},\ldots,\theta^{(d)}) \in \mathcal{S}_p^{d,+}$, $c_1 = \pi^{-1}\Gamma(1+\alpha)\sin(\pi\alpha)$, $\boldsymbol{v}_{i,y}^{p,\boldsymbol{\theta}}$ is the vector from (5.12) and

$$\gamma_i(p,x) := \frac{1}{p}\log\Big(e^{px}\big(1-(\theta^{(i)})^p\big) + (1+e^x\theta^{(i)})^p\Big).$$

Proof We first note that for $1 \le i \le d$, $\boldsymbol{\theta} = (\theta^{(1)}, \dots, \theta^{(d)}) \in \mathcal{S}_p^{d,+}$, $x \in \mathbb{R}$ and $y > 0$,

$$\|e^x \boldsymbol{\theta} + y\boldsymbol{e}_i\|_p = \Big(e^{px} \sum_{j \neq i} (\theta^{(j)})^p + (e^x\theta^{(i)} + y)^p\Big)^{1/p} = \Big(e^{px}\big(1 - (\theta^{(i)})^p\big) + (e^x\theta^{(i)} + y)^p\Big)^{1/p},$$

where the last equality comes from the fact that $\|\boldsymbol{\theta}\|_p = 1$. We introduce the following (already familiar) coordinate system:

$$(x, \boldsymbol{\theta}) = (\log \|\boldsymbol{w}\|_p, \arg(\boldsymbol{w})), \qquad \boldsymbol{w} \in [0,\infty)^d \setminus \{\mathbf{0}_d\}.$$

Since $\boldsymbol{w} = e^x\boldsymbol{\theta}$, for $\boldsymbol{w} \in [0,\infty)^d \setminus \{\mathbf{0}_d\}$, Lemma 6.3.6, in the above coordinate system says that

$$\frac{\partial}{\partial z_i}\big(\|\boldsymbol{\theta}\|_p\big) = e^{-x}(\theta^{(i)})^{p-1}.$$

Plugging this expression in, and then making the change-of-variables

$$z = \frac{1}{p}\log\Big(e^{px}\big(1-(\theta^{(i)})^p\big) + (e^x\theta^{(i)}+y)^p\Big), \qquad y = \Big(e^{pz} - e^{px}\big(1-(\theta^{(i)})^p\big)\Big)^{1/p} - e^x\theta^{(i)},$$

$$\frac{dy}{dz} = e^{pz}\Big(e^{pz} - e^{px}\big(1-(\theta^{(i)})^p\big)\Big)^{\frac{1-p}{p}},$$

the operator A from Theorem 6.3.5 (with $\|\cdot\| = \|\cdot\|_p$) becomes

$$\begin{aligned}
&Af(x,\boldsymbol{\theta}) \\
&= c_1 e^{\alpha x}\int_x^\infty \Bigg\{\sum_{i=1}^d \Bigg(f\big(z, \boldsymbol{v}^{p,\boldsymbol{\theta}}_{i,z-x}\big) - f(x,\boldsymbol{\theta}) \\
&\qquad - \Big[\Big((\theta^{(i)})^{p-1}\boldsymbol{e}_0 + (1-(\theta^{(i)})^p)\boldsymbol{e}_i - (\theta^{(i)})^{p-1}\sum_{\substack{1\le j\le d\\ j\neq i}} \theta^{(j)}\boldsymbol{e}_j\Big)\cdot \boldsymbol{\nabla} f(x,\boldsymbol{\theta})\Big]\times \\
&\qquad\qquad \times \Big(\Big(e^{p(z-x)} - 1 + (\theta^{(i)})^p\Big)^{1/p} - \theta^{(i)}\Big)\mathbb{1}_{\left(x,\gamma_i(p,x)\right)}(z)\Bigg)\Bigg\}\times \\
&\qquad\qquad\qquad \times \frac{e^{pz}\Big(e^{pz} - e^{px}\big(1-(\theta^{(i)})^p\big)\Big)^{\frac{1-p}{p}}}{\Big(\Big(e^{pz} - e^{px}\big(1-(\theta^{(i)})^p\big)\Big)^{1/p} - e^x\theta^{(i)}\Big)^{1+\alpha}}dz,
\end{aligned}$$

where $\gamma_i(p,x) := \frac{1}{p}\log\Big(e^{px}\big(1-(\theta^{(i)})^p\big) + (1+e^x\theta^{(i)})^p\Big)$. By then performing the simple change-of-variables $y = z - x$, the above expression can be further evaluated to

$$Af(x,\boldsymbol{\theta})$$

$$= c_1 e^{\alpha x} \int_0^\infty \Bigg\{ \sum_{i=1}^d \Bigg(f\big(x+y, \boldsymbol{v}_{i,y}^{p,\boldsymbol{\theta}}\big) - f(x,\boldsymbol{\theta})$$

$$- \Big[\Big((\theta^{(i)})^{p-1}\boldsymbol{e}_0 + (1-(\theta^{(i)})^p)\boldsymbol{e}_i - (\theta^{(i)})^{p-1} \sum_{\substack{1\le j\le d\\ j\neq i}} \theta^{(j)}\boldsymbol{e}_j\Big) \cdot \boldsymbol{\nabla} f(x,\boldsymbol{\theta})\Big] \times$$

$$\times \Big(\Big(e^{py} - 1 + (\theta^{(i)})^p\Big)^{1/p} - \theta^{(i)}\Big) \mathbb{1}_{\left(0,\gamma(p,x)-x\right)}(y)\Bigg)\Bigg\} \times$$

$$\times \frac{e^{p(x+y)}\Big(e^{p(x+y)} - e^{px}\big(1-(\theta^{(i)})^p\big)\Big)^{\frac{1-p}{p}}}{\Big(\Big(e^{p(x+y)} - e^{px}\big(1-(\theta^{(i)})^p\big)\Big)^{1/p} - e^x\theta^{(i)}\Big)^{1+\alpha}} dy.$$

Finally, by observing that

$$e^{\alpha x} \cdot \frac{e^{p(x+y)}\Big(e^{p(x+y)} - e^{px}\big(1-(\theta^{(i)})^p\big)\Big)^{\frac{1-p}{p}}}{\Big(\Big(e^{p(x+y)} - e^{px}\big(1-(\theta^{(i)})^p\big)\Big)^{1/p} - e^x\theta^{(i)}\Big)^{1+\alpha}} = \frac{e^{py}e^{-x(1-p)}\Big(e^{p(x+y)} - e^{px}\big(1-(\theta^{(i)})^p\big)\Big)^{\frac{1-p}{p}}}{\Big(\Big(e^{py} - 1 + (\theta^{(i)})^p\Big)^{1/p} - \theta^{(i)}\Big)^{1+\alpha}}$$

$$= \frac{e^{py}\Big(e^{py} - 1 + (\theta^{(i)})^p\Big)^{\frac{1-p}{p}}}{\Big(\Big(e^{py} - 1 + (\theta^{(i)})^p\Big)^{1/p} - \theta^{(i)}\Big)^{1+\alpha}}$$

we are able to bring our previous expression of the operator A to the one given in the statement of the corollary. ∎

As usual, the $p = 1$ version of the above result brings about a cleaner expression for the generator A.

Corollary 6.3.8 *Let (ξ^R, Ξ^R) be the underlying MAP of R from (6.15) with respect to the norm $\|\cdot\|_1$ on $\mathbb{R}^d$. Then, for $f \in C_b^2(\mathbb{R} \times \mathcal{S}_1^{d,+})$ of class $\mathcal{D}_{\|\cdot\|_1}^d$, the process*

$$f(\xi_t^R, \Xi_t^R) - f(\xi_0^R, \Xi_0^R) - \int_0^t Af(\xi_s^R, \Xi_s^R) ds, \qquad t \geq 0,$$

is a martingale, where the operator A is given on this class of functions by

$$Af(x,\boldsymbol{\theta})$$

$$= c_1 \int_0^\infty \Bigg\{ \sum_{i=1}^d \Bigg(f\big(y+x, \boldsymbol{v}_{i,y}^{1,\boldsymbol{\theta}}\big) - f(x,\boldsymbol{\theta})$$

$$- (e^y-1)\Big[\Big(\boldsymbol{e}_0 + (1-\theta^{(i)})\boldsymbol{e}_i - \sum_{j\neq i}\theta^{(j)}\boldsymbol{e}_j\Big)\cdot \boldsymbol{\nabla} f(x,\boldsymbol{\theta})\Big]\mathbb{1}_{\left(0,\log(1+e^x)-x\right)}(y)\Bigg)\Bigg\}\frac{e^y}{(e^y-1)^{1+\alpha}}dy,$$

for $x\in\mathbb{R}$, $\boldsymbol{\theta}=(\theta^{(1)},\dots,\theta^{(d)})\in\mathcal{S}_1^{d,+}$, $c_1=\pi^{-1}\Gamma(1+\alpha)\sin(\pi\alpha)$, *and* $\boldsymbol{v}^{1,\boldsymbol{\theta}}_{j,y}$ *is the vector from (5.12).*

And the result becomes even more interesting and explicit in dimension $d=2$:

Lemma 6.3.9 *Let* $(\xi^R,\Xi^R)=(\xi^R_t,\Xi^R_t)_{t\geq 0}$ *be the underlying MAP of R (with* $d=2$*) from (6.15) with respect to the* L^1 *norm on* $\mathbb{R}^2$*. Then, for functions* $f\in C^2_b(\mathbb{R}\times\mathcal{S}_1^{2,+})$ *satisfying the following (reflecting) boundary conditions*

$$\frac{\partial f}{\partial x_0}(w,0,1)+\frac{\partial f}{\partial x_1}(w,0,1)-\frac{\partial f}{\partial x_2}(w,0,1)=0=\frac{\partial f}{\partial x_0}(w,1,0)-\frac{\partial f}{\partial x_1}(w,1,0)+\frac{\partial f}{\partial x_2}(w,1,0),\qquad w\in\mathbb{R},$$

the process

$$f(\xi^R_t,\Xi^R_t)-f(\xi^R_0,\Xi^R_0)-\int_0^t Af(\xi^R_s,\Xi^R_s)ds,\qquad t\geq 0,$$

is a martingale, where

$$\begin{aligned}
&Af(x,\boldsymbol{\theta})\\
&=c_1\int_0^\infty\Bigg\{f\left(y+x,\begin{pmatrix}\theta^{(1)}+(1-\mathrm{e}^{-y})\theta^{(2)}\\ \mathrm{e}^{-y}\theta^{(2)}\end{pmatrix}\right)-f(x,\boldsymbol{\theta})\\
&\qquad-\left(\frac{\partial f}{\partial x}(x,\boldsymbol{\theta})+\theta^{(2)}\frac{\partial f}{\partial\theta^{(1)}}(x,\boldsymbol{\theta})-\theta^{(2)}\frac{\partial f}{\partial\theta^{(2)}}(x,\boldsymbol{\theta})\right)(e^y-1)\mathbb{1}_{(0,\log(1+\mathrm{e}^x)-x)}(y)\Bigg\}\frac{e^y}{(e^y-1)^{1+\alpha}}dy\\
&+c_1\int_0^\infty\Bigg\{f\left(y+x,\begin{pmatrix}\mathrm{e}^{-y}\theta^{(1)}\\ \theta^{(2)}+(1-\mathrm{e}^{-y})\theta^{(1)}\end{pmatrix}\right)-f(x,\boldsymbol{\theta})\\
&\qquad-\left(\frac{\partial f}{\partial x}(x,\boldsymbol{\theta})-\theta^{(1)}\frac{\partial f}{\partial\theta^{(1)}}(x,\boldsymbol{\theta})+\theta^{(1)}\frac{\partial f}{\partial\theta^{(2)}}(x,\boldsymbol{\theta})\right)(e^y-1)\mathbb{1}_{(0,\log(1+\mathrm{e}^x)-x)}(y)\Bigg\}\frac{e^y}{(e^y-1)^{1+\alpha}}dy,
\end{aligned}$$

with $c_1=\pi^{-1}\Gamma(1+\alpha)\sin(\pi\alpha)$, $x\in\mathbb{R}$ *and* $\boldsymbol{\theta}=(\theta^{(1)},\theta^{(2)})\in\mathcal{S}_1^{2,+}$.

Chapter 7

Reflected Brownian Motion in the Positive Orthant of $\mathbb{R}^d$

7.1 Introduction

By using the methods from Section 6.3, in this chapter we derive the generator of the underlying MAP, $(\rho, \Theta) = (\rho_t, \Theta_t)_{t\geq 0}$, of *Skorokhod-reflected d-dimensional Brownian motion* (up to the time of its absorption at $\mathbf{0}_d \in \mathbb{R}^d$, of course). The latter is defined as the process $\mathcal{R} = (\mathcal{R}_t)_{t\geq 0}$, where

$$\mathcal{R}_t := (B_t^{(1)} - \underline{B}_t^{(1)}, \ldots, B_t^{(d)} - \underline{B}_t^{(d)})\mathbb{1}_{\{t<\upsilon\}}, \qquad t \geq 0, \tag{7.1}$$

where $\upsilon := \inf\{t > 0 : \mathcal{R}_t = \mathbf{0}_d\}$ and the $B^{(i)}$ are independent one-dimensional Brownian motions. Throughout this chapter we will focus our attention on the norm $\|\cdot\|_p$, for $p \geq 1$. Additionally, in the special case of $p = 1$ and $d = 2$ we obtain, in Section 7.2.2, the SDE of which said MAP is the (unique) weak solution.

The Skorokhod problem for Brownian motion in domains such as the orthant $[0, \infty)^d$ has been extensively studied and is by now well understood; see, for example, [28] and [43]. The results in the aforementioned papers ensure that the process $\mathcal{R}$ from (7.1) is well defined and enjoys uniqueness properties. While the aforementioned papers guarantee existence and uniqueness of the reflected diffusion itself, they do not directly provide information about the generator or structural properties of the underlying MAP (ρ, Θ) governing the evolution of the radial and angular components (with respect to a chosen norm).

The difficulty in describing (ρ, Θ) lies in the fact that a nonlinear transformation of the reflected process $\mathcal{R}$ is involved. More specifically, the radial component $\rho_t = \log \|\mathcal{R}_{I_t}\|_p$ and the angular component $\Theta_t = \mathcal{R}_{I_t}/\|\mathcal{R}_{I_t}\|_p$, where we recall that I_t is the right-continuous inverse of $t \mapsto \int_0^t \|\mathcal{R}_s\|_p^{-2} ds$, evolve in a coupled manner that depends on both the geometry induced by the norm and the behaviour of the reflected process near the boundary of the orthant. Although the dynamics of $\mathcal{R}$ are well understood, translating them in order to extract those of (ρ, Θ) requires careful analysis of how reflection affects the radial and angular components. This transformation is non-trivial and is not covered by the classical theory. In particular, we remark that the method we had employed in Section 6.3 to derive the generator of the underlying MAP of the respective ssMp through first solving the associated martingale problem for the latter and then via basic calculus and the Volkonskii formula 'converting' the solution (i.e., performing a change of variables and a time-change) becomes a much more computationally demanding procedure in the current setting with Brownian motion, as we shall indeed see in the proof of Theorem 7.2.1.

7.2 Underlying MAP of Skorokhod-Reflected d-dimensional Brownian motion

7.2.1 Infinitesimal Generator

As discussed in the introduction, we employ the method we had used in Section 6.3 to derive the generator of (ρ, Θ). We recall that this involves first deriving the generator (with boundary conditions) of $\mathcal{R}$ (which in fact is already well-known, but nonetheless can also be readily identified from Theorem 6.3.3), and then through basic calculus and the Volkonskii formula 'converting' these mathematical objects into those that the underlying MAP (ρ, Θ) must possess. As mentioned in the introduction, this particular procedure in the current setting involves much more strenuous calculus than the analogous one we saw in Section 6.3.

Theorem 7.2.1 *Fix $p \geq 1$. Let $(\rho, \Theta) = (\rho_t, \Theta_t)_{t \geq 0}$ be the underlying MAP of $\mathcal{R}$ from (7.1) with respect to the norm $\|\cdot\|_p$ on $\mathbb{R}^d$. Then, for $f \in C_b^2(\mathbb{R} \times \mathcal{S}_p^{d,+})$ of class $\mathcal{D}^d_{\|\cdot\|_p}$, the process*

$$f(\rho_t, \Theta_t) - f(\rho_0, \Theta_0) - \int_0^t Af(\rho_s, \Theta_s) ds, \qquad t \geq 0,$$

is a martingale, where the operator A is given on this class of functions by

$$Af(x, \boldsymbol{\theta}) = \sum_{i=0}^{d} b_i(\boldsymbol{\theta}) \frac{\partial f}{\partial y_i}(x, \boldsymbol{\theta}) + \frac{1}{2} \sum_{i,j=0}^{d} a_{ij}(\boldsymbol{\theta}) \frac{\partial^2 f}{\partial y_i \partial y_j}(x, \boldsymbol{\theta}),$$

where

$$b_0(\boldsymbol{\theta}) = \frac{1}{2}\sum_{k=1}^{d}((p-1)\theta_k^{p-2} - p\theta_k^{2p-2});$$

$$b_i(\boldsymbol{\theta}) = \frac{1}{2}\Big((p+1)(\theta_i^{2p-1} - \theta_i^{p-1}) + \theta_i \sum_{k\neq i}((p+1)\theta_k^{2p-2} - (p-1)\theta_k^{p-2})\Big), \qquad i \in \{1,\dots,d\};$$

$$a_{00}(\boldsymbol{\theta}) = \sum_{k=1}^{d}\theta_k^{2p-2};$$

$$a_{ii}(\boldsymbol{\theta}) = (1-\theta_i^p)^2 + \theta_i^2\sum_{k\neq i}\theta_k^{2p-2}, \qquad i \in \{1,\dots,d\};$$

$$a_{0i}(\boldsymbol{\theta}) = a_{i0}(\boldsymbol{\theta}) = \theta_i^{p-1} - \theta_i^{2p-1} - \theta_i\sum_{k\neq i}\theta_k^{2p-2}, \qquad i \in \{1,\dots,d\};$$

$$a_{ij}(\boldsymbol{\theta}) = a_{ji}(\boldsymbol{\theta}) = \theta_i(\theta_j^{2p-1} - \theta_j^{p-1}) + \theta_j(\theta_i^{2p-1} - \theta_i^{p-1}) + \theta_i\theta_j\sum_{k\neq i,j}\theta_k^{2p-2}, \qquad i,j \in \{1,\dots,d\},\ i \neq j,$$

where $x \in \mathbb{R}$ *and* $\boldsymbol{\theta} = (\theta_1,\dots,\theta_d) \in \mathcal{S}_p^{d,+}$.

Proof By plugging in $\sigma_i = 1$, $b_i = 0$, and $\Pi^{(i)} \equiv 0$ in Theorem 6.3.3, we see that for $g \in C_b^2([0,\infty)^d)$ satisfying the boundary conditions of the aforementioned theorem, the process

$$g(\mathcal{R}_t) - g(\mathcal{R}_0) - \int_0^t (\frac{1}{2}\sum_{i=1}^{d}\frac{\partial^2 g}{\partial z_i^2}(\mathcal{R}_s))ds, \qquad t \geq 0,$$

is a martingale.

Using the fact that $\mathcal{R}$ has index of self-similarity 2 (since Brownian motion does), Volkonskii's formula tells us, in the same way it did in our proof of Theorem 6.3.5, that for the above function g, the process

$$g(\mathcal{R}_{I_t}) - g(\mathcal{R}_0) - \int_0^t (\frac{\|\mathcal{R}_{I_s}\|_p^2}{2}\sum_{i=1}^{d}\frac{\partial^2 g}{\partial z_i^2}(\mathcal{R}_{I_s}))ds, \qquad t \geq 0, \tag{7.2}$$

is a martingale, where I_t is the right-continuous inverse of $t \mapsto \int_0^t \|\mathcal{R}_s\|_p^{-2}ds$. We recall that (ρ,Θ) is the following space-time transformation of $\mathcal{R}$:

$$(\rho_t,\Theta_t) = (\log\|\mathcal{R}_{I_t}\|_p, \arg(\mathcal{R}_{I_t})).$$

Consider the $C_b^2(\mathbb{R}\times\mathcal{S}_p^{d,+})$-function $f \in \mathcal{D}^d_{\|\cdot\|_p}$ from the statement of the theorem. By plugging in the $C_b^2([0,\infty)^d)$ function

$$g(\boldsymbol{w}) = f(\log\|\boldsymbol{w}\|_p, \arg(\boldsymbol{w})), \qquad \boldsymbol{w} \in [0,\infty)^d \setminus \{\mathbf{0}_d\},$$

in (7.2) and rewriting the expression in the coordinate system given by

$$(x, \boldsymbol{\theta}) = (\log\|\boldsymbol{w}\|_p, \arg(\boldsymbol{w})), \qquad \boldsymbol{w} \in [0,\infty)^d \setminus \{\mathbf{0}_d\}, \tag{7.3}$$

the result will follow. Indeed, our first step is to establish that, under the theorem's boundary conditions on f (i.e., the fact that it is of class $\mathcal{D}^d_{\|\cdot\|_p}$), the above function g satisfies the boundary conditions of Theorem 6.3.3. We have already seen that this is indeed the case in our proof of Theorem 6.3.5, however. Moreover, in the proof of Corollary 6.3.7 we had seen, via Lemma 6.3.6, that for $\boldsymbol{w} \in [0,\infty)^d \setminus \{\mathbf{0}_d\}$ and $i \in \{1\ldots,d\}$,

$$\begin{aligned}
\frac{\partial g}{\partial z_i}(\boldsymbol{w}) = \|\boldsymbol{w}\|_p^{-1}\Big(&\frac{\partial}{\partial z_i}(\|\boldsymbol{w}\|_p) f_{y_0}(\log\|\boldsymbol{w}\|_p, \arg(\boldsymbol{w})) + f_{y_i}(\log\|\boldsymbol{w}\|_p, \arg(\boldsymbol{w})) \\
&- \frac{\partial}{\partial z_i}(\|\boldsymbol{w}\|_p)\sum_{j=1}^d \arg(\boldsymbol{w})^{(j)} f_{y_j}(\log\|\boldsymbol{w}\|_p, \arg(\boldsymbol{w}))\Big) \\
= \|\boldsymbol{w}\|_p^{-1}\Big(&(\arg(\boldsymbol{w})^{(i)})^{p-1} f_{y_0}(\log\|\boldsymbol{w}\|_p, \arg(\boldsymbol{w})) + (1 - (\arg(\boldsymbol{w})^{(i)})^p) f_{y_i}(\log\|\boldsymbol{w}\|_p, \arg(\boldsymbol{w})) \\
&- (\arg(\boldsymbol{w})^{(i)})^{p-1}\sum_{j\neq i} \arg(\boldsymbol{w})^{(j)} f_{y_j}(\log\|\boldsymbol{w}\|_p, \arg(\boldsymbol{w}))\Big).
\end{aligned}$$

In order to compute the desired

$$\frac{\|\boldsymbol{w}\|_p^2}{2}\sum_{i=1}^d \frac{\partial^2 g}{\partial z_i^2}(\boldsymbol{w}), \qquad \boldsymbol{w} \in [0,\infty)^d \setminus \{\mathbf{0}_d\}, \tag{7.4}$$

we need to compute, for $1 \le i \le d$, the quantity $\frac{\partial^2 g}{\partial z_i^2}(\boldsymbol{w})$. This will be done by differentiating the derived quantity $\frac{\partial g}{\partial z_i}(\boldsymbol{w})$ from above with respect to the z_i coordinate with the help of the usual chain/product rules from calculus. To this end, to avoid calculations with heavy notation, we will make temporary use of the following abbreviations in our below computations: for $\boldsymbol{w} = (w_1,\ldots,w_d) \in [0,\infty)^d \setminus \{\mathbf{0}_d\}$,

$$\|\boldsymbol{w}\| \coloneqq \|\boldsymbol{w}\|_p, \qquad a(\boldsymbol{w}) \coloneqq \arg(\boldsymbol{w}) = \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|_p} = (\frac{w_1}{\|\boldsymbol{w}\|},\ldots,\frac{w_d}{\|\boldsymbol{w}\|})^T,$$

and we will denote the i-th entry, for $1 \le i \le d$, of the vector $a(\boldsymbol{w})$ by $a(\boldsymbol{w})_i$. Let us fix $i \in \{1,\ldots,d\}$. Then, we find

$$\frac{\partial^2 g}{\partial z_i^2}(\boldsymbol{w}) = A_i(\boldsymbol{w}) + B_i(\boldsymbol{w}) + C_i(\boldsymbol{w}) + D_i(\boldsymbol{w}),$$

where

$$A_i(\boldsymbol{w}) = -\|\boldsymbol{w}\|^{-2} \cdot \frac{\partial}{\partial z_i}(\|\boldsymbol{w}\|) \cdot \Big(a(\boldsymbol{w})_i^{p-1} f_{y_0}\big(\log\|\boldsymbol{w}\|, a(\boldsymbol{w})\big) + \big(1 - a(\boldsymbol{w})_i^p\big) f_{y_i}\big(\log\|\boldsymbol{w}\|, a(\boldsymbol{w})\big) - a(\boldsymbol{w})_i^{p-1} \sum_{j\neq i} a(\boldsymbol{w})_j f_{y_j}\big(\log\|\boldsymbol{w}\|, a(\boldsymbol{w})\big)\Big);$$

and, by observing that $\frac{\partial}{\partial z_i}(a(\boldsymbol{w})_i^{p-1}) = \frac{\partial^2}{\partial z_i^2}(\|\boldsymbol{w}\|)$,

$$\begin{aligned} &B_i(\boldsymbol{w}) \\ &= \|\boldsymbol{w}\|^{-1}\Bigg(\frac{\partial^2}{\partial z_i^2}(\|\boldsymbol{w}\|) f_{y_0}\big(\log\|\boldsymbol{w}\|, a(\boldsymbol{w})\big) + a(\boldsymbol{w})_i^{p-1}\|\boldsymbol{w}\|^{-1}\Big(a(\boldsymbol{w})_i^{p-1} f_{y_0^2}\big(\log\|\boldsymbol{w}\|, a(\boldsymbol{w})\big) \\ &\qquad + \big(1 - a(\boldsymbol{w})_i^p\big) f_{y_0 y_i}\big(\log\|\boldsymbol{w}\|, a(\boldsymbol{w})\big) - a(\boldsymbol{w})_i^{p-1} \sum_{j\neq i} a(\boldsymbol{w})_j f_{y_0 y_j}\big(\log\|\boldsymbol{w}\|, a(\boldsymbol{w})\big)\Big)\Bigg); \end{aligned}$$

$$\begin{aligned} &C_i(\boldsymbol{w}) \\ &= \|\boldsymbol{w}\|^{-1}\Bigg(-\frac{\partial}{\partial z_i}(a(\boldsymbol{w})_i^p) f_{y_i}\big(\log\|\boldsymbol{w}\|, a(\boldsymbol{w})\big) + \big(1 - a(\boldsymbol{w})_i^p\big)\|\boldsymbol{w}\|^{-1}\Big(a(\boldsymbol{w})_i^{p-1} f_{y_0 y_i}\big(\log\|\boldsymbol{w}\|, a(\boldsymbol{w})\big) \\ &\qquad + \big(1 - a(\boldsymbol{w})_i^p\big) f_{y_i^2}\big(\log\|\boldsymbol{w}\|, a(\boldsymbol{w})\big) - a(\boldsymbol{w})_i^{p-1} \sum_{j\neq i} a(\boldsymbol{w})_j f_{y_i y_j}\big(\log\|\boldsymbol{w}\|, a(\boldsymbol{w})\big)\Big)\Bigg); \end{aligned}$$

and

$$\begin{aligned} &D_i(\boldsymbol{w}) \\ &= \|\boldsymbol{w}\|^{-1}\Bigg(-\frac{\partial^2}{\partial z_i^2}(\|\boldsymbol{w}\|) \sum_{j\neq i} a(\boldsymbol{w})_j f_{y_j}\big(\log\|\boldsymbol{w}\|, a(\boldsymbol{w})\big) \\ &\qquad - a(\boldsymbol{w})_i^{p-1} \sum_{j\neq i} \Big[-\frac{\partial}{\partial z_i}(\|\boldsymbol{w}\|) \cdot \frac{w_j}{\|\boldsymbol{w}\|^2} f_{y_j}\big(\log\|\boldsymbol{w}\|, a(\boldsymbol{w})\big) + a(\boldsymbol{w})_j \cdot \|\boldsymbol{w}\|^{-1} \times \\ &\qquad\quad \times \Big(a(\boldsymbol{w})_i^{p-1} f_{y_j y_0}\big(\log\|\boldsymbol{w}\|, a(\boldsymbol{w})\big) + \big(1 - a(\boldsymbol{w})_i^p\big) f_{y_j y_i}\big(\log\|\boldsymbol{w}\|, a(\boldsymbol{w})\big) \\ &\qquad\qquad - a(\boldsymbol{w})_i^{p-1} \sum_{k\neq i} a(\boldsymbol{w})_k f_{y_k y_j}\big(\log\|\boldsymbol{w}\|, a(\boldsymbol{w})\big)\Big)\Big]\Bigg). \end{aligned}$$

We now apply Lemma 6.3.6 on the above expressions and subsequently express the resulting quantities in the other coordinate system given in (7.3):

$$A_i(x,\boldsymbol{\theta}) = -e^{-2x}\theta_i^{p-1}\Big(\theta_i^{p-1} f_{y_0}(x,\boldsymbol{\theta}) + (1-\theta_i^p) f_{y_i}(x,\boldsymbol{\theta}) - \theta_i^{p-1}\sum_{j\neq i}\theta_j f_{y_j}(x,\boldsymbol{\theta})\Big);$$

$$\begin{aligned} &B_i(x,\boldsymbol{\theta}) \\ &\quad = e^{-x}\Bigg((p-1)e^{-x}\big(\theta_i^{p-2} - \theta_i^{2p-2}\big) f_{y_0}(x,\boldsymbol{\theta}) \\ &\qquad\qquad + e^{-x}\theta_i^{p-1}\Big(\theta_i^{p-1} f_{y_0^2}(x,\boldsymbol{\theta}) + \big(1-\theta_i^p\big) f_{y_0 y_i}(x,\boldsymbol{\theta}) - \theta_i^{p-1}\sum_{j\neq i}\theta_j f_{y_0 y_j}(x,\boldsymbol{\theta})\Big)\Bigg); \end{aligned}$$

$$\begin{aligned} &C_i(x,\boldsymbol{\theta}) \\ &\quad = e^{-x}\Bigg(-pe^{-x}\big(\theta_i^{p-1} - \theta_i^{2p-1}\big) f_{y_i}(x,\boldsymbol{\theta}) \\ &\qquad\qquad + e^{-x}(1-\theta_i^p)\Big(\theta_i^{p-1} f_{y_0 y_i}(x,\boldsymbol{\theta}) + \big(1-\theta_i^p\big) f_{y_i^2}(x,\boldsymbol{\theta}) - \theta_i^{p-1}\sum_{j\neq i}\theta_j f_{y_i y_j}(x,\boldsymbol{\theta})\Big)\Bigg); \end{aligned}$$

$$\begin{aligned} &D_i(x,\boldsymbol{\theta}) \\ &\quad = e^{-2x}\Bigg((p-1)\big(\theta_i^{2p-2} - \theta_i^{p-2}\big)\sum_{j\neq i}\theta_j f_{y_j}(x,\boldsymbol{\theta}) + \theta_i^{2p-2}\sum_{j\neq i}\theta_j f_{y_j}(x,\boldsymbol{\theta}) - \theta_i^{2p-2}\sum_{j\neq i}\theta_j f_{y_j y_0}(x,\boldsymbol{\theta}) \\ &\qquad + \big(\theta_i^{2p-1} - \theta_i^{p-1}\big)\sum_{j\neq i}\theta_j f_{y_j y_i}(x,\boldsymbol{\theta}) + \theta_i^{2p-2}\sum_{j\neq i}\theta_j^2 f_{y_j^2}(x,\boldsymbol{\theta}) + \theta_i^{2p-2}\sum_{j\neq i}\theta_j\sum_{k\neq j,i}\theta_k f_{y_k y_j}(x,\boldsymbol{\theta})\Bigg). \end{aligned}$$

In order to work out the desired operator A, it remains to write the quantity from (7.4) in the other coordinate system, which is precisely the sum over all $1 \leq i \leq d$ of the expression

$$\frac{e^{2x}}{2}\Big(A_i(x,\boldsymbol{\theta}) + B_i(x,\boldsymbol{\theta}) + C_i(x,\boldsymbol{\theta}) + D_i(x,\boldsymbol{\theta})\Big). \tag{7.5}$$

By inspecting the expressions of $A_i(x,\boldsymbol{\theta})$, $B_i(x,\boldsymbol{\theta})$, $C_i(x,\boldsymbol{\theta})$, $D_i(x,\boldsymbol{\theta})$ from above, we find that the coefficient of $f_{y_0}(x,\boldsymbol{\theta})$ in the expression from (7.5) is precisely

$$\frac{1}{2}\Big((p-1)\theta_i^{p-2} - p\theta_i^{2p-2}\Big); \tag{7.6}$$

the coefficient of $f_{y_i}(x,\boldsymbol{\theta})$ in the expression from (7.5) is precisely

$$\frac{p+1}{2}\big(\theta_i^{2p-1} - \theta_i^{p-1}\big); \tag{7.7}$$

the coefficient of $f_{y_j}(x,\boldsymbol{\theta})$, for $j \in \{1,\ldots,d\} \setminus \{i\}$, in the expression from (7.5) is precisely

$$\frac{1}{2}\Big((p+1)\theta_j\theta_i^{2p-2} - (p-1)\theta_j\theta_i^{p-2}\Big); \tag{7.8}$$

the coefficient of $f_{y_0^2}(x,\boldsymbol{\theta})$ in the expression from (7.5) is precisely

$$\frac{1}{2}\theta_i^{2p-2}; \tag{7.9}$$

the coefficient of $f_{y_i^2}(x,\boldsymbol{\theta})$ in the expression from (7.5) is precisely

$$\frac{1}{2}\big(1-\theta_i^p\big)^2; \tag{7.10}$$

the coefficient of $f_{y_j^2}(x,\boldsymbol{\theta})$, for $j \in \{1,\ldots,d\} \setminus \{i\}$, in the expression from (7.5) is precisely

$$\frac{1}{2}\theta_j^2\theta_i^{2p-2}; \tag{7.11}$$

the coefficient of $f_{y_0 y_i}(x,\boldsymbol{\theta})$ in the expression from (7.5) is precisely

$$\theta_i^{p-1} - \theta_i^{2p-1}; \tag{7.12}$$

the coefficient of $f_{y_j y_0}(x,\boldsymbol{\theta})$, for $j \in \{1,\ldots,d\} \setminus \{i\}$, in the expression from (7.5) is precisely

$$-\theta_j\theta_i^{2p-2}; \tag{7.13}$$

the coefficient of $f_{y_j y_i}(x,\boldsymbol{\theta})$, for $j \in \{1,\ldots,d\} \setminus \{i\}$, in the expression from (7.5) is precisely

$$\theta_j\big(\theta_i^{2p-1} - \theta_i^{p-1}\big); \tag{7.14}$$

and lastly, the coefficient of $f_{y_j y_k}(x,\boldsymbol{\theta})$, for $j \in \{1,\ldots,d\} \setminus \{i\}$ and $k \in \{1,\ldots,d\} \setminus \{i,j\}$, in the expression from (7.5) is precisely

$$\theta_j\theta_k\theta_i^{2p-2}. \tag{7.15}$$

And so then, it is easy to see that $b_0(\boldsymbol{\theta})$ from the statement of the theorem is none other than the sum over all $1 \le i \le d$ of the expression from (7.6).

To derive $b_i(\boldsymbol{\theta})$, for $i \in \{1,\ldots,d\}$, from the statement of the theorem we first interchange j with i in (7.8), then sum the resulting expression over all $j \in \{1\ldots,d\} \setminus \{i\}$, and then add the expression from (7.7).

It is also easy to see that $a_{00}(\boldsymbol{\theta})$ from the statement of the theorem is none other than the sum over all $1 \le i \le d$ of the expression from (7.9) multiplied by 2.

In a similar way, we can derive the quantity $a_{ii}(\boldsymbol{\theta})$, for $i \in \{1,\dots,d\}$, from the statement of the theorem by first interchanging i with j in (7.11), then summing the resulting expression over all $j \in \{1\dots,d\} \setminus \{i\}$, and then adding the expression from (7.10), and finally multiplying the result by 2.

In like fashion, to derive $a_{0i}(\boldsymbol{\theta}) = a_{i0}(\boldsymbol{\theta})$, for $i \in \{1,\dots,d\}$, from the statement of the theorem, we just need to interchange i with j in (7.13), then sum the resulting expression over all $j \in \{1\dots,d\} \setminus \{i\}$, and finally add the expression from (7.12).

It remains to derive $a_{ij}(\boldsymbol{\theta}) = a_{ji}(\boldsymbol{\theta})$, for $i,j \in \{1,\dots,d\}$ and $i \neq j$, from the statement of the theorem. For this we need to first interchange i with k in (7.15) and then sum the resulting expression over all $k \in \{1,\dots,d\} \setminus \{i,j\}$, and finally add the expression from (7.14) as well as the latter expression with i and j interchanged. ∎

The generator significantly cleans up in the case of $p = 1$:

Corollary 7.2.2 *Let (ρ,Θ) be the underlying MAP of $\mathcal{R}$ from (7.1) with respect to the norm $\|\cdot\|_1$ on $\mathbb{R}^d$. Then, for functions $f \in C_b^2(\mathbb{R}\times\mathcal{S}_1^{d,+})$ of class $\mathcal{D}^d_{\|\cdot\|_1}$ the process*

$$f(\rho_t,\Theta_t) - f(\rho_0,\Theta_0) - \int_0^t Af(\rho_s,\Theta_s)ds, \qquad t \ge 0,$$

is a martingale, where the operator A is given on this class of functions by

$$Af(x,\boldsymbol{\theta}) = \sum_{i=0}^d b_i(\boldsymbol{\theta})\frac{\partial f}{\partial y_i}(x,\boldsymbol{\theta}) + \frac{1}{2}\sum_{i,j=0}^d a_{ij}(\boldsymbol{\theta})\frac{\partial^2 f}{\partial y_i \partial y_j}(x,\boldsymbol{\theta}),$$

where for $\boldsymbol{\theta} = (\theta_1,\dots,\theta_d) \in \mathcal{S}_1^{d,+}$ and $1 \le i \le d$,

$$b_0(\boldsymbol{\theta}) = -\frac{d}{2}, \quad b_i(\boldsymbol{\theta}) = d\theta_i - 1, \quad a_{00}(\boldsymbol{\theta}) = d, \quad a_{ii}(\boldsymbol{\theta}) = (1-\theta_i)^2 + (d-1)\theta_i^2, \quad a_{0i}(\boldsymbol{\theta}) = a_{i0}(\boldsymbol{\theta}) = 1 - d\theta_i,$$

$$a_{ij}(\boldsymbol{\theta}) = a_{ji}(\boldsymbol{\theta}) = d\theta_i\theta_j - \theta_i - \theta_j, \qquad i,j \in \{1,\dots,d\},\ i \neq j.$$

And the above result becomes even more explicit in dimension $d = 2$:

Corollary 7.2.3 *Let $(\rho,\Theta) = (\rho_t,\Theta_t)_{t\ge 0}$ be the underlying MAP of $\mathcal{R}$ (with $d = 2$) from (7.1) with respect to the norm $\|\cdot\|_1$ on $\mathbb{R}^2$. For every $f \in C_b^2(\mathbb{R}\times\mathcal{S}_1^{2,+})$ satisfying the boundary conditions* *i* $(1,1,-1)\cdot\boldsymbol{\nabla} f(w,0,1)=0 \quad \forall w\in\mathbb{R}$;

ii $(1,-1,1)\cdot\boldsymbol{\nabla} f(w,1,0)=0 \quad \forall w\in\mathbb{R}$,

the process

$$f(\rho_t,\Theta_t)-f(\rho_0,\Theta_0)-\int_0^t Af(\rho_s,\Theta_s)ds, \qquad t\geq 0,$$

is a martingale, where the operator A is given on this class of functions by

$$\begin{aligned}Af(x,\boldsymbol{\theta}) = &-f_{y_0}(x,\boldsymbol{\theta})+(\theta_1-\theta_2)f_{y_1}(x,\boldsymbol{\theta})+(\theta_2-\theta_1)f_{y_2}(x,\boldsymbol{\theta})+f_{y_0^2}(x,\boldsymbol{\theta})+\frac{\theta_1^2+\theta_2^2}{2}f_{y_1^2}(x,\boldsymbol{\theta})\\ &+\frac{\theta_1^2+\theta_2^2}{2}f_{y_2^2}(x,\boldsymbol{\theta})+(\theta_2-\theta_1)f_{y_0y_1}(x,\boldsymbol{\theta})+(\theta_1-\theta_2)f_{y_0y_2}(x,\boldsymbol{\theta})-(\theta_1^2+\theta_2^2)f_{y_1y_2}(x,\boldsymbol{\theta}),\end{aligned}$$

where $x\in\mathbb{R}$ and $\boldsymbol{\theta}=(\theta_1,\theta_2)\in\mathcal{S}_1^{2,+}$.

Proof The result follows by setting $d=2$ in Corollary 7.2.2 and making use of the fact that

$$\boldsymbol{\theta}=(\theta_1,\theta_2)\in\mathcal{S}_1^{2,+} \iff \theta_1+\theta_2=1 \iff (\theta_1+\theta_2)^2=1 \iff 2\theta_1\theta_2=1-\theta_1^2+\theta_2^2.$$

Indeed, the above equations allow us to further evaluate some of the terms from said corollary:

$$b_1(\boldsymbol{\theta})=d\theta_1-1=2\theta_1-1=\theta_1-(1-\theta_1)=\theta_1-\theta_2, \qquad b_2(\boldsymbol{\theta})=d\theta_2-1=\theta_2-\theta_1, \qquad \boldsymbol{\theta}\in\mathcal{S}_1^{2,+};$$

$$a_{01}(\boldsymbol{\theta})=a_{10}(\boldsymbol{\theta})=1-d\theta_1=\theta_2-\theta_1, \qquad a_{02}(\boldsymbol{\theta})=a_{20}(\boldsymbol{\theta})=1-d\theta_2=\theta_1-\theta_2, \qquad \boldsymbol{\theta}\in\mathcal{S}_1^{2,+};$$

and

$$a_{12}(\boldsymbol{\theta})=a_{21}(\boldsymbol{\theta})=d\theta_1\theta_2-\theta_1-\theta_2=2\theta_1\theta_2-\theta_1-\theta_2=(1-\theta_1)-\theta_1^2-\theta_2^2-\theta_2=-(\theta_1^2+\theta_2^2), \qquad \boldsymbol{\theta}\in\mathcal{S}_1^{2,+}.$$

From which the expression of the operator A given in the corollary immediately follows. ∎

7.2.2 The Unique Weak Solution to an SDE in Dimension $d=2$

Because we are dealing with a reflected MAP, the SDE that it solves is expected to be modulated and include some reflecting boundary conditions. The theory of stochastic differential equations for diffusions with reflecting boundary conditions was largely developed during the 1970s.; cf., [60], [26], [62], [64], [65]. The (relatively) recent book by Pilipenko, [47], provides a thorough summary of all the developments. Specifically, in Chapter 3.2 therein, an explicit method is presented – which is essentially a synthesis of some old results of Stroock and Varadhan from the paper [60] –

for going from the generator of a reflected diffusion to the SDE that it solves (we remind the reader that we have also detailed this method in our preliminary Section 2.2.6). Following this recipe in conjunction with our Corollary 7.2.3, in this subsection we derive the SDE that (ρ, Θ) weakly – and uniquely, we shall prove later on – solves.

Due to the lack of any conducive/slick identities in normed spaces $(\mathbb{R}^d, \|\cdot\|_p)$, for $d > 2$ and/or $p > 1$, that can be exploited for the purposes of shortening/simplifying various complex expressions emerging from subsequent calculations involving elements of $\mathcal{S}_p^{d,+}$, we solely consider $d = 2$ and $p = 1$, where, as we shall see, seemingly messy expressions can be made remarkably simple thanks to the many fruitful identities that stem from the one we have used many times, that is $\theta_1 + \theta_2 = 1$, for $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathcal{S}_1^{2,+}$.

Corollary 7.2.4 *Let $(\rho, \Theta) = (\rho_t, \Theta_t)_{t \geq 0}$ be the underlying MAP of $\mathcal{R}$ from (7.1) in dimension $d = 2$ with respect to the L^1 norm on $\mathbb{R}^2$. Then there exists a continuous, adapted, non-decreasing process $l = (l_t)_{t \geq 0}$ such that the triple (ρ, Θ, l) is the unique weak solution to the reflected SDE*

$$d \begin{pmatrix} \rho_t \\ \Theta_t \end{pmatrix} = \boldsymbol{a}(\Theta_t)\, dt + \boldsymbol{b}(\Theta_t)\, d\boldsymbol{W}_t + \boldsymbol{\gamma}(\Theta_t)\, dl_t,$$

where $\boldsymbol{W}$ is a 2-dimensional Brownian motion, l is an adapted process of bounded variation such that $l_0 = 0$ which increases only when $\Theta_t \in \partial \mathcal{S}_1^{2,+} (= \{(0,1), (1,0)\})$, $\boldsymbol{\gamma}$ specifies the inward reflection direction

$$\boldsymbol{\gamma}(\boldsymbol{\theta}) := \begin{pmatrix} 1 \\ \mathbb{1}_{\{(0,1)\}}(\boldsymbol{\theta}) - \mathbb{1}_{\{(1,0)\}}(\boldsymbol{\theta}) \\ \mathbb{1}_{\{(1,0)\}}(\boldsymbol{\theta}) - \mathbb{1}_{\{(0,1)\}}(\boldsymbol{\theta}) \end{pmatrix}, \qquad \boldsymbol{a}(\boldsymbol{\theta}) := \begin{pmatrix} -1 \\ \theta_1 - \theta_2 \\ \theta_2 - \theta_1 \end{pmatrix}, \qquad \boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathcal{S}_1^{2,+},$$

and

$$\boldsymbol{b}(\boldsymbol{\theta}) := \begin{pmatrix} \frac{2\theta_1\theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})}} + \frac{\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})}}{2} & -\frac{\theta_1 - \theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})}} \\ -\frac{\theta_1 - \theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})}} & \frac{\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})}}{4} - \frac{\theta_1\theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})}} \\ \frac{\theta_1 - \theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})}} & \frac{\theta_1\theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})}} - \frac{\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})}}{4} \end{pmatrix}, \qquad \boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathcal{S}_1^{2,+},$$

where $\lambda_2(\boldsymbol{\theta}) = 1 + \theta_1^2 + \theta_2^2 + \sqrt{(1 + \theta_1^2 + \theta_2^2)^2 - 2}$, and $\lambda_3(\boldsymbol{\theta}) = 1 + \theta_1^2 + \theta_2^2 - \sqrt{(1 + \theta_1^2 + \theta_2^2)^2 - 2}$.

Uniqueness holds in the sense of weak solutions in the class of processes satisfying the above reflection condition.

Proof Consistent with the notation and terminology of Chapter 2.2 of [47] (reviewed also in our preliminary Section 2.2.6), what we are dealing with here is a particular example of the Skorokhod problem in the (3-dimensional) domain $D = \mathbb{R} \times \mathcal{S}_1^{2,+} \subseteq \mathbb{R}^3$, which has boundary $\partial D = \mathbb{R} \times \{(0,1),(1,0)\}$, and the following reflecting directions:

$$K_{(w,0,1)^T} = \{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\} \text{ and } K_{(w,1,0)^T} = \{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\}, \qquad w \in \mathbb{R}. \tag{7.16}$$

The two equations from (7.16) plainly say that as the modulator Θ traverses continuously (i.e., without jumping) in the interior of the simplex in the first quadrant and hits either boundary point $(0,1)$ or $(1,0)$, it immediately bounces, again in a continuous fashion, back into the interior of the simplex, from which it came.

We now follow the (time-homogeneous version of) recipe illustrated at the beginning of Chapter 3.2 of [47]. More precisely, we first observe that, in our case, the analogous operator L from (3.21) therein is the operator A presented in our Theorem 7.2.1 with $d = 2$ and $p = 1$, with the analogous a_i and σ_{ij} being, for $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathcal{S}_1^{2,+}$,

$$a_0(\boldsymbol{\theta}) = -1, \qquad a_1(\boldsymbol{\theta}) = \theta_1 - \theta_2, \qquad a_2(\boldsymbol{\theta}) = \theta_2 - \theta_1;$$

and

$$\sigma_{00}(\boldsymbol{\theta}) = 2, \qquad \sigma_{11}(\boldsymbol{\theta}) = \theta_1^2 + \theta_2^2, \qquad \sigma_{22}(\boldsymbol{\theta}) = \theta_1^2 + \theta_2^2, \qquad \sigma_{01}(\boldsymbol{\theta}) = \sigma_{10}(\boldsymbol{\theta}) = \theta_2 - \theta_1,$$
$$\sigma_{02}(\boldsymbol{\theta}) = \sigma_{20}(\boldsymbol{\theta}) = \theta_1 - \theta_2, \qquad \sigma_{21}(\boldsymbol{\theta}) = \sigma_{12}(\boldsymbol{\theta}) = -(\theta_1^2 + \theta_2^2);$$

and the analogous operator J from (3.20) of [47] that describes the reflecting boundary conditions is, in our case, the operator $J : C_b^2(\mathbb{R} \times \mathcal{S}_1^{2,+}) \to C_b^2(\mathbb{R} \times \mathcal{S}_1^{2,+})$, supported on the boundary, given by

$$Jf(x,\boldsymbol{\theta}) = \begin{cases} (1,1,-1) \cdot \boldsymbol{\nabla} f(x,\boldsymbol{\theta}), & \text{if } (x,\boldsymbol{\theta}) \in \mathbb{R} \times \{(0,1)\} \\ (1,-1,1) \cdot \boldsymbol{\nabla} f(x,\boldsymbol{\theta}), & \text{if } (x,\boldsymbol{\theta}) \in \mathbb{R} \times \{(1,0)\} \end{cases};$$

and therefore, the analogous γ_i are, for $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathcal{S}_1^{2,+}$,

$$\gamma_0(\boldsymbol{\theta}) = 1, \qquad \gamma_1(\boldsymbol{\theta}) = \mathbb{1}_{\{(0,1)\}}(\boldsymbol{\theta}) - \mathbb{1}_{\{(1,0)\}}(\boldsymbol{\theta}), \qquad \gamma_2(\boldsymbol{\theta}) = \mathbb{1}_{\{(1,0)\}}(\boldsymbol{\theta}) - \mathbb{1}_{\{(0,1)\}}(\boldsymbol{\theta}).$$

Our Theorem 7.2.1 states that, for $f \in C_b^2(\mathbb{R} \times \mathcal{S}_1^{2,+})$ such that $Jf(x,\boldsymbol{\theta}) = 0$, $(x,\boldsymbol{\theta}) \in \partial D$, the process

$$f(\rho_t, \Theta_t) - f(\rho_0, \Theta_0) - \int_0^t Af(\rho_s, \Theta_s) ds, \qquad t \geq 0,$$

is a martingale. In other words, the analogous (3.21) from [47] holds for every $f \in C_b^2(\mathbb{R} \times \mathcal{S}_1^{2,+})$

satisfying our analogous (3.20) from the same piece of text. This ensures (ρ, Θ) as a weak solution to the following (reflecting) SDE:

$$d\begin{pmatrix}\rho_t\\ \Theta_t\end{pmatrix} = \boldsymbol{a}(\Theta_t)dt + \boldsymbol{b}(\Theta_t)d\boldsymbol{W}_t + \boldsymbol{\gamma}(\Theta_t)dl_t, \tag{7.17}$$

where, for $\boldsymbol{\theta} \in \mathcal{S}_1^{2,+}$, $\boldsymbol{b}(\boldsymbol{\theta})$ is a matrix $(b_{ij}(\boldsymbol{\theta}))_{i,j=0}^2$ satisfying

$$\sigma_{ij}(\boldsymbol{\theta}) = \sum_{k=0}^{2} b_{ki}(\boldsymbol{\theta})b_{kj}(\boldsymbol{\theta}) \tag{7.18}$$

for every $i, j \in \{0, 1, 2\}$; $\boldsymbol{a}(\boldsymbol{\theta})$ is the vector $(a_0(\boldsymbol{\theta}), a_1(\boldsymbol{\theta}), a_2(\boldsymbol{\theta}))$; $\boldsymbol{\gamma}(\boldsymbol{\theta})$ is the vector $(\gamma_0(\boldsymbol{\theta}), \gamma_1(\boldsymbol{\theta}), \gamma_2(\boldsymbol{\theta}))$; $\boldsymbol{W}$ is 3-dimensional Brownian motion, and l is local time of Θ at the boundary ∂D.

As was explained in Theorem 2.2 of [61] (albeit in the setting where there are no boundary conditions present, however the same argument still holds even when there are), one can simply choose the matrix $\boldsymbol{b}(\boldsymbol{\theta})$ to be the unique positive-definite symmetric square-root of the matrix $\boldsymbol{\sigma}(\boldsymbol{\theta}) = (\sigma_{ij}(\boldsymbol{\theta}))_{i,j=0}^2$. To work out this desired matrix $\boldsymbol{b}(\boldsymbol{\theta})$, we use some basic results from linear algebra. Namely, we note that, since $\boldsymbol{\sigma}(\boldsymbol{\theta})$ is a symmetric matrix, it is diagonalizable, that is, it is similar to a diagonal matrix $\boldsymbol{\Lambda}(\boldsymbol{\theta})$. Moreover, it is known that

$$\boldsymbol{\Lambda}(\boldsymbol{\theta}) = \mathrm{diag}(\lambda_1(\boldsymbol{\theta}), \lambda_2(\boldsymbol{\theta}), \lambda_3(\boldsymbol{\theta})),$$

where the $\lambda_i(\boldsymbol{\theta})$ are the distinct eigenvalues of $\boldsymbol{\sigma}(\boldsymbol{\theta})$. We also know that the associated change-of-basis matrix $\boldsymbol{S}(\boldsymbol{\theta})$ is the matrix that has i-th column being the eigenvector of $\boldsymbol{\sigma}(\boldsymbol{\theta})$ associated with the eigenvalue $\lambda_i(\boldsymbol{\theta})$, which we denote by $\boldsymbol{x}^{(i)}(\boldsymbol{\theta})$. By setting $\boldsymbol{\Lambda}^{1/2}(\boldsymbol{\theta}) := \mathrm{diag}(\sqrt{\lambda_1(\boldsymbol{\theta})}, \sqrt{\lambda_2(\boldsymbol{\theta})}, \sqrt{\lambda_3(\boldsymbol{\theta})})$, it is then easy to see that

$$(\boldsymbol{\Lambda}^{1/2}(\boldsymbol{\theta}))^2 = \boldsymbol{\Lambda}(\boldsymbol{\theta}) = \boldsymbol{S}(\boldsymbol{\theta})^{-1}\boldsymbol{\sigma}(\boldsymbol{\theta})\boldsymbol{S}(\boldsymbol{\theta}),$$

and thus,

$$\boldsymbol{\sigma}(\boldsymbol{\theta}) = \boldsymbol{S}(\boldsymbol{\theta})(\boldsymbol{\Lambda}^{1/2}(\boldsymbol{\theta}))^2\boldsymbol{S}(\boldsymbol{\theta})^{-1} = (\boldsymbol{S}(\boldsymbol{\theta})\boldsymbol{\Lambda}^{1/2}(\boldsymbol{\theta})\boldsymbol{S}(\boldsymbol{\theta})^{-1})^2.$$

Therefore, it remains to find

$$\boldsymbol{b}(\boldsymbol{\theta}) := \boldsymbol{S}(\boldsymbol{\theta})\boldsymbol{\Lambda}^{1/2}(\boldsymbol{\theta})\boldsymbol{S}(\boldsymbol{\theta})^{-1}.$$

After some elementary linear algebra in conjunction with the identities

$$\theta_1 + \theta_2 = 1 \iff (\theta_1 + \theta_2)^2 = 1 \iff 2\theta_1\theta_2 = 1 - (\theta_1^2 + \theta_2^2), \qquad \boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathcal{S}_1^{2,+}, \tag{7.19}$$

one will find

$$\lambda_1(\boldsymbol{\theta}) = 0, \qquad \lambda_2(\boldsymbol{\theta}) = 1+\theta_1^2+\theta_2^2+\sqrt{(1+\theta_1^2+\theta_2^2)^2-2}, \qquad \lambda_3(\boldsymbol{\theta}) = 1+\theta_1^2+\theta_2^2-\sqrt{(1+\theta_1^2+\theta_2^2)^2-2};$$

and

$$\boldsymbol{x}^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \ \boldsymbol{x}^{(2)} = \begin{pmatrix} 2(\theta_1-\theta_2) \\ 2-\lambda_2(\boldsymbol{\theta}) \\ \lambda_2(\boldsymbol{\theta})-2 \end{pmatrix}, \ \boldsymbol{x}^{(3)} = \begin{pmatrix} 2(\theta_1-\theta_2) \\ 2-\lambda_3(\boldsymbol{\theta}) \\ \lambda_3(\boldsymbol{\theta})-2 \end{pmatrix}.$$

Therefore,

$$\boldsymbol{S}(\boldsymbol{\theta}) = \begin{pmatrix} 0 & 2(\theta_1-\theta_2) & 2(\theta_1-\theta_2) \\ 1 & 2-\lambda_2(\boldsymbol{\theta}) & 2-\lambda_3(\boldsymbol{\theta}) \\ 1 & \lambda_2(\boldsymbol{\theta})-2 & \lambda_3(\boldsymbol{\theta})-2 \end{pmatrix},$$

and

$$\boldsymbol{S}(\boldsymbol{\theta})^{-1} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ \frac{2-\lambda_3(\boldsymbol{\theta})}{4(\theta_1-\theta_2)\sqrt{(1+\theta_1^2+\theta_2^2)^2-2}} & -\frac{1}{4\sqrt{(1+\theta_1^2+\theta_2^2)^2-2}} & \frac{1}{4\sqrt{(1+\theta_1^2+\theta_2^2)^2-2}} \\ \frac{\lambda_2(\boldsymbol{\theta})-2}{4(\theta_1-\theta_2)\sqrt{(1+\theta_1^2+\theta_2^2)^2-2}} & \frac{1}{4\sqrt{(1+\theta_1^2+\theta_2^2)^2-2}} & -\frac{1}{4\sqrt{(1+\theta_1^2+\theta_2^2)^2-2}} \end{pmatrix}.$$

Finally, by exploiting identities (7.19) and the well-known

$$\sqrt{x+a} - \sqrt{x+b} = \frac{a-b}{\sqrt{x+a}+\sqrt{x+b}}, \tag{7.20}$$

one obtains $\boldsymbol{b}(\boldsymbol{\theta})$ in the following closed form:

$$\boldsymbol{b}(\boldsymbol{\theta}) = \begin{pmatrix} \frac{2\theta_1\theta_2}{\sqrt{\lambda_2}+\sqrt{\lambda_3}} + \frac{\sqrt{\lambda_2}+\sqrt{\lambda_3}}{2} & -\frac{\theta_1-\theta_2}{\sqrt{\lambda_2}+\sqrt{\lambda_3}} & \frac{\theta_1-\theta_2}{\sqrt{\lambda_2}+\sqrt{\lambda_3}} \\ -\frac{\theta_1-\theta_2}{\sqrt{\lambda_2}+\sqrt{\lambda_3}} & \frac{\sqrt{\lambda_2}+\sqrt{\lambda_3}}{4} - \frac{\theta_1\theta_2}{\sqrt{\lambda_2}+\sqrt{\lambda_3}} & \frac{\theta_1\theta_2}{\sqrt{\lambda_2}+\sqrt{\lambda_3}} - \frac{\sqrt{\lambda_2}+\sqrt{\lambda_3}}{4} \\ \frac{\theta_1-\theta_2}{\sqrt{\lambda_2}+\sqrt{\lambda_3}} & \frac{\theta_1\theta_2}{\sqrt{\lambda_2}+\sqrt{\lambda_3}} - \frac{\sqrt{\lambda_2}+\sqrt{\lambda_3}}{4} & \frac{\sqrt{\lambda_2}+\sqrt{\lambda_3}}{4} - \frac{\theta_1\theta_2}{\sqrt{\lambda_2}+\sqrt{\lambda_3}} \end{pmatrix}.$$

We shall provide a detailed derivation of just the entries $(1,1)$, $(1,2)$, $(2,1)$ and $(2,2)$ of the above matrix $\boldsymbol{b}(\boldsymbol{\theta})$, for the rest are derived in much the same way using the same algebraic tricks which we shall disclose shortly. The following three identities involving $\lambda_2 = \lambda_2(\boldsymbol{\theta})$ and $\lambda_3 = \lambda_3(\boldsymbol{\theta})$ will be used extensively, the first of which is a direct consequence of the identity (7.20) from above, and the last two immediate from the very definitions of λ_2 and λ_3:

$$\sqrt{\lambda_2(\boldsymbol{\theta})} - \sqrt{\lambda_3(\boldsymbol{\theta})} = \frac{2\sqrt{(1+\theta_1^2+\theta_2^2)^2-2}}{\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})}}; \tag{7.21}$$

$$\lambda_3(\boldsymbol{\theta}) = \lambda_2(\boldsymbol{\theta}) - 2\sqrt{(1+\theta_1^2+\theta_2^2)^2-2}; \tag{7.22}$$

$$\lambda_2(\boldsymbol{\theta}) + \lambda_3(\boldsymbol{\theta}) = 2(1 + \theta_1^2 + \theta_2^2). \tag{7.23}$$

We start off by deriving entry $(1,1)$ of $\boldsymbol{b}(\boldsymbol{\theta})$, which we denote by $\boldsymbol{b}_{11}(\boldsymbol{\theta})$:

$$\begin{aligned}
\boldsymbol{b}_{11}(\boldsymbol{\theta}) &= 2(\theta_1 - \theta_2)\sqrt{\lambda_2} \cdot \frac{2 - \lambda_3}{4(\theta_1 - \theta_2)\sqrt{(1 + \theta_1^2 + \theta_2^2)^2 - 2}} \\
&\qquad + 2(\theta_1 - \theta_2)\sqrt{\lambda_3} \cdot \frac{\lambda_2 - 2}{4(\theta_1 - \theta_2)\sqrt{(1 + \theta_1^2 + \theta_2^2)^2 - 2}} \\
&= \frac{2(\sqrt{\lambda_2} - \sqrt{\lambda_3}) - \lambda_3\sqrt{\lambda_2} + \lambda_2\sqrt{\lambda_3}}{2\sqrt{(1 + \theta_1^2 + \theta_2^2)^2 - 2}} \\
&= \frac{2}{\sqrt{\lambda_2} + \sqrt{\lambda_3}} - \frac{\left(\lambda_2 - 2\sqrt{(1 + \theta_1^2 + \theta_2^2)^2 - 2}\right)\sqrt{\lambda_2} - \lambda_2\sqrt{\lambda_3}}{2\sqrt{(1 + \theta_1^2 + \theta_2^2)^2 - 2}} && \text{(by (7.21) and (7.22))} \\
&= \frac{2}{\sqrt{\lambda_2} + \sqrt{\lambda_3}} - \frac{\lambda_2(\sqrt{\lambda_2} - \sqrt{\lambda_3})}{2\sqrt{(1 + \theta_1^2 + \theta_2^2)^2 - 2}} + \sqrt{\lambda_2} \\
&= \frac{2}{\sqrt{\lambda_2} + \sqrt{\lambda_3}} - \frac{\lambda_2}{\sqrt{\lambda_2} + \sqrt{\lambda_3}} + \sqrt{\lambda_2} && \text{(by (7.21))} \\
&= \frac{2 + \sqrt{\lambda_2\lambda_3}}{\sqrt{\lambda_2} + \sqrt{\lambda_3}} \\
&= \frac{4 + (\sqrt{\lambda_2} + \sqrt{\lambda_3})^2 - (\lambda_2 + \lambda_3)}{2(\sqrt{\lambda_2} + \sqrt{\lambda_3})} \\
&= \frac{2(1 - \theta_1^2 - \theta_2^2)}{2(\sqrt{\lambda_2} + \sqrt{\lambda_3})} + \frac{\sqrt{\lambda_2} + \sqrt{\lambda_3}}{2} && \text{(by (7.23))} \\
&= \frac{2\theta_1\theta_2}{\sqrt{\lambda_2} + \sqrt{\lambda_3}} + \frac{\sqrt{\lambda_2} + \sqrt{\lambda_3}}{2} && \text{(by (7.19)).}
\end{aligned}$$

Next, we derive entry $(1,2)$ of $\boldsymbol{b}(\boldsymbol{\theta})$, which we denote by $\boldsymbol{b}_{12}(\boldsymbol{\theta})$:

$$\boldsymbol{b}_{12}(\boldsymbol{\theta}) = \frac{2(\theta_1 - \theta_2)\sqrt{\lambda_3}}{4\sqrt{(1 + \theta_1^2 + \theta_2^2)^2 - 2}} - \frac{2(\theta_1 - \theta_2)\sqrt{\lambda_2}}{4\sqrt{(1 + \theta_1^2 + \theta_2^2)^2 - 2}} = \frac{(\theta_2 - \theta_1)(\sqrt{\lambda_2} - \sqrt{\lambda_3})}{2\sqrt{(1 + \theta_1^2 + \theta_2^2)^2 - 2}} = \frac{\theta_2 - \theta_1}{\sqrt{\lambda_2} + \sqrt{\lambda_3}},$$

where the last equality follows from (7.21).

Next, we derive entry $(2,1)$ of $\boldsymbol{b}(\boldsymbol{\theta})$, which we denote by $\boldsymbol{b}_{21}(\boldsymbol{\theta})$:

$$\begin{aligned}
\boldsymbol{b}_{21}(\boldsymbol{\theta}) &= \frac{(2 - \lambda_2)(2 - \lambda_3)(\sqrt{\lambda_2} - \sqrt{\lambda_3})}{4(\theta_1 - \theta_2)\sqrt{(1 + \theta_1^2 + \theta_2^2)^2 - 2}} = \frac{(2 - \lambda_2)(2 - \lambda_3)}{2(\theta_1 - \theta_2)(\sqrt{\lambda_2} + \sqrt{\lambda_3})} = \frac{4 - 2(\lambda_2 + \lambda_3) + \lambda_2\lambda_3}{2(\theta_1 - \theta_2)(\sqrt{\lambda_2} + \sqrt{\lambda_3})} \\
&= \frac{1 - 2(\theta_1^2 + \theta_2^2)}{(\theta_1 - \theta_2)(\sqrt{\lambda_2} + \sqrt{\lambda_3})} \\
&= \frac{2\theta_1\theta_2 - (\theta_1^2 + \theta_2^2)}{(\theta_1 - \theta_2)(\sqrt{\lambda_2} + \sqrt{\lambda_3})} = \frac{-(\theta_1 - \theta_2)^2}{(\theta_1 - \theta_2)(\sqrt{\lambda_2} + \sqrt{\lambda_3})} = -\frac{\theta_1 - \theta_2}{\sqrt{\lambda_2} + \sqrt{\lambda_3}},
\end{aligned}$$

where the second equality follows from (7.21), the equality in the second line follows from (7.23) in conjunction with the trivial fact that $\lambda_2\lambda_3 = 2$, and the equality in the third line follows from (7.19).

Finally, we derive entry $(2,2)$ of $\boldsymbol{b}(\boldsymbol{\theta})$, which we denote by $\boldsymbol{b}_{22}(\boldsymbol{\theta})$:

$$\begin{aligned}
\boldsymbol{b}_{22}(\boldsymbol{\theta}) &= \frac{(2-\lambda_3)\sqrt{\lambda_3}}{4\sqrt{(1+\theta_1^2+\theta_2^2)^2-2}} - \frac{(2-\lambda_2)\sqrt{\lambda_2}}{4\sqrt{(1+\theta_1^2+\theta_2^2)^2-2}} \\
&= -\frac{2(\sqrt{\lambda_2}-\sqrt{\lambda_3})+\lambda_3\sqrt{\lambda_3}-\lambda_2\sqrt{\lambda_2}}{4\sqrt{(1+\theta_1^2+\theta_2^2)^2-2}} \\
&= -\frac{1}{\sqrt{\lambda_2}+\sqrt{\lambda_3}} + \frac{\lambda_2\sqrt{\lambda_2}-(\lambda_2-2\sqrt{(1+\theta_1^2+\theta_2^2)^2-2})\sqrt{\lambda_3}}{4\sqrt{(1+\theta_1^2+\theta_2^2)^2-2}} && \text{(by (7.21) and (7.22))} \\
&= -\frac{1}{\sqrt{\lambda_2}+\sqrt{\lambda_3}} + \frac{\lambda_2(\sqrt{\lambda_2}-\sqrt{\lambda_3})}{4\sqrt{(1+\theta_1^2+\theta_2^2)^2-2}} + \frac{\sqrt{\lambda_3}}{2} \\
&= -\frac{1}{\sqrt{\lambda_2}+\sqrt{\lambda_3}} + \frac{\lambda_2}{2(\sqrt{\lambda_2}+\sqrt{\lambda_3})} + \frac{\sqrt{\lambda_3}}{2} && \text{(by (7.21))} \\
&= \frac{\lambda_2+\lambda_3+\sqrt{\lambda_2\lambda_3}-2}{2(\sqrt{\lambda_2}+\sqrt{\lambda_3})} \\
&= \frac{2(\theta_1^2+\theta_2^2)+\sqrt{\lambda_2\lambda_3}}{2(\sqrt{\lambda_2}+\sqrt{\lambda_3})} && \text{(by (7.23))} \\
&= \frac{2(1+\theta_1^2+\theta_2^2)+2\sqrt{\lambda_2\lambda_3}-4\theta_1\theta_2}{4(\sqrt{\lambda_2}+\sqrt{\lambda_3})} && \text{(by (7.19))} \\
&= \frac{\lambda_2+\lambda_3+2\sqrt{\lambda_2\lambda_3}-4\theta_1\theta_2}{4(\sqrt{\lambda_2}+\sqrt{\lambda_3})} && \text{(by (7.23))} \\
&= \frac{(\sqrt{\lambda_2}+\sqrt{\lambda_3})^2-4\theta_1\theta_2}{4(\sqrt{\lambda_2}+\sqrt{\lambda_3})} = \frac{\sqrt{\lambda_2}+\sqrt{\lambda_3}}{4} - \frac{\theta_1\theta_2}{\sqrt{\lambda_2}+\sqrt{\lambda_3}}.
\end{aligned}$$

We now define a new (2-dimensional) Brownian motion, $\tilde{\boldsymbol{W}} = (\tilde{\boldsymbol{W}}^{(1)}, \tilde{\boldsymbol{W}}^{(2)})$, in terms of our 3-dimensional Brownian motion, $\boldsymbol{W} = (\boldsymbol{W}^{(1)}, \boldsymbol{W}^{(2)}, \boldsymbol{W}^{(3)})$, by setting $\tilde{\boldsymbol{W}}^{(1)} := \boldsymbol{W}^{(1)}$ and $\tilde{\boldsymbol{W}}^{(2)} := \boldsymbol{W}^{(2)} - \boldsymbol{W}^{(3)}$. Finally, observe how we can rewrite the SDE from (7.17) in the following way:

$$d\begin{pmatrix}\rho_t \\ \Theta_t\end{pmatrix} = \boldsymbol{a}(\Theta_t)dt + \tilde{\boldsymbol{b}}(\Theta_t)d\tilde{\boldsymbol{W}}_t + \boldsymbol{\gamma}(\Theta_t)dl_t,$$

where

$$\tilde{\boldsymbol{b}}(\boldsymbol{\theta}) := \begin{pmatrix} \frac{2\theta_1\theta_2}{\sqrt{\lambda_2}+\sqrt{\lambda_3}} + \frac{\sqrt{\lambda_2}+\sqrt{\lambda_3}}{2} & -\frac{\theta_1-\theta_2}{\sqrt{\lambda_2}+\sqrt{\lambda_3}} \\ -\frac{\theta_1-\theta_2}{\sqrt{\lambda_2}+\sqrt{\lambda_3}} & \frac{\sqrt{\lambda_2}+\sqrt{\lambda_3}}{4} - \frac{\theta_1\theta_2}{\sqrt{\lambda_2}+\sqrt{\lambda_3}} \\ \frac{\theta_1-\theta_2}{\sqrt{\lambda_2}+\sqrt{\lambda_3}} & \frac{\theta_1\theta_2}{\sqrt{\lambda_2}+\sqrt{\lambda_3}} - \frac{\sqrt{\lambda_2}+\sqrt{\lambda_3}}{4} \end{pmatrix}.$$

And so we discern that the source of randomness of the SDE our MAP solves is actually 2-dimensional. ∎

Finally, we prove our claim regarding uniqueness.

Lemma 7.2.5 *The weak solution of the SDE from the statement of Corollary 7.2.4 is unique.*

Proof The result essentially follows by verifying that the coefficients satisfy the hypotheses of Theorem 2.2.19. In other words, it suffices to check that the conditions from Theorem 2.1.1 of [47] are satisfied by the functions $\boldsymbol{a}$ and $\boldsymbol{b}$ from the statement of our corollary (as remarked in Exercise 2.2.1 of [47], the conditions for uniqueness listed in Theorem 2.1.1 therein also carry over into our setup with domain $D = \mathbb{R} \times \mathcal{S}_1^{2,+}$). More specifically, we need to show that

(i) there exists an $L > 0$ such that $|\boldsymbol{a}(\boldsymbol{\theta}) - \boldsymbol{a}(\boldsymbol{\phi})| + |\boldsymbol{b}_1(\boldsymbol{\theta}) - \boldsymbol{b}_1(\boldsymbol{\phi})| + |\boldsymbol{b}_2(\boldsymbol{\theta}) - \boldsymbol{b}_2(\boldsymbol{\phi})| \leq L|\boldsymbol{\theta} - \boldsymbol{\phi}|$ for every $\boldsymbol{\theta}, \boldsymbol{\phi} \in \mathcal{S}_1^{2,+}$;

(ii) there exists a $C > 0$ such that $|\boldsymbol{a}(\boldsymbol{\theta})| + |\boldsymbol{b}_1(\boldsymbol{\theta})| + |\boldsymbol{b}_2(\boldsymbol{\theta})| \leq C(1 + |\boldsymbol{\theta}|)$ for every $\boldsymbol{\theta} \in \mathcal{S}_1^{2,+}$,

where $\boldsymbol{a}(\boldsymbol{\theta})$ is the function from the statement of our corollary; $\boldsymbol{b}_1(\boldsymbol{\theta})$ and $\boldsymbol{b}_2(\boldsymbol{\theta})$ denote the first and second columns, respectively, of the matrix $\boldsymbol{b}(\boldsymbol{\theta})$ from the statement of the corollary, and $|\cdot|$ denotes the classic Euclidean distance in $\mathbb{R}^2$ (i.e., the L^2 norm).

We start by proving condition (i): let $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathcal{S}_1^{2,+}$ and $\boldsymbol{\phi} = (\phi_1, \phi_2) \in \mathcal{S}_1^{2,+}$, then

$$|\boldsymbol{a}(\boldsymbol{\theta}) - \boldsymbol{a}(\boldsymbol{\phi})|^2 = 2((\theta_1 - \phi_1) - (\theta_2 - \phi_2))^2 \leq 4((\theta_1 - \phi_1)^2 + (\theta_2 - \phi_2)^2) = 4|\boldsymbol{\theta} - \boldsymbol{\phi}|^2, \tag{7.24}$$

where we have used the well-known inequality

$$(a \pm b)^2 \leq 2a^2 + 2b^2, \qquad a, b \in \mathbb{R}. \tag{7.25}$$

We also have

$$\begin{aligned}
&|\boldsymbol{b}_1(\boldsymbol{\theta}) - \boldsymbol{b}_1(\boldsymbol{\phi})|^2 \\
&= ((\frac{2\theta_1\theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})}} - \frac{2\phi_1\phi_2}{\sqrt{\lambda_2(\boldsymbol{\phi})} + \sqrt{\lambda_3(\boldsymbol{\phi})}}) + (\frac{\sqrt{\lambda_2(\boldsymbol{\theta})} - \sqrt{\lambda_2(\boldsymbol{\phi})}}{2} + \frac{\sqrt{\lambda_3(\boldsymbol{\theta})} - \sqrt{\lambda_3(\boldsymbol{\phi})}}{2}))^2 \\
&\quad + 2(\frac{\theta_1 - \theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})}} - \frac{\phi_1 - \phi_2}{\sqrt{\lambda_2(\boldsymbol{\phi})} + \sqrt{\lambda_3(\boldsymbol{\phi})}})^2 \\
&\leq 8(\frac{\theta_1\theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})}} - \frac{\phi_1\phi_2}{\sqrt{\lambda_2(\boldsymbol{\phi})} + \sqrt{\lambda_3(\boldsymbol{\phi})}})^2 + (\sqrt{\lambda_2(\boldsymbol{\theta})} - \sqrt{\lambda_2(\boldsymbol{\phi})})^2 + (\sqrt{\lambda_3(\boldsymbol{\theta})} - \sqrt{\lambda_3(\boldsymbol{\phi})})^2
\end{aligned}$$

$$+ 2(\frac{\theta_1 - \theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})}} - \frac{\phi_1 - \phi_2}{\sqrt{\lambda_2(\boldsymbol{\phi})} + \sqrt{\lambda_3(\boldsymbol{\phi})}})^2,$$

where we have used (7.25) yet again. To proceed we need to find a sufficiently tight lower bound for the function $\mathcal{S}_1^{2,+} \ni (x_1, x_2) = \boldsymbol{x} \mapsto \sqrt{\lambda_2(\boldsymbol{x})} + \sqrt{\lambda_3(\boldsymbol{x})}$. First note that

$$\frac{1}{2} \leq x_1^2 + x_2^2 = x_1^2 + (1 - x_1)^2 = 2x_1^2 - 2x_1 + 1 \leq 1, \qquad \boldsymbol{x} = (x_1, x_2) \in \mathcal{S}_1^{2,+}. \tag{7.26}$$

This implies

$$\lambda_3(\boldsymbol{x}) = 1 + x_1^2 + x_2^2 - \sqrt{(1 + x_1^2 + x_2^2)^2 - 2} \geq \frac{3}{2} - \sqrt{2} =: M_0 > 0, \qquad \boldsymbol{x} = (x_1, x_2) \in \mathcal{S}_1^{2,+}. \tag{7.27}$$

Therefore,

$$\sqrt{\lambda_2(\boldsymbol{x})} + \sqrt{\lambda_3(\boldsymbol{x})} \geq 1 + \sqrt{M_0} > 1 > \sqrt{M_0}, \qquad \boldsymbol{x} = (x_1, x_2) \in \mathcal{S}_1^{2,+}. \tag{7.28}$$

We now derive yet another useful inequality for $\sqrt{\lambda_2(\boldsymbol{x})} + \sqrt{\lambda_3(\boldsymbol{x})}$:

$$\begin{aligned}
(\sqrt{\lambda_2(\boldsymbol{x})} + \sqrt{\lambda_3(\boldsymbol{x})})^2 &= \lambda_2(\boldsymbol{x}) + \lambda_3(\boldsymbol{x}) + 2\sqrt{\lambda_2(\boldsymbol{x})\lambda_3(\boldsymbol{x})} \\
&= 2(1 + x_1^2 + x_2^2) + 2\sqrt{\lambda_2(\boldsymbol{x})\lambda_3(\boldsymbol{x})} \\
&\leq 2(1 + x_1^2 + x_2^2) + (\lambda_2(\boldsymbol{x}) + \lambda_3(\boldsymbol{x})) \qquad\qquad \text{(by equation (7.25))} \\
&= 4(1 + x_1^2 + x_2^2), \qquad \boldsymbol{x} = (x_1, x_2) \in \mathcal{S}_1^{2,+},
\end{aligned}$$

that is,

$$(\sqrt{\lambda_2(\boldsymbol{x})} + \sqrt{\lambda_3(\boldsymbol{x})})^2 \leq 4(1 + |\boldsymbol{x}|^2), \qquad \boldsymbol{x} = (x_1, x_2) \in \mathcal{S}_1^{2,+}. \tag{7.29}$$

Next we derive a bound for the quantity $\theta_1\theta_2 - \phi_1\phi_2$ which we will be using many times:

$$|\theta_1\theta_2 - \phi_1\phi_2| = \frac{1}{2}|(\phi_1^2 - \theta_1^2) + (\phi_2^2 - \theta_2^2)| = \frac{1}{2}|(\phi_1 + \theta_1)(\phi_1 - \theta_1) + (\phi_2 + \theta_2)(\phi_2 - \theta_2)| \leq \|\boldsymbol{\phi} - \boldsymbol{\theta}\|_{L^1},$$

where we have used (7.19), the triangle inequality, the definition of the L^1 norm, and the fact that $\theta_i, \phi_i \leq 1$. Now, since all norms in $\mathbb{R}^2$ are equivalent, there exists a constant $M_1 > 0$ such that

$$|\theta_1\theta_2 - \phi_1\phi_2| \leq \|\boldsymbol{\phi} - \boldsymbol{\theta}\|_1 \leq M_1|\boldsymbol{\phi} - \boldsymbol{\theta}|. \tag{7.30}$$

Now, without loss of generality, assume $\lambda_2(\boldsymbol{\theta}) \geq \lambda_2(\boldsymbol{\phi})$. Then,

$$\begin{aligned}
&\sqrt{\lambda_2(\boldsymbol{\theta})} - \sqrt{\lambda_2(\boldsymbol{\phi})} \\
&= \frac{\lambda_2(\boldsymbol{\theta}) - \lambda_2(\boldsymbol{\phi})}{\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_2(\boldsymbol{\phi})}}
\end{aligned}$$

$$
\begin{aligned}
&\le \lambda_2(\boldsymbol{\theta}) - \lambda_2(\boldsymbol{\phi}) \\
&\quad = (\theta_1^2 - \phi_1^2) + (\theta_2^2 - \phi_2^2) + (\sqrt{(1+\theta_1^2+\theta_2^2)^2 - 2} - \sqrt{(1+\phi_1^2+\phi_2^2)^2 - 2}) \\
&\quad = (\theta_1+\phi_1)(\theta_1-\phi_1) + (\theta_2+\phi_2)(\theta_2-\phi_2) + \frac{(1+\theta_1^2+\theta_2^2)^2 - (1+\phi_1^2+\phi_2^2)^2}{\sqrt{(1+\theta_1^2+\theta_2^2)^2 - 2} + \sqrt{(1+\phi_1^2+\phi_2^2)^2 - 2}} \\
&\le 2\|\boldsymbol{\theta} - \boldsymbol{\phi}\|_{L^1} + (1+\theta_1^2+\theta_2^2)^2 - (1+\phi_1^2+\phi_2^2)^2,
\end{aligned}
$$

where the first inequality follows from (7.28), and the second inequality from the trivial fact that for every $\boldsymbol{x} = (x_1, x_2) \in \mathcal{S}_1^{2,+}$, we have $x_1, x_2 \le 1$ and

$$
\sqrt{(1+x_1^2+x_2^2)^2 - 2} \ge \frac{1}{2}, \qquad \boldsymbol{x} = (x_1, x_2) \in \mathcal{S}_1^{2,+},
$$

as a result of (7.26). Now, using the fact that all norms on $\mathbb{R}^2$ are equivalent, and hence the Euclidean norm $|\cdot|$ is equivalent to the L^1 norm $\|\cdot\|_1$, there exists a constant $M_1 > 0$ such that

$$
\begin{aligned}
\sqrt{\lambda_2(\boldsymbol{\theta})} - \sqrt{\lambda_2(\boldsymbol{\phi})} &\le 2M_1|\boldsymbol{\theta} - \boldsymbol{\phi}| + (2+\theta_1^2+\theta_2^2+\phi_1^2+\phi_2^2)((\theta_1^2-\phi_1^2) + (\theta_2^2-\phi_2^2)) \\
&\le 10M_1|\boldsymbol{\theta} - \boldsymbol{\phi}|,
\end{aligned}
$$

where the last inequality is an easy consequence of (7.26) and the inequality for $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathcal{S}_1^{2,+}$ and $\boldsymbol{\phi} = (\phi_1, \phi_2) \in \mathcal{S}_1^{2,+}$,

$$
(\theta_1^2-\phi_1^2) + (\theta_2^2-\phi_2^2) \le (\theta_1+\phi_1)(\theta_1-\phi_1) + (\theta_2+\phi_2)(\theta_2-\phi_2) \le 2\|\boldsymbol{\theta}-\boldsymbol{\phi}\|_1 \le 2M_1|\boldsymbol{\theta}-\boldsymbol{\phi}|
$$

we had derived earlier. Therefore, there exists a constant $C_1 > 0$ such that

$$
(\sqrt{\lambda_2(\boldsymbol{\theta})} - \sqrt{\lambda_2(\boldsymbol{\phi})})^2 \le C_1|\boldsymbol{\theta}-\boldsymbol{\phi}|^2. \tag{7.31}
$$

Now, without loss of generality, assume $\lambda_3(\boldsymbol{\theta}) \ge \lambda_3(\boldsymbol{\phi})$. Thanks to (7.27),

$$
\sqrt{\lambda_3(\boldsymbol{\theta})} - \sqrt{\lambda_3(\boldsymbol{\phi})} = \frac{\lambda_3(\boldsymbol{\theta}) - \lambda_3(\boldsymbol{\phi})}{\sqrt{\lambda_3(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\phi})}} \le (2M_0)^{-1}(\lambda_3(\boldsymbol{\theta}) - \lambda_3(\boldsymbol{\phi}));
$$

and the difference $\lambda_3(\boldsymbol{\theta}) - \lambda_3(\boldsymbol{\phi})$ can be analysed in virtually the same way as $\lambda_2(\boldsymbol{\theta}) - \lambda_2(\boldsymbol{\phi})$ above. So, again there exists some constant $C_2 > 0$ such that

$$
(\sqrt{\lambda_3(\boldsymbol{\theta})} - \sqrt{\lambda_3(\boldsymbol{\phi})})^2 \le C_2|\boldsymbol{\theta}-\boldsymbol{\phi}|^2. \tag{7.32}
$$

Next, we seek for an upper bound of the form $C_3|\boldsymbol{\theta}-\boldsymbol{\phi}|^2$ for the quantity

$$
(\frac{\theta_1\theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})}} - \frac{\phi_1\phi_2}{\sqrt{\lambda_2(\boldsymbol{\phi})} + \sqrt{\lambda_3(\boldsymbol{\phi})}})^2.
$$

Let's first examine the case when $\theta_1\theta_2 = \phi_1\phi_2$. In this case, $\lambda_2(\boldsymbol{\theta}) = \lambda_2(\boldsymbol{\phi})$ and $\lambda_3(\boldsymbol{\theta}) = \lambda_3(\boldsymbol{\phi})$. Indeed, by using (7.19), it is not difficult to see that for $\boldsymbol{x} = (x_1, x_2) \in \mathcal{S}_1^{2,+}$,

$$\lambda_2(\boldsymbol{x}) = 2(1 - x_1x_2) + \sqrt{4(1 - x_1x_2)^2 - 2} \text{ and } \lambda_3(\boldsymbol{x}) = 2(1 - x_1x_2) - \sqrt{4(1 - x_1x_2)^2 - 2},$$

from which the claim follows. Therefore, in this case, the inequality we seek trivially holds:

$$(\frac{\theta_1\theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})}} - \frac{\phi_1\phi_2}{\sqrt{\lambda_2(\boldsymbol{\phi})} + \sqrt{\lambda_3(\boldsymbol{\phi})}})^2 = 0 = (\theta_1\theta_2 - \phi_1\phi_2)^2 \leq M_1^2|\boldsymbol{\theta} - \boldsymbol{\phi}|^2,$$

where we have used (7.30) in the last inequality. Let's now examine the case when $\theta_1\theta_2 \neq \phi_1\phi_2$. In this case,

$$\begin{aligned}
&(\frac{\theta_1\theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})}} - \frac{\phi_1\phi_2}{\sqrt{\lambda_2(\boldsymbol{\phi})} + \sqrt{\lambda_3(\boldsymbol{\phi})}})^2 \\
&\qquad = (\theta_1\theta_2 - \phi_1\phi_2)^2 \cdot (\frac{\theta_1\theta_2(\sqrt{\lambda_2(\boldsymbol{\phi})} + \sqrt{\lambda_3(\boldsymbol{\phi})}) - \phi_1\phi_2(\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})})}{(\sqrt{\lambda_2(\boldsymbol{\phi})} + \sqrt{\lambda_3(\boldsymbol{\phi})})(\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})})(\theta_1\theta_2 - \phi_1\phi_2)})^2 \\
&\qquad \leq M_2 \cdot (\theta_1\theta_2 - \phi_1\phi_2)^2,
\end{aligned}$$

for some constant $M_2 > 0$, since the quantity above on the right is clearly bounded above thanks to (7.28), the fact that both functions λ_2 and λ_3 are bounded above, and our restriction $\theta_1\theta_2 \neq \phi_1\phi_2$. It then follows, thanks to (7.30), that there exists a constant $C_3 > 0$ such that

$$(\frac{\theta_1\theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})}} - \frac{\phi_1\phi_2}{\sqrt{\lambda_2(\boldsymbol{\phi})} + \sqrt{\lambda_3(\boldsymbol{\phi})}})^2 \leq C_3|\boldsymbol{\theta} - \boldsymbol{\phi}|^2. \tag{7.33}$$

Lastly, we also seek for an upper bound of the form $C_4|\boldsymbol{\theta} - \boldsymbol{\phi}|^2$ for the quantity

$$(\frac{\theta_1 - \theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})}} - \frac{\phi_1 - \phi_2}{\sqrt{\lambda_2(\boldsymbol{\phi})} + \sqrt{\lambda_3(\boldsymbol{\phi})}})^2.$$

The case when $\boldsymbol{\theta} = \boldsymbol{\phi}$ is trivial. So let's assume $\boldsymbol{\theta} \neq \boldsymbol{\phi}$. Then,

$$\begin{aligned}
&|\frac{\theta_1 - \theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})}} - \frac{\phi_1 - \phi_2}{\sqrt{\lambda_2(\boldsymbol{\phi})} + \sqrt{\lambda_3(\boldsymbol{\phi})}}| \\
&\qquad = |(\theta_1 - \phi_1) + (\phi_2 - \theta_2)| \cdot |\frac{(\theta_1 - \theta_2)(\sqrt{\lambda_2(\boldsymbol{\phi})} + \sqrt{\lambda_3(\boldsymbol{\phi})}) - (\phi_1 - \phi_2)(\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})})}{(\sqrt{\lambda_2(\boldsymbol{\theta})} + \sqrt{\lambda_3(\boldsymbol{\theta})})(\sqrt{\lambda_2(\boldsymbol{\phi})} + \sqrt{\lambda_3(\boldsymbol{\phi})})(\theta_1 - \phi_1 + \phi_2 - \theta_2)}| \\
&\qquad \leq \|\boldsymbol{\theta} - \boldsymbol{\phi}\|_1 \cdot M_3
\end{aligned}$$

for some constant $M_3 > 0$, where we have made use of the triangle inequality, the definition of the L^1 norm, and the fact that the quantity above on the right is clearly bounded above under our restriction $\boldsymbol{\theta} \neq \boldsymbol{\phi}$ (by much the same reasoning we gave earlier for a similar quantity and situation).

It then follows, thanks to the fact that the L^1 norm is equivalent to the Euclidean norm in $\mathbb{R}^2$, that there exists a constant $C_4 > 0$ such that

$$(\frac{\theta_1-\theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})}}-\frac{\phi_1-\phi_2}{\sqrt{\lambda_2(\boldsymbol{\phi})}+\sqrt{\lambda_3(\boldsymbol{\phi})}})^2 \leq C_4|\boldsymbol{\theta}-\boldsymbol{\phi}|^2. \tag{7.34}$$

By (7.31) - (7.34), we have

$$|\boldsymbol{b}_1(\boldsymbol{\theta})-\boldsymbol{b}_1(\boldsymbol{\phi})|^2 \leq C'|\boldsymbol{\theta}-\boldsymbol{\phi}|^2, \tag{7.35}$$

where $C' \coloneqq 8C_3 + C_1 + C_2 + 2C_4 > 0$.

By inspecting the second column $\boldsymbol{b}_2(\boldsymbol{\theta})$ of the matrix $\boldsymbol{b}(\boldsymbol{\theta})$ from the statement of the corollary, one will see that the method we used to derive (7.35) can be applied in an analogous way to derive a constant $C'' > 0$ such that

$$|\boldsymbol{b}_2(\boldsymbol{\theta})-\boldsymbol{b}_2(\boldsymbol{\phi})|^2 \leq C''|\boldsymbol{\theta}-\boldsymbol{\phi}|^2. \tag{7.36}$$

Equations (7.24), (7.35) and (7.36) then prove the desired global Lipschitz condition given in (i) above.

It remains to prove the linear growth condition given in (ii): let $\boldsymbol{\theta}=(\theta_1,\theta_2) \in \mathcal{S}_1^{2,+}$. Then,

$$\begin{aligned}
&|\boldsymbol{a}(\boldsymbol{\theta})|+|\boldsymbol{b}_1(\boldsymbol{\theta})|+|\boldsymbol{b}_2(\boldsymbol{\theta})| \\
&= (1+2(\theta_1-\theta_2)^2)^{1/2} + ((\frac{2\theta_1\theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})}}+\frac{\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})}}{2})^2+2(\frac{\theta_1-\theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})}})^2)^{1/2} \\
&\quad + (2(\frac{\theta_1\theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})}}-\frac{\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})}}{4})^2+(\frac{\theta_1-\theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})}})^2)^{1/2}.
\end{aligned}$$

Now,

$$\begin{aligned}
(1+2(\theta_1-\theta_2)^2)^{1/2} &= (1+2(\theta_1^2+\theta_2^2)-4\theta_1\theta_2)^{1/2} \\
&= (1+2(\theta_1^2+\theta_2^2)+2\theta_1^2+2\theta_2^2-2)^{1/2} \qquad \text{(by equation (7.19))} \\
&= (4|\boldsymbol{\theta}|^2-1)^{1/2} \\
&\leq (4|\boldsymbol{\theta}|^2)^{1/2} = 2|\boldsymbol{\theta}| \leq 2(1+|\boldsymbol{\theta}|).
\end{aligned} \tag{7.37}$$

Also,

$$\begin{aligned}
&((\frac{2\theta_1\theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})}}+\frac{\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})}}{2})^2+2(\frac{\theta_1-\theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})}})^2)^{1/2} \\
&\leq (\frac{4\theta_1^2\theta_2^2}{(\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})})^2}+2\theta_1\theta_2+\frac{(\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})})^2}{4}+2\frac{\theta_1^2+\theta_2^2-2\theta_1\theta_2}{(\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})})^2})^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&= \Big(\frac{(1-|\boldsymbol{\theta}|^2)^2}{(\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})})^2} + (1-|\boldsymbol{\theta}|^2) + \frac{(\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})})^2}{4} + \frac{4|\boldsymbol{\theta}|^2-2}{(\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})})^2}\Big)^{1/2} \quad \text{(by (7.19))} \\
&\le \Big(\frac{(1-|\boldsymbol{\theta}|^2)^2}{M_0} + (1-|\boldsymbol{\theta}|^2) + (1+|\boldsymbol{\theta}|^2) + \frac{4|\boldsymbol{\theta}|^2-2}{M_0}\Big)^{1/2} \quad \text{(by (7.28), (7.29))} \\
&= \Big((2-\frac{1}{M_0}) + \frac{2}{M_0}|\boldsymbol{\theta}|^2 + \frac{|\boldsymbol{\theta}|^4}{M_0}\Big)^{1/2}.
\end{aligned}
$$

We know $2-\frac{1}{M_0}$ is a negative quantity; and, due to the fact that $\theta_i \le 1$, we also have

$$
\frac{|\boldsymbol{\theta}|^4}{M_0} = \frac{|\boldsymbol{\theta}|^2}{M_0}\cdot|\boldsymbol{\theta}|^2 = \frac{\theta_1^2+\theta_2^2}{M_0}\cdot|\boldsymbol{\theta}|^2 \le \frac{2}{M_0}|\boldsymbol{\theta}|^2.
$$

Therefore,

$$
\Big(\big(\frac{2\theta_1\theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})}} + \frac{\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})}}{2}\big)^2 + 2\big(\frac{\theta_1-\theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})}}\big)^2\Big)^{1/2} \le \frac{2}{\sqrt{M_0}}|\boldsymbol{\theta}| \le \frac{2}{\sqrt{M_0}}(1+|\boldsymbol{\theta}|). \tag{7.38}
$$

Using this very method, one can likewise derive a constant $L' > 0$ such that

$$
\Big(2\big(\frac{\theta_1\theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})}} - \frac{\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})}}{4}\big)^2 + \big(\frac{\theta_1-\theta_2}{\sqrt{\lambda_2(\boldsymbol{\theta})}+\sqrt{\lambda_3(\boldsymbol{\theta})}}\big)^2\Big)^{1/2} \le L'(1+|\boldsymbol{\theta}|). \tag{7.39}
$$

Equations (7.37), (7.38) and (7.39) prove condition (ii). ∎

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