A martingale review of some fluctuation theory for spectrally negative Lévy processes

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Summary. We give a review of some fluctuation theory for spectrally negative Lévy processes using for the most part martingale theory. The methodology is based on the techniques found in Kyprianou and Palmowski (2003) which deals with similar issues for a general class of Markov additive processes.

1 Introduction

Two and one sided exit problems for spectrally negative Lévy processes have been the object of several studies over the last 40 years. Significant contributions have come from Zolotarev (1964), Takács (1967), Emery (1973), Bingham (1975) Rogers (1990) and Bertoin (1996a, 1996b, 1997). The principal tools of analysis of these authors are the Wiener–Hopf factorization and Itô's excursion theory.

In recent years, the study of Lévy processes has enjoyed rejuvenation. This has resulted in many applied fields such as the theory of mathematical finance, risk and queues adopting more complicated models which involve an underlying Lévy process. The aim of this text is to give a reasonably self contained approach to some elementary fluctuation theory which avoids the use of the Wiener–Hopf factorization and Itô's excursion theory and rely mainly on martingale arguments together with the Strong Markov property. None of the results we present are new but for the most part, the proofs approach the results from a new angle following Kyprianou and Palmowski (2003) who also used them to handle a class of Markov additive processes.

2 Spectrally negative Lévy processes

We start by briefly reviewing what is meant by a spectrally negative Lévy process. The reader is referred to Bertoin (1996a) and Sato (1999) for a complete discussion.

Suppose that $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is a filtered probability space with filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ satisfying the usual conditions of right continuity and completion. In this text, we take as our definition of a Lévy process for $(\Omega, \mathcal{F}, \mathbb{F}, P)$, the strong Markov, $\mathbb{F}$-adapted process $X = \{X_t : t \geq 0\}$ with right continuous paths having the properties that $P(X_0 = 0) = 1$ and for each $0 \leq s \leq t$, the increment $X_t - X_s$ is independent of $\mathcal{F}_s$ and has the same distribution as X_{t-s}. In this sense, it is said that a Lévy process has stationary independent increments.

On account of the fact that the process has stationary independent increments, it is not too difficult to show that

$$E\left(\mathrm{e}^{\mathrm{i}\theta X_t}\right) = \mathrm{e}^{t\Psi(\theta)},$$

where $\Psi(\theta) = \log E(\exp\{\mathrm{i}\theta X_1\})$. The Lévy–Khinchine formula gives the general form of the function $\Psi(\theta)$. That is,

$$\Psi(\theta) = \mathrm{i}\mu\theta - \frac{\sigma^2}{2}\theta^2 + \int_{(-\infty,\infty)} \left(\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x\,\mathbf{1}_{|x|<1}\right) \Pi(\mathrm{d}x) \qquad (1)$$

for every $\theta \in \mathbb{R}$ where $\mu \in \mathbb{R}$, $\sigma > 0$ and Π is a measure on $\mathbb{R}\backslash\{0\}$ such that $\int (1 \wedge x^2)\, \Pi(\mathrm{d}x) < \infty$.

Finally, we say that X is spectrally negative if the measure Π is supported only on $(-\infty, 0)$. We exclude from the discussion however the case of a descending subordinator, that is a spectrally negative Lévy process with monotone decreasing paths. Included in the discussion however are descending subordinators plus an upward drift (such as one might use when modelling an insurance risk process, dam and storage models or a virtual waiting time process in an $M/G/1$ queue) and a Brownian motion with drift. Also included are processes such as asymmetric α-stable processes for $\alpha \in (1,2)$ which have unbounded variation and zero quadratic variation. By adding independent copies of any of the above (spectrally negative) processes together one still has a spectrally negative Lévy process.

For spectrally negative Lévy processes it is possible to talk of the Laplace exponent $\psi(\lambda)$ defined by

$$E\left(\mathrm{e}^{\lambda X_t}\right) = \mathrm{e}^{\psi(\lambda)t}, \qquad (2)$$

in other words, $\psi(\lambda) = \Psi(-\mathrm{i}\lambda)$. Since Π has negative support, we can safely say that $\psi(\lambda)$ exists at least for all $\lambda \geq 0$. Further, it is easy to check that ψ is strictly convex and tends to infinity as λ tends to infinity. This allows us to define for $q \in \mathbb{R}$,

$$\Phi(q) = \sup\{\lambda \geq 0 : \psi(\lambda) = q\},$$

the largest root of the equation $\psi(\lambda) = q$ when it exits. Note that there exist at most two roots for a given q and precisely one root when $q > 0$. Further we can identify $\psi'(0^+) = E(X_1) \in [-\infty, \infty)$ which, as we shall see in the next section, determines the long term behaviour of the process.

Suppose now the probabilities $\{P_x : x \in \mathbb{R}\}$ correspond to the conditional version of P where $X_0 = x$ is given. We simply write $P_0 = P$. The equality (2) allows for a Girsanov-type change of measure to be defined, namely via

$$\left.\frac{\mathrm{d}P_x^c}{\mathrm{d}P_x}\right|_{\mathcal{F}_t} = \frac{\mathcal{E}_t(c)}{\mathcal{E}_0(c)}$$

for any $c \geq 0$ where $\mathcal{E}_t(c) = \exp\{cX_t - \psi(c)t\}$ is the exponential martingale under P_x. Note that the fact that $\mathcal{E}_t(c)$ is a martingale follows from the fact that X has stationary independent increments together with (2). It is easy to check that under this change of measure, X remains within the class of spectrally negative processes and the Laplace exponent of X under P_x^c is given by

$$\psi_c(\theta) = \psi(\theta + c) - \psi(c)$$

for $\theta \geq -c$.

3 Exit problems

Let us now turn to the one and two sided exit problems for spectrally negative Lévy processes. The exit problems essentially consist of characterizing the Laplace transforms of τ_a^+, τ_0^- and $\tau_a^+ \wedge \tau_0^-$ where

$$\tau_0^- = \inf\{t \geq 0 : X_t \leq 0\} \qquad \text{and} \qquad \tau_a^+ = \inf\{t \geq 0 : X_t \geq a\}$$

for any $a > 0$. Note that X will hit the point a when crossing upwards as it can only move continuously upwards. On the other hand, it may either hit 0 or jump over zero when crossing 0 from above depending on the components of the process.

It has turned out (cf. Zolotarev (1964), Takács (1967), Emery (1973), Bingham (1975), Rogers (1990) and Bertoin (1996a, 1996b, 1997)) that one and two sided exit problems of spectrally negative Lévy processes can be characterized by the exponential function together with two families, $\{W^{(q)}(x) : q \geq 0, x \in \mathbb{R}\}$ and $\{Z^{(q)}(x) : q \geq 0, x \in \mathbb{R}\}$ known as the scale functions which we defined in the following main theorem of this text.

Theorem 1. *There exist a family of functions* $W^{(q)} : \mathbb{R} \to [0, \infty)$ *and*

$$Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)\,\mathrm{d}y, \qquad \text{for } x \in \mathbb{R}$$

defined for each $q \geq 0$ such that the following hold (for short we shall write $W^{(0)} = W$).

One sided exit above. *For any $x \leq a$ and $q \geq 0$,*

$$E_x\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{(\tau_a^+<\infty)}\right) = \mathrm{e}^{-\Phi(q)(a-x)}. \tag{3}$$

One sided exit below. *For any $x \in \mathbb{R}$ and $q \geq 0$,*

$$E_x\left(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{(\tau_0^-<\infty)}\right) = Z^{(q)}(x) - \frac{q}{\Phi(q)}W^{(q)}(x), \tag{4}$$

where we understand $q/\Phi(q)$ in the limiting sense for $q = 0$, so that

$$P_x(\tau_0^- < \infty) = \begin{cases} 1 - \psi'(0)W(x) & \text{if } \psi'(0) > 0 \\ 1 & \text{if } \psi'(0) \leq 0. \end{cases}$$

Two sided exit. *For any $x \leq a$ and $q \geq 0$,*

$$E_x\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{(\tau_0^->\tau_a^+)}\right) = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{5}$$

and

$$E_x\left(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{(\tau_0^-<\tau_a^+)}\right) = Z^{(q)}(x) - Z^{(q)}(a)\frac{W^{(q)}(x)}{W^{(q)}(a)}. \tag{6}$$

Further, for any $q \geq 0$, we have $W^{(q)}(x) = 0$ for $x \leq 0$ and $W^{(q)}$ is characterized on $(0, \infty)$ by the unique left continuous function whose Laplace transform satisfies

$$\int_0^\infty \mathrm{e}^{-\beta x}W^{(q)}(x)\,\mathrm{d}x = \frac{1}{\psi(\beta) - q} \qquad \text{for } \beta > \Phi(q). \tag{7}$$

Remark 2. Let us make a historical note on the appearance of these formulae. Identity (3) can be found in Emery (1973) and Bertoin (1996a). Identity (4) appears in the form of a Fourier transform again in Emery (1973). Identity (5) first appeared for the case $q = 0$ in Zolotarev (1964) followed by Takács (1967) and then with a short proof in Rogers (1990). The case $q > 0$ was first given in Bertoin (1996b) for the case of a purely asymmetric stable process and then again for a general spectrally negative Lévy process in Bertoin (1997) (who refered to a method used for the case $q = 0$ in Bertoin (1996a)). Finally (6) belongs originally to Suprun (1976) with a more modern proof given in Bertoin (1997).

Remark 3. By changing measure using the exponential martingale, one may extract identities from the above expressions giving the joint Laplace transform of the time to overshoot and overshoot itself. For example we have for any v with $\psi(v) < \infty$, $u \geq \psi(v) \vee 0$ and $x \in \mathbb{R}$,

$$E_x\left(\mathrm{e}^{-u\tau_0^- + vX_{\tau_0^-}}\mathbf{1}_{(\tau_0^-<\infty)}\right) = \mathrm{e}^{vx}\left(Z_v^{(p)}(x) - \frac{p}{\Phi(p)}W_v^{(p)}(x)\right)$$

where $W_v^{(p)}$ and $Z_v^{(p)}$ are scale functions with respect to the measure P^v, $p = u - \psi(v)$ and $p/\Phi(p)$ is understood in the limiting sense if $p = 0$, as in (3). In fact, it was shown in Bertoin (1997) that for each $x \in \mathbb{R}$, $W^{(q)}(x)$ is analytically extendable, as a function in q, to the whole complex plane; and hence the same is true of $Z^{(q)}(x)$. In which case arguing again by analytic extention one may weaken the requirement that $u \geq \psi(v) \vee 0$ to simply $u \geq 0$.

The proof we give of (3) is not new and follows as an easy consequence of Doob's optional stopping theorem applied to the exponential martingale; a technique traditionally attributed to *Wald*. The proof of the remaining results in Theorem 1 are a direct consequence of a special martingale which we shall discuss in Section 5. The proofs of (5), (4) and (6) are given in Sections 6, 7 and 8 respectively. The structure of this text is based on new results and methodology for a general class of Markov additive processes given in Kyprianou and Palmowski (2003).

4 Proof: one sided exit above

Assume that $x \leq a$ and $q > 0$. Since $t \wedge \tau_a^+ \leq t$ is a bounded stopping time and $X_{t\wedge\tau_a^+} \leq a$, it follows from Doob's Optional Stopping Theorem that

$$E_x\left(\frac{\mathcal{E}_{t\wedge\tau_a^+}(\Phi(q))}{\mathcal{E}_0(\Phi(q))}\right) = E_x\left(\mathrm{e}^{\Phi(q)(X_{t\wedge\tau_a^+}-x)-q(t\wedge\tau_a^+)}\right) = 1.$$

By dominated convergence and the fact that $X_{\tau_a^+} = a$ on $\tau_a^+ < \infty$ we have,

$$E_x\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{(\tau_a^+<\infty)}\right) = \mathrm{e}^{-\Phi(q)(a-x)}. \tag{8}$$

The case for $q = 0$ is dealt with by taking the limit as $q \downarrow 0$ in the above identity.

5 The Kella–Whitt martingale

As already mentioned in the introduction, we shall base our proofs for the most part on martingale arguments. A martingale which plays a fundamental role in our calculations is the Kella–Whitt martingale, introduced in Kella and Whitt (1992). This martingale has close links to so called Kennedy martingales (cf. Kennedy (1976)). For completeness we shall introduce the Kella–Whitt martingale in the following theorem.

Theorem 4. *Let $\overline{X}_t = \sup_{0\le u\le t} X_u$, and $Z_t = \overline{X}_t - X_t$, then for $\alpha \geq 0$*

$$M_t := \psi(\alpha)\int_0^t \mathrm{e}^{-\alpha Z_s}\,\mathrm{d}s + 1 - \mathrm{e}^{-\alpha Z_t} - \alpha\overline{X}_t, \qquad t \geq 0 \tag{9}$$

is a martingale.

Proof. Let $\mathcal{E}_t(\alpha) = \exp\{\alpha X_t - \psi(\alpha)t\}$ and note that

$$\begin{aligned}\mathrm{d}\mathcal{E}_t(\alpha) &= \mathcal{E}_{t-}(\alpha)\big(\alpha\,\mathrm{d}X_t - \psi(\alpha)\,\mathrm{d}t\big) + \frac{1}{2}\,\alpha^2\mathcal{E}_{t-}(\alpha)\,\mathrm{d}[X,X]_t^c\\ &\quad + \{\triangle\mathcal{E}_t(\alpha) - \alpha\mathcal{E}_{t-}(\alpha)\triangle X_t\}.\end{aligned}$$

Note also that

$$\begin{aligned}\mathrm{d}M_t &= \psi(\alpha)\mathrm{e}^{-\alpha Z_{t-}}\,\mathrm{d}t + \alpha\mathrm{e}^{-\alpha Z_{t-}}\,\mathrm{d}Z_t - \frac{1}{2}\,\alpha^2\mathrm{e}^{-\alpha Z_{t-}}\,\mathrm{d}[X,X]_t^c\\ &\quad - \{\triangle\mathrm{e}^{-\alpha Z_t} + \alpha\triangle Z_t\} - \alpha\,\mathrm{d}\overline{X}_t\\ &= \mathrm{e}^{-\alpha\overline{X}_t+\psi(\alpha)t}\Big[\psi(\alpha)\mathcal{E}_{t-}(\alpha)\,\mathrm{d}t + \alpha\mathcal{E}_{t-}(\alpha)\big(\mathrm{d}\overline{X}_t - \mathrm{d}X_t\big)\\ &\qquad - \frac{1}{2}\,\alpha^2\mathcal{E}_{t-}(\alpha)\,\mathrm{d}[X,X]_t^c\\ &\qquad - \mathcal{E}_{t-}(\alpha)\{\mathrm{e}^{\alpha\triangle X_t} - 1 - \alpha\triangle X_t\}\\ &\qquad - \alpha\mathrm{e}^{\alpha\overline{X}_t-\psi(\alpha)t}\,\mathrm{d}\overline{X}_t\Big]\\ &= \mathrm{e}^{-\alpha\overline{X}_t+\psi(\alpha)t}\Big\{-\mathrm{d}\mathcal{E}_t(\alpha) + \alpha\big(\mathcal{E}_t(\alpha) - \mathrm{e}^{\alpha\overline{X}_t-\psi(\alpha)t}\big)\,\mathrm{d}\overline{X}_t\Big\},\end{aligned}$$

where we have used that $\overline{X}_{t-} = \overline{X}_t$. Since $\overline{X}_t = X_t$ if and only if $\overline{X}_t$ increases, we may write

$$\begin{aligned}\mathrm{d}M_t &= \mathrm{e}^{-\alpha\overline{X}_t+\psi(\alpha)t}\Big\{-\mathrm{d}\mathcal{E}_t(\alpha) + \alpha\big(\mathcal{E}_t(\alpha) - \mathrm{e}^{\alpha\overline{X}_t-\psi(\alpha)t}\big)\mathbf{1}_{(\overline{X}_t=X_t)}\,\mathrm{d}\overline{X}_t\Big\}\\ &= -\mathrm{e}^{-\alpha\overline{X}_t+\psi(\alpha)t}\,\mathrm{d}\mathcal{E}_t(\alpha)\end{aligned}$$

showing that M_t is a local martingale since $\mathcal{E}_t(\alpha)$ is a martingale. To prove that M is a martingale, it suffices to show that for each $t > 0$,

$$E\left(\sup_{s\le t}|M_s|\right) < \infty.$$

To this end note that since the events $\{\overline{X}_{\mathbf{e}_q} > x\}$ and $\{\tau_x^+ < \mathbf{e}_q\}$ are almost surely equivalent where $\mathbf{e}_q$ is an exponential distribution with intensity $q > 0$ independent of X, it follows from (8)

$$P\big(\overline{X}_{\mathbf{e}_q} > x\big) = E\big(\mathrm{e}^{-q\tau_x^+}\mathbf{1}_{(\tau_x^+<\infty)}\big) = \mathrm{e}^{-\Phi(q)x}$$

showing that $\overline{X}_{\mathbf{e}_q}$ is exponentially distributed with parameter $\Phi(q)$. It follows that

$$E\left(\overline{X}_{\mathbf{e}_q}\right) = \int_0^\infty q\mathrm{e}^{-qt}\, E\left(\overline{X}_t\right) \mathrm{d}t = \frac{1}{\Phi(q)} < \infty$$

and hence, since $\overline{X}_t$ is an increasing process, we have $E\left(\overline{X}_t\right) < \infty$ for all t. Now note by the positivity of the process Z and again since $\overline{X}$ increases,

$$E\left(\sup_{s\leq t}|M_s|\right) \leq \psi(\alpha)t + 2 + \alpha\, E\left(\overline{X}_t\right) < \infty$$

for each finite $t > 0$. □

An application involving this martingale, brings us to an identity which is effectively the Wiener–Hopf factorization in disguise. Alternatively one may say that the Wiener–Hopf factorization for spectrally negative Lévy processes brings one to the same conclusion.

Theorem 5. *Let* $\underline{X}_t = \inf_{0\leq u\leq t} X_u$ *and suppose that* $\mathbf{e}_q$ *is an exponentially distributed random variable with parameter* $q > 0$ *independent of the process* X. *Then for* $\alpha > 0$,

$$E\left(\mathrm{e}^{\alpha \underline{X}_{\mathbf{e}_q}}\right) = \frac{q(\alpha - \Phi(q))}{\Phi(q)(\psi(\alpha) - q)}\,. \tag{10}$$

Proof. We begin by noting some facts which will be used in conjunction with the martingale (9). Recall that $\mathbf{e}_q$ is an exponentially distributed random variable with parameter $q > 0$ independent of the process X.

First note that by an application of Fubini's theorem,

$$E\int_0^{\mathbf{e}_q} \mathrm{e}^{-\alpha Z_s}\,\mathrm{d}s = \int_0^\infty \mathrm{e}^{-qs}\, E\left(\mathrm{e}^{-\alpha Z_s}\right) \mathrm{d}s = \frac{1}{q}\, E\left(\mathrm{e}^{-\alpha Z_{\mathbf{e}_q}}\right).$$

Next we recall a well known result, known as the Duality Lemma, which can best be verified with a diagram. That is by defining the process $\{\widetilde{X}_s = X_{(t-s)^-} - X_t : 0 \leq s \leq t\}$ as the time reversed Lévy process from the fixed moment, t, the law of $\widetilde{X}$ and $\{-X_s : 0 \leq s \leq t\}$ are the same. In particular, this means that

$$-\inf_{0\leq s\leq t} X_s \overset{d}{=} \sup_{0\leq s\leq t} \widetilde{X}_s = \overline{X}_t - X_t.$$

From Theorem 4 we have that $E\left(M_{\mathbf{e}_q}\right) = E\, M_0 = 0$ and hence using the last two observations we obtain

$$\frac{\psi(\alpha) - q}{q}\, E\left(\mathrm{e}^{\alpha \underline{X}_{\mathbf{e}_q}}\right) = \alpha\, E\left(\overline{X}_{\mathbf{e}_q}\right) - 1.$$

Recall from the proof of Theorem 4 that $\overline{X}_{\mathbf{e}_q}$ is exponentially distributed with parameter $\Phi(q)$. It now follows that

$$\frac{\psi(\alpha) - q}{q}\, E\left(\mathrm{e}^{\alpha \underline{X}_{\mathbf{e}_q}}\right) = \frac{\alpha - \Phi(q)}{\Phi(q)} \tag{11}$$

and the theorem is now proved. □

Remark 6. Recall that $\overline{X}_{\mathbf{e}_q}$ is exponentially distributed with parameter $\Phi(q)$. It thus follows that for $\alpha < \Phi(q)$

$$E\left(\mathrm{e}^{\alpha \overline{X}_{\mathbf{e}_q}}\right) = \frac{\Phi(q)}{\Phi(q) - \alpha} \tag{12}$$

and hence (11) reads

$$E\left(\mathrm{e}^{\alpha \overline{X}_{\mathbf{e}_q}}\right) E\left(\mathrm{e}^{\alpha \underline{X}_{\mathbf{e}_q}}\right) = \frac{q}{q - \psi(\alpha)} = E\big(\mathrm{e}^{\alpha X_{\mathbf{e}_q}}\big).$$

which is a conclusion that also follows from the Wiener–Hopf factorization.

In the previous section it was remarked that $\psi'(0^+)$ characterizes the asymptotic behaviour of X. We may now use the results of the previous Remark and Theorem to elaborate on this point. We do so in the form of a Lemma.

Lemma 7. *We have that*

(i) $\overline{X}_\infty$ *and* $-\underline{X}_\infty$ *are either infinite almost surely or finite almost surely,*
(ii) $\overline{X}_\infty = \infty$ *if and only if* $\psi'(0^+) \geq 0$,
(iii) $\underline{X}_\infty = -\infty$ *if and only if* $\psi'(0^+) \leq 0$.

Proof. On account of the strict convexity ψ it follows that $\Phi(0) > 0$ if and only if $\psi'(0^+) < 0$ and hence

$$\lim_{q\downarrow 0} \frac{q}{\Phi(q)} = \begin{cases} 0 & \text{if } \psi'(0^+) \leq 0 \\ \psi'(0^+) & \text{if } \psi'(0^+) > 0. \end{cases}$$

By taking q to zero in the identity (10) we now have that

$$E\big(\mathrm{e}^{\alpha \underline{X}_\infty}\big) = \begin{cases} 0 & \text{if } \psi'(0^+) \leq 0 \\ \psi'(0^+)\alpha/\psi(\alpha) & \text{if } \psi'(0^+) > 0. \end{cases}$$

Next, recall from (12) that for $\alpha > 0$

$$E\left(\mathrm{e}^{-\alpha \overline{X}_{\mathbf{e}_q}}\right) = \frac{\Phi(q)}{\Phi(q) + \alpha}$$

and hence by taking the limit of both sides as q tends to zero,

$$E\left(\mathrm{e}^{-\alpha \overline{X}_\infty}\right) = \begin{cases} (\alpha/\Phi(0) + 1)^{-1} & \text{if } \psi'(0^+) < 0 \\ 0 & \text{if } \psi'(0^+) \geq 0. \end{cases}$$

Parts (i)–(iii) follow immidiately from the previous two identities by considering their limits as $\alpha \downarrow 0$. □

6 Proof: two sided exit above

Our proof first deals with the case that $\psi'(0^+) > 0$ and $q = 0$, then the case that $q > 0$ (no restriction on $\psi'(0^+)$) or $q = 0$ and $\psi'(0) < 0$. Finally the case that $\psi'(0^+) = 0$ and $q = 0$ is achieved by passing to the limit as q tends to zero.

Assume then that $\psi'(0^+) > 0$ so that $-\underline{X}_\infty$ is almost surely finite. As earlier seen in the proof of Lemma 7, by taking q to zero in (10) it follows that

$$E\big(\mathrm{e}^{\alpha \underline{X}_\infty}\big) = \psi'(0)\,\frac{\alpha}{\psi(\alpha)}.$$

Integration by parts shows that

$$\begin{aligned} E\big(\mathrm{e}^{\alpha \underline{X}_\infty}\big) &= \int_{[0,\infty)} \mathrm{e}^{-\alpha x}\, P(-\underline{X}_\infty \in \mathrm{d}x) \\ &= \alpha \int_0^\infty \mathrm{e}^{-\alpha x} P(-\underline{X}_\infty < x)\,\mathrm{d}x \\ &= \alpha \int_0^\infty \mathrm{e}^{-\alpha x} P_x(\underline{X}_\infty > 0)\,\mathrm{d}x. \end{aligned}$$

Now define the function

$$W(x) = \frac{1}{\psi'(0^+)}\, P_x(\underline{X}_\infty > 0). \tag{13}$$

Clearly $W(x) = 0$ for $x \leq 0$, is left continuous since it is also equal to the left continuous distribution function $P(-\underline{X}_\infty < x)$ and therefore is uniquely determined by its Laplace transform, $1/\psi(\alpha)$ for all $\alpha > 0$. [Note that this shows the existence of the scale function when $\psi'(0^+) > 0$ and $q = 0$]. A simple argument using the law of total probability and the Strong Markov Property now yields for $x \in (0, a)$

$$\begin{aligned} &P_x(\underline{X}_\infty > 0) \\ &\quad = E_x\, P_x(\underline{X}_\infty > 0 \,|\, \mathcal{F}_{\tau_a^+}) \\ &\quad = E_x\Big(\mathbf{1}_{(\tau_a^+ < \tau_0^-)} P_a(\underline{X}_\infty > 0)\Big) + E_x\Big(\mathbf{1}_{(\tau_a^+ > \tau_0^-)} P_{X_{\tau_0^-}}(\underline{X}_\infty > 0)\Big) \\ &\quad = P_a(\underline{X}_\infty > 0)\, P_x(\tau_a^+ < \tau_0^-), \end{aligned} \tag{14}$$

where the second term in the second equality disappears as $X_{\tau_0^-} \leq 0$ and $P_x(\underline{X}_\infty > 0) = 0$ for $x \leq 0$. That is to say

$$P_x(\tau_a^+ < \tau_0^-) = \frac{W(x)}{W(a)} \tag{15}$$

and clearly the same equality holds even when $x \leq 0$.

Now assume that $q > 0$ or $\psi'(0) < 0$ and $q = 0$. In this case, by convexity of ψ, we know that $\Phi(q) > 0$ and hence $\psi'_{\Phi(q)}(0) = \psi'(\Phi(q)) > 0$ (again by convexity). Changing measure using the Girsanov density, we have for $x \in (0, a)$

$$
\begin{aligned}
E_x\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{(\tau_a^+<\tau_0^-)}\right) &= E_x\left(\frac{\mathcal{E}_{\tau_a^+}(\Phi(q))}{\mathcal{E}_0(\Phi(q))}\mathbf{1}_{(\tau_a^+<\tau_0^-)}\right)\mathrm{e}^{-\Phi(q)(a-x)} \\
&= \mathrm{e}^{-\Phi(q)(a-x)}\,P_x^{\Phi(q)}\left(\tau_a^+ < \tau_0^-\right).
\end{aligned}
$$

According to our previous calculations for the case that $q = 0$ and $\psi'(0^+) > 0$, we can now identify

$$
E_x\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{(\tau_a^+<\tau_0^-)}\right) = \frac{W^{(q)}(x)}{W^{(q)}(a)} \tag{16}
$$

such that $W^{(q)}(x) = \mathrm{e}^{\Phi(q)x}W_{\Phi(q)}(x)$ where $W_{\Phi(q)}(x)$ is identically zero on $(-\infty, 0]$, is left continuous and has Laplace transform $1/\psi_{\Phi(q)}(\alpha)$ for all $\alpha > 0$. Taking Laplace transforms of $W^{(q)}(x)$ it appears now that for $\alpha > \Phi(q)$,

$$
\begin{aligned}
\int_0^\infty \mathrm{e}^{-\alpha x}W^{(q)}(x)\,\mathrm{d}x &= \int_0^\infty \mathrm{e}^{-(\alpha-\Phi(q))x}W_{\Phi(q)}(x)\,\mathrm{d}x \\
&= \frac{1}{\psi_{\Phi(q)}(\alpha - \Phi(q))} \\
&= \frac{1}{\psi(\alpha) - q}\,,
\end{aligned} \tag{17}
$$

where in the last equality we have used the fact that for $c > 0$, $\psi_c(\theta) = \psi(\theta+c) - \psi(c)$. [Note again that this last calculation again justifies that $W^{(q)}$ exists for the regime that we are considering.]

As mentioned at the beginning of the proof, the final missing case of X not drifting to infinity (ie $\psi'(0^+) = 0$) and $q = 0$ is achieved by passing to the limit as $q \downarrow 0$. Since $W_{\Phi(q)}$ has Laplace transform $1/\psi_{\Phi(q)}$ for $q > 0$, integration by parts reveals that

$$
\int_{(0,\infty)} \mathrm{e}^{-\beta x}\,W_{\Phi(q)}(\mathrm{d}x) = \frac{\beta}{\psi_{\Phi(q)}(\beta)}\,. \tag{18}
$$

One may appeal to the Extended Continuity Theorem for Laplace Transforms, see Feller (1971) Theorem XIII.1.2a, and (18) to deduce that since

$$
\lim_{q\downarrow 0}\int_{(0,\infty)} \mathrm{e}^{-\beta x}\,W_{\Phi(q)}(\mathrm{d}x) = \frac{\beta}{\psi(\beta)}
$$

then there exists a measure W^* such that in the weak sense $W^* = \lim_{q\downarrow 0} W_{\Phi(q)}$ and

$$\int_{(0,\infty)} \mathrm{e}^{-\beta x}\, W^*(\mathrm{d}x) = \frac{\beta}{\psi(\beta)}.$$

Integration by parts shows that its left continuous distribution,

$$W(x) := W^*(-\infty, x) = \lim_{q\downarrow 0} W^{(q)}(x)$$

satisfies

$$\int_0^\infty \mathrm{e}^{-\beta x} W(x)\,\mathrm{d}x = \frac{1}{\psi(\beta)} \tag{19}$$

for $\beta > 0$. Considering the limit as $q \downarrow 0$ in (16) and remembering that $W^{(q)}(x) = \mathrm{e}^{\Phi(q)x} W_{\Phi(q)}(x)$ we recover the required identity (15).

7 Proof: one sided exit below

Taking (17) and (18) into account, we can interpret (10) as saying that

$$P\big(-\underline{X}_{\mathbf{e}_q} \in \mathrm{d}x\big) = \frac{q}{\Phi(q)}\, W^{(q)}(\mathrm{d}x) - qW^{(q)}(x)\,\mathrm{d}x$$

and hence with an easy manipulation, for $x > 0$

$$\begin{aligned} E_x\big(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{(\tau_0^-<\infty)}\big) &= P_x\big(\mathbf{e}_q > \tau_0^-\big) \\ &= P_x\big(\underline{X}_{\mathbf{e}_q} < 0\big) \\ &= 1 + q\int_0^x W^{(q)}(y)\,\mathrm{d}y - \frac{q}{\Phi(q)}\, W^{(q)}(x) \\ &= Z^{(q)}(x) - \frac{q}{\Phi(q)}\, W^{(q)}(x). \end{aligned} \tag{20}$$

Note that since $Z^{(q)}(x) = 1$ and $W^{(q)}(x) = 0$ for all $x \in (-\infty, 0]$, the statement is valid for all $x \in \mathbb{R}$. The proof is now complete for the case that $q > 0$.

Recalling that $\lim_{q\downarrow 0} q/\Phi(q)$ is either $\psi'(0^+)$ or zero, the proof is completed by taking the limit in q.

8 Proof: two sided exit below

Fix $q > 0$. The Strong Markov Property together with the identity (20) give us that

$$\begin{aligned} &P_x\Big(\underline{X}_{\mathbf{e}_q} < 0 \,\Big|\, \mathcal{F}_{t\wedge\tau_a^+\wedge\tau_0^-}\Big) \\ &= \mathrm{e}^{-q(t\wedge\tau_a^+\wedge\tau_0^-)}\, P_{X_{t\wedge\tau_a^+\wedge\tau_0^-}}\big(\underline{X}_{\mathbf{e}_q} < 0\big) \\ &= \mathrm{e}^{-q(t\wedge\tau_a^+\wedge\tau_0^-)}\Big(Z^{(q)}\Big(X_{t\wedge\tau_a^+\wedge\tau_0^-}\Big) - \frac{q}{\Phi(q)} W^{(q)}\Big(X_{t\wedge\tau_a^+\wedge\tau_0^-}\Big)\Big) \end{aligned}$$

showing that the right hand side is a martingale for $t \geq 0$. Note also that with a similar methodology we have (using that $W^{(q)}(X_{\tau_0^- \wedge \tau_a^+}) = \mathbf{1}_{(\tau_a^+ < \tau_0^-)} W^{(q)}(a)$)

$$
\begin{aligned}
&E_x\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{(\tau_a^+<\tau_0^-)} \,\middle|\, \mathcal{F}_{t\wedge\tau_a^+\wedge\tau_0^-}\right)\\
&= \mathbf{1}_{(t<\tau_0^-\wedge\tau_a^+)}\mathrm{e}^{-qt}E_{X_t}\left(\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{(\tau_a^+<\tau_0^-)}\right) + \mathbf{1}_{(t>\tau_0^-\wedge\tau_a^+)}\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{(\tau_a^+<\tau_0^-)}\\
&= \mathbf{1}_{(t<\tau_0^-\wedge\tau_a^+)}\mathrm{e}^{-qt}\,\frac{W^{(q)}(X_t)}{W^{(q)}(a)} + \mathbf{1}_{(t>\tau_0^-\wedge\tau_a^+)}\mathrm{e}^{-q(\tau_0^-\wedge\tau_a^+)}\,\frac{W^{(q)}(X_{\tau_0^-\wedge\tau_a^+})}{W^{(q)}(a)}\\
&= \mathrm{e}^{-q(t\wedge\tau_a^+\wedge\tau_0^-)}\,\frac{W^{(q)}(X_{t\wedge\tau_0^-\wedge\tau_a^+})}{W^{(q)}(a)}
\end{aligned}
$$

showing again that the right hand side is a martingale for $t \geq 0$.

Now it follows by linearity that

$$
\mathrm{e}^{-q(t\wedge\tau_a^+\wedge\tau_0^-)}\left(Z^{(q)}\left(X_{t\wedge\tau_a^+\wedge\tau_0^-}\right) - \frac{Z^{(q)}(a)}{W^{(q)}(a)}\,W^{(q)}\left(X_{t\wedge\tau_a^+\wedge\tau_0^-}\right)\right)
$$

is also a martingale for $t \geq 0$. In fact it is a uniformly integrable martingale and hence its terminal expectation is equal to its initial expectation. That is to say

$$
\begin{aligned}
E_x\left(\mathrm{e}^{-q(\tau_a^+\wedge\tau_0^-)}\left(Z^{(q)}\left(X_{\tau_a^+\wedge\tau_0^-}\right) - \frac{Z^{(q)}(a)}{W^{(q)}(a)}\,W^{(q)}\left(X_{\tau_a^+\wedge\tau_0^-}\right)\right)\right)&\\
&= E_x\left(\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{(\tau_a^+>\tau_0^-)}\right)\\
&= Z^{(q)}(x) - \frac{Z^{(q)}(a)}{W^{(q)}(a)}\,W^{(q)}(x),
\end{aligned}
$$

where as usual we have used the fact that

$$
Z^{(q)}\left(X_{\tau_a^+\wedge\tau_0^-}\right) = 1 \quad \text{and} \quad W^{(q)}\left(X_{\tau_a^+\wedge\tau_0^-}\right) = 0 \qquad \text{if } \tau_0^- < \tau_a^+,
$$

and

$$
Z^{(q)}\left(X_{\tau_a^+\wedge\tau_0^-}\right) = Z^{(q)}(a) \quad \text{and} \quad W^{(q)}\left(X_{\tau_a^+\wedge\tau_0^-}\right) = W^{(q)}(a) \qquad \text{if } \tau_0^- > \tau_a^+.
$$

For the case that $q = 0$, we again take limits as q tends to zero.

9 Final Remarks

We conclude with some final remarks concerning some more subtle points of the calculations we have made which are not necessarily immidiately obvious. The definition of $\tau_x^- = \inf\{t \geq 0 : X_t \leq x\}$ requiring *weak* first passage below the level x forces the definition of W proportional to $P_x(\underline{X}_\infty > 0)$ in the case that $q = 0$ and $\psi'(0^+) > 0$ in (13). This in turn determines the left continuity of $W^{(q)}$ for all $q \geq 0$, a fact which is seen to be of importance in the calculation (14) as well as later, for example in Section 8, where it is stated that $W^{(q)}(X_{\tau_0^- \wedge \tau_a^+}) = \mathbf{1}_{(\tau_a^+ < \tau_0^-)}$. However, Bertoin (1997) works with a definition of *strong* first downward passage equivalent to $\tau_x^- = \inf\{t \geq 0 : X_t < x\}$. Following the analysis here one sees in (13) that W should then be taken as

$$W(x) = \frac{1}{\psi'(0^+)} P_x\left(\underline{X}_\infty \geq 0\right) = \frac{1}{\psi'(0^+)} P\left(-\underline{X}_\infty \leq x\right).$$

But then, if 0 is irregular for $(-\infty, 0)$ for X we have $P(\tau_0^- > \tau_a^+) > 0$, which itself is a result of the definition of τ_0^- in the strong sense. The effect of this definition is that $W^{(q)}$ is now right continuous. None the less, with very subtle adjustments, all the arguments go through as presented. An example of a calculation which needs a little extra care is (14).

In this case, it is possible that $X_{\tau_0^-} = 0$ with positive probability, that is to say X may creep downwards over zero, and hence in principle the second term in (14) may not be zero. However, it is known that spectrally negative Lévy processes may only creep downwards if and only if a Gaussian component is present (cf. Bertoin (1996a) p. 175). In this case $P(\underline{X}_\infty \geq 0) = 0$ anyway and the calculation goes through.

To some extent, it is more natural to want work with the right continuous version of $W^{(q)}$ because one captures the probability of starting at the origin and escaping at a before entering $(-\infty, 0)$ in the expression $W(0)/W(a)$ as opposed to $W(0^+)/W(a)$ for the left continuous case. However we promised in the introduction a self contained approach to our results which avoids the use of the Wiener–Hopf factorization. Hence we have opted to present the case of left continuity in $W^{(q)}$ thus avoiding the deeper issue of creeping, which is intimately connected to the Wiener–Hopf factorization.

For other recent perspectives and new proofs of existing results concerning fluctuation theory of spectrally negative Lévy processes see Doney (2004), Pistorius (2004) and Nguyen-Ngoc and Yor (2004).

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