When k-tribes go to war (a point is all that you can score)

Andreas Kyprianou Joint work with:
Tim Rogers (Bath)
Lizabeth Peñaloza (UMAR)


ART OF ZTT.COM
FRANKIE GOES TO HOLLYWOOD
TWO TRIBES
"WE DON'T WANT TO DIE"
ZTT

What happened on Rapa Nui (Easter Island)? The (European) tale of the Tu'u and the 'Otu 'Itu

k-TRIBES GO TO WAR

- Consider a population consisting of $k \geq 2$ tribes
- Members of each tribe kill one another in conflict (including within-tribe conflict)
- Model the numbers in each tribe as a Markov chain

  $$\boldsymbol{n}(t) = (n_1(t), \cdots, n_k(t)), \qquad t \geq 0,$$

  where $n_i(t)$ is the number of individuals alive in tribe i at time $t \geq 0$
- The MC $\boldsymbol{n}$ lives on

  $$\mathbb{N}_*^k = \left\{ \boldsymbol{\nu} \in \mathbb{N}_0^k : \sum_{i=1}^{k} \nu_i \geq 1 \right\} = \mathbb{N}_0^k \setminus \{\mathbf{0}\},$$

  where $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$ where $n_i(t)$ is the number of individuals alive in tribe i at time $t \geq 0$.

TRIBAL DYNAMICS

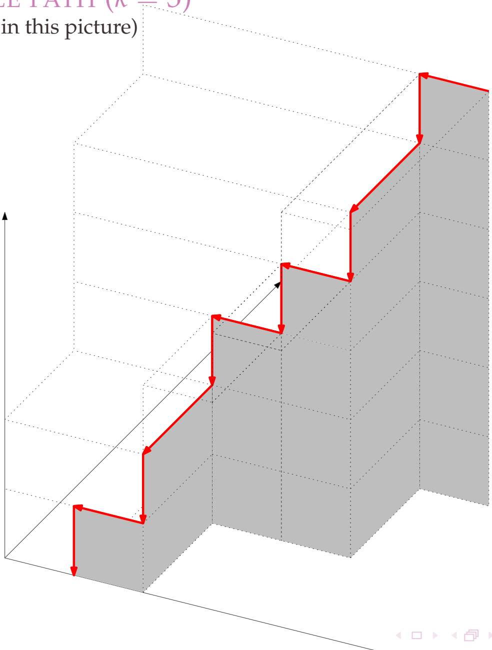
- Given that $\boldsymbol{n}(t) = (n_1, \cdots, n_k) \in \mathbb{N}_*^k$ with $\sum_{i=1}^k n_i > 1$:
  - For $i \in \{1, \cdots, k\}$ any two individuals from tribe i will fight at rate $C_{i,i}$, one of them will kill the other and hence the total rate at which tribe i loses an individual through infighting is $C_{i,i}\binom{n_i}{2}$.
  - For $i \neq j$, both selected from $\{1, \cdots, k\}$, an individual from tribe i will encounter an individual of tribe j and will fight resulting in the former killing the latter. This occurs at rate $C_{i,j}$. Hence the total rate at which someone from tribe i will kill some from tribe j is

$$\sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} C_{i,j} n_i n_j.$$

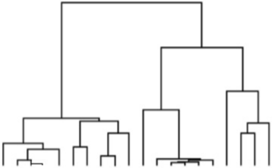
- The k states for which $\sum_{i=1}^k n_i = 1$ are absorbing, representing the end of the process in which there is one final surviving block.
- For reasons that will become apparent later on, we refer to the process $(\boldsymbol{n}(t), t \geq 0)$ as the **replicator coalescent**

Example sample path ($k = 3$)

(There is no time in this picture)



Similar Models



Kingman's Coalescent:

- Block counting process $(n(t), t \geq 0)$, any two blocks collide and merge at rate $c > 0$ so that if $n(t) = n$ then total merge rate is $c\binom{n}{2}$.
- Has the advantage of relation to exchangeable partition structures on $\mathbb{N}$.
- Fundamental result: coming down from infinity

$$\lim_{t \downarrow 0} t n(t) = \frac{c}{2}$$

SIMILAR MODELS

OK Corral:

- Model for the number of surviving shooters in a famous 19th Century Arizona 30 second shoot-out between lawmen and outlaws in 1881, in a town called Tombstone.
- $\boldsymbol{n}(t) = (n_1(t), n_2(t))$, $t = 0, 1, 2, \ldots$, where $n_1(t)$ are the number of surviving lawmen and $n_2(t)$ are the number of surviving outlaws
- $$\mathbb{P}\Big(n_1(t+1) = n_1(t) - 1,\ \ n_2(t+1) = n_2(t)\ \mid\ (n_1(t), n_2(t))\Big) = \frac{n_2(t)}{n_1(t) + n_2(t)}$$
  $$\mathbb{P}\Big(n_1(t+1) = n_1(t),\ \ n_2(t+1) = n_2(t) - 1\ \mid\ (n_1(t), n_2(t))\Big) = \frac{n_1(t)}{n_1(t) + n_2(t)}$$
- Kingman (1999), Kingman & Volkov (2003, 2019) Investigate the probability of the number of surviving gunmen and whether they are outlaws or lawmen

Questions

- Does the replicator coalescent come down from infinity? ✓
- How does it come down from infinity? ✓
- What is the distribution on $\{1, \ldots, k\}$ of the tribe of the surviving individual? $\times$

POLAR DECOMPOSITION

- Prefer to describe it via an L^1-*polar decomposition*.

$$\sigma(t) = ||\boldsymbol{n}(t)||_1 = n_1(t) + \cdots + n_k(t) \in \mathbb{N}$$

  and

$$\boldsymbol{r}(t) := \sigma(t)^{-1}\boldsymbol{n}(t) \in \mathcal{S}^k = \left\{ (x_1, x_2, ..., x_k) \in \mathbb{R}^k : \sum_{i=1}^{k} x_i = 1, x_i \geq 0 \; \forall i \right\}$$

  is the k-dimensional simplex.

- As such, we often refer to the process $\boldsymbol{n}$ as $(\sigma, \boldsymbol{r})$.
- In particular, if $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_k) \in \mathbb{N}_*^k$, then, we will use $\mathbb{P}_{\boldsymbol{\eta}}$ for the law of the replicator coalescent issued from state $\boldsymbol{n}(0) = \boldsymbol{\eta}$.
- We think of such laws on the standard Skorokhod space $(\mathbb{D}, \mathcal{D})$, where $\mathbb{D}$ is the space of càdlàg paths from $[0, \infty)$ to $\mathbb{N}_*^k$, equivalently on $\mathbb{N} \circ \mathcal{S}^k := \{(n, \boldsymbol{x}) \in \mathbb{N} \times \mathcal{S}^k : n \times \boldsymbol{x} \in \mathbb{N}_*^k\}$, and $\mathcal{D}$ is the Borel sigma algebra on $\mathbb{D}$ generated from the usual Skorokhod metric.

- What are the possible infinities to come down from? They are $(\infty, \boldsymbol{r})$ with

$$\boldsymbol{r} \in \mathcal{S}_+^k := \left\{ (x_1, x_2, ..., x_k) \in \mathbb{R}^k : \sum_{i=1}^{k} x_i = 1, x_i > 0 \,\forall i \right\}.$$

Theorem

For any sequence $(\boldsymbol{\eta}^N, N \geq 1)$ *in* $\mathbb{N}_*^k$ *such that* $(||\boldsymbol{\eta}^N||_1, \arg(\boldsymbol{\eta}^N)) \to (\infty, \boldsymbol{r})$ *with* $\boldsymbol{r} \in \mathcal{S}_+^k$, *the replicator coalescent comes down from infinity in the sense that*

$$\lim_{N\to\infty} \mathbb{P}_{\boldsymbol{\eta}^N}(0 < \gamma_m < \infty), \qquad m \in \mathbb{N},$$

and

$$\lim_{m\to\infty} \lim_{N\to\infty} \mathbb{P}_{\boldsymbol{\eta}^N}(\gamma_m < \varepsilon) = 1, \qquad \varepsilon > 0,$$

where $\gamma_m = \inf\{t > 0 : \sigma(t) = n\}$.

How does it come down from infinity?

- In the field of evolutionary game theory, there is a theory of a k-typed population described as a continuum $\boldsymbol{x}(t) = (x_1(t), \dots, x_k(t))$, $t \geq 0$ in $\mathcal{S}^k$.
- The evolution of $(\boldsymbol{x}(t), t \geq 0)$ is described by a dynamical system (the **replicator equations**)

$$\dot{x}_i(t) = x_i(t)(f_i(\boldsymbol{x}(t)) - \bar{f}(\boldsymbol{x}(t))), \qquad i = 1, \cdots, k,\ t \geq 0,$$

  where $f_i(\boldsymbol{x})$ is the fitness of type i and $\bar{f}(\boldsymbol{x}) = \sum_{i=1}^{n} x_i f_i(\boldsymbol{x})$ is the average population fitness, when the population density is given by $\boldsymbol{x}$.
- Fitness is often assumed to depend linearly upon the population distribution
- Population types survive according to their accumulated payoff in an iterated two-player game with payoff matrix A. The entry $A_{i,j}$ describes the reward per unit of population of type i player whose per unit of opponent is of type j.
- As such

$$f_i(\boldsymbol{x}) = \sum_{j=1}^{n} A_{i,j} x_j.$$

- The replicator equation simplifies to the $\boldsymbol{A}$-replicator equation

$$\dot{x}_i(t) = x_i(t)([\boldsymbol{A}\boldsymbol{x}(t)]_i - \boldsymbol{x}(t)^T \boldsymbol{A}\boldsymbol{x}(t)), \qquad i = 1, \cdots, k,\ t \geq 0,$$

THE $\boldsymbol{A}$-REPLICATOR EQUATION

- If the replicator equations can be solved by a fixed point in the simplex, i.e. $x_i(t) = x_i^*, i = 1, \cdots, k$, for some vector $\boldsymbol{x}^* = (x_1^*, \cdots, x_k^*) \in \mathcal{S}^k$, then we see that, necessarily,

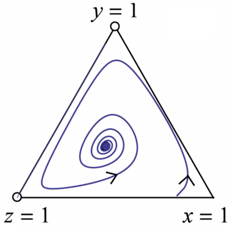
$$[\boldsymbol{A}\boldsymbol{x}^*]_i = (\boldsymbol{x}^*)^T \boldsymbol{A}\boldsymbol{x}^* \quad i = 1, \cdots, k.$$

- If $\boldsymbol{x}^*$ satisfies the relation

$$(\boldsymbol{x}^*)^T \boldsymbol{A}\boldsymbol{x} > \boldsymbol{x}^T \boldsymbol{A}\boldsymbol{x}, \qquad \boldsymbol{x} \in \mathcal{S}^k,$$

  then it is called an *evolutionary stable state* (ESS) and standard evolutionary game theory gives us that

$$\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{x}^*.$$



- If we were to try and match the replicator coalescent as closely as possible to a replicator equation dynamical system, what would $\boldsymbol{A}$ look like?

- Let us henceforth define

$$A_{i,j} = -\left(C_{j,i} \mathbf{1}_{j \neq i} + \frac{1}{2} C_{i,i} \right).$$

- Thanks to some standard theory and some mild assumptions on the matrix $\boldsymbol{C}$, this offers us the existence of an ESS, $\boldsymbol{x}^*$

Coming down from infinity: at the very beginning

Theorem

For a sequence $(\boldsymbol{\eta}^N, N \geq 1)$ *in* $\mathbb{N}_*^k$ *such that* $(||\boldsymbol{\eta}^N||_1, \arg(\boldsymbol{\eta}^N)) \to (\infty, \boldsymbol{r})$ *with* $\boldsymbol{r} \in \mathcal{S}_+^k$*, let* $\boldsymbol{R}(t) = \boldsymbol{r}(\tau(t)), t \geq 0$ *where*

$$\tau(t) = \inf\{s > 0 : \int_0^s \sigma(u)\, \mathrm{d}\, u > t\}, \qquad t \geq 0.$$

Suppose $\boldsymbol{x}(t) = (x_1(t), \cdots, x_k(t))$ *solves the corresponding* $\boldsymbol{A}$*-replicator equation, then*

$$\lim_{N\to\infty} \mathbb{E}_{\boldsymbol{\eta}^N}[|R_i(t) - x_i(t)|] = 0, \qquad i = 1, \cdots, k,$$

and in particular,

$$\lim_{t\uparrow\infty} \lim_{N\to\infty} \mathbb{E}_{\boldsymbol{\eta}^N}[|R_i(t) - x_i^*|] = 0, \qquad i = 1, \cdots, k.$$

Theorem

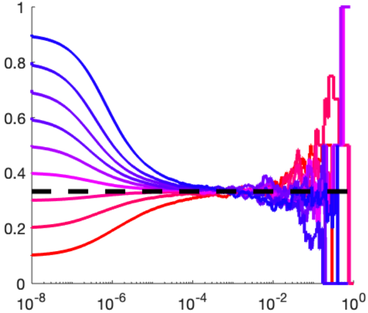
For any sequence $(\boldsymbol{\eta}^N, N \geq 1)$ *in* $\mathbb{N}_*^k$ *such that* $(||\boldsymbol{\eta}^N||_1, \arg(\boldsymbol{\eta}^N)) \to (\infty, \boldsymbol{r})$ *and* $\boldsymbol{r} \in \mathcal{S}_+^k$ *and for* $i = 1, \cdots, k$*,*

$$\lim_{m\to\infty} \lim_{N\to\infty} \mathbb{E}_{\boldsymbol{\eta}^N}[|r_i(\gamma_m) - \boldsymbol{x}_i^*|] = 0, \qquad i = 1, \cdots, k,$$

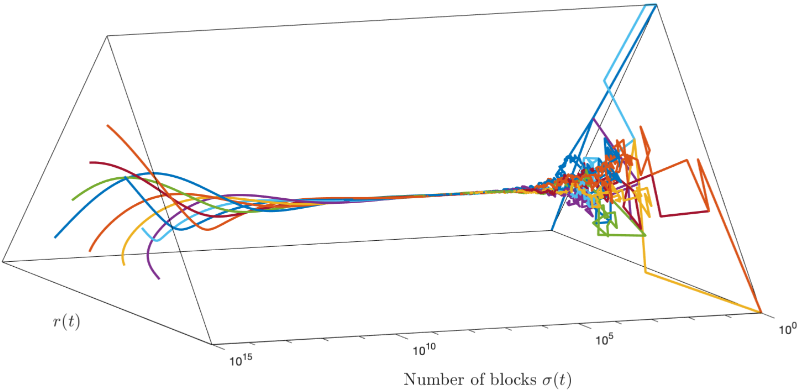
where

$$\gamma_m = \inf\{t > 0 : \sigma(t) = m\} \qquad m \in \mathbb{N}.$$

SIMULATIONS WITH $k = 2$



SIMULATIONS WITH $k = 3$



Some remarks on the proofs

- Comparison with Kingman's coalescent:
  total rate of mergers given by

$$\lambda(\boldsymbol{n}) = \sum_{i=1}^{k} \left(\sum_{j \neq i} C_{ji} n_j n_i + \frac{C_{i,i}}{2} n_i (n_i - 1) \right), \qquad \boldsymbol{n} \in \mathbb{N}_*^k$$

  and

$$\underline{C} \binom{||\boldsymbol{n}||_1}{2} < \lambda(\boldsymbol{n}) < \overline{C} \binom{||\boldsymbol{n}||_1}{2} \qquad \boldsymbol{n} \in \mathbb{N}_*^k.$$

- Appealing to the skip free nature of the process $(\sigma(t), t \geq 0)$ we can pathwise compare it to a Kingman coalescent $(\nu^+(t), t \geq 0)$ with the lower rate $\underline{C}$ and Kingman coalescent $(\nu^-(t), t \geq 0)$ with the upper rate $\overline{C}$, both on the same space such that

$$\lim_{\theta \to 0} \lim_{N \to \infty} \mathbb{E}_{\boldsymbol{\eta}^N} [\mathrm{e}^{-\theta \gamma_m}] \geq \lim_{\theta \to 0} \lim_{N \to \infty} \mathbb{E} \left[\mathrm{e}^{-\theta \beta_m^+} \middle| \nu^+(0) = ||\boldsymbol{\eta}^N||_1 \right] = 1,$$

$$\lim_{m \to \infty} \lim_{N \to \infty} \mathbb{E}_{\boldsymbol{\eta}^N} [\mathrm{e}^{-\theta \gamma_m}] \geq \lim_{m \to \infty} \lim_{N \to \infty} \mathbb{E} \left[\mathrm{e}^{-\theta \beta_m^+} \middle| \nu^+(0) = ||\boldsymbol{\eta}^N||_1 \right] = 1.$$

  and

$$\mathbb{P}_{\boldsymbol{\eta}} [\gamma_m > 0] \geq \mathbb{P}[\beta_m^- > 0 | \nu(0) = \sigma(0)], \qquad \theta \geq 0,$$

Some remarks on the proofs

- Understanding the relationship with the $\boldsymbol{A}$-replicator equations and the ESS $\boldsymbol{x}^*$ revolves around the following semimartingale decomposition:

$$\boldsymbol{y}(t) = \begin{pmatrix} \boldsymbol{r}(t \wedge \gamma_1) \\ 1/\sigma(t \wedge \gamma_1) \end{pmatrix}, \qquad t \geq 0,$$

where $\gamma_1 = \inf\{t > 0 : \sigma(t) = 1\}$.

$$\boldsymbol{y}(t) = \boldsymbol{y}(0) + \boldsymbol{m}(t) + \boldsymbol{\alpha}(t), \qquad t \geq 0$$

where $(\boldsymbol{m}(t), t \geq 0)$ is a martingale taking the form

$$\boldsymbol{m}(t) = \sum_{s \leq t \wedge \gamma_1} \Delta \boldsymbol{y}(s) - \boldsymbol{\alpha}(t), \qquad t \geq 0,$$

such that $\Delta \boldsymbol{y}(t) = \boldsymbol{y}(t) - \boldsymbol{y}(t-)$ and $(\boldsymbol{\alpha}(t), t \geq 0)$ is a compensator taking the form

$$\boldsymbol{\alpha}(t) = \int_0^{t \wedge \gamma_1} \frac{\sigma(s)}{\sigma(s) - 1} \sum_{i=1}^{k} \begin{pmatrix} \sigma(s)(\boldsymbol{r}(s) - \boldsymbol{e}_i) \\ 1 \end{pmatrix} r_i(s)[\sigma(s)^{-1}\mathrm{diag}(\boldsymbol{A})\mathbf{1} - \boldsymbol{A}\boldsymbol{r}(s)]_i \, \mathrm{d}\, s.$$

Some remarks on the proofs

- Define the sequence of stopping times $(\tau(t), t \geq 0)$, which are defined by the right inverse,

$$\tau(t) = \inf\{u > 0 : \int_0^u \sigma(s)\,\mathrm{d}s > t\}, \qquad t \geq 0.$$

- Then $\boldsymbol{y}^\tau := \boldsymbol{y} \circ \tau$ has semimartingale decomposition $\boldsymbol{y}^\tau = \boldsymbol{m}^\tau + \boldsymbol{\alpha}^\tau$, where $\boldsymbol{m}^\tau := \boldsymbol{m} \circ \tau$ is a martingale and, for $t \geq 0$,

$$\boldsymbol{\alpha}^\tau(t) = \int_0^{t \wedge \tau^{-1}(\gamma_1)} \frac{\sigma(\tau(s))}{\sigma(\tau(s)) - 1} \sum_{i=1}^k \begin{pmatrix} \boldsymbol{r}(\tau(s)) - \boldsymbol{e}_i \\ \dfrac{1}{\sigma(\tau(s))} \end{pmatrix} r_i(\tau(s))[\sigma(\tau(s))^{-1}\mathrm{diag}(\boldsymbol{A})\mathbf{1} - \boldsymbol{A}\boldsymbol{r}(\tau(s))]_i \,\mathrm{d}s$$

- We can show $\boldsymbol{m}^\tau \to 0$ in L^2
- Recall $\boldsymbol{y}^\tau = (\boldsymbol{r} \circ \tau, 1/\sigma \circ \tau)$, focusing on the entry $\boldsymbol{R} := \boldsymbol{r} \circ \tau$ as we take limits under $\mathbb{P}_{\boldsymbol{\eta}_N}$ as $N \to \infty$, roughly speaking we see that

$$R_i(t) \approx r_i(0) + \int_0^t R_i(s)\left([\boldsymbol{A}\boldsymbol{R}(s)]_i - \boldsymbol{R}(s)^T \boldsymbol{A}\boldsymbol{R}(s)\right)\mathrm{d}s,$$

which is the $\boldsymbol{A}$-replicator equations in integral form.

OPEN PROBLEMS

- Can we make better sense of ‘coming down from infinity’? Does an entrance law exist in the sense that there exists a law on $\mathbb{P}^\infty$ on $(\mathbb{D}, \mathcal{D})$ which is consistent with $\mathbb{P}$ in the sense that,

$$\mathbb{P}^\infty(\boldsymbol{n}(t+s)=\boldsymbol{n}) = \sum_{\substack{\boldsymbol{n}' \in \mathbb{N}_*^k \\ ||\boldsymbol{n}'||_1 \geq ||\boldsymbol{n}||_1}} \mathbb{P}^\infty(\boldsymbol{n}(t)=\boldsymbol{n}')\mathbb{P}_{\boldsymbol{n}'}(\boldsymbol{n}(s)=\boldsymbol{n}), \qquad s,t>0, \boldsymbol{n}\in\mathbb{N}_*^k,$$

  $\mathbb{P}^\infty(\sigma(t)<\infty)=1$, for all $t>0$, and $\mathbb{P}^\infty(\lim_{t\downarrow 0}\sigma(t)=\infty)=1$.

- Eventually everyone is dead except one individual. The tribal index I of that individual is distributed on $\{1,\cdots,k\}$

- Problem (very hard):
  Given initial configuration $\boldsymbol{n}\in\mathbb{N}_*^k$, what is the distribution of I?

- Problem (hard):
  What is the distribution of I when the process comes down from infinity?

†Footnote: What really happened on Rapa Nui?

There are conflicting theories as to the collapse of society on Rapa Nui as there is very poor recorded observational evidence of what happened on the island.

- Deforestation may have been the result of uncontrolled an Polynesian rat population, a species that was brought with the first Polynesian settlers
- Deforestation did not necessarily lead to societal collapse and there is evidence that the Rapa Nui people adapted farming methods
- Accounts of the first Dutch explorers did not indicate starvation nor warfare
- After a second Spanish explorers visit in 1770 in which the island was surveyed, there was a noted population decrease but still no evidence of warfare
- The British (James Cook) 1774 arrived thereafter noting that virtually all of the Moai statues had been toppled since the Spanish visit
- The first Dutch expedition brought disease, which killed up to 80% of the population, which was reinforced two-three generations later by the Spanish visitors. After the Spanish visit, the islanders lost faith in their spiritual beliefs, which saw the Moai monoliths (protective guardians) toppled.

Thank you!