STOCHASTIC METHODS FOR THE NEUTRON TRANSPORT EQUATION I: LINEAR SEMIGROUP ASYMPTOTICS

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The neutron transport equation (NTE) describes the flux of neutrons through an inhomogeneous fissile medium. In this paper, we reconnect the NTE to the physical model of the spatial Markov branching process which describes the process of nuclear fission, transport, scattering, and absorption. By reformulating the NTE in its mild form and identifying its solution as an expectation semigroup, we use modern techniques to develop a Perron–Fröbenius (PF) type decomposition, showing that growth is dominated by a leading eigenfunction and its associated left and right eigenfunctions. In the spirit of results for spatial branching and fragmentation processes, we use our PF decomposition to show the existence of an intrinsic martingale and associated spine decomposition. Moreover, we show how criticality in the PF decomposition dictates the convergence of the intrinsic martingale. The mathematical difficulties in this context come about through unusual piecewise linear motion of particles coupled with an infinite type-space which is taken as neutron velocity. The fundamental nature of our PF decomposition also plays out in accompanying work (Harris, Horton and Kyprianou (2020), Cox et al. (2020)).

1. Introduction. The neutron transport equation (NTE) describes the flux of neutrons across a planar cross-section in an inhomogeneous fissile medium (measured in number of neutrons per cm^2 per second). Neutron flux is described as a function of time, t, Euclidian location, r, direction, Ω and neutron energy E. It is not uncommon in the physics literature to assume that velocity is a function of both direction and energy, thereby reducing the number of variables by one. This allows us to describe the dependency of flux more simply in terms of time and, what we call, the *configuration variables* $(r, \upsilon) \in D \times V$ where $D \subseteq \mathbb{R}^3$ is a nonempty, open, smooth, bounded and convex domain such that ∂D has zero Lebesgue measure, and V is the velocity space, which we take to be the three-dimensional annulus $V = \{\upsilon \in \mathbb{R}^3 : \mathtt{v_{min}} \leq |\upsilon| \leq \mathtt{v_{max}}\}$, where $0 < \mathtt{v_{min}} < \mathtt{v_{max}} < \infty$.

As a *backward equation*, the NTE is written in the form

$$
\frac{\partial}{\partial t} \psi_t(r, \upsilon) = \upsilon \cdot \nabla \psi_t(r, \upsilon) - \sigma(r, \upsilon) \psi_t(r, \upsilon)
$$

$$
(1.1) \qquad\qquad + \sigma_{\mathrm{s}}(r, \upsilon) \int_V \psi_t(r, \upsilon') \pi_{\mathrm{s}}(r, \upsilon, \upsilon') \, \mathrm{d}\upsilon'
$$

$$
+ \sigma_{\mathrm{f}}(r, \upsilon) \int_V \psi_t(r, \upsilon') \pi_{\mathrm{f}}(r, \upsilon, \upsilon') \, \mathrm{d}\upsilon',
$$

where the different components (or *cross-sections* as they are known in the nuclear physics literature) have the following interpretation:

$\sigma_{\mathrm{s}}(r, \upsilon)$: the rate at which scattering occurs from incoming velocity υ,

$\sigma_{\mathrm{f}}(r, \upsilon)$: the rate at which fission occurs from incoming velocity υ,

$\sigma(r, \upsilon)$: the sum of the rates $\sigma_{\mathrm{f}} + \sigma_{\mathrm{s}}$ and is known as the total cross section,

$\pi_{\mathrm{s}}(r, \upsilon, \upsilon')\, d\upsilon'$: the scattering yield at velocity υ' from incoming velocity υ,
satisfying $\int_V \pi_{\mathrm{s}}(r, \upsilon, \upsilon')\, d\upsilon' = 1$, and

$\pi_{\mathrm{f}}(r, \upsilon, \upsilon')\, d\upsilon'$: the neutron yield at velocity υ' from fission with incoming velocity υ,
satisfying $\int_V \pi_{\mathrm{f}}(r, \upsilon, \upsilon')\, d\upsilon' < \infty$.

Some or all of the three assumptions below will be used from time to time in our main results.

(H1): *Cross-sections σ_{s}, σ_{f}, π_{s} and π_{f} are uniformly bounded away from infinity.*

(H2): *We have $\sigma_{\mathrm{s}}\pi_{\mathrm{s}} + \sigma_{\mathrm{f}}\pi_{\mathrm{f}} > 0$ on $D \times V \times V$.*

(H3): *There is an open ball B compactly embedded in D such that $\sigma_{\mathrm{f}}\pi_{\mathrm{f}} > 0$ on $B \times V \times V$.*

It is also usual to insist on the physical boundary conditions

(1.2)
$$\begin{cases} \psi_0(r, \upsilon) = g(r, \upsilon) & \text{for } r \in D, \upsilon \in V, \\ \psi_t(r, \upsilon) = 0 & \text{for } t \geq 0 \text{ and } r \in \partial D \text{ if } \upsilon \cdot \mathbf{n}_r > 0, \end{cases}$$

where $\mathbf{n}_r$ is the outward unit normal at $r \in \partial D$ and $g : D \times V \to [0, \infty)$ is a bounded, measurable function on which we will later impose further conditions. Physically, this boundary condition mean that any neutron starting on the boundary of the reactor with velocity pointing outwards will be "killed".

Formally speaking, (1.1) as stated is ill defined (because of regularity issues associated to the transport operator $\upsilon \cdot \nabla$) and has traditionally otherwise appeared in applied mathematics and physics literature in the form of an *abstract Cauchy problem* on $L_2(D \times V)$, the space of square integrable functions on $D \times V$. This has formed the principle historical outlook of the analysis of the NTE, appealing to c_0-semigroup theory; see, for example, the classical works of [2, 11–13, 21, 25, 27–30, 36–38].

The connection of the NTE via semigroup theory to an underlying stochastic process has, in contrast, received a very limited amount of attention; cf [11, 27, 32]. Accordingly the stochastic analysis of (1.1) has seen very little development in light of recent innovations in the relevant theory of stochastic processes.

In the current article, we are more interested in exploring how NTE can be interpreted as a *mild equation*, describing the mean semigroup evolution of the stochastic process that models the underlying physical process of neutron fission, transport, scatter and absorption. More precisely, we have two main contributions: (i) to develop a new precise statement of the form $\psi_t = e^{\lambda_* t} c_g \varphi + o(e^{\lambda_* t})$, where λ_* and φ are a leading eigenvalue and eigenfunction associated to the NTE and c_g is a constant that depends on the initial data g; (ii) to make the first step in understanding how the growth of the solution to the NTE relative to its lead eigenfunction plays out in terms of the aforementioned physical stochastic process and an associated martingale.

This paper follows the review article [10] which consolidates the existing c_0-semigroup approach and how it relates to the stochastic representation. A deeper subsequent analysis in the direction of our second objective is continued in the accompanying paper [20]. Further numerical and Monte Carlo considerations based on our stochastic approach will also appear in forthcoming work [9].

In order to consider the probabilistic perspective, we start by defining the underlying stochastic processes which mimics the physics of neutron fission, transport, scattering and absorption.

2. The physical process and the mild NTE. Consider a neutron branching process (NBP), which at time $t \geq 0$ is represented by a configuration of particles which are specified via their physical location and velocity in $D \times V$, say $\{(r_i(t), \upsilon_i(t)) : i = 1, \ldots, N_t\}$, where N_t is the number of particles alive at time $t \geq 0$. In order to describe the process, we will represent it as a process in the space of finite atomic measures

$$(2.1) \qquad X_t(A) = \sum_{i=1}^{N_t} \delta_{(r_i(t), \upsilon_i(t))}(A), \quad A \in \mathcal{B}(D \times V), t \geq 0,$$

where δ is the Dirac measure, defined on $\mathcal{B}(D \times V)$, the Borel subsets of $D \times V$. The evolution of $(X_t, t \geq 0)$ is a stochastic process valued in the space of atomic measures $\mathcal{M}(D \times V) := \{\sum_{i=1}^{n} \delta_{(r_i, \upsilon_i)} : n \in \mathbb{N}, (r_i, \upsilon_i) \in D \times V, i = 1, \ldots, n\}$ which evolves randomly as follows.

A particle positioned at r with velocity υ will continue to move along the trajectory $r + \upsilon t$, until one of the following things happens.

(i) The particle leaves the physical domain D, in which case it is instantaneously killed.

(ii) Independently of all other neutrons, a scattering event occurs when a neutron comes in close proximity to an atomic nucleus and, accordingly, makes an instantaneous change of velocity. For a neutron in the system with position and velocity (r, υ), if we write T_s for the random time that scattering may occur, then independently of any other physical event that may affect the neutron, $\Pr(T_s > t) = \exp\{-\int_0^t \sigma_s(r + \upsilon s, \upsilon) \, ds\}$, for $t \geq 0$.

When scattering occurs at space-velocity (r, υ), the new velocity is selected in V independently with probability $\pi_s(r, \upsilon, \upsilon') \, d\upsilon'$.

(iii) Independently of all other neutrons, a fission event occurs when a neutron smashes into an atomic nucleus. For a neutron in the system with initial position and velocity (r, υ), if we write T_f for the random time that fission may occur, then independently of any other physical event that may affect the neutron, $\Pr(T_f > t) = \exp\{-\int_0^t \sigma_f(r + \upsilon s, \upsilon) \, ds\}$, for $t \geq 0$.

When fission occurs, the smashing of the atomic nucleus produces lower mass isotopes and releases a random number of neutrons, say $N \geq 0$, which are ejected from the point of impact with randomly distributed, and possibly correlated, velocities, say $(\upsilon_i : i = 1, \ldots, N)$. The outgoing velocities are described by the atomic random measure

$$(2.2) \qquad \mathcal{Z}(A) := \sum_{i=1}^{N} \delta_{\upsilon_i}(A), \quad A \in \mathcal{B}(V).$$

When fission occurs at location $r \in \mathbb{R}^d$ from a particle with incoming velocity $\upsilon \in V$, we denote by $\mathcal{P}_{(r,\upsilon)}$ the law of $\mathcal{Z}$. The probabilities $\mathcal{P}_{(r,\upsilon)}$ are such that, for $\upsilon' \in V$, for bounded and measurable $g : V \to [0, \infty)$,

$$(2.3) \qquad \int_V g(\upsilon') \pi_f(r, \upsilon, \upsilon') \, d\upsilon' = \mathcal{E}_{(r,\upsilon)} \left[\int_V g(\upsilon') \mathcal{Z}(d\upsilon') \right] =: \mathcal{E}_{(r,\upsilon)}[\langle g, \mathcal{Z} \rangle].$$

Note, the possibility that $\Pr(N = 0) > 0$, which will be tantamount to neutron capture (that is, where a neutron slams into a nucleus but no fission results and the neutron is absorbed into the nucleus).

In essence, one may think of the process $X := (X_t, t \geq 0)$ as a typed spatial Markov branching process, where the type of each particle is the velocity $\upsilon \in V$ and the underlying Markov motion is nothing more than movement in a straight line at velocity υ.

REMARK 2.1. It is worth noting how the assumptions (H1)–(H3) play out in the construction of the NBP. While they serve as sufficient conditions, they are not necessary. For

example, one could equally assume that for example, there are two open domains B_s and B_f (which may or may not intersect) contained in D on which $\sigma_f(r, \upsilon)\pi_f(r, \upsilon, \upsilon') > 0$, for $B_s \times V \times V$ and $\sigma_f(r, \upsilon)\pi_f(r, \upsilon, \upsilon') > 0$, for $B_f \times V \times V$, respectively. This would ensure that, at least starting from some configurations $(r, \upsilon) \in D \times V$, the NBP could access regions where scatter or fission occurs with positive probability. From there, the particle system will thus propagate by allowing further opportunities for scatter or fission. That said, there will also be some initial configurations $(r, \upsilon) \in D \times V$ for which the particles will neither scatter nor undergo fission and head straight for the boundary ∂D, whereupon they are killed.

This example informally alerts us to the notion of "*irreducibility*" of the state space. For contrast, and to highlight the issue further, it is worth comparing the situation to, for example, a branching Brownian motion on a smooth, convex, bounded domain of $D \subseteq \mathbb{R}^d$ for which the branching rate is supported only on a subdomain B strictly contained in D. In that setting, the Brownian motion of a given particle would always be able to "find" the region B with positive probability, where branching can occur (thus propagating the stochastic process in a nontrivial way). Through this comparison, we see that the piecewise linear spatial paths of neutrons in the NBP, although simpler to depict than the path of a Brownian motion, are significantly more irregular. The assumption (H2) may thus be thought of as a sufficient condition to ensure irreducibility of the state space by enforcing the possibility of either scatter or fission (but not necessarily the possibility of both), whereas assumption (H3) ensures that there is at least one area of the domain where fission occurs. The condition (H1) simply ensures that activity (scatter and fission) cannot happen too fast, and hence the eventuality of explosion in finite time does not appear in our forthcoming calculations.

REMARK 2.2.   The NBP is parameterised by the quantities σ_s, π_s, σ_f and the family of measures $\mathcal{P} = (\mathcal{P}_{(r,\upsilon)}, r \in D, \upsilon \in V)$ and accordingly we refer to it as a $(\sigma_s, \pi_s, \sigma_f, \mathcal{P})$-NBP. It is associated to the NTE via the relation (2.3), and, although a $(\sigma_s, \pi_s, \sigma_f, \mathcal{P})$-NBP is uniquely defined, a NBP specified by $(\sigma_s, \pi_s, \sigma_f, \pi_f)$ alone is not.

What is of importance for the purpose of our analysis, however, is that for the given quadruple $(\sigma_s, \pi_s, \sigma_f, \pi_f)$, at least one $(\sigma_s, \pi_s, \sigma_f, \mathcal{P})$-NBP exists such that (2.3) holds. It is relatively easy to construct an example of $\mathcal{P}$ as such.

Indeed, let us suppose (H1) and (H3) hold. Then, for a given π_f, define

$$n_{max} = \min\left\{k \geq 1 : \sup_{(r,\upsilon)\in D\times V}\int_V \pi_f(r, \upsilon, \upsilon')\,d\upsilon' \leq k\right\}.$$

The ensemble $(\upsilon_i, i = 1, \ldots, N)$ is such that: (i) $N \in \{0, n_{max}\}$; (ii) the probability of the event $\{N = n_{max}\}$ under $\mathcal{P}_{(r,\upsilon)}$ is given by $\int_V \pi_f(r, \upsilon, \upsilon'')\,d\upsilon''/n_{max}$; (iii) on the event $\{N = n_{max}\}$, each of the n_{max} neutrons are released with the same velocities $\upsilon_1 = \cdots = \upsilon_{n_{max}}$; (iv) the distribution of this common velocity is given by

$$\mathcal{P}_{(r,\upsilon)}(\upsilon_i \in d\upsilon' | N = n_{max}) = \frac{\pi_f(r, \upsilon, \upsilon')}{\int_V \pi_f(r, \upsilon, \upsilon'')\,d\upsilon''}\,d\upsilon',$$

for $i = 1, \ldots, n_{max}$.

With the construction (i)–(iv) for $\mathcal{P}_{(r,\upsilon)}$, we have for bounded and measurable $g : V \to [0, \infty)$,

$$\int_V g(\upsilon')\pi_f(r, v, \upsilon')\,d\upsilon'$$

$$= 0 \times (1 - \mathcal{P}_{(r,\upsilon)}(N = n_{max}))$$

$$+ \mathcal{P}_{(r,\upsilon)}(N = n_{max})n_{max}\int_V g(\upsilon')\mathcal{P}_{(r,\upsilon)}(\upsilon_i \in d\upsilon' | N = n_{max})$$

$$= \frac{\int_V \pi_{\mathrm{f}}(r, \upsilon, \upsilon'') \, \mathrm{d}\upsilon''}{n_{\max}} n_{\max} \int_V g(\upsilon') \frac{\pi_{\mathrm{f}}(r, \upsilon, \upsilon')}{\int_V \pi_{\mathrm{f}}(r, \upsilon, \upsilon'') \, \mathrm{d}\upsilon''} \, \mathrm{d}\upsilon'$$

$$= \int_V g(\upsilon') \pi_{\mathrm{f}}(r, \upsilon, \upsilon') \, \mathrm{d}\upsilon',$$

thus matching (2.3), as required.

It is interesting to note that the construction above is precisely what happens in industrial models of nuclear reactor cores (for which only the cross-sections $(\sigma_{\mathrm{s}}, \pi_{\mathrm{s}}, \sigma_{\mathrm{f}}, \pi_{\mathrm{f}})$ are known) when it comes to Monte Carlo simulation; see further discussion below as well as [9].

The maximum number of neutrons that can be emitted during a fission event with positive probability (e.g., in an environment where the heaviest nucleus is *Uranium-235*, there are at most 143 neutrons that can be released in a fission event, albeit, in reality it is more likely that 2 or 3 are released). We will thus occasionally work with:

(H4): *Fission offspring are bounded in number by the constant $n_{\max} > 1$.*

In particular this means that $\sup_{r \in D, \upsilon \in V} \int_V \pi_{\mathrm{f}}(r, \upsilon, \upsilon') \, \mathrm{d}\upsilon' \le n_{\max}$.

Write $\mathbb{P}_{\delta_{(r,\upsilon)}}$ for the the law of X when issued from a single particle with space-velocity configuration $(r, \upsilon) \in D \times V$. More generally, for $\mu \in \mathcal{M}(D \times V)$, we understand $\mathbb{P}_\mu := \mathbb{P}_{\delta_{(r_1, \upsilon_1)}} \otimes \cdots \otimes \mathbb{P}_{\delta_{(r_n, \upsilon_n)}}$ when $\mu = \sum_{i=1}^n \delta_{(r_i, \upsilon_i)}$. In other words, the process X when issued from initial configuration μ, is equivalent to issuing n independent copies of X, each with configuration (r_i, υ_i), $i = 1, \ldots, n$.

Like all spatial Markov branching processes, $(X, \mathbb{P})$, where $\mathbb{P} := (\mathbb{P}_\mu, \mu \in \mathcal{M}(D \times V))$, respects the *Markov branching property* with respect to the filtration $\mathcal{F}_t := \sigma((r_i(s), \upsilon_i(s)) : i = 1, \ldots, N_s, s \le t)$, $t \ge 0$. That is to say, for all bounded and measurable $g : D \times V \to [0, \infty)$ and $\mu \in \mathcal{M}(D \times V)$ written $\mu = \sum_{i=1}^n \delta_{(r_i, \upsilon_i)}$, we have $\mathbb{E}_\mu[\prod_{i=1}^n g(r_i, \upsilon_i)] = \prod_{i=1}^n u_t[g](r_i, \upsilon_i)$, for $t \ge 0, r \in D, \upsilon \in V$, where $u_t[g](r, \upsilon) := \mathbb{E}_{\delta_{(r,\upsilon)}}[\prod_{i=1}^{N_t} g(r_i(t), \upsilon_i(t))]$. *In this setting it is also customary to work with the notion that the empty product is valued as unity*; see [22–24].

What is of particular interest to us in the context of the NTE is the *expectation semigroup* of the neutron branching process. More precisely, and with pre-emptive notation, we are interested in

$$(2.4) \qquad \psi_t[g](r, \upsilon) := \mathbb{E}_{\delta_{(r,\upsilon)}}[\langle g, X_t \rangle], \quad t \ge 0, r \in \bar{D}, \upsilon \in V,$$

for $g \in L_\infty^+(D \times V)$, the space of nonnegative uniformly bounded measurable functions on $D \times V$. Here we have made a slight abuse of notation (see $\langle \cdot, \cdot \rangle$ as it appears in (2.3)) and written $\langle g, X_t \rangle$ to mean $\int_{D \times V} g(r, \upsilon) X_t(\mathrm{d}r, \mathrm{d}\upsilon)$.

To see why $(\psi_t, t \ge 0)$ deserves the name of expectation semigroup, it is a straightforward exercise with the help of the Markov branching property to show that

$$(2.5) \qquad \psi_{t+s}[g](r, \upsilon) = \psi_t[\psi_s[g]](r, \upsilon), \quad s, t \ge 0.$$

The connection of the expectation semigroup (2.4) with the NTE (1.1) was explored in the recent article [10] (see also older work in [11, 27]). In order to present the relevant findings, let us momentarily introduce some notation. The deterministic evolution $\mathrm{U}_t[g](r, \upsilon) = g(r + \upsilon t, \upsilon) \mathbf{1}_{\{t < \kappa_{r,\upsilon}^D\}}$, $t \ge 0$, and $\kappa_{r,\upsilon}^D := \inf\{t > 0 : r + \upsilon t \notin D\}$, represents the advection semigroup associated with a single neutron travelling at velocity υ from r. The backwards scatter operator is denoted by

$$(2.6) \qquad \mathrm{S}f(r, \upsilon) = \sigma_{\mathrm{s}}(r, \upsilon) \int_V f(r, \upsilon') \pi_{\mathrm{s}}(r, \upsilon, \upsilon') \, \mathrm{d}\upsilon' - \sigma_{\mathrm{s}}(r, \upsilon) f(r, \upsilon)$$

and the backwards fission operator is given by

$$(2.7) \qquad \mathtt{F}f(r, \upsilon) = \sigma_{\mathtt{f}}(r, \upsilon) \int_V f(r, \upsilon') \pi_{\mathtt{f}}(r, \upsilon, \upsilon') \, \mathrm{d}\upsilon' - \sigma_{\mathtt{f}}(r, \upsilon) f(r, \upsilon),$$

for $f \in L_\infty^+(D \times V)$, such that both $\mathtt{S}$ and $\mathtt{F}$ are defined on $D \times V$ and zero otherwise.

LEMMA 2.1 ([10]). *Under* (H1) *and* (H2), *for* $g \in L_\infty^+(D \times V)$, *there exist constants* $C_1, C_2 > 0$ *such that* $\psi_t[g]$, *as given in* (2.4), *is uniformly bounded by* $C_1 \exp(C_2 t)$, *for all* $t \geq 0$. *Moreover,* $(\psi_t[g], t \geq 0)$ *is the unique solution, which is bounded in time, to the so-called mild equation* (*also called a Duhamel solution in the PDE literature*):

$$(2.8) \qquad \psi_t[g] = \mathtt{U}_t[g] + \int_0^t \mathtt{U}_s \big[(\mathtt{S} + \mathtt{F}) \psi_{t-s}[g] \big] \, \mathrm{d}s, \quad t \geq 0,$$

for which (1.2) *holds.*

The fact that (2.4) solves (2.8) is a simple matter of conditioning the expression in (2.4) on the first fission or scatter event (whichever occurs first) and rearranging the resulting equation. Uniqueness is a matter of working in the right way with Grönwall's Lemma. The association of (2.8) with (1.1) in this way was also explored in Theorem 7.1 [10], where it was shown that the unique solution to (1.1) when seen as an abstract Cauchy problem on $L_2(D \times V)$ agrees with the unique solution to (2.8) in the $L_2(D \times V)$ norm.

The reader should note that we do not need (H3) or (H4) as the result does not require information about the pathwise behaviour of any associated underlying stochastic processes. Nor does it distinguish between the settings that $\mathtt{F}$ is present or not.

3. Perron–Frobenius asymptotics. As alluded to above, one of the classical ways in which neutron flux is understood is to look for the leading eigenvalue and associated ground state eigenfunction. Roughly speaking, this means looking for an associated triple of eigenvalue $\lambda_* \in \mathbb{R}$, positive right eigenfunction $\varphi : D \times V \to [0, \infty)$, a left eigenmeasure $\tilde{\varphi}(r, \upsilon) \, \mathrm{d}r \, \mathrm{d}\upsilon$ on $D \times V$ in $L_2^+(D \times V)$ (the cone of nonnegative square integrable functions on $D \times V$) such that $\langle f, \psi_t[\varphi] \rangle = \mathrm{e}^{\lambda_* t} \langle f, \varphi \rangle$ and $\langle \tilde{\varphi}, \psi_t[f] \rangle = \mathrm{e}^{\lambda_* t} \langle \tilde{\varphi}, f \rangle$, for $t \geq 0$. Here, we again abuse our notation (see the use of $\langle \cdot, \cdot \rangle$ in (2.3) and (2.4)) and write, for $f, g \in L_2^+(D \times V)$, $\langle f, g \rangle = \int_{D \times V} f(r, \upsilon) g(r, \upsilon) \, \mathrm{d}r \, \mathrm{d}\upsilon$. With the eigentriple in hand, it is a common point of analysis that, to leading order, the NTE (1.1) is solved through the approximation

$$(3.1) \qquad \psi_t(r, \upsilon) = \mathrm{e}^{\lambda_* t} \langle g, \tilde{\varphi} \rangle \varphi(r, \upsilon) + o(\mathrm{e}^{\lambda_* t}), \quad t \geq 0, r \in D, \upsilon \in V,$$

where the sense of the equality depends on how one interprets the NTE (i.e., as an abstract Cauchy problem or in its mild form).

The eigenfunction $\tilde{\varphi}$ is called the *importance map* and offers a quasi-stationary profile of radioactive activity (unless $\lambda_* = 0$, in which case it is a stationary profile). Indeed, in modern nuclear reactor design and safety regulation, it is usually the case that virtual reactor models such as the one seen in Figure 1 (an example of a uranium pebble bed reactor) are designed such that $\lambda_* = 0$ and the behaviour of $\tilde{\varphi}$ within spatial domains on the human scale remains within regulated levels. Existing physics and engineering literature with focus on applications in the nuclear regulation industry, has largely been concerned with different numerical methods for estimating the value of the eigenvalue λ_* as well as the eigenfunction φ and eigenmeasure $\tilde{\varphi}(r, \upsilon) \, \mathrm{d}r \, \mathrm{d}\upsilon$. Giving a sensible meaning to (3.1) will play an important part in unraveling the analysis of stochastic representations of solutions to the NTE as well. Moreover, in additional forthcoming work [9], we will also see that our asymptotic (3.1), together
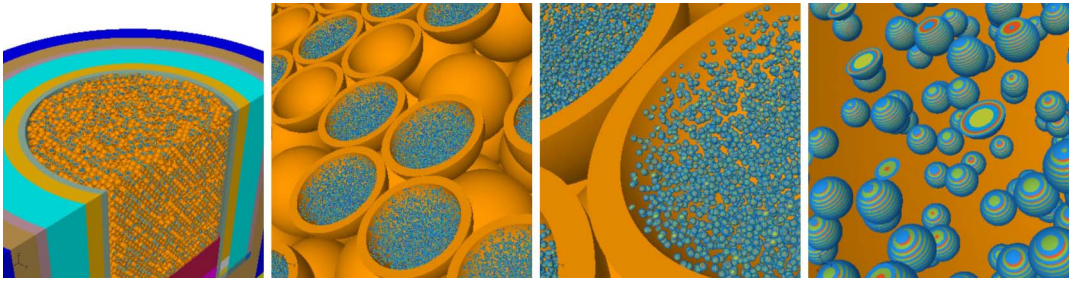
FIG. 1. *Zooming into a virtual model of a pebble bed nuclear reactor core made from tennis ball sized orbs which encapsulate uranium pellets. From left to right, the diagrams illustrate detail from metres down to millimetres. Colour indicating the different regions in which the cross sections σ_s, σ_f, π_s, π_f are constant. The structural design of such a reactor can easily be stored as a virtual environment (i.e., storing the coordinates of the different geometrical domains and the material properties in each domain) with just circa 150 MB of data, on to which extensive data libraries of industrial numerical values for the respective quantities σ_s, σ_f, π_s, π_f can be mapped.*

with the accompanying stochastic analysis developed here, has influence on a number of completely new Monte Carlo methods associated with the NTE that, in turn, bears relevance to the applied NTE literature.

The approximation (3.1) can be seen as a functional version of the Perron–Frobenius Theorem, in particular when noting via (2.8) that we can understand $\psi_t[g]$ as a semigroup. Many attempts have been made to generalise the notion of the Perron–Frobenius decomposition to semigroups of Markov processes with countable and uncountable state spaces, as well as with killing and mass creation (see e.g., [14, 33–35]), using what has come to be known as R-theory. The conditions there seem difficult to verify in the current setting.

More recently, [7, 8] have provided an alternative approach to the R-theory presented in aforementioned works. In the current context, Theorem 2.1 and Proposition 2.3 of [7] will help us to achieve the global result, given below. To state it we need to introduce the quantity

$$(3.2) \quad \alpha(r, \upsilon)\pi(r, \upsilon, \upsilon') = \sigma_s(r, \upsilon)\pi_s(r, \upsilon, \upsilon') + \sigma_f(r, \upsilon)\pi_f(r, \upsilon, \upsilon'), \quad r \in D, \upsilon, \upsilon' \in V,$$

where π is taken to be a probability density. As such it necessarily follows that

$$(3.3) \qquad \alpha(r, \upsilon) = \sigma_s(r, \upsilon) + \sigma_f(r, \upsilon) \int_V \pi_f(r, \upsilon, \upsilon')\, d\upsilon'.$$

THEOREM 3.1. *Suppose that* (H1) *holds as well as*

(H2)*: $\inf_{r \in D, \upsilon, \upsilon' \in V} \alpha(r, \upsilon)\pi(r, \upsilon, \upsilon') > 0$.

Then, for the semigroup $(\psi_t, t \geq 0)$ *identified by* (2.8)*, there exists a* $\lambda_* \in \mathbb{R}$*, a positive*[1] *right eigenfunction* $\varphi \in L_\infty^+(D \times V)$ *and a left eigenmeasure which is absolutely continuous with respect to Lebesgue measure on* $D \times V$ *with density* $\tilde\varphi \in L_\infty^+(D \times V)$*, both having associated eigenvalue* $e^{\lambda_* t}$*, and such that* φ *(resp.* $\tilde\varphi$*) is uniformly (resp. a.e. uniformly) bounded away from zero on each compactly embedded subset of* $D \times V$*. In particular, for all* $g \in L_\infty^+(D \times V)$*,*

$$(3.4) \qquad \langle \tilde\varphi, \psi_t[g] \rangle = e^{\lambda_* t} \langle \tilde\varphi, g \rangle \qquad (resp.\ \psi_t[\varphi] = e^{\lambda_* t}\varphi), \quad t \geq 0.$$

Moreover, there exists $\varepsilon > 0$ *such that*

$$(3.5) \qquad \sup_{g \in L_\infty^+(D \times V): \|g\|_\infty \leq 1} \left\| e^{-\lambda_* t}\varphi^{-1}\psi_t[g] - \langle \tilde\varphi, g \rangle \right\|_\infty = O(e^{-\varepsilon t}) \quad as\ t \to \infty.$$

---

[1]To be precise, by a positive eigenfunction, we mean a mapping from $D \times V \to (0, \infty)$. This does not prevent it being valued zero on ∂D, as D is an open bounded, convex domain.

This result differs significantly from what is already in the literature principally through the assumptions on the cross-sections, the strict positivity properties and the uniform boundedness of φ, $\tilde{\varphi}$ and uniformity in the mode of convergence. In existing literature (3.5) is usually given in the L_p setting, where $1 < p < \infty$ is strictly enforced due to the nature of the c_0-semigroup perturbation analysis; cf. [12, 36] and the discussion in [10].

The proof of Theorem 3.1 is a nontrivial application of the recent theory of [7, 8] in that verifying their assumptions (which essentially leads to the full statement of Theorem 3.1) is highly technical, taking account of the dimension of the system and the piecewise linear (and hence irregular) nature of the neutron paths in the underlying NBP.

Once again the assumptions (H3) and (H4) are unnecessary. As we shall shortly see, the result relies on the treatment of the sum of the operators $S + F$ as a single object, re-written as a scattering generator with action

$$S' f(r, \upsilon) = \int_V (f(r, \upsilon') - f(r, \upsilon)) \alpha(r, \upsilon) \pi(r, \upsilon, \upsilon') \, d\upsilon', \quad r \in D, \upsilon \in V.$$

The assumption (H2)* is a condition on the intensity of this new generator. In this sense, the need for fission or for control of the pathwise behaviour of number of offspring (other than through their mean) is not needed.

4. Neutron random walk and many-to-one methodology. There is a second stochastic representation of the unique solution to (2.8) which will form the basis of our proof of Theorem 3.1. In order to describe it, we need to introduce the notion of a *neutron random walk* (NRW).

A NRW on D is defined by its scatter rates, $\varsigma(r, \upsilon), r \in D, \upsilon \in V$, and scatter probability densities $\varpi(r, \upsilon, \upsilon'), r \in D, \upsilon, \upsilon' \in V$ such that $\int_V \varpi(r, \upsilon, \upsilon') \, d\upsilon' = 1$ for all $r \in D, \upsilon \in V$. Simply, when issued from r with a velocity υ, the NRW will propagate linearly with that velocity until either it exits the domain D, in which case it is killed, or at the random time T_S a scattering occurs, where $\Pr(T_S > t) = \exp\{-\int_0^t \varsigma(r + \upsilon s, \upsilon) \, ds\}$, for $t \geq 0$. When the scattering event occurs in position-velocity configuration (r, υ), a new velocity υ' is selected with probability $\varpi(r, \upsilon, \upsilon') \, d\upsilon'$. If we denote by $(R, \Upsilon) = ((R_t, \Upsilon_t), t \geq 0)$, the position-velocity of the resulting continuous-time random walk on $D \times V$ with an additional cemetery state $\{\dagger\}$ for when it leaves the domain D, then it is easy to show that (R, Υ) is a Markov process. Note, neither R nor Υ alone is Markovian. We call the process (R, Υ) an $\varsigma\varpi$-NRW. It is worth remarking that when $\varsigma\varpi$ is given as a single rate function, the density ϖ, and hence the rate ς, is uniquely identified by normalising of the given product from $\varsigma\varpi$ to make it a probability distribution.

To describe the second stochastic representation of (2.8), we define

$$(4.1) \qquad \beta(r, \upsilon) = \sigma_f(r, \upsilon) \left(\int_V \pi_f(r, \upsilon, \upsilon') \, d\upsilon' - 1 \right) \geq - \sup_{r \in D, \upsilon \in V} \sigma_f(r, \upsilon) > -\infty,$$

where the lower bound is due to assumption (H1). The following result was established in Lemma 7.1 of [10].

LEMMA 4.1 (Many-to-one formula, [10]). *Under the assumptions of Lemma* 2.1, *we have the second representation*

$$(4.2) \qquad \psi_t[g](r, \upsilon) = \mathbf{E}_{(r,\upsilon)} \big[e^{\int_0^t \beta(R_s, \Upsilon_s) \, ds} g(R_t, \Upsilon_t) \mathbf{1}_{\{t < \tau^D\}} \big], \quad t \geq 0, r \in D, \upsilon \in V,$$

where $\tau^D = \inf\{t > 0 : R_t \notin D\}$ *and* $\mathbf{P}_{(r,\upsilon)}$ *for the law of the* $\alpha\pi$-*NRW starting from a single neutron with configuration* (r, υ).

Noting that $\bar{\beta} := \sup_{r \in D, \upsilon \in V} \beta(r, \upsilon) < \infty$ thanks to (H1), let us introduce $\mathtt{P}^\dagger := (\mathtt{P}_t^\dagger, t \geq 0)$ for the expectation semigroup of the $\alpha\pi$-neutron random walk with potential β, such as is represented by the semigroup (4.2), but now killed at rate $(\bar{\beta} - \beta)$. More precisely, for $g \in L_\infty^+(D \times V)$,

(4.3)
$$\begin{aligned}
\mathtt{P}_t^\dagger[g](r, \upsilon) &= \psi_t[g](r, \upsilon) e^{-\bar{\beta}t} \\
&= \mathbf{E}_{(r,\upsilon)} \left[e^{\int_0^t (\beta(R_s, \Upsilon_s) - \bar{\beta}) \, ds} g(R_t, \Upsilon_t) \mathbf{1}_{\{t < \tau^D\}} \right] \\
&= \mathbf{E}_{(r,\upsilon)} \left[g(R_t, \Upsilon_t) \mathbf{1}_{\{t < \mathtt{k}\}} \right] \\
&=: \mathbf{E}_{(r,\upsilon)}^\dagger \left[g(R_t, \Upsilon_t) \right], \quad t \geq 0, r \in D, \upsilon \in V,
\end{aligned}$$

where

(4.4)
$$\mathtt{k} = \inf \left\{ t > 0 : \int_0^t (\bar{\beta} - \beta(R_s, \Upsilon_s)) \, ds > \mathbf{e} \right\} \wedge \tau^D,$$

and $\mathbf{e}$ is an independent exponentially distributed random variable with mean 1.

We will naturally write $\mathbf{P}_{(r,\upsilon)}^\dagger$ for the (sub)probability measure associated to $\mathbf{E}_{(r,\upsilon)}^\dagger$, $r \in \bar{D}$, $\upsilon \in V$. The family $\mathbf{P}^\dagger := (\mathbf{P}_{(r,\upsilon)}^\dagger, r \in \bar{D}, \upsilon \in V)$ now defines a Markov family of probability measures on the path space of the neutron random walk with cemetery state $\{\dagger\}$, which is where the path is sent when hitting the boundary ∂D or the clock associated to the killing rate $\bar{\beta} - \beta$ rings. *We note for future calculations that we can extend the domain of functions on $D \times V$ to accommodate taking a value on $\{\dagger\}$ by insisting that this value is always 0.*

Our strategy for proving Theorem 3.1 thus boils down to understanding the evolution of the semigroup of the NRW $((R, \Upsilon), \mathbf{P}^\dagger)$. In this sense, we see that the essence of Theorem 3.1 is, roughly speaking, a classical Perron–Frobenius-type problem for the semigroup of a Markov process; namely $((R, \Upsilon), \mathbf{P}^\dagger)$.

It is also worthy of note that, given the role the $\alpha\pi$-NRW in the proof of Theorem 3.1, we can also interpret the role of the assumptions (H1) and (H2)* in terms of this process. The condition (H1) ensures that scattering cannot occur too fast. We can describe the condition (H2)* by saying that it offers "strong irreducibility" of the $\alpha\pi$-NRW (where e.g., we could say that (H2) only offers "weak irreducibility").

As alluded to previously, we can also see why the absence of the assumption (H3) is not a problem. In the event that, for example, $\sigma_f \pi_f$ is identically zero, the original NBP is nothing more than a $\sigma_s \pi_s$-NRW, that is, $\alpha\pi = \sigma_s \pi_s$ and $\beta \leq 0$. As such the analysis in the proof of Theorem 3.1, which fundamentally concerns a NRW with a "strictly irreducible" state space and killing is still valid. Similarly, the inclusion of (H4) is not necessary as we only need control over the kernel $\alpha\pi$ for the purpose of analysing the associated NRW and not the pathwise behaviour of the otherwise associated NBP.

5. The ground state martingale. As an application of the Perron–Frobenius behaviour of the linear semigroups discussed in Theorem 3.1, we complete the summary of the main results of this paper by discussing how the existence of the right eigenfunction φ plays directly into the existence of a classical (ground state) martingale for the underlying physical process. Analogues of this martingale appear in the setting of all spatial branching processes and is sometimes referred to there as "the additive martingale" (see e.g., the recent monograph [39] which discusses the analogous setting for branching random walks, or [3] for fragmentation processes).

Under the assumptions of Theorem 3.1 thanks to the semigroup property of (2.4) and the invariance of φ in Theorem 3.1, it is now easy to see that

(5.1)
$$W_t := e^{-\lambda_* t} \frac{\langle \varphi, X_t \rangle}{\langle \varphi, \mu \rangle}, \quad t \geq 0,$$

is a unit mean martingale under $\mathbb{P}_\mu$ where $\mu \in \mathcal{M}(D \times V)$. It is worth noting that this claim is not so easy to make under analogues of Theorem 3.1 found in previous literature (cf. [11, 12, 36]) as the setting of the eigenfunction φ in an L_p space would make it difficult to make sense of expectations of inner product $\langle \varphi, X_t \rangle$ without saying more about the mean semigroup of $(X_t, t \geq 0)$.

As a nonnegative martingale, the almost sure limit of (5.1) is assured. Our second main result tells us precisely when this martingale limit is nonzero. Before stating the theorem, we require one more assumption on the fission rate and kernel, which is a stricter version of (H3).

(H3)*: There exists an open ball B, compactly embedded in D, such that

$$\inf_{r \in B, \upsilon, \upsilon' \in V} \sigma_{\mathrm{f}}(r, \upsilon) \pi_{\mathrm{f}}(r, \upsilon, \upsilon') > 0.$$

THEOREM 5.1.   *For the $(\sigma_{\mathrm{s}}, \pi_{\mathrm{s}}, \sigma_{\mathrm{f}}, \mathcal{P})$-NBP satisfying (H1), (H2)* and (H4), we have the following cases for the martingale $W = (W_t, t \geq 0)$:*

  (i) *If $\lambda_* > 0$ and (H3) holds, then W is $L_1(\mathbb{P})$ convergent;*
 (ii) *If $\lambda_* < 0$ and (H3) holds, then $W_\infty = 0$ almost surely;*
(iii) *If $\lambda_* = 0$ and (H3)* holds, then $W_\infty = 0$ almost surely.*

Although a subtle inclusion, the following theorem also frames Theorem 5.1 in a more tidy way, showing the zero set of the martingale limit agrees with extinction.

THEOREM 5.2.   *In each of the three cases of Theorem 5.1, we also have that the events $\{W_\infty = 0\}$ and $\{\zeta < \infty\}$ almost surely agree, where $\zeta = \inf\{t > 0 : \langle 1, X_t \rangle = 0\}$ is the time of extinction of the NBP. In particular, there is almost sure extinction if and only if $\lambda_* \leq 0$.*

As we can see from the above theorem, the critical case requires slightly more stringent conditions than the super- or sub-critical cases. However, it we assume the conditions of the critical case across the board, we get the aesthetically more pleasing corollary below.

COROLLARY 5.3.   *For the $(\sigma_{\mathrm{s}}, \pi_{\mathrm{s}}, \sigma_{\mathrm{f}}, \mathcal{P})$-NBP satisfying (H1), (H2)*, (H3)* and (H4), the martingale W is $L_1(\mathbb{P})$ convergent if and only if $\lambda_* > 0$ and otherwise $W_\infty = 0$. Irrespective of λ_*, $\{W_\infty = 0\} = \{\zeta < \infty\}$ almost surely.*

Note that, unlike many spatial branching process (e.g., the classical result of [6]), there is no "$x \log x$" condition thanks to the assumption (H2) and a precise dichotomy on λ_* emerges. The result mimics a behavioural trait that has been observed for branching diffusions in compact domains in, for example, [18]. In essence it states that in the competing physical processes of fission, transport, scattering and absorption, it is the lead eigenvalue which dictates growth or decay of mass. In this respect we can also mimic other similar results in the spatial branching process literature (cf. [1, 19, 26]), the proof of which falls out of the proof of Theorem 5.1.

COROLLARY 5.4.   *For the $(\sigma_{\mathrm{s}}, \pi_{\mathrm{s}}, \sigma_{\mathrm{f}}, \mathcal{P})$-NBP satisfying the assumptions (H1), (H2)*, (H3) and (H4), when $\lambda_* > 0$, the martingale $(W_t, t \geq 0)$ is $L_2(\mathbb{P})$ convergent.*

It is particularly interesting to note that in the setting of a critical system, $\lambda_* = 0$, which is typically what is envisaged for a nuclear reactor, the above results evidences the hypothesis that the fission process eventually dies out (similarly to other examples of critical branching processes).

To verify the aforementioned hypothesis rigorously, one needs an almost sure growth result for the particle system which would take the format

$$
(5.2) \qquad \lim_{t \to \infty} \mathrm{e}^{-\lambda_* t} \frac{\langle g, X_t \rangle}{\langle \varphi, \mu \rangle} = \langle g, \tilde{\varphi} \rangle W_\infty,
$$

$\mathbb{P}_\mu$-almost surely, for all $g \in L_\infty^+(D \times V)$. This is a much more difficult result than the one stated in Theorem 5.1 and is addressed in a second instalment to this paper; see [20]. The reader should note that (5.2) verifies what has been known in the nuclear industry for a long time. Namely that critical nuclear reactors will not persist in energy generation, but will eventually cease working, corresponding to the case that $W_\infty = 0$.

6. Neutron random walk and spine decomposition. As with many spatial branching processes, the most efficient way to analyse martingale convergence is through the pathwise behaviour of the particle system (known as a *spine decomposition*) when considered under a change of measure induced by the martingale itself. While classical in the branching process literature, it is unknown in the setting of neutron transport. We will devote the remainder of this section to describing the pathwise spine decomposition of the physical process, our final main contribution.

We are interested in the change of measure

$$
(6.1) \qquad \left. \frac{\mathrm{d}\mathbb{P}_\mu^\varphi}{\mathrm{d}\mathbb{P}_\mu} \right|_{\mathcal{F}_t} = W_t, \quad t \geq 0,
$$

for the NBP with characteristics $\sigma_{\mathrm{s}}, \pi_{\mathrm{s}}, \sigma_{\mathrm{f}}, \mathcal{P}$ (cf. Remark 2.2), where μ belongs to the space of finite atomic measures $\mathcal{M}(D \times V)$.

In the next theorem we will formalise an understanding of this change of measure in terms of another $\mathcal{M}(D \times V)$-valued stochastic process

$$
(6.2) \qquad X^\varphi := (X_t^\varphi, t \geq 0) \quad \text{with probabilities } \tilde{\mathbb{P}}^\varphi := (\tilde{\mathbb{P}}_\mu^\varphi, \mu \in \mathcal{M}(D \times V)),
$$

which we will now describe through an algorithmic construction.

1. From the initial configuration $\mu \in \mathcal{M}(D \times V)$ with an arbitrary enumeration of particles, the ith neutron is selected and marked "*spine*" with empirical probability

$$
\frac{\varphi(r_i, \upsilon_i)}{\langle \varphi, \mu \rangle}.
$$

2. The neutrons $j \neq i$ in the initial configuration that are not marked "*spine*", each issue independent copies of $(X, \mathbb{P}_{\delta_{(r_j, \upsilon_j)}})$ respectively.

3. For the marked neutron, issue a NRW characterised by the rate function

$$
\sigma_{\mathrm{s}}(r, \upsilon) \frac{\varphi(r, \upsilon')}{\varphi(r, \upsilon)} \pi_{\mathrm{s}}(r, \upsilon, \upsilon'), \quad r \in D, \upsilon, \upsilon' \in V.
$$

4. The marked neutron undergoes fission at the accelerated rate $\varphi(r, \upsilon)^{-1}(\mathrm{F} + \sigma_{\mathrm{f}}\mathrm{I}) \times \varphi(r, \upsilon)$, when in physical configuration $(r, \upsilon) \in D \times V$, at which point, it scatters a random number of particles according to the random measure on V given by $(\mathcal{Z}, \mathcal{P}_{(r,\upsilon)}^\varphi)$ where

$$
(6.3) \qquad \frac{\mathrm{d}\mathcal{P}_{(r,\upsilon)}^\varphi}{\mathrm{d}\mathcal{P}_{(r,\upsilon)}} = \frac{\langle \varphi, \mathcal{Z} \rangle}{\mathcal{E}_{(r,\upsilon)}[\langle \varphi, \mathcal{Z} \rangle]}.
$$

5. When fission of the marked neutron occurs in physical configuration $(r, \upsilon) \in D \times V$, set

$$\mu = \sum_{i=1}^{n} \delta_{(r, \upsilon_i)} \quad \text{where, in the previous step, } \mathcal{Z} = \sum_{i=1}^{n} \delta_{\upsilon_i},$$

and repeat step 1.

The process X_t^{φ} describes the physical configuration (position and velocity) of all the particles in the system at time t, for $t \geq 0$ (i.e., ignoring the marked genealogy). We will also be interested in the configuration of the single genealogical line of descent which has been marked "*spine*". This process, referred to simply as *the spine*, will be denoted by $(R^{\varphi}, \Upsilon^{\varphi}) := ((R_t^{\varphi}, \Upsilon_t^{\varphi}), t \geq 0)$. Together, the processes $(X^{\varphi}, (R^{\varphi}, \Upsilon^{\varphi}))$ make a Markov pair, whose probabilities we will denote by $(\tilde{\mathbb{P}}_{\mu,(r,\upsilon)}^{\varphi}, \mu \in \mathcal{M}(D \times V), (r, \upsilon) \in V \times D)$. Note in particular that

$$\tilde{\mathbb{P}}_{\mu}^{\varphi} = \sum_{i=1}^{n} \frac{\varphi(r_i, \upsilon_i)}{\langle \varphi, \mu \rangle} \tilde{\mathbb{P}}_{\mu,(r_i,\upsilon_i)}^{\varphi}$$

when $\mu = \sum_{i=1}^{n} \delta_{(r_i, \upsilon_i)}$.

THEOREM 6.1.   *Under assumptions* (H1), (H2) *and* (H4), *the process* $(X^{\varphi}, \tilde{\mathbb{P}}^{\varphi})$ *is Markovian and equal in law to* $(X, \mathbb{P}^{\varphi})$, *where* $\mathbb{P}^{\varphi} = (\mathbb{P}_{\mu}^{\varphi}, \mu \in \mathcal{M}(D \times V))$.

It is also worth understanding the dynamics of the spine $(R^{\varphi}, \Upsilon^{\varphi})$. For convenience, let us denote the family of probabilities of the latter by $\tilde{\mathbf{P}}^{\varphi} = (\tilde{\mathbf{P}}_{(r,\upsilon)}^{\varphi}, (r, \upsilon) \in D \times V)$, in other words, the marginals of $(\tilde{\mathbb{P}}_{\mu,(r,\upsilon)}^{\varphi}, \mu \in \mathcal{M}(D \times V), (r, \upsilon) \in V \times D)$.

Next we define the probabilities $\mathbf{P}^{\varphi} := (\mathbf{P}_{(r,\upsilon)}^{\varphi}, (r, \upsilon) \in D \times V)$ to describe the law of an $\alpha^{\varphi} \pi^{\varphi}$-NRW, where

(6.4) $$\alpha^{\varphi}(r, \upsilon) \pi^{\varphi}(r, \upsilon, \upsilon') = \frac{\varphi(r, \upsilon')}{\varphi(r, \upsilon)} \left(\sigma_{\mathrm{s}}(r, \upsilon) \pi_{\mathrm{s}}(r, \upsilon, \upsilon') + \sigma_{\mathrm{f}}(r, \upsilon) \pi_{\mathrm{f}}(r, \upsilon, \upsilon') \right),$$

for $r \in D$, $\upsilon, \upsilon' \in V$ We are now ready to identify the spine.

LEMMA 6.1.   *Under assumptions* (H1), (H2) *and* (H4), *the process* $((R^{\varphi}, \Upsilon^{\varphi}), \tilde{\mathbf{P}}^{\varphi})$ *is a NRW equal in law to* $((R, \Upsilon), \mathbf{P}^{\varphi})$ *and, moreover,*

(6.5) $$\left. \frac{d\mathbf{P}_{(r,\upsilon)}^{\varphi}}{d\mathbf{P}_{(r,\upsilon)}} \right|_{\mathcal{F}_t} = e^{-\lambda_* t + \int_0^t \beta(R_s, \Upsilon_s) \, ds} \frac{\varphi(R_t, \Upsilon_t)}{\varphi(r, \upsilon)} \mathbf{1}_{\{t < \tau^D\}}, \quad t \geq 0, r \in D, \upsilon \in V,$$

from which we deduce that $((R, \Upsilon), \mathbf{P}^{\varphi})$ *is conservative with a stationary distribution* $\varphi \tilde{\varphi}(r, \upsilon) \, dr \, d\upsilon$ *on* $D \times V$. (*Recall that* (R, Υ) *under* $\mathbf{P}$ *is the* $\alpha \pi$-NRW *that appears in the many-to-one Lemma* 4.1.)

Now that we have stated all of our main results, it is worth noting that, in places, the analysis echoes very similar issues that have very recently appeared in the analysis of growth-fragmentation equations (see, e.g., [5] and [4]) and for good reason. Growth-fragmentation equations, although dealing with a particle system in which particles' mass is positive-valued and for which there is no consideration of classical "velocity", the dynamics of fragmentation shares the phenomenon of nonlocal branching. This explains the appearance of integral operators. Moreover, a combination of Lévy-type and piecewise linear movement of particles in the growth-fragmentation setting also mirrors the phenomenon of advection and scattering in the NTE and the associated operators.

In the rest of the paper we prove Theorem 3.1, Theorem 6.1, Lemma 6.1, Theorem 5.1 and Corollary 5.4 in that order.

7. Proof of Theorem 3.1. Our approach to proving Theorem 3.1 will be to extract the existence of the eigentriple λ_*, φ and $\tilde\varphi$ for the expectation semigroup $(\psi_t, t \geq 0)$ from the existence of a similar triple of the semigroup $(\mathrm{P}_t^\dagger, t \geq 0)$ defined in (4.3). Indeed, from (4.3), it is clear that when the latter exists, the eigenfunctions of the former are the same and the eigenvalues differ only by the constant $\overline\beta$.

Throughout this section, we assume the assumptions of Theorem 3.1 *are in force.*

As alluded to earlier, what lies at the core of our proof is the general result of Theorem 2.1 and Proposition 2.3 of [7] and Theorem 2.1 and the discussion around (1.5) of [8], which, combined in the current context, reads as follows.

THEOREM 7.1. *Suppose that there exists a probability measure ν on $D \times V$ such that*

(A1) *there exist $t_0, c_1 > 0$ such that for each $(r, \upsilon) \in D \times V$,*

$$\mathbf{P}_{(r,\upsilon)}\big((R_{t_0}, \Upsilon_{t_0}) \in \cdot\,|t_0 < \Bbbk\big) \geq c_1 \nu(\cdot);$$

(A2) *there exists a constant $c_2 > 0$ such that for each $(r, \upsilon) \in D \times V$ and for every $t \geq 0$,*

$$\mathbf{P}_\nu(t < \Bbbk) \geq c_2 \mathbf{P}_{(r,\upsilon)}(t < \Bbbk),$$

where $\Bbbk$ was defined in (4.4). *Then, there exists $\lambda_c < 0$ such that, there exists an eigenmeasure η on $D \times V$ and a positive right eigenfunction φ of $\mathrm{P}^\dagger$ with eigenvalue $\mathrm{e}^{\lambda_c t}$, such that η is a probability measure and $\varphi \in L_\infty^+(D \times V)$, that is, for all $g \in L_\infty(D \times V)$*

(7.1) $$\eta\big[\mathrm{P}_t^\dagger[g]\big] = \mathrm{e}^{\lambda_c t}\eta[g] \quad and \quad \mathrm{P}_t^\dagger[\varphi] = \mathrm{e}^{\lambda_c t}\varphi t \geq 0.$$

Moreover, there exist $C, \varepsilon > 0$ such that, for $g \in L_\infty^+(D \times V)$ and t sufficiently large (*which does not depend on g*),

(7.2) $$\big\|\mathrm{e}^{-\lambda_c t}\varphi^{-1}\mathrm{P}_t^\dagger[g] - \eta[g]\big\|_\infty \leq C\mathrm{e}^{-\varepsilon t}\|g\|_\infty.$$

In particular, setting $g \equiv 1$, as $t \to \infty$,

(7.3) $$\big\|\mathrm{e}^{-\lambda_c t}\varphi^{-1}\mathbf{P}_\cdot(t < \Bbbk) - 1\big\|_\infty \leq C\mathrm{e}^{-\varepsilon t}.$$

We aim to prove that assumptions (A1) and (A2) are satisfied, so that the conclusions of the above theorem hold. Then we prove that φ is uniformly bounded away from 0 on each compactly embedded subset of $D \times V$ and that η admits a positive bounded density with respect to the Lebesgue measure on $D \times V$ (see Lemma 7.4), which concludes the proof of Theorem 3.1. In order to do so, we start by introducing two alternative assumptions to (A1) and (A2):

There exists an $\varepsilon > 0$ such that

(B1) $D_\varepsilon := \{r \in D : \inf_{y \in \partial D} |r - y| > \varepsilon\mathrm{v}_{\max}\}$ is nonempty and connected.

(B2) there exist $0 < s_\varepsilon < t_\varepsilon$ and $\gamma > 0$ such that, for all $r \in D \setminus D_\varepsilon$, there exists $K_r \subset V$ measurable such that $\mathrm{Vol}(K_r) \geq \gamma > 0$ and for all $\upsilon \in K_r$, $r + \upsilon s \in D_\varepsilon$ for every $s \in [s_\varepsilon, t_\varepsilon]$ and $r + \upsilon s \notin \partial D$ for all $s \in [0, s_\varepsilon]$.

It is easy to verify that (B1) and (B2) are implied when we assume that D is a nonempty convex domain, as we have done in the Introduction. They are also satisfied if, for example, the boundary of D is a smooth, connected, compact manifold and ε is sufficiently small. Geometrically, (B2) means that each of the sets

(7.4) $$L_r := \left\{z \in \mathbb{R}^3 : \frac{\|z - r\|}{\|\upsilon\|} \in [s_\varepsilon, t_\varepsilon], \upsilon \in K_r\right\}, \quad r \in D \setminus D_\varepsilon$$

is included in D_ε and has Lebesgue measure at least $\gamma(t_\varepsilon^2 - s_\varepsilon^2)/2$. Roughly speaking, for each $r \in D$ which is within εv_{max} of the boundary ∂D, L_r is the set of points from which one can issue a neutron with a velocity chosen from $v \in K_r$ such that (ignoring scattering and fission) we can ensure that it passes through $D \setminus D_\varepsilon$ during the time interval $[s_\varepsilon, t_\varepsilon]$.

Our proof of Theorem 3.1 thus consists of proving that assumptions (B1) and (B2) imply assumptions (A1) and (A2). Our method is motivated by [7], Section 4.2, however, we note that our approach accommodates for the more general setting we have here (e.g., $V \subset \mathbb{R}^3$ is bounded and $d = 3$) at the cost of greater technicalities.

We begin by considering several technical lemmas. The first is a straightforward consequence of D being a bounded subset of $\mathbb{R}^3$.

LEMMA 7.1. *Let $B(r, v)$ be the ball in $\mathbb{R}^3$ centred at r with radius v.*

(i) *There exists an integer $\mathfrak{n} \geq 1$ and $r_1, \ldots, r_n \in D_\varepsilon$ such that $D_\varepsilon \subset \bigcup_{i=1}^{\mathfrak{n}} B(r_i, v_{max}\varepsilon/32)$ and $D_\varepsilon \cap B(r_i, v_{max}\varepsilon/32) \neq \varnothing$ for each $i \in \{1, \ldots, \mathfrak{n}\}$.*

(ii) *For all $r, r' \in D_\varepsilon$, there exists $m \leq n$ and $i_1, \ldots, i_m$ distinct in $\{1, \ldots, \mathfrak{n}\}$ such that $r \in B(r_{i_1}, v_{max}\varepsilon/32)$, $r' \in B(r_{i_m}, v_{max}\varepsilon/32)$ and for all $1 \leq j \leq m-1$, $B(r_{i_j}, v_{max}\varepsilon/32) \cap B(r_{i_{j+1}}, v_{max}\varepsilon/32) \neq \varnothing$.*

Heuristically, the above lemma ensures that there is a universal covering of D_ε by the balls $B(r_i, v_{max}\varepsilon/32)$, $1 \leq i \leq \mathfrak{n}$ such that between any two points r, r' in D_ε, there is a sequence of overlapping balls $B(r_{i_1}, v_{max}\varepsilon/32), \ldots, B(r_{i_m}, v_{max}\varepsilon/32)$ that one may pass through in order to get from r to r'.

The next lemma provides a minorization of the law of (R_t, Υ_t) under $\mathbf{P}^\dagger$. The result is similar to [7], Lemma 4.5, however, we provide a less geometrical proof by considering a change of variables from Cartesian to polar coordinates. In the statement of the lemma, we use $\text{dist}(r, \partial D)$ for the distance of r from the boundary ∂D.

Define $\underline{\alpha} = \inf_{r \in D, v \in V} \alpha(r, v) > 0$ and $\underline{\pi} = \inf_{r \in D, v, v' \in V} \pi(r, v, v')$. We will also similarly write $\overline{\alpha}$ and $\overline{\pi}$ with obvious meanings. We note that due to the assumption (H1) we have $\overline{\alpha} < \infty$ and $\overline{\pi} < \infty$ and hence, combining this with (H2)* it follows that,

$$\underline{\alpha} = \frac{1}{\overline{\pi}} \inf_{r \in D, v \in V} \alpha(r, v)\overline{\pi} \geq \frac{1}{\overline{\pi}} \inf_{r \in D, v, v' \in V} \alpha(r, v)\pi(r, v, v') > 0,$$

and a similar calculation shows that $\underline{\pi} > 0$.

LEMMA 7.2. *For all $r \in D$, $v \in V$ and $t > 0$ such that $v_{max}t < \text{dist}(r, \partial D)$, the law of (R_t, Υ_t) under $\mathbf{P}^\dagger_{(r,v)}$, defined in (4.3), satisfies*

$$
\begin{aligned}
(7.5) \quad \mathbf{P}^\dagger_{(r,v)}(R_t \in dz, \Upsilon_t \in dv) \geq{}& \frac{Ce^{-\overline{\alpha}t}}{t^2}\left[\frac{t}{2}\left(v_{max}^2 - \left(v_{min} \vee \frac{|z-r|}{t}\right)^2\right)\right. \\
&\left. - |z-r|\left(v_{max} - v_{min} \vee \frac{|z-r|}{t}\right)\right]\mathbf{1}_{\{z \in B(r, v_{max}t)\}}\,dz\,dv,
\end{aligned}
$$

where $C > 0$ is a positive constant.

PROOF. Fix $r_0 \in D$. Let J_k denote the kth jump time of (R_t, Υ_t) under $\mathbf{P}^\dagger_{(r,v)}$ and let Υ_0 be uniformly distributed on V. Assuming that $v_{max}t < \text{dist}(r_0, \partial D)$, we first give a minorization of the density of (R_t, Υ_t), with initial configuration (r_0, Υ_0), on the event $\{J_1 \leq t < J_2\}$. Note that, on this event, we have

$$R_t = r_0 + J_1\Upsilon_0 + (t - J_1)\Upsilon_{J_1},$$

where Υ_{J_1} is the velocity of the process after the first jump. Then

(7.6)
$$
\begin{aligned}
& \mathbf{E}^{\dagger}_{(r_0, \Upsilon_0)}\big[f(R_t, \Upsilon_t)\mathbf{1}_{\{J_1 \le t < J_2\}}\big] \\
& = \int_0^t ds \int_V d\upsilon_0 \int_V d\upsilon_1 \alpha(r_0 + \upsilon_0 s, \upsilon_0) e^{-\int_0^s \alpha(r_0 + \upsilon_0 u, \upsilon_0)\, du} e^{-\int_0^{t-s} \alpha(r_0 + \upsilon_0 s + \upsilon_1 u, \upsilon_1)\, du} \\
& \quad \times \pi(r_0 + \upsilon_0 s, \upsilon_0, \upsilon_1) f\big(r_0 + \upsilon_0 s + (t-s)\upsilon_0, \upsilon_1\big) \\
& \ge \underline{\alpha} e^{-\bar{\alpha} t} \underline{\pi} \int_V d\upsilon_1 \int_0^t ds \int_V d\upsilon_0 f\big(r_0 + s\upsilon_0 + (t-s)\upsilon_1, \upsilon_1\big),
\end{aligned}
$$

where we have used the bounds on α and π. We now make the change of variables $\upsilon_0 \mapsto (\rho_0, \theta_0, \varphi_0)$ and $\upsilon_1 \mapsto (\rho_1, \theta_1, \varphi_1)$ so that (7.6) becomes

(7.7)
$$
\begin{aligned}
& \mathbf{E}^{\dagger}_{(r_0, \Upsilon_0)}\big[f(R_t, \Upsilon_t)\mathbf{1}_{\{J_1 \le t < J_2\}}\big] \\
& \ge C_1 \underline{\alpha} e^{-\bar{\alpha} t} \underline{\pi} \int_0^t ds \int_{\upsilon_{\min}}^{\upsilon_{\max}} d\rho_1 \int_0^{\pi} d\varphi_1 \int_0^{2\pi} d\theta_1 \int_{\upsilon_{\min}}^{\upsilon_{\max}} d\rho_0 \int_0^{\pi} d\varphi_0 \int_0^{2\pi} d\theta_0 \\
& \quad f\big(r_0 + \Theta_{\rho_0, \rho_1, \theta_1, \varphi_1}(s, \theta_0, \varphi_0), \widetilde{\Theta}(\rho_1, \theta_1, \varphi_1)\big) \Delta(\rho_0, \theta_0, \varphi_0) \Delta(\rho_1, \theta_1, \varphi_1),
\end{aligned}
$$

where

(7.8)
$$
\Theta_{\rho_0, \rho_1, \theta_1, \varphi_1}(s, \theta_0, \varphi_0) = \begin{bmatrix} s\rho_0 \sin\varphi_0 \cos\theta_0 + (t-s)\rho_1 \sin\varphi_1 \cos\theta_1 \\ s\rho_0 \sin\varphi_0 \sin\theta_0 + (t-s)\rho_1 \sin\varphi_1 \sin\theta_1 \\ s\rho_0 \cos\varphi_0 + (t-s)\rho_1 \cos\varphi_1 \end{bmatrix},
$$

represents the spatial variable $s\upsilon_0 + (t-s)\upsilon_1$ in polar coordinates,

(7.9)
$$
\widetilde{\Theta}(\rho_1, \theta_1, \varphi_1) = \begin{bmatrix} \rho_1 \sin\varphi_1 \cos\theta_1 \\ \rho_1 \sin\varphi_1 \sin\theta_1 \\ \rho_1 \cos\varphi_1 \end{bmatrix}
$$

represents υ_1 in polar coordinates,

(7.10)
$$
\Delta(\rho, \theta, \varphi) = \rho^2 \sin\varphi,
$$

is the determinant of the Jacobian matrix for the change of variables from Cartesian to polar coordinates, and C_1 is an unimportant normalising constant.

For fixed ρ_0, ρ_1, θ_1 and φ_1, we first consider the part of (7.7) given by

(7.11)
$$
\begin{aligned}
& (s, \theta_0, \varphi_0) \\
& \mapsto \int_0^t ds \int_0^{\pi} d\varphi_0 \int_0^{2\pi} d\theta_0 f\big(r_0 + \Theta_{\rho_0, \rho_1, \theta_1, \varphi_1}(s, \theta_0, \varphi_0), \widetilde{\Theta}(\rho_1, \theta_1, \varphi_1)\big) \Delta(\rho_0, \theta_0, \varphi_0).
\end{aligned}
$$

The Jacobian of $\Theta_{\rho_0, \rho_1, \theta_1, \varphi_1}$, as a function of (s, θ_0, φ_0), is given by

$$
\begin{bmatrix} \rho_0 \cos\theta_0 \sin\varphi_0 - \rho_1 \cos\theta_1 \sin\varphi_1 & -s\rho_0 \sin\theta_0 \sin\varphi_0 & s\rho_0 \cos\varphi_0 \cos\theta_0 \\ \rho_0 \sin\theta_0 \sin\varphi_0 - \rho_1 \sin\theta_1 \sin\varphi_1 & s\rho_0 \cos\theta_0 \sin\varphi_0 & s\rho_0 \cos\varphi_0 \sin\theta_0 \\ \rho_0 \cos\varphi_0 - \rho_1 \cos\varphi_1 & 0 & -s\rho_0 \sin\varphi_0 \end{bmatrix},
$$

whose determinant, $\det(D_{\rho_0, \rho_1, \theta_1, \varphi_1}(s, \theta_0, \varphi_0))$ satisfies

$$
\frac{\Delta(\rho_0, \theta_0, \varphi_0)}{\det(D_{\rho_0, \rho_1, \theta_1, \varphi_1}(s, \theta_0, \varphi_0))} \ge \frac{1}{4s^2 \mathrm{v}_{\max}^3} \ge \frac{1}{4t^2 \mathrm{v}_{\max}^3}, \quad s \le t.
$$

We thus have the following lower bound for (7.11)

$$(7.12) \quad \frac{1}{4t^2 \mathrm{v}_{\max}^3} \int_0^t \mathrm{d}s \int_0^\pi \mathrm{d}\varphi_0 \int_0^{2\pi} \mathrm{d}\theta_0 \, f\big(r_0 + \Theta_{\rho_0,\rho_1,\theta_1,\varphi_1}(s,\theta_0,\varphi_0), \widetilde{\Theta}(\rho_1,\theta_1,\varphi_1)\big)$$
$$\times \det\big(D_{\rho_0,\rho_1,\theta_1,\varphi_1}(s,\theta_0,\varphi_0)\big).$$

Making another change of variables $(s,\theta_0,\varphi_0) \mapsto r \in \mathbb{R}^3$ and using the fact that, regardless of the values of ρ_1, θ_1 and φ_1, $\Theta_{\rho_0,\rho_1,\theta_1,\varphi_1}$ maps $(0,t) \times (0,\pi) \times (0,2\pi)$ surjectively onto a set that contains $B(\rho_0 t)$, where $B(r)$ is the ball in $\mathbb{R}^3$ of radius r centred at the origin, (7.12), and hence (7.11), is bounded below by

$$(7.13) \quad \frac{1}{4t^2 v_{\max}^3} \int_{B(\rho_0 t)} f\big(r, \widetilde{\Theta}(\rho_1,\theta_1,\varphi_1)\big) \, \mathrm{d}r.$$

Substituting this equation back into (7.7) and changing $(\rho_1,\theta_1,\varphi_1)$ back to Cartesian coordinates, we have

$$(7.14) \quad \mathbf{E}^\dagger_{(r_0,\Upsilon_0)}\big[f(R_t,\Upsilon_t)\mathbf{1}_{\{J_1 \le t < J_2\}}\big] \ge \frac{C_2 e^{-\overline{\alpha} t}}{t^2} \int_{v_{\min}}^{v_{\max}} \mathrm{d}\rho_0 \int_{B(\rho_0 t)} \mathrm{d}r \int_V \mathrm{d}v_1 \, f(r,v_1),$$

where $C_2 = \underline{\alpha} \pi C_1/(4 v_{\max}^3)$.

Now suppose we fix an initial configuration $(r_0, v_0) \in D \times V$, with $t v_{\max} < \mathrm{dist}(r_0, \partial D)$. By considering the event $\{J_2 \le t < J_3\}$ and noting that the scattering kernel is bounded below by $\underline{\pi}$, we may apply the Markov property together with (7.14) to the process at time J_1 before choosing the new velocity. Using the bounds on α and π as before, and recalling that Υ_0 is uniformly distributed, we have

$$\mathbf{E}^\dagger_{(r_0,v_0)}\big[f(R_t,\Upsilon_t)\mathbf{1}_{\{J_2 \le t < J_3\}}\big]$$

$$\ge \int_0^t \mathrm{d}s \underline{\alpha} e^{-\overline{\alpha} s} \underline{\pi} \mathbf{E}^\dagger_{(r_0 + s v_0, \Upsilon_0)}\big[f(R_{t-s},\Upsilon_{t-s})\mathbf{1}_{\{J_1 \le t-s < J_2\}}\big]$$

$$(7.15) \quad \ge \int_0^t \mathrm{d}s \underline{\alpha} e^{-\overline{\alpha} s} \underline{\pi} \frac{C_2 e^{-\overline{\alpha}(t-s)}}{(t-s)^2} \int_V \mathrm{d}v_1 \int_{v_{\min}}^{v_{\max}} \mathrm{d}\rho_0 \int_{\rho_0(t-s)B} \mathrm{d}r \, f(r_0 + s v_0 + r, v_1)$$

$$\ge \frac{C_3 e^{-\overline{\alpha} t}}{t^2} \int_0^t \mathrm{d}s \int_V \mathrm{d}v_1 \int_{v_{\min}}^{v_{\max}} \mathrm{d}\rho_0 \int_{\rho_0(t-s)B} \mathrm{d}r \, f(r_0 + s v_0 + r, v_1)$$

$$= \frac{C_3 e^{-\overline{\alpha} t}}{t^2} \int_0^t \mathrm{d}s \int_V \mathrm{d}v_1 \int_{v_{\min}}^{v_{\max}} \mathrm{d}\rho_0 \int_{r_0 + s v_0 + \rho_0(t-s)B} \mathrm{d}y \, f(y, v_1),$$

where we have used the substitution $y = r_0 + s v_0 + r$ to obtain the final line and C_3 is another constant in $(0,\infty)$. Now note that for $s \le \rho_0 t/(\rho_0 + v_{\max})$ we have $r_0 + B(\rho_0 t - (\rho_0 + v_{\max})s) \subset r_0 + s v_0 + B(\rho_0(t-s))$. Combining this with (7.15) and using Fubini, we have

$$\mathbf{E}^\dagger_{(r_0,v_0)}\big[f(R_t,\Upsilon_t)\mathbf{1}_{\{J_2 \le t < J_3\}}\big]$$

$$\ge \frac{C_3 e^{-\overline{\alpha} t}}{t^2} \int_V \mathrm{d}v_1 \int_{v_{\min}}^{v_{\max}} \mathrm{d}\rho_0 \int_{\mathbb{R}} \mathbf{1}_{\{0 \le s \le \frac{\rho_0}{\rho_0 + v_{\max}} t\}} \mathrm{d}s \int_{\mathbb{R}^3} \mathrm{d}y \mathbf{1}_{\{|y - r_0| \le \rho_0 t - (\rho_0 + v_{\max})s\}}$$

$$(7.16) \quad \times f(y, v_1)$$

$$= \frac{C_3 e^{-\overline{\alpha} t}}{t^2} \int_V \mathrm{d}v_1 \int_{v_{\min}}^{v_{\max}} \mathrm{d}\rho_0 \int_{\mathbb{R}} \mathrm{d}s \int_{\mathbb{R}^3} \mathrm{d}y \mathbf{1}_{\{0 \le s \le \frac{\rho_0 t - |y - r_0|}{\rho_0 + v_{\max}}\}} f(y, v_1)$$

$$= \frac{C_3 e^{-\overline{\alpha} t}}{t^2} \int_V \mathrm{d}v_1 \int_{v_{\min}}^{v_{\max}} \mathrm{d}\rho_0 \int_{\mathbb{R}^3} \mathrm{d}y \mathbf{1}_{\{|y - r_0| \le \rho_0 t\}} \left(\frac{\rho_0 t - |y - r_0|}{\rho_0 + v_{\max}}\right) f(y, v_1).$$

We finally compute the integral with respect to $\rho_0 \in (\mathrm{v}_{\min}, \mathrm{v}_{\max})$. In order to do so, we first note that since $\rho_0 < \mathrm{v}_{\max}$, the integrand in (7.16) is bounded below by

$$\frac{\rho_0 t - |y - r_0|}{2\mathrm{v}_{\max}}.$$

Absorbing $1/2\mathrm{v}_{\max}$ into the constant C_3, applying Fubini and computing the ρ_0 integral yields

$$
(7.17) \quad
\begin{aligned}
\mathbf{E}^\dagger_{(r_0, \upsilon_0)}[f(R_t, \Upsilon_t)] \geq {}& \frac{C_3 e^{-\overline{\alpha} t}}{t^2} \int_V d\upsilon_1 \int_{\mathbb{R}^3} dy \left[\frac{t}{2} \left(\mathrm{v}_{\max}^2 - \left(\mathrm{v}_{\min} \vee \frac{|y - r|}{t} \right)^2 \right) \right. \\
& \left. - |y - r| \left(\mathrm{v}_{\max} - \mathrm{v}_{\min} \vee \frac{|y - r|}{t} \right) \right] \mathbf{1}_{\{|y - r_0| \leq \mathrm{v}_{\max} t\}} f(y, \upsilon_1),
\end{aligned}
$$

as required. $\square$

We now turn to the proof of (A1) under the assumptions of (B1) and (B2).

PROOF OF (A1). In this proof, we will follow a similar strategy to the one presented in [7], Section 4.2. We therefore start by proving (A1) for initial configurations in $D_\varepsilon \times V$.

To this end, fix $(r, \upsilon) \in D_\varepsilon \times V$. From Lemma 7.1, there exists an $i \in \{1, \ldots, \mathfrak{n}\}$ such that $r \in B(r_i, \mathrm{v}_{\max} \varepsilon / 32) \cap D_\varepsilon$. Then, for each $t \in [\varepsilon/2, \varepsilon)$, Lemma 7.2 yields

$$
(7.18) \quad
\begin{aligned}
\mathbf{P}^\dagger_{(r, \upsilon)}(R_t \in dz, \Upsilon_t \in dw) \geq {}& \frac{C e^{-\overline{\alpha} t}}{t^2} \left[\frac{t}{2} \left(\mathrm{v}_{\max}^2 - \left(\mathrm{v}_{\min} \vee \frac{|z - r|}{t} \right)^2 \right) \right. \\
& \left. - |z - r| \left(\mathrm{v}_{\max} - \mathrm{v}_{\min} \vee \frac{|z - r|}{t} \right) \right] \\
& \times \mathbf{1}_{\{z \in B(r, \mathrm{v}_{\max} t)\}} \, dz \, dw.
\end{aligned}
$$

Now, if $j \in \{1, \ldots, \mathfrak{n}\}$ is such that $B(r_i, \mathrm{v}_{\max}\varepsilon/32) \cap B(r_j, \mathrm{v}_{\max}\varepsilon/32) \neq \varnothing$, the triangle inequality implies that $D_\varepsilon \cap (B(r_i, \mathrm{v}_{\max}\varepsilon/32) \cup B(r_j, \mathrm{v}_{\max}\varepsilon/32)) \subset B(r, \mathrm{v}_{\max}\varepsilon/8) \subset B(r, \mathrm{v}_{\max} t)$, with the latter inclusion following from the fact that $t \in [\varepsilon/2, \varepsilon)$.

Hence, for $z \in B(r_i, \mathrm{v}_{\max}\varepsilon/32) \cup B(r_j, \mathrm{v}_{\max}\varepsilon/32)$ and $t \in [\varepsilon/2, \varepsilon)$, the density on the right-hand side of (7.18) is bounded below by a constant $C_\varepsilon > 0$, which is independent of r, υ, i and j. Hence,

$$(7.19) \quad \mathbf{P}^\dagger_{(r, \upsilon)}(R_t \in dz, \Upsilon_t \in dw) \geq C_\varepsilon \mathbf{1}_{\{z \in D_\varepsilon \cap (B(r_i, \varepsilon/32) \cup B(r_j, \varepsilon/32))\}} \, dz \, dw, \quad z \in D, w \in V.$$

Now let $t \geq (\mathfrak{n} + 1)\varepsilon/2$. By writing $t = k\varepsilon/2 + t'$, for some $k \geq \mathfrak{n}$ and $t' \in [\varepsilon/2, \varepsilon)$. We will demonstrate that a repeated application of (7.19) will lead to the inequality

$$(7.20) \quad \mathbf{P}^\dagger_{(r, \upsilon)}(R_t \in dz, \Upsilon_t \in dw) \geq C_\varepsilon c_\varepsilon^k \mathbf{1}_{\{z \in D_\varepsilon\}} \, dz \, dw, \quad z \in D, w \in V,$$

for $(r, \upsilon) \in D_\varepsilon \times V$, where $c_\varepsilon > 0$ is another unimportant constant which depends only on ε and is defined in the following analysis.

To this end, we start by noting that, since $r \in D_\varepsilon$ and $\upsilon \in V$, there exists $i_0, i_1 \in \{1, \ldots, \mathfrak{n}\}$ such that $r \in B(r_{i_0}, \mathrm{v}_{\max}\varepsilon/32)$ and $B(r_{i_0}, \mathrm{v}_{\max}\varepsilon/32) \cap B(r_{i_1}, \mathrm{v}_{\max}\varepsilon/32) \cap D_\varepsilon \neq \varnothing$. Applying

(7.19) at time t' (recall that we have identified $t = k\varepsilon/2 + t'$ for some $k \geq \mathfrak{n}$) we obtain,

$$\mathbf{P}^{\dagger}_{(r,\upsilon)}(R_t \in dz, \Upsilon_t \in dw)$$

$$= \mathbf{P}^{\dagger}_{(r,\upsilon)}(R_{t'+k\varepsilon/2} \in dz, \Upsilon_{t'+k\varepsilon/2} \in dw)$$

$$\geq \mathbf{E}^{\dagger}_{(r,\upsilon)}\big[\mathbf{1}_{\{R_{t'} \in B(r_{i_1}, \mathrm{v}_{\max}\varepsilon/32) \cap D_\varepsilon, \Upsilon_{t'} \in V\}}\mathbf{P}^{\dagger}_{(R_{t'}, \Upsilon_{t'})}(R_{k\varepsilon/2} \in dz, \Upsilon_{k\varepsilon/2} \in dw)\big]$$

$$= \int_{B(r_{i_1}, \mathrm{v}_{\max}\varepsilon/32) \cap D_\varepsilon} \int_V \mathbf{P}^{\dagger}_{(r',\upsilon')}(R_{k\varepsilon/2} \in dz, \Upsilon_{k\varepsilon/2} \in dw)$$

(7.21)

$$\times \mathbf{P}^{\dagger}_{(r,\upsilon)}(R_{t'} \in dr', \Upsilon_{t'} \in d\upsilon')$$

$$\geq C_\varepsilon \int_{B(r_{i_1}, \mathrm{v}_{\max}\varepsilon/32) \cap D_\varepsilon} \int_V \mathbf{P}^{\dagger}_{(r',\upsilon')}(R_{k\varepsilon/2} \in dz, \Upsilon_{k\varepsilon/2} \in dw)$$

$$\times \mathbf{1}_{\{r' \in (B(r_{i_0}, \mathrm{v}_{\max}\varepsilon/32) \cup B(r_{i_1}, \mathrm{v}_{\max}\varepsilon/32)) \cap D_\varepsilon\}} \, dr' \, d\upsilon'$$

$$= C_\varepsilon \int_{B(r_{i_1}, \mathrm{v}_{\max}\varepsilon/32) \cap D_\varepsilon} \int_V \mathbf{P}^{\dagger}_{(r',\upsilon')}(R_{k\varepsilon/2} \in dz, \Upsilon_{k\varepsilon/2} \in dw) \, dr' \, d\upsilon'.$$

We now turn our attention to $\mathbf{P}^{\dagger}_{(r',\upsilon')}(R_{k\varepsilon/2} \in dz, \Upsilon_{k\varepsilon/2} \in dw)$, for $(r', \upsilon') \in (B(r_{i_1}, \mathrm{v}_{\max}\varepsilon/32) \cap D_\varepsilon) \times V$ and $k \geq \mathfrak{n}$. Thanks to Lemma 7.1, for all $i_{k+1} \in \{1, \ldots, \mathfrak{n}\}$, there exist $i_2, \ldots, i_k \in \{1, \ldots, \mathfrak{n}\}$ such that $B(r_{i_j}, \varepsilon/32) \cap B(r_{i_{j+1}}, \varepsilon/32) \neq \varnothing$ for every $j \in \{1, \ldots, k\}$. Note, here we see the importance of choosing $k \geq \mathfrak{n}$, to ensure the validity of the previous statement.

Applying (7.19) and following the same steps that lead to (7.21), we obtain

(7.22)

$$\mathbf{P}^{\dagger}_{(r',\upsilon')}(R_{k\varepsilon/2} \in dz, \Upsilon_{k\varepsilon/2} \in dw)$$

$$\geq C_\varepsilon \int_{B(r_{i_2}, \varepsilon/32) \cap D_\varepsilon} \int_V \mathbf{P}^{\dagger}_{(r'',\upsilon'')}(R_{(k-1)\varepsilon/2} \in dz, \Upsilon_{(k-1)\varepsilon/2} \in dw) \, dr'' \, d\upsilon''.$$

Iterating this step a further $k - 2$ times, we obtain

(7.23)

$$\mathbf{P}^{\dagger}_{(r',\upsilon')}(R_{k\varepsilon/2} \in dz, \Upsilon_{k\varepsilon/2} \in dw)$$

$$\geq C_\varepsilon c_\varepsilon^{k-2} \int_{B(r_{i_k}, \mathrm{v}_{\max}\varepsilon/32) \cap D_\varepsilon} \int_V \mathbf{P}^{\dagger}_{(r'',\upsilon'')}(R_{\varepsilon/2} \in dz, \Upsilon_{\varepsilon/2} \in dw) \, dr'' \, d\upsilon'',$$

where $c_\varepsilon = C_\varepsilon \mathrm{Vol}(V) \min_{i=1,\ldots,n} \mathrm{Vol}(B(r_i, \mathrm{v}_{\max}\varepsilon/32) \cap D_\varepsilon)$. Using this inequality to bound the right-hand side of (7.21) yields

(7.24)

$$\mathbf{P}^{\dagger}_{(r,\upsilon)}(R_t \in dz, \Upsilon_t \in dw)$$

$$\geq C_\varepsilon c_\varepsilon^{k-1} \int_{B(r_{i_k}, \varepsilon/32) \cap D_\varepsilon} \int_V \mathbf{P}^{\dagger}_{(r',\upsilon')}(R_{\varepsilon/2} \in dz, \Upsilon_{\varepsilon/2} \in dw) \, dr' \, d\upsilon'.$$

We now apply (7.19) a final time at time $\varepsilon/2$ to obtain

(7.25) $$\mathbf{P}^{\dagger}_{(r,\upsilon)}(R_t \in dz, \Upsilon_t \in dw) \geq C_\varepsilon c_\varepsilon^k \mathbf{1}_{\{z \in B(r_{i_{k+1}}, \varepsilon/2) \cap D_\varepsilon\}} \, dz \, dw.$$

Since this inequality holds for every $i_{k+1} \in \{1, \ldots, \mathfrak{n}\}$, it also follows that

$$\mathbf{P}^{\dagger}_{(r,\upsilon)}(R_t \in dz, \Upsilon_t \in dw) \geq C_\varepsilon c_\varepsilon^k \sup_{i_{k+1} \in \{1,\ldots,\mathfrak{n}\}} \mathbf{1}_{\{z \in B(r_{i_{k+1}}, \varepsilon/2) \cap D_\varepsilon\}} \, dz \, dw$$

$$\geq C_\varepsilon c_\varepsilon^k \mathbf{1}_{\{z \in D_\varepsilon\}} \, dz \, dw,$$

where the final line follows from Lemma 7.1 since $k + 1 > \mathfrak{n}$. This is the lower bound claimed in (7.20).

Finally, noting that for any two events A, B, $\Pr(A|B) = \Pr(A \cap B)/\Pr(B) \geq \Pr(A \cap B)$, we have that for initial conditions $(r, \upsilon) \in D_\varepsilon \times V$, any $t_0 \geq (\mathfrak{n} + 1)\varepsilon/2$ and ν equal to Lebesgue measure on $D_\varepsilon \times V$, there exists a constant $c_1 \in (0, \infty)$ such that

$$\mathbf{P}_{(r,\upsilon)}\big((R_{t_0}, \Upsilon_{t_0}) \in \cdot \,|t_0 < \mathrm{k}\big) \geq c_1 \nu(\cdot),$$

as required by (A1).

We now prove (A1) for initial conditions in $(D \setminus D_\varepsilon) \times V$. Once again, we recall that assumptions (B1) and (B2) are in force.

Choose $r \in D \setminus D_\varepsilon$, $\upsilon \in V$ and define the (deterministic) time

$$\kappa_{r,\upsilon}^{D \setminus D_\varepsilon} := \inf\{t > 0 : r + t\upsilon \notin \partial D \setminus D_\varepsilon\},$$

which is the time it would take a neutron released at r with velocity υ to hit the boundary of $D \setminus D_\varepsilon$ if no scatter or fission took place. Note in particular that $\kappa_{r,\upsilon}^{D \setminus D_\varepsilon}$ is not a random time but entirely deterministic. We first consider the case $r + \kappa_{r,\upsilon}^{D \setminus D_\varepsilon} \upsilon \in \partial D_\varepsilon$

$$(7.26) \qquad \mathbf{P}_{(r,\upsilon)}^\dagger(R_{\kappa_{r,\upsilon}^{D \setminus D_\varepsilon}} \in \partial D_\varepsilon) \geq \mathrm{e}^{-\bar{\alpha} \kappa_{r,\upsilon}^{D \setminus D_\varepsilon}} \geq \mathrm{e}^{-\bar{\alpha} \operatorname{diam}(D)/\mathrm{v}_{\min}}.$$

Combining this with (7.20) and the Markov property, for all $t \geq (\mathfrak{n} + 1)\varepsilon/2$

$$
(7.27) \quad
\begin{aligned}
&\mathbf{P}_{(r,\upsilon)}\big(R_{\kappa_{r,\upsilon}^{D \setminus D_\varepsilon}+t} \in \mathrm{d}z, \Upsilon_{\kappa_{r,\upsilon}^{D \setminus D_\varepsilon}+t} \in \mathrm{d}w | \kappa_{r,\upsilon}^{D \setminus D_\varepsilon}+t < \mathrm{k}\big) \\
&\qquad \geq \mathbf{P}_{(r,\upsilon)}^\dagger(R_{\kappa_{r,\upsilon}^{D \setminus D_\varepsilon}+t} \in \mathrm{d}z, \Upsilon_{\kappa_{r,\upsilon}^{D \setminus D_\varepsilon}+t} \in \mathrm{d}w) \\
&\qquad \geq \mathrm{e}^{-\bar{\alpha} \operatorname{diam}(D)/\mathrm{v}_{\min}} C_\varepsilon c_\varepsilon^k \mathbf{1}_{\{z \in D_\varepsilon\}} \, \mathrm{d}z \, \mathrm{d}w,
\end{aligned}
$$

where $k \geq \mathfrak{n}$ is such that $t = k\varepsilon/2 + t'$ for some $t' \in [\varepsilon/2, \varepsilon)$.

On the other hand, suppose $r + \kappa_{r,\upsilon}^{D \setminus D_\varepsilon} \upsilon \in \partial D$. Then, recalling the assumptions (B1) and (B2) it follows that $\{J_1 < \kappa_{r,\upsilon}^{D \setminus D_\varepsilon} \wedge (t_\varepsilon - s_\varepsilon), \Upsilon_{J_1} \in K_{r+\upsilon J_1}, J_2 > t_\varepsilon\} \subset \{R_{t_\varepsilon} \in D_\varepsilon, t_\varepsilon < \mathrm{k}\}$. Heuristically speaking, this is because if the first jump occurs before time $\kappa_{r,\upsilon}^{D \setminus D_\varepsilon} \wedge (t_\varepsilon - s_\varepsilon)$, then the process hasn't hit the boundary and there are still (at least) s_ε units of time left until t_ε. By then choosing the new velocity, Υ_{J_1}, from $K_{r+\upsilon J_1}$, thanks to the assumption (B1) and the remarks around (7.4), this implies that the process will remain in $D \setminus D_\varepsilon$ for s_ε units of time, at some point in time after which, it will move into D_ε, providing the process doesn't jump again before entering D_ε. Combining this with the usual bounds on α, and recalling from (B2) that $\operatorname{Vol}(K_r) > \gamma > 0$ for all $r \in D \setminus D_\varepsilon$ and $\upsilon \in V$, we have

$$
(7.28) \quad
\begin{aligned}
\mathbf{P}_{(r,\upsilon)}(R_{t_\varepsilon} \in D_\varepsilon, t_\varepsilon < \mathrm{k}) &\geq \mathbf{P}_{(r,\upsilon)}^\dagger\big(J_1 < \kappa_{r,\upsilon}^{D \setminus D_\varepsilon} \wedge (t_\varepsilon - s_\varepsilon), \Upsilon_{J_1} \in K_{r+\upsilon J_1}, J_2 > t_\varepsilon\big) \\
&\geq \underline{\pi} \gamma \mathrm{e}^{-\bar{\alpha} t_\varepsilon} \mathbf{P}_{(r,\upsilon)}^\dagger\big(J_1 < \kappa_{r,\upsilon}^{D \setminus D_\varepsilon} \wedge (t_\varepsilon - s_\varepsilon)\big).
\end{aligned}
$$

Along with (7.20), this implies that, for all $r \in D \setminus D_\varepsilon$, $\upsilon \in V$ and $t \geq (\mathfrak{n} + 1)\varepsilon/2$ such that $t + t_\varepsilon \geq \kappa_{r,\upsilon}^{D \setminus D_\varepsilon}$

$$
(7.29) \quad
\begin{aligned}
&\mathbf{P}_{(r,\upsilon)}(R_{t+t_\varepsilon} \in \mathrm{d}z, \Upsilon_{t+t_\varepsilon} \in \mathrm{d}w | t + t_\varepsilon < \mathrm{k}) \\
&\quad \geq \frac{\mathbf{P}_{(r,\upsilon)}(R_{t_\varepsilon} \in D_\varepsilon, t_\varepsilon < \mathrm{k}, R_{t+t_\varepsilon} \in \mathrm{d}z, \Upsilon_{t+t_\varepsilon} \in \mathrm{d}w)}{\mathbf{P}_{(r,\upsilon)}(t + t_\varepsilon < \mathrm{k})} \\
&\quad = \frac{\mathbf{P}_{(r,\upsilon)}^\dagger(R_{t+t_\varepsilon} \in \mathrm{d}z; \Upsilon_{t+t_\varepsilon} \in \mathrm{d}w | R_{t_\varepsilon} \in D_\varepsilon, t_\varepsilon < \mathrm{k})\mathbf{P}_{(r,\upsilon)}(R_{t_\varepsilon} \in D_\varepsilon, t_\varepsilon < \mathrm{k})}{\mathbf{P}_{(r,\upsilon)}(t + t_\varepsilon < \mathrm{k})}
\end{aligned}
$$

$$\geq \inf_{r \in D_\varepsilon, \upsilon \in V} \mathbf{P}^\dagger_{(r,\upsilon)}(R_t \in dz; \Upsilon_t \in dw)$$

$$\times \frac{\mathbf{P}^\dagger_{(r,\upsilon)}(J_1 < \kappa_{r,\upsilon}^{D \setminus D_\varepsilon} \wedge (t_\varepsilon - s_\varepsilon))}{\mathbf{P}_{(r,\upsilon)}(t + t_\varepsilon < \mathbf{k})} \underline{\pi} \gamma e^{-\overline{\alpha} t_\varepsilon} c_\varepsilon^k \mathbf{1}_{\{z \in D_\varepsilon\}} dz \, dw$$

$$\geq \frac{\mathbf{P}^\dagger_{(r,\upsilon)}(J_1 < \kappa_{r,\upsilon}^{D \setminus D_\varepsilon} \wedge (t_\varepsilon - s_\varepsilon))}{\mathbf{P}_{(r,\upsilon)}(t + t_\varepsilon < \mathbf{k})} \underline{\pi} \gamma e^{-\overline{\alpha} t_\varepsilon} C_\varepsilon c_\varepsilon^k \mathbf{1}_{\{z \in D_\varepsilon\}} dz \, dw.$$

Now, since we are considering the case $r + \kappa_{r,\upsilon}^{D \setminus D_\varepsilon} \upsilon \in \partial D$ and $t + t_\varepsilon \geq \kappa_{r,\upsilon}^{D \setminus D_\varepsilon}$, it follows that $\{t + t_\varepsilon < \mathbf{k}\} \subset \{J_1 < \kappa_{r,\upsilon}^{D \setminus D_\varepsilon}\}$. Then,

(7.30)
$$\frac{\mathbf{P}^\dagger_{(r,\upsilon)}(J_1 < \kappa_{r,\upsilon}^{D \setminus D_\varepsilon} \wedge (t_\varepsilon - s_\varepsilon))}{\mathbf{P}_{(r,\upsilon)}(t + t_\varepsilon < \mathbf{k})} \geq \frac{\mathbf{P}^\dagger_{(r,\upsilon)}(J_1 < \kappa_{r,\upsilon}^{D \setminus D_\varepsilon} \wedge (t_\varepsilon - s_\varepsilon))}{\mathbf{P}_{(r,\upsilon)}(J_1 < \kappa_{r,\upsilon}^{D \setminus D_\varepsilon})}$$

$$\geq \frac{1 - e^{-\underline{\alpha}(\kappa_{r,\upsilon}^{D \setminus D_\varepsilon} \wedge (t_\varepsilon - s_\varepsilon))}}{1 - e^{-\overline{\alpha} \kappa_{r,\upsilon}^{D \setminus D_\varepsilon}}},$$

with the bound on the right-hand side above being itself bounded below by a constant that does not depend on (r, υ). Substituting this back into (7.29), this proves (A1) with ν taken as Lebesgue measure on $D_\varepsilon \times V$ as before, t_0 can be sufficiently taken as $(\mathfrak{n} + 1)\varepsilon/2 + \text{diam}(D)/\mathrm{v}_{\min}$ and we may start with any initial configurations in $D \setminus D_\varepsilon \times V$. $\square$

In order to prove (A2) we require the following lemma, the proof of which will be given after that of (A2).

LEMMA 7.3. *For all $r \in D$ and $\upsilon \in V$, recalling that J_k denotes the kth jump time of the process (R, Υ), we have*

(7.31)
$$\mathbf{P}^\dagger_{(r,\upsilon)}(J_7 < \mathbf{k}, R_{J_7} \in dz) \leq C \mathbf{1}_{\{z \in D\}} \, dz,$$

for some constant $C > 0$, and

(7.32)
$$\mathbf{P}^\dagger_\nu(J_1 < \mathbf{k}, R_{J_1} \in dz) \geq c \mathbf{1}_{\{z \in D\}} \, dz,$$

for another constant $c > 0$, where ν, from the proof of (A1), is Lebesgue measure on $D_\varepsilon \times V$.

PROOF OF (A2). Again, we follow the proof given by the authors in [7]. Let $t \geq 7 \, \text{diam}(D)/\mathrm{v}_{\min}$ and note that on the event $\{\mathbf{k} > t\}$, we have $J_7 \leq 7 \, \text{diam}(D)/\mathrm{v}_{\min}$ almost surely. This inequality along with the strong Markov property imply that,

(7.33)
$$\mathbf{P}_{(r,\upsilon)}(t < \mathbf{k}) \leq \mathbf{E}^\dagger_{(r,\upsilon)}\left[\mathbf{1}_{\{J_7 < t\}} \mathbf{P}_{(R_{J_7}, \Upsilon_{J_7})}(t - s < \mathbf{k})_{s = J_7}\right]$$

$$\leq \mathbf{E}^\dagger_{(r,\upsilon)}\left[\mathbf{P}_{(R_{J_7}, \Upsilon_{J_7})}\left(t - \frac{7 \, \text{diam}(D)}{\mathrm{v}_{\min}} < \mathbf{k}\right)\right].$$

Since π is uniformly bounded above, conditional on $\{J_7 < \infty, R_{J_7} \in dz\}$, the density of Υ_{J_7} is bounded above by $\overline{\pi}$ multiplied by Lebesgue measure on V. Combining this with (7.31) and (7.33), we obtain

(7.34)
$$\mathbf{P}_{(r,\upsilon)}(t < \mathbf{k}) \leq C' \int_D \int_V \mathbf{P}_{(z,w)}\left(t - \frac{7 \, \text{diam}(D)}{\mathrm{v}_{\min}} < \mathbf{k}\right) dw \, dz,$$

for some $C' \in (0, \infty)$ Similarly, for $t \geq \text{diam}(D)/\mathrm{v}_{\min}$, equation (7.32), the fact that the inclusion $\{t < \mathbf{k}\} \subset \{J_1 \leq \text{diam}(D)/\mathrm{v}_{\min}\}$, the strong Markov property and the fact that π is

uniformly bounded below entail that,

$$\mathbf{P}_\nu(t < \mathrm{k}) = \mathbf{E}_\nu^\dagger\big[\mathbf{1}_{\{J_1 \leq \mathrm{k}\}}\mathbf{P}_{(R_{J_1}, \Upsilon_{J_1})}(t - s < \mathrm{k})_{s=J_1}\big]$$

$$\geq \mathbf{E}_\nu^\dagger\big[\mathbf{1}_{\{J_1 \leq \mathrm{k}\}}\mathbf{P}_{(R_{J_1}, \Upsilon_{J_1})}(t < \mathrm{k})\big]$$

$$\geq c' \int_D \int_V \mathbf{P}_{(z,w)}(t < \mathrm{k})\,\mathrm{d}w\,\mathrm{d}z,$$

for some $c' \in (0, \infty)$, where ν is Lebesgue measure on $D_\varepsilon \times V$. Putting (7.33) and (7.34) together, for all $t \geq 8\,\mathrm{diam}(D)/\mathrm{v}_{\min}$, we have

$$(7.35) \qquad \mathbf{P}_{(r,\upsilon)}(t < \mathrm{k}) \leq \frac{C'}{c'}\mathbf{P}_\nu\left(t - \frac{7\,\mathrm{diam}(D)}{\mathrm{v}_{\min}} < \mathrm{k}\right).$$

Now, recalling t_0 and ν from the proof of (A1), it follows from (A1) that

$$(7.36) \qquad \mathbf{P}_\nu^\dagger\big((R_{t_0}, \Upsilon_{t_0}) \in \cdot\big) \geq c_1\mathbf{P}_\nu(t_0 < \mathrm{k})\nu(\cdot).$$

The event $\{t < \mathrm{k}\}$ occurs if the particle has either been killed on the boundary of D or if it has been absorbed by fissile material, which occurs at rate $\bar{\beta} - \beta$. Since t_0 and ν are fixed, and $\bar{\beta} - \beta \leq \bar{\beta} + 1 < \infty$ by assumption, $\mathbf{P}_\nu(t_0 < \mathrm{k}) \geq K$ for some constant $K > 0$. Thus, keeping $t \geq 8\,\mathrm{diam}(D)/\mathrm{v}_{\min}$, using (7.36)

$$(7.37) \quad \begin{aligned} \mathbf{P}_\nu\left(t - \frac{7\,\mathrm{diam}(D)}{\mathrm{v}_{\min}} + t_0 < \mathrm{k}\right) &= \mathbf{E}_\nu\left[\mathbf{1}_{\{t_0 < \mathrm{k}\}}\mathbf{P}_{(R_{t_0}, \Upsilon_{t_0})}^\dagger\left(t - \frac{7\,\mathrm{diam}(D)}{\mathrm{v}_{\min}} < \mathrm{k}\right)\right] \\ &\geq \tilde{c}_1\mathbf{P}_\nu\left(t - \frac{7\,\mathrm{diam}(D)}{\mathrm{v}_{\min}} < \mathrm{k}\right), \end{aligned}$$

where $\tilde{c}_1 = Kc_1$.

Now define $N = \lceil 7\,\mathrm{diam}(D)/(\mathrm{v}_{\min}t_0)\rceil$. Then, for any $t > 0$, $t - 7\,\mathrm{diam}(D)/\mathrm{v}_{\min} + Nt_0 \geq t$ so that, trivially,

$$(7.38) \qquad \mathbf{P}_\nu(t < \mathrm{k}) \geq \mathbf{P}_\nu\left(t - \frac{7\,\mathrm{diam}(D)}{\mathrm{v}_{\min}} + Nt_0 < \mathrm{k}\right).$$

Applying (7.37) N times implies that

$$(7.39) \qquad \mathbf{P}_\nu(t < \mathrm{k}) \geq \tilde{c}_1^N\mathbf{P}_\nu\left(t - \frac{4\,\mathrm{diam}(D)}{\mathrm{v}_{\min}} < \mathrm{k}\right).$$

Combining this with (7.35) completes the proof of (A2). □

PROOF OF LEMMA 7.3. Let us first prove (7.31). Again, following the proof given in [7], we couple the neutron transport random walk in D with one on the whole of $\mathbb{R}^3$. Denote by $(\hat{R}_t, \hat{\Upsilon}_t)$ the neutron random walk in $\hat{D} = \mathbb{R}^3$, coupled with (R, Υ) such that $\hat{R}_t = R_t$ and $\hat{\Upsilon}_t = \Upsilon_t$ for all $t < \mathrm{k}$ and $(R_0, \Upsilon_0) = (\hat{R}_0, \hat{\Upsilon}_0) = (r, \upsilon)$, for $r \in D$, $\upsilon \in V$. Denote by $\hat{J}_1 < \hat{J}_2 < \cdots$ the jump times of $\hat{\Upsilon}_t$. Then for each $k \geq 1$ such that $J_k < \mathrm{k}$, we have $\hat{J}_k = J_k$. Due to the inequality

$$(7.40) \qquad \mathbf{E}_{(r,\upsilon)}^\dagger\big[f(R_{J_7}); J_7 < \mathrm{k}\big] \leq \mathbf{E}_{(r,\upsilon)}\big[f(\hat{R}_{\hat{J}_7})\big], \quad r \in D, \upsilon \in V,$$

we will consider the distribution of $\hat{R}_{\hat{J}_i}$ for $i \geq 2$. We first look at the case when $i = 2$. For $(r, \upsilon) \in D \times V$ and non-negative, bounded, measurable functions f,

$$(7.41) \quad \begin{aligned} \mathbf{E}_{(r,\upsilon)}\big[f(\hat{R}_{\hat{J}_2})\big] &= \mathbf{E}_{(r,\upsilon)}\big[f(r + \upsilon\hat{J}_1 + \hat{\Upsilon}_{\hat{J}_1}(\hat{J}_2 - \hat{J}_1))\big] \\ &\leq \bar{\alpha}^2\bar{\pi}\int_0^\infty \mathrm{d}j_1 \int_V \mathrm{d}\upsilon_1 \int_0^\infty \mathrm{d}j_2 e^{-\underline{\alpha}(j_1+j_2)} f(r + \upsilon j_1 + \upsilon_1 j_2). \end{aligned}$$

For j_1 fixed, we consider the integrals over υ_1 and j_2 in (7.41). Making the change of variables $\upsilon_1 \mapsto (\rho, \varphi, \theta)$, we have

$$
\begin{aligned}
(7.42) \qquad & \int_V d\upsilon_1 \int_0^\infty dj_2 e^{-\underline{\alpha} j_2} f(r + \upsilon j_1 + \upsilon_1 j_2) \\
& \leq \int_{v_{\min}}^1 d\rho \int_0^{2\pi} d\theta \int_0^\pi d\varphi \int_0^\infty dj_2 e^{-\underline{\alpha} j_2} f(r + \upsilon j_1 + \widetilde{\Theta}(\rho j_2, \theta, \varphi)) \rho^2 \sin\varphi,
\end{aligned}
$$

where $\widetilde{\Theta}$ was defined in (7.9). Now making the substitution $u = \rho j_2$ in (7.42),

$$
\begin{aligned}
(7.43) \qquad & \int_V d\upsilon_1 \int_0^\infty dj_2 e^{-\underline{\alpha} j_2} f(r + \upsilon j_1 + \upsilon_1 j_2) \\
& \leq \int_{v_{\min}}^{v_{\max}} d\rho \int_0^{2\pi} d\theta \int_0^\pi d\varphi \int_0^\infty du\, e^{-\underline{\alpha} u/\rho} f(r + \upsilon j_1 + \widetilde{\Theta}(u, \theta, \varphi)) \rho \sin\varphi \\
& \leq C \int_0^{2\pi} d\theta \int_0^\pi d\varphi \int_0^\infty du\, e^{-\underline{\alpha} u/v_{\max}} f(r + \upsilon j_1 + \widetilde{\Theta}(u, \theta, \varphi)) \sin\varphi,
\end{aligned}
$$

where $C = v_{\max}(v_{\max} - v_{\min})$. Making a final change of variables $(u, \theta, \varphi) \mapsto x \in \mathbb{R}^3$, we have

$$
(7.44) \qquad \int_V d\upsilon_1 \int_0^\infty dj_2 e^{-\underline{\alpha} j_2} f(r + \upsilon j_1 + \upsilon_1 j_2) \leq C \int_{\mathbb{R}^3} dx\, f(r + \upsilon j_1 + x) \frac{e^{-\underline{\alpha} |x|/v_{\max}}}{|x|^2}.
$$

Substituting this back into (7.41) yields

$$
(7.45) \qquad \mathbf{E}_{(r,\upsilon)}[f(\hat{R}_{\hat{J}_2})] \leq \bar{\alpha} K \int_0^\infty dj_1 e^{-\underline{\alpha} j_1} \int_{\mathbb{R}^3} dx\, f(r + \upsilon j_1 + x) \frac{e^{-\underline{\alpha}|x|/v_{\max}}}{|x|^2},
$$

where $K = \bar{\alpha} \bar{\pi} C$. Iterating this process over the next five jumps of the process gives

$$
\begin{aligned}
(7.46) \qquad & \mathbf{E}_{(r,\upsilon)}[f(\hat{R}_{\hat{J}_7})] \\
& \leq \bar{\alpha} K^6 \int_0^\infty dj_1 e^{-\underline{\alpha} j_1} \int_{\mathbb{R}^3} dx_1 \cdots \int_{\mathbb{R}^3} dx_6\, f(r + \upsilon j_1 + x_1 + \cdots + x_6) g(x_1) \cdots g(x_6),
\end{aligned}
$$

where $g(x) = e^{-\underline{\alpha}|x|/v_{\max}}/|x|^2$, $x \in \mathbb{R}^3$. Now, $g \in L^p(\mathbb{R}^3)$ for each $p < 3/2$ so that, in particular, $g \in L^{6/5}(\mathbb{R}^3)$. Hence, repeatedly applying Young's inequality implies that the six-fold convolution $*^6 g \in L^\infty(\mathbb{R}^3)$. (The reader will note that this is the fundamental reason we have focused our calculations around the 7th jump time J_7, rather than it being an arbitrary choice.) Making the substitution $x = x_1 + \cdots + x_6$,

$$
(7.47) \qquad \mathbf{E}^\dagger_{(r,\upsilon)}[f(\hat{R}_{\hat{J}_7})] \leq \bar{\alpha} K^6 \|*^6 g\|_\infty \int_0^\infty dj_1 e^{-\underline{\alpha} j_1} \int_{\mathbb{R}^3} dx_1 \cdots \int_{\mathbb{R}^3} dx_6\, f(r + \upsilon j_1 + x).
$$

Finally, setting $z = r + \upsilon j_1 + x$ yields

$$
(7.48) \qquad \mathbf{E}^\dagger_{(r,\upsilon)}[f(R_{J_7}); J_7 < \Bbbk] \leq \mathbf{E}_{(r,\upsilon)}[f(\hat{R}_{\hat{J}_7})] \leq C' \int_{\mathbb{R}^3} f(z) dz,
$$

where $C' = \bar{\alpha} K^6 \|g * \cdots * g\|_\infty$, which completes the proof of (7.31).

We now prove (7.32). For $r, r' \in \mathbb{R}^3$, let $[r, r']$ denote the line segment between r and r'. For all $f \in \mathcal{B}(\mathbb{R}^3)$, recalling the definition of ν from the proof of (A1) and using the usual bounds on α,

$$
\begin{aligned}
(7.49) \qquad & \mathbf{E}_\nu[f(R_{J_1}); J_1 < \Bbbk] \\
& \geq \int_{D_\varepsilon} \frac{dr}{\mathrm{Vol}(D_\varepsilon)} \int_V \frac{d\upsilon}{\mathrm{Vol}(V)} \int_0^\infty ds\, \mathbf{1}_{\{[r, r+s\upsilon] \subset D\}} \underline{\alpha} e^{-\bar{\alpha} s} f(r + s\upsilon),
\end{aligned}
$$

where $\mathrm{Vol}(D_\varepsilon) = \int_{D_\varepsilon} \mathrm{d}r$ and $\mathrm{Vol}(V) = \int_V \mathrm{d}\upsilon$. Following a similar method to those employed in the proof of Lemma 7.2 and (7.31) and changing first to polar coordinates via $\upsilon \mapsto (\rho, \theta, \varphi)$, followed by the substitution $u = s\rho$, and finally changing back to Cartesian coordinates via $(u, \theta, \varphi) \mapsto x$, the right-hand side of (7.49) is bounded below by

$$(7.50) \qquad C \int_{D_\varepsilon} \mathrm{d}r \int_{\mathbb{R}^3} \mathrm{d}x \mathbf{1}_{\{[r, r+x] \subset D\}} \frac{\alpha \mathrm{e}^{-\overline{\alpha}s/\mathrm{v}_{\min}}}{|x|^2} f(r+x),$$

where $C > 0$ is a constant. Making a final substitution of $x = z - r$, yields

$$(7.51) \qquad \mathbf{E}_\upsilon[f(R_{J_1}); J_1 < \mathrm{k}] \geq C \int_D \mathrm{d}z \mathbf{1}_{[r,z] \subset D} \frac{\alpha \mathrm{e}^{-\overline{\alpha}|z-r|/\mathrm{v}_{\min}}}{|z-r|^2} f(z)$$

$$\geq C \frac{\mathrm{v}_{\min}^2 \underline{\alpha} \mathrm{e}^{-\overline{\alpha} \,\mathrm{diam}(D)/\upsilon_{\min}^2}}{(\mathrm{diam}(D))^2} \int_D \mathrm{d}z \mathbf{1}_{\{[r,z] \subset D\}} f(z).$$

For all $z \in D \setminus D_\varepsilon$, (B1) and the discussion thereafter now imply that

$$(7.52) \qquad \int_{D_\varepsilon} \mathbf{1}_{\{[r,z] \subset D\}} \mathrm{d}r \geq \mathrm{Vol}(L_z) \geq \frac{\gamma}{2}(t_\varepsilon^2 - s_\varepsilon^2),$$

where s_ε and t_ε are defined in (B2), and L_z is defined in (7.4). On the other hand, for all $z \in D_\varepsilon$,

$$(7.53) \qquad \int_{D_\varepsilon} \mathbf{1}_{\{[r,z] \subset D\}} \mathrm{d}r \geq \mathrm{Vol}(D_\varepsilon \cap B(r, \varepsilon)).$$

Since the map $z \mapsto \mathrm{Vol}(D_\varepsilon \cap B(z, \varepsilon))$ is continuous and positive on the compact set $\bar{D}_\varepsilon$, the latter equation is uniformly bounded below by a strictly positive constant. It then follows that for every $z \in D$, the integral $\int_{D_\varepsilon} \mathrm{d}r \mathbf{1}_{\{[r,z] \subset D\}}$ is bounded below by a positive constant. Using this to bound the right-hand side of (7.51) yields the result. $\quad\square$

We thus have proved that the conclusions of Theorem 7.1 are valid under our assumptions. In order to conclude that Theorem 3.1 holds true, it remains to prove that φ is uniformly bounded away from 0 on each compactly embedded subset of $D \times V$ and the existence of a positive bounded density for the left eigenmeasure η.

LEMMA 7.4. *The right eigenfunction φ is uniformly bounded away from 0 on each compactly embedded subset of $D \times V$ and the probability measure η admits a positive density with respect to the Lebesgue measure on $D \times V$, which corresponds to the quantity $\tilde{\varphi}$ and which is uniformly bounded from above and a.e. uniformly bounded from below on each compactly embedded subset of $D \times V$.*

PROOF. For all $\varepsilon > 0$, we deduce from the eigenfunction property of φ (cf. Theorem 7.1) and from (7.20) that there exist a time $t_\varepsilon > 0$ and a constant $\tilde{C}_\varepsilon > 0$ such that

$$\varphi(r, \upsilon) = \mathrm{e}^{-\lambda_c t_\varepsilon} \mathrm{P}_{t_\varepsilon}[\varphi](r, \upsilon) \geq \mathrm{e}^{-\lambda t_\varepsilon} \tilde{C}_\varepsilon \int_{D_\varepsilon \times V} \varphi(z, w) \, \mathrm{d}z \, \mathrm{d}w > 0,$$

for all $(r, \upsilon) \in D_\varepsilon \times V$. It follows that φ is uniformly bounded away from 0 on each compactly embedded domain of $D \times V$.

Using the same notations as in the proof of Lemma 7.3, we consider the neutron transport random walk $(\hat{R}_t, \hat{\Upsilon}_t)$ in $\hat{D} = \mathbb{R}^3$, coupled with (R, Υ) such that $\hat{R}_t = R_t$ and $\hat{\Upsilon}_t = \Upsilon_t$ for all $t < \mathrm{k}$. We also denote by $\hat{J}_1 < \hat{J}_2 < \cdots$ the jump times of $(\hat{\Upsilon}_t)_{t \geq 0}$. Let $T \geq 0$ be a random time independent of $(\hat{R}, \hat{\Upsilon})$ with uniform law on $[\underline{T}, \bar{T}]$, where $\underline{T} < \bar{T}$ are fixed and $\underline{T} \geq 7 \,\mathrm{diam}(D)/\mathrm{v}_{\min}$. We first prove that the law of $(\hat{R}_T, \hat{\Upsilon}_T)$ after the 7th jump admits a

uniformly bounded density with respect to the Lebesgue measure. We conclude by using the coupling with (R, Υ) and the quasi-stationary property of η in (7.1).

For all $k \geq 7$ and for any positive, bounded and measurable function f vanishing outside of $D \times V$, we have

$$
\mathbf{E}[f(\hat{R}_T, \hat{\Upsilon}_T)\mathbf{1}_{\{\hat{J}_k \leq T < \hat{J}_{k+1}\}} \mid \hat{R}_0, \hat{\Upsilon}_0, T]
$$
$$
= \mathbf{E}[f(\hat{R}_0 + \hat{J}_1\hat{\Upsilon}_0 + \cdots + \hat{J}_k\hat{\Upsilon}_{k-1} + (T - \hat{J}_1 - \cdots - \hat{J}_k)\hat{\Upsilon}_k, \hat{\Upsilon}_k)\mathbf{1}_{\{\hat{J}_k \leq T < \hat{J}_{k+1}\}} \mid
$$
$$
\hat{R}_0, \hat{\Upsilon}_0, T]
$$
$$
= \int_0^T ds_1 \alpha(\hat{R}_0 + \upsilon_0 s_1, \upsilon_0) e^{-\int_0^{s_1} \alpha(\hat{R}_0 + \upsilon_0 u, \upsilon_0)\, du}
$$
$$
\times \int_V d\upsilon_1 \pi(r_0 + \upsilon_0 s_1, \upsilon_0, \upsilon_1)
$$
$$
\times \int_0^{T-s_1} ds_2 \alpha(\hat{R}_0 + \upsilon_0 s_1 + \upsilon_1 s_2, \upsilon_1) e^{-\int_0^{s_2} \alpha(\hat{R}_0 + \upsilon_0 s_1 + \upsilon_1 u, \upsilon_1)\, du}
$$
$$
\times \cdots
$$
$$
\times \int_V d\upsilon_{k-1} \pi(\hat{R}_0 + \upsilon_0 s_1 + \cdots + \upsilon_{k-2} s_{k-1}, \upsilon_{k-2}, \upsilon_{k-1})
$$
$$
\times \int_0^{T-s_1-\cdots-s_{k-1}} ds_k \alpha(\hat{R}_0 + \upsilon_0 s_1 + \cdots + \upsilon_{k-1} s_k, \upsilon_{k-1})
$$
$$
\times e^{-\int_0^{s_k} \alpha(\hat{R}_0 + \upsilon_0 s_1 + \cdots + \upsilon_{k-2} s_{k-1} + \upsilon_{k-1} u, \upsilon_{k-1})\, du}
$$
$$
\times \int_V d\upsilon_k \pi(\hat{R}_0 + \upsilon_0 s_1 + \cdots + \upsilon_{k-1} s_k, \upsilon_{k-1}, \upsilon_k)
$$
$$
\times e^{-\int_0^{T-s_1-\cdots-s_k} \alpha(\hat{R}_0 + \upsilon_0 s_1 + \cdots + \upsilon_{k-1} s_k + \upsilon_k u, \upsilon_k)\, du}
$$
$$
\times f(\hat{R}_0 + \upsilon_0 s_1 + \cdots + \upsilon_{k-1} s_k + \upsilon_k(t - s_1 - \cdots - s_k), \upsilon_k).
$$

Henceforth

$$
\mathbf{E}[f(\hat{R}_T, \hat{\Upsilon}_T)\mathbf{1}_{\{\hat{J}_k \leq T < \hat{J}_{k+1}\}} \mid \hat{R}_0, \hat{\Upsilon}_0, T]
$$
$$
\leq \bar{\alpha}^k \bar{\pi}^k e^{-T\underline{\alpha}} \int_0^T ds_1 \int_V d\upsilon_1 \cdots \int_0^{T-s_1-\cdots-s_{k-1}} ds_k \int_V d\upsilon_k
$$
$$
\times f(\hat{R}_0 + \upsilon_0 s_1 + \cdots + \upsilon_{k-1} s_k + \upsilon_k(T - s_1 - \cdots - s_k), \upsilon_k).
$$

Taking the expectation with respect to T, we obtain

$$
\mathbf{E}[f(\hat{R}_T, \hat{\Upsilon}_T)\mathbf{1}_{\{\hat{J}_k \leq T < \hat{J}_{k+1}\}} \mid \hat{R}_0, \hat{\Upsilon}_0]
$$
$$
\leq \frac{\bar{\alpha}^k \bar{\pi}^k}{\bar{T}} \int_0^{\bar{T}} dt \int_0^t ds_1 \int_V d\upsilon_1 \cdots \int_0^{t-s_1-\cdots-s_{k-1}} ds_k \int_V d\upsilon_k
$$
$$
\times f(\hat{R}_0 + \upsilon_0 s_1 + \cdots + \upsilon_{k-1} s_k + \upsilon_k(t - s_1 - \cdots - s_k), \upsilon_k).
$$

Using the change of variable $(u_1, \ldots, u_k, u_{k+1}) = (s_1, \ldots, s_k, t - s_1 - \cdots - s_k)$ yields

$$\mathbf{E}[f(\hat{R}_T, \hat{\Upsilon}_T)\mathbf{1}_{\{\hat{j}_k \le T < \hat{j}_{k+1}\}} \mid \hat{R}_0, \hat{\Upsilon}_0]$$

$$\le \frac{\bar{\alpha}^k \bar{\pi}^k}{\bar{T}} \int_{[0,\bar{T}]^{k+1}} du \mathbf{1}_{0 \le u_1 + \cdots + u_{k+1} \le \bar{T}} \int_{V^k} d\upsilon$$

$$\times f(\hat{R}_0 + \upsilon_0 u_1 + \cdots + \upsilon_{k-1} u_k + \upsilon_k u_{k+1}, \upsilon_k).$$

The same approach as in Lemma 7.3 shows that there exists a constant $C > 0$ (which does not depend on $\hat{R}_0$ nor on $\hat{\Upsilon}_0$) such that, for all measurable function $g : \mathbb{R}^3 \to [0, \infty)$,

$$\int_{[0,\bar{T}]^7} du \int_{V^6} d\upsilon g(\hat{R}_0 + \hat{\Upsilon}_0 u_1 + \cdots + \upsilon_6 u_7) \le C \int_{\mathbb{R}^d} dx g(x).$$

Hence,

$$\mathbf{E}[f(\hat{R}_T, \hat{\Upsilon}_T)\mathbf{1}_{\{\hat{j}_k \le T < \hat{j}_{k+1}\}} \mid \hat{R}_0, \hat{\Upsilon}_0]$$

$$\le \frac{C\bar{\alpha}^k \bar{\pi}^k}{\bar{T}} \int_{[0,\bar{T}]^{k+1-7}} du \mathbf{1}_{0 \le u_8 + \cdots + u_{k+1} \le \bar{T}} \int_{V^{k-6}} d\upsilon$$

$$\times \int_{\mathbb{R}^3} dx f(x + \upsilon_7 u_8 + \cdots + \upsilon_k u_{k+1}, \upsilon_k)$$

$$= \frac{C\bar{\alpha}^k \bar{\pi}^k}{\bar{T}} \int_{[0,\bar{T}]^{k+1-7}} du \mathbf{1}_{0 \le u_8 + \cdots + u_{k+1} \le \bar{T}} \int_{V^{k-6}} d\upsilon \int_{\mathbb{R}^3} dy f(y, \upsilon_k)$$

$$= C\bar{\alpha}^k \bar{\pi}^k \operatorname{Vol}(V)^{k-8} \frac{\bar{T}^{k+1-8}}{(k+1-7)!} \int_D dy \int_V d\upsilon_k f(y, \upsilon_k),$$

where we used the change of variable $y = x + \upsilon_7 u_8 + \cdots + \upsilon_k u_{k+1}$ and the fact that f vanishes outside $D \times V$. Summing over $k \ge 7$, we deduce that there exists a constant $C' > 0$ (which only depends on C, $\bar{\alpha}$, $\bar{\pi}$ and $\bar{T}$) such that

$$\mathbf{E}[f(\hat{R}_T, \hat{\Upsilon}_T)\mathbf{1}_{\{\hat{j}_7 \le T\}} \mid \hat{R}_0, \hat{\Upsilon}_0] \le C' \int_D dy \int_V d\upsilon f(y, \upsilon).$$

Similarly as in the proof of (A2), we chose $\underline{T} \ge 7 \operatorname{diam}(D)/\mathrm{v}_{\min}$, so that, on the event $\{\mathrm{k} > T\}$, we have $J_7 \le 7 \operatorname{diam}(D)/\mathrm{v}_{\min} \le T$ almost surely. Hence, we obtain that, for any $(r_0, \upsilon_0) \in D \times V$,

$$\mathbf{E}^{\dagger}_{(r_0, \upsilon_0)}[f(R_T, \Upsilon_T); T < \mathrm{k}] = \mathbf{E}^{\dagger}_{(r_0, \upsilon_0)}[f(R_T, \Upsilon_T); T < \mathrm{k}, J_7 \le T]$$

$$\le \mathbf{E}_{(r_0, \upsilon_0)}[f(\hat{R}_T, \hat{\Upsilon}_T); \hat{J}_7 \le T]$$

$$\le C' \int_D dy \int_V d\upsilon f(y, \upsilon).$$

Integrating with respect to η and using the quasi-stationary property (7.1) and Fubini's Theorem (recall that T and the process (R, Υ) are independent), we obtain

(7.54)
$$\frac{1}{\bar{T} - \underline{T}} \int_{\underline{T}}^{\bar{T}} dt e^{\lambda_c t} \eta[f] = \frac{1}{\bar{T} - \underline{T}} \int_{\underline{T}}^{\bar{T}} dt \, \mathbf{E}^{\dagger}_{\eta}[f(R_t, \Upsilon_t); t < \mathrm{k}]$$

$$= \mathbf{E}^{\dagger}_{\eta}[f(R_T, \Upsilon_T); T < \mathrm{k}]$$

$$\le C' \int_D dy \int_V d\upsilon f(y, \upsilon).$$

Since f was chosen arbitrarily, this proves that η admits a uniformly bounded density (from above) with respect to the Lebesgue measure on $D \times V$.

Finally, using the quasi-stationarity of η from (7.1) and integrating inequality (7.19) with respect to η implies that (here the time t and the constants k, C_ε, c_ε depend on ε as in inequality (7.20)), for all bounded measurable functions f on $D \times E$,

$$e^{\lambda_c t} \int_{D \times V} f(x) \eta(\mathrm{d}x) = \mathbf{E}_\eta^\dagger[f(R_t, \Upsilon_t); t < \mathrm{k}]$$

$$\geq \eta(D_\varepsilon \times V) C_\varepsilon c_\varepsilon^k \int_{D_\varepsilon \times V} f(z, w) \, \mathrm{d}z \, \mathrm{d}w.$$

This implies that $\tilde{\varphi}$ is a.e. lower bounded by $e^{-\lambda_c t} \eta(D_\varepsilon \times V) C_\varepsilon c_\varepsilon^k$ on $D_\varepsilon \times V$. Since this inequality can be proved for any $\varepsilon > 0$ small enough, one deduces that, on any subset $D_\varepsilon \times V$ with $\varepsilon > 0$ and hence on any compactly embedded subset of $D \times V$, $\tilde{\varphi}$ is a.e. uniformly bounded away from zero. $\quad \square$

8. Proof of Theorem 6.1.

There are three main steps to the proof. The first is to characterise the law of transitions of the Markov process $(X, \mathbb{P}^\varphi)$, defined in the change of measure (6.1); note that the latter ensures the Markov property is preserved. The second step is to show that they agree with those of $(X^\varphi, \tilde{\mathbb{P}}^\varphi)$. The third step is to show that $(X^\varphi, \tilde{\mathbb{P}}^\varphi)$ is Markovian. Together these three imply the statement of the theorem.

Step 1. Next we look at the multiplicative semigroup which characterises uniquely the transitions of $(X^\varphi, \mathbb{P}^\varphi)$ (cf. [22–24])

$$(8.1) \qquad u_t^\varphi[g](r, \upsilon) := \mathbb{E}_{\delta_{(r,\upsilon)}} \left[W_t \prod_{i=1}^{N_t} g(R_t^i, \Upsilon_t^i) \right] = \mathbb{E}_{\delta_{(r,\upsilon)}} \left[e^{-\lambda_* t} \frac{\langle \varphi, X_t \rangle}{\varphi(r, \upsilon)} e^{\langle \log g, X_t \rangle} \right]$$

for $t \geq 0$ and $g \in L_\infty^+(D \times V)$ which is uniformly bounded by unity. Note, we keep to the convention that an empty product is understood as 1, however we also define the empty inner product as zero (corresponding to all functions scoring zero when particles arrive at the cemetery state $\{\dagger\}$). As such, if we are to extend the domain of test functions in the product to include the cemetery state $\{\dagger\}$, we need to insist on the default value $g(\{\dagger\}) = 1$; see [22–24].

We start in the usual way by splitting the expectation in the second equality of (8.1) according to whether a scattering or fission event occurs. (The reader may wish to recall the role of the quantities $\sigma_\mathrm{s}, \sigma_\mathrm{f}, \sigma = \sigma_\mathrm{s} + \sigma_\mathrm{f}, \pi_\mathrm{s}$ and σ_f in (1.1)). We get

$$
\begin{aligned}
u_t^\varphi[g](r, \upsilon) = {}& g\big(r + \upsilon(t \wedge \kappa_{(r,\upsilon)}^D), \upsilon\big) \frac{\varphi(r + \upsilon t, \upsilon)}{\varphi(r, \upsilon)} e^{-\int_0^t \lambda_* + \sigma(r + \upsilon s, \upsilon) \, \mathrm{d}s} \mathbf{1}_{\{t < \kappa_{(r,\upsilon)}^D\}} \\
& + \int_0^{t \wedge \kappa_{(r,\upsilon)}^D} \sigma_\mathrm{s}(r + \upsilon s, \upsilon) e^{-\int_0^s \lambda_* + \sigma(r + \upsilon \ell, \upsilon) \, \mathrm{d}\ell} \frac{\varphi(r + \upsilon s, \upsilon)}{\varphi(r, \upsilon)} \\
& \times \int_V u_{t-s}^\varphi[g](r + \upsilon s, \upsilon') \frac{\varphi(r + \upsilon s, \upsilon')}{\varphi(r + \upsilon s, \upsilon)} \pi_\mathrm{s}(r + \upsilon s, \upsilon, \upsilon') \, \mathrm{d}\upsilon' \\
& + \int_0^{t \wedge \kappa_{(r,\upsilon)}^D} \sigma_\mathrm{f}(r + \upsilon s, \upsilon) e^{-\int_0^s \lambda_* + \sigma(r + \upsilon \ell, \upsilon) \, \mathrm{d}\ell} \frac{\varphi(r + \upsilon s, \upsilon)}{\varphi(r, \upsilon)} \mathcal{E}_{(r + \upsilon s, \upsilon)} \\
& \otimes \mathbb{E}_{\delta_{(r,\upsilon)}} \left[\sum_{i=1}^N \frac{\varphi(r + \upsilon s, \upsilon_i)}{\varphi(r + \upsilon s, \upsilon)} W_{t-s}^i(r + \upsilon s, \upsilon_i) \prod_{j=1}^N e^{\langle \log g, X_{t-s}^j(r + \upsilon s, \upsilon_i) \rangle} \right],
\end{aligned}
$$

(8.2)

where, for $r \in D$, $v \in V$, $W^i(r, \upsilon)$ and $X^i(r, \upsilon)$ are independent copies of the pair W and X under $\mathbb{P}_{\delta_{(r,\upsilon)}}$. Note that the first term on the right-hand side of (8.2) contains includes

$g(r + \upsilon(t \wedge \kappa^D_{(r,\upsilon)}), \upsilon)$ to account for the fact that $g(\{\dagger\}) = 1$. Before developing the right-hand side above any further, we need to make two additional observations and to introduce some more notation.

The first observation is that, since W is a martingale, by sampling at the time of the first scattering event, fission event or when it leaves the domain D, whichever happens first, thanks to Doob's Optional Sampling Theorem, its mean must remain equal to 1 and we get the functional equation

$$
\begin{aligned}
\varphi(r, \upsilon) &= \varphi(r + \upsilon t, \upsilon) e^{-\int_0^t \lambda_* + \sigma(r+\upsilon s, \upsilon)\, ds} \mathbf{1}_{\{t < \kappa^D_{r,\upsilon}\}} \\
&\quad + \int_0^{t \wedge \kappa^D_{r,\upsilon}} e^{-\int_0^s \lambda_* + \sigma(r+\upsilon\ell, \upsilon)\, d\ell} \sigma_{\mathrm{f}}(r + \upsilon s, \upsilon) \\
&\quad \times \int_V \frac{\varphi(r + \upsilon s, \upsilon')}{\varphi(r + \upsilon s, \upsilon)} \pi_{\mathrm{f}}(r + \upsilon s, \upsilon, \upsilon') \varphi(r + \upsilon s, \upsilon)\, ds \\
&\quad + \int_0^{t \wedge \kappa^D_{r,\upsilon}} e^{-\int_0^s \lambda_* + \sigma(r+\upsilon\ell, \upsilon)\, d\ell} \sigma_{\mathrm{s}}(r + \upsilon s, \upsilon) \\
&\quad \times \int_V \frac{\varphi(r + \upsilon s, \upsilon')}{\varphi(r + \upsilon s, \upsilon)} \pi_{\mathrm{s}}(r + \upsilon s, \upsilon, \upsilon') \varphi(r + \upsilon s, \upsilon)\, ds
\end{aligned}
$$

(8.3)

for $r \in D$, $\upsilon \in V$. Now appealing to Lemma 1.2, Chapter 4 in [16], to treat the last two terms of (8.3) as potentials, with a little bit of algebra we can otherwise write the above as

$$
\varphi(r, \upsilon) = \varphi(r + \upsilon t, \upsilon) \exp\left\{ \int_0^t \frac{(\mathrm{S} + \mathrm{F} - \lambda_* \mathrm{I})\varphi(r + \upsilon s, \upsilon)}{\varphi(r + \upsilon s, \upsilon)}\, ds \right\},
$$

for $t < \kappa^D_{r,\upsilon}$, where I is the identity operator, which is to say, for $t < \kappa^D_{r,\upsilon}$,

(8.4)
$$
\begin{aligned}
&\frac{\varphi(r + \upsilon t, \upsilon)}{\varphi(r, \upsilon)} e^{-\int_0^t \lambda_* + \sigma(r+\upsilon s, \upsilon)\, ds} \\
&= \exp\left\{ -\int_0^t \frac{(\mathrm{S} + \mathrm{F} + \sigma \mathrm{I})\varphi(r + \upsilon s, \upsilon)}{\varphi(r + \upsilon s, \upsilon)}\, ds \right\}.
\end{aligned}
$$

Our second observation pertains to the manipulation of the expectation on the right-hand side of (8.2). Define for $g \in L^+_\infty(D \times V)$, $(r, \upsilon) \in D \times V$ and $t \geq 0$,

(8.5)
$$
u_t[g](r, \upsilon) := \mathbb{E}_{\delta_{(r,\upsilon)}}\left[\prod_{i=1}^{N_t} g(R^i_t, \Upsilon^i_t) \right].
$$

We have that for all $(r, \upsilon) \in D \times V$,

$$
\sigma_{\mathrm{f}}(r, \upsilon)\mathcal{E}_{(r+\upsilon s, \upsilon)} \otimes \mathbb{E}_{\delta_{(r,\upsilon)}}\left[\sum_{i=1}^N \frac{\varphi(r, \upsilon_i)}{\varphi(r, \upsilon)} W^i_{t-s}(r, \upsilon_i) \prod_{j=1}^N e^{\langle \log g, X^j_{t-s}(r, \upsilon_i) \rangle} \right]
$$

$$
= \sigma_{\mathrm{f}}(r, \upsilon)\mathcal{E}_{(r+\upsilon s, \upsilon)}
$$

$$
\otimes \mathbb{E}_{\delta_{(r,\upsilon)}}\left[\frac{\langle \varphi, \mathcal{Z} \rangle}{\varphi(r, \upsilon)} \sum_{i=1}^N \frac{\varphi(r, \upsilon_i)}{\langle \varphi, \mathcal{Z} \rangle} W^i_{t-s}(r, \upsilon_i) e^{\langle \log g, X^i_{t-s}(r, \upsilon_i) \rangle} \prod_{\substack{j=1 \\ i \neq j}}^N e^{\langle \log g, X^j_{t-s}(r, \upsilon_i) \rangle} \right]
$$

(8.6)
$$
= \sigma_{\mathrm{f}}(r, \upsilon) \frac{\mathcal{E}_{(r,\upsilon)}[\langle \varphi, \mathcal{Z} \rangle]}{\varphi(r, \upsilon)}
$$

$$\times \mathcal{E}_{(r,\upsilon)}\left[\frac{\langle \varphi, \mathcal{Z}\rangle}{\mathcal{E}_{(r,\upsilon)}[\langle \varphi, \mathcal{Z}\rangle]} \sum_{i=1}^{N} \frac{\varphi(r,\upsilon_i)}{\langle \varphi, \mathcal{Z}\rangle} u_{t-s}^{\varphi}[g](r,\upsilon_i) \prod_{\substack{j=1 \\ i\neq j}}^{N} u_{t-s}[g](r,\upsilon_j)\right]$$

$$= G_{\mathrm{f}}^{\varphi}[u_{t-s}^{\varphi}[g], u_{t-s}[g]](r,\upsilon) + \frac{(\mathrm{F}+\sigma_{\mathrm{f}}\mathrm{I})\varphi(r,\upsilon)}{\varphi(r,\upsilon)} u_{t-s}^{\varphi}[g](r,\upsilon),$$

where in the penultimate equality we have taken expectations conditional on the fission event and

(8.7)
$$G_{\mathrm{f}}^{\varphi}[f, g](r,\upsilon) := \frac{(\mathrm{F}+\sigma_{\mathrm{f}}\mathrm{I})\varphi(r,\upsilon)}{\varphi(r,\upsilon)} \mathcal{E}_{(r,\upsilon)}^{\varphi}\left[\sum_{i=1}^{N} \frac{\varphi(r,\upsilon_i)}{\langle \varphi, \mathcal{Z}\rangle} f(r,\upsilon_i) \prod_{\substack{j=1 \\ i\neq j}}^{N} g(r,\upsilon_j)\right]$$

$$- \frac{(\mathrm{F}+\sigma_{\mathrm{f}}\mathrm{I})\varphi(r,\upsilon)}{\varphi(r,\upsilon)} f(r,\upsilon)$$

for $f, g \in L_{\infty}^{+}(D \times V)$, which are uniformly bounded by unity, and for $r \in D$, $\upsilon \in V$, where we recall that $\mathcal{P}_{(r,\upsilon)}^{\varphi}$ was defined in (6.3). Note in particular that

(8.8)
$$\frac{\mathcal{E}_{(r,\upsilon)}[\langle \varphi, \mathcal{Z}\rangle]}{\varphi(r,\upsilon)} = \int_{V} \frac{\varphi(r,\upsilon')}{\varphi(r,\upsilon)} \pi_{\mathrm{f}}(r,\upsilon,\upsilon')\,\mathrm{d}\upsilon' = \frac{(\mathrm{F}+\sigma_{\mathrm{f}}\mathrm{I})\varphi(r,\upsilon)}{\sigma_{\mathrm{f}}(r,\upsilon)\varphi(r,\upsilon)}.$$

We will also make use of the notation

(8.9)
$$G_{\mathrm{f}}[f](r,\upsilon) = \sigma_{\mathrm{f}}(r,\upsilon)\mathcal{E}_{(r,\upsilon)}\left[\prod_{j=1}^{N} g(r,\upsilon_j) - g(r,\upsilon)\right],$$

for $r \in D$, $\upsilon \in V$ and $g \in L_{\infty}^{+}(D \times V)$, which is uniformly bounded by unity. Recall that the empty product in the definition (8.5) is defined as unity.

In a similar manner to (8.2) we can break the expectation over the event of scattering or fission in (8.5), which defines of $u_t[g]$, to see that the operator G_{f} appears in the decomposition

(8.10)
$$u_t[g](r,\upsilon) = \hat{\mathrm{U}}_t[g] + \int_0^t \mathrm{U}_s[\mathrm{S}u_{t-s}[g] + G_{\mathrm{f}}[u_{t-s}[g]]\,\mathrm{d}s, \quad t \geq 0,$$

for $g \in L_{\infty}^{+}(D \times V)$, which is uniformly bounded by unity. Here, we have adjusted the definition of the semigroup U to

(8.11)
$$\hat{\mathrm{U}}_t[g](r,\upsilon) = g(r + \upsilon(t \wedge \kappa_{r,\upsilon}^{D}),\upsilon), \quad t \geq 0, r \in D, \upsilon \in V.$$

Now returning to (8.2) with the above observations and definitions in hand, while again appealing to Lemma 1.2, Chapter 4 in [16], we have

$$u_t^{\varphi}[g](r,\upsilon) = g(r + \upsilon(t \wedge \kappa_{r,\upsilon}^{D}),\upsilon)$$

$$+ \int_0^{t \wedge \kappa_{(r,\upsilon)}^{D}} \sigma_{\mathrm{s}}(r + \upsilon s,\upsilon) \int_V u_{t-s}^{\varphi}[g](r + \upsilon s,\upsilon')$$

$$\times \frac{\varphi(r + \upsilon s,\upsilon')}{\varphi(r + \upsilon s,\upsilon)} \pi_{\mathrm{s}}(r + \upsilon s,\upsilon,\upsilon')\,\mathrm{d}\upsilon'\,\mathrm{d}s$$

$$+ \int_0^{t \wedge \kappa_{(r,\upsilon)}^{D}} G_{\mathrm{f}}^{\varphi}[u_{t-s}^{\varphi}[g], u_{t-s}[g]](r + \upsilon s,\upsilon)$$

$$+ \frac{(\mathrm{F}+\sigma_{\mathrm{f}}\mathrm{I})\varphi(r + \upsilon s,\upsilon)}{\varphi(r + \upsilon s,\upsilon)} u_{t-s}^{\varphi}[g](r + \upsilon s,\upsilon)\,\mathrm{d}s$$

$$(8.12) \qquad -\int_0^{t \wedge \kappa_{(r,\upsilon)}^D} \frac{(S + F + \sigma I)\varphi(r + \upsilon s, \upsilon)}{\varphi(r + \upsilon s, \upsilon)} u_{t-s}^\varphi[g](r + \upsilon s, \upsilon)\, ds$$

$$(8.13) \qquad = \hat{U}_t[g] + \int_0^t U_s[S_\varphi u_{t-s}^\varphi[g]]\, ds + \int_0^t U_s[G_{\mathrm{f}}^\varphi[u_{t-s}^\varphi[g], u_{t-s}[g]]]\, ds,$$

where

$$S_\varphi f(r, \upsilon) := \int_V [f(r, \upsilon') - f(r, \upsilon)] \sigma_{\mathrm{s}}(r, \upsilon) \frac{\varphi(r, \upsilon')}{\varphi(r, \upsilon)} \pi_{\mathrm{s}}(r, \upsilon, \upsilon')\, d\upsilon'$$

on $D \times V$ and otherwise equal to zero,

Step 2. Define

$$(8.14) \qquad \tilde{u}_t^\varphi[g](r, \upsilon) = \tilde{\mathbb{E}}_{\delta_{(r,\upsilon)}}^\varphi \left[\prod_{i=1}^{N_t} g(R_t^i, \Upsilon_t^i) \right], \quad t \geq 0,$$

for $g \in L_\infty^+(D \times V)$, where $\{(R_t^i, \Upsilon_t^i) : i = 1, \ldots, N_t\}$ are the physical configurations of the particles alive in the system at time $t \geq 0$ in X^φ.

By conditioning $\tilde{u}_t^\varphi$ on the first time a scattering of fission event occurs, it is a straightforward exercise to show that it also solves (8.13). For the sake of brevity, we leave this as an exercise to the reader as the arguments are similar to those mentioned previously. In order to show that (8.13) has a unique solution, we consider $v_t^\varphi[g] := \varphi u_t^\varphi[g]$ and $\tilde{v}_t^\varphi[g] := \varphi \tilde{u}_t^\varphi[g]$. Since u_t^φ and $\tilde{u}_t^\varphi$ both satisfy (8.12), applying [16], Lemma 1.2, Chapter 4, along with (8.4), it is straightforward to show that v_t^φ and $\tilde{v}_t^\varphi$ both satisfy

$$v_t[g](r, \upsilon) = g\big(r + \upsilon(t \wedge \kappa_{(r,\upsilon)}^D), \upsilon\big) \varphi(r + \upsilon t, \upsilon) \mathrm{e}^{-\int_0^t \lambda_* + \sigma(r + \upsilon s, \upsilon)\, ds} \mathbf{1}_{\{t < \kappa_{(r,\upsilon)}^D\}}$$

$$(8.15) \qquad + \int_0^{t \wedge \kappa_{(r,\upsilon)}^D} \mathrm{e}^{-\int_0^s \lambda_* + \sigma(r + \upsilon l, \upsilon)\, dl} U_s\big[(S + \sigma_{\mathrm{s}}) v_{t-s}[g]\big]\, ds$$

$$\qquad + \int_0^{t \wedge \kappa_{(r,\upsilon)}^D)} \mathrm{e}^{-\int_0^s \lambda_* + \sigma(r + \upsilon l, \upsilon)\, dl}$$

$$\qquad \times U_s\big[\tilde{G}_{\mathrm{f}}[v_{t-s}[g], u_{t-s}[g]] + \sigma_{\mathrm{f}} v_{t-s}[g]\big]\, ds,$$

where

$$\tilde{G}_{\mathrm{f}}[f, g](r, \upsilon) = \sigma_{\mathrm{f}}(r, \upsilon) \left\{ \mathcal{E}_{(r,\upsilon)} \left[\sum_{i=1}^N f(r, \upsilon_i) \prod_{\substack{j=1 \\ j \neq i}}^N g(r, \upsilon_j) \right] - f(r, \upsilon) \right\}.$$

Due to the assumptions (H1) and (H4), an application of Grönwall's inequality implies uniqueness of (8.15), which in turn implies uniqueness of (8.13). We leave this as an exercise to the reader as it is a relatively standard computation and very similar to the calculations given in [10].

Step 3. We start by noting that the joint process $(X^\varphi, (R^\varphi, \Upsilon^\varphi))$ is, by construction, Markovian under $\tilde{\mathbb{P}}^\varphi$, we thus need to show that the marginalisation of the coupled system to just X^φ retains the Markov property. We do this by showing that for $f \in L_\infty^+(D \times V)$, $\mu \in \mathcal{M}(D \times V)$ and $(r, \upsilon) \in D \times V$,

$$(8.16) \qquad \tilde{\mathbb{E}}_\mu^\varphi[f(R_t^\varphi, \Upsilon_t^\varphi) | X_t^\varphi] = \frac{\langle f\varphi, X_t^\varphi \rangle}{\langle \varphi, X_t^\varphi \rangle}, \quad t \geq 0.$$

This says that knowing X_t^φ only allows one to construct the law of $(R_t^\varphi, \Upsilon_t^\varphi)$ through an empirical distribution using φ. Hence, for $g \in L_\infty^+(D \times V)$ which is bounded by unity and

$\mu \in \mathcal{M}(D \times V)$,

$$\tilde{\mathbb{E}}^{\varphi}_{\mu}\big[\mathrm{e}^{\langle \log g, X^{\varphi}_{t+s}\rangle}|\mathcal{F}_t\big] = \tilde{\mathbb{E}}^{\varphi}_{\mu}\Bigg[\sum_{i=1}^{N_t} \frac{\varphi(R^i_t, \Upsilon^i_t)}{\langle \varphi, X^{\varphi}_t\rangle} \tilde{\mathbb{E}}^{\varphi}_{\mu',(r,\upsilon)}\big[\mathrm{e}^{\langle \log g, X^{\varphi}_s\rangle}\big]_{\mu'=X^{\varphi}_t,(r,\upsilon)(R^i_t,\Upsilon^i_t)}\Bigg]$$

$$= \tilde{\mathbb{E}}^{\varphi}_{\mu}\big[\tilde{\mathbb{E}}^{\varphi}_{\mu'}\big[\mathrm{e}^{\langle \log g, X^{\varphi}_s\rangle}\big]_{\mu'=X^{\varphi}_t}\big],$$

where we have written $X^{\varphi}_t = \sum_{i=1}^{N_t} \delta_{(R^i_t, \Upsilon^i_t)}$, and thus the Markov property of X^{φ}, $\tilde{\mathbb{P}}^{\varphi}$ follows.

We are thus left with proving (8.16) to complete this step. To do so we note that it suffices to show that for $f, g \in L^+_{\infty}(D \times V)$ bounded by unity, $\mu \in \mathcal{M}(D \times V)$ and $(r, \upsilon) \in D \times V$,

$$(8.17) \qquad \tilde{\mathbb{E}}^{\varphi}_{\mu}\big[f(R^{\varphi}_t, \Upsilon^{\varphi}_t)\mathrm{e}^{\langle \log g, X^{\varphi}_t\rangle}\big] = \tilde{\mathbb{E}}^{\varphi}_{\mu}\Bigg[\frac{\langle f\varphi, X^{\varphi}_t\rangle}{\langle \varphi, X^{\varphi}_t\rangle}\mathrm{e}^{\langle \log g, X^{\varphi}_t\rangle}\Bigg], \quad t \geq 0.$$

On the left-hand side of (8.17), we have

$$\tilde{\mathbb{E}}^{\varphi}_{\mu}\big[f(R^{\varphi}_t, \Upsilon^{\varphi}_t)\mathrm{e}^{\langle \log g, X^{\varphi}_t\rangle}\big]$$

$$= \tilde{\mathbb{E}}^{\varphi}_{\mu}\big[\tilde{\mathbb{E}}^{\varphi}_{\mu}\big[f(R^{\varphi}_t, \Upsilon^{\varphi}_t)\mathrm{e}^{\langle \log g, X^{\varphi}_t\rangle}|R^{\varphi}_t, \Upsilon^{\varphi}_t\big]\big]$$

$$= \sum_{k=1}^{n} \frac{\varphi(r_k, \upsilon_k)}{\langle \varphi, \mu\rangle} \tilde{\mathbb{E}}^{\varphi}_{\delta_{(r_k,\upsilon_k)}}\Bigg[f(R^{\varphi}_t, \Upsilon^{\varphi}_t) \prod_{i\geq 1:T_i\leq t} \prod_{j=1}^{N^i} \frac{u_{t-T_i}[g](R^{\varphi}_{T_i}, \upsilon^i_j)}{u_{t-T_i}[g](R^{\varphi}_{T_i}, \Upsilon^{\varphi}_{T_i})}\Bigg],$$

where $\mu = \sum_{k=1}^{n} \delta_{(r_i, \upsilon_i)}$, T_i, $i \geq 1$ are the times of fission along the spine at which point N^i particles are issued at υ^i_j, $j = 1 \cdots, N^i_j$. On the right-hand side of of (8.17), we may appeal to Step 1 and Step 2 to deduce that

$$\tilde{\mathbb{E}}^{\varphi}_{\mu}\Bigg[\frac{\langle f\varphi, X^{\varphi}_t\rangle}{\langle \varphi, X^{\varphi}_t\rangle}\mathrm{e}^{\langle \log g, X^{\varphi}_t\rangle}\Bigg] = \mathrm{e}^{-\lambda_* t}\mathbb{E}_{\mu}\Bigg[\frac{\langle f\varphi, X_t\rangle}{\langle \varphi, \mu\rangle}\mathrm{e}^{\langle \log g, X_t\rangle}\Bigg]$$

$$= \sum_{i=1}^{n} \frac{\varphi(r_i, \upsilon_i)}{\langle \varphi, \mu\rangle}\mathrm{e}^{-\lambda_* t}\mathbb{E}_{\delta_{(r_i,\upsilon_i)}}\Bigg[\frac{\langle f\varphi, X_t\rangle}{\varphi(r_i, \upsilon_i)}\mathrm{e}^{\langle \log g, X_t\rangle}\Bigg].$$

The proof of this final step is thus complete as soon as we can show that

$$(8.18) \qquad \begin{aligned} &\tilde{\mathbb{E}}^{\varphi}_{\delta_{(r_k,\upsilon_k)}}\Bigg[f(R^{\varphi}_t, \Upsilon^{\varphi}_t) \prod_{1\leq i:T_i\leq t} \prod_{j=1}^{N^i} \frac{u_{t-T_i}[g](R^{\varphi}_{T_i}, \upsilon^i_j)}{u_{t-T_i}[g](R^{\varphi}_{T_i}, \Upsilon^{\varphi}_{T_i})}\Bigg] \\ &= \mathrm{e}^{-\lambda_* t}\mathbb{E}_{\delta_{(r_i,\upsilon_i)}}\Bigg[\frac{\langle f\varphi, X_t\rangle}{\varphi(r_i, \upsilon_i)}\mathrm{e}^{\langle \log g, X_t\rangle}\Bigg]. \end{aligned}$$

To this end, we note that splitting the expectation on the right-hand of (8.18) side at either a scattering or fission event results in a calculation that is almost identical to the one above that concludes with (8.13). More precisely, the expectation on the right-hand side solves (8.13) albeit the role of $\hat{\mathsf{U}}_t[g]$ is replaced by $\mathsf{U}_t[fg]$. Similarly splitting the expectation on the left-hand side of (8.18) also results in a solution to (8.13) (with the aforementioned adjustment). The uniqueness of (8.13), with $\hat{\mathsf{U}}$ replaced by U, follows from the same arguments and hence the equality in (8.18) now follows, as required.

9. Proof of Lemma 6.1. The fact that the spine is Markovian is immediate from its definition of $(R^{\varphi}, \Upsilon^{\varphi})$. Indeed, once given its initial configuration, it evolves as the NRW associated to the rate $\varphi^{-1}(r, \upsilon)\sigma_{\mathsf{s}}(r, \upsilon)\varphi(r, \upsilon')\pi_{\mathsf{s}}(r, \upsilon, \upsilon')$. Moreover, when in configuration

$(r, \upsilon) \in D \times V$, at rate $\varphi(r, \upsilon)^{-1}(\mathrm{F} + \sigma_{\mathrm{f}}\mathrm{I})\varphi(r, \upsilon)$, it experiences an additionally scattering with new velocity υ', with distribution

$$\mathcal{E}_{(r,\upsilon)}\left[\frac{\langle \varphi, \mathcal{Z} \rangle}{\mathcal{E}_{(r,\upsilon)}[\langle \varphi, \mathcal{Z} \rangle]} \frac{\langle \varphi \mathbf{1}_{(\cdot \in \mathrm{d}\upsilon')}, \mathcal{Z} \rangle}{\langle \varphi, \mathcal{Z} \rangle}\right] = \frac{\sigma_{\mathrm{f}}(r, \upsilon)}{(\mathrm{F} + \sigma_{\mathrm{f}}\mathrm{I})\varphi(r, \upsilon)} \varphi(r, \upsilon')\pi_{\mathrm{f}}(r, \upsilon, \upsilon')\,\mathrm{d}\upsilon',$$

for $\upsilon' \in V$, where we have used (8.8). The total scatter rate is thus

$$\sigma_{\mathrm{s}}(r, \upsilon)\frac{\varphi(r, \upsilon')}{\varphi(r, \upsilon)}\pi_{\mathrm{s}}(r, \upsilon, \upsilon') + \sigma_{\mathrm{f}}(r, \upsilon)\frac{\varphi(r, \upsilon')}{\varphi(r, \upsilon)}\pi_{\mathrm{f}}(r, \upsilon, \upsilon')\,\mathrm{d}\upsilon'$$

(9.1)
$$= \alpha(r, \upsilon)\frac{\varphi(r, \upsilon')}{\varphi(r, \upsilon)}\pi(r, \upsilon, \upsilon')$$

$$= \alpha^{\varphi}(r, \upsilon)\pi^{\varphi}(r, \upsilon, \upsilon')$$

as required.

For the second statement, write

$$(9.2) \qquad \psi_t^{\varphi}[g](r, \upsilon) := \mathbf{E}_{(r,\upsilon)}\left[\mathrm{e}^{-\lambda_* t + \int_0^t \beta(R_s, \Upsilon_s)\,\mathrm{d}s}\frac{\varphi(R_t, \Upsilon_t)}{\varphi(r, \upsilon)}g(R_t, \Upsilon_t)\mathbf{1}_{\{t < \tau^D\}}\right].$$

By conditioning the expectation on the right-hand side on the first scattering event we have, for $t \geq 0$, $r \in D$ and $\upsilon \in V$,

$$\psi_t^{\varphi}[g](r, \upsilon) = \mathrm{e}^{-\lambda_* t + \int_0^t \beta(r + \upsilon\ell, \upsilon) - \alpha(r + \upsilon\ell, \upsilon)\,\mathrm{d}\ell}\frac{\varphi(r + \upsilon t, \upsilon)}{\varphi(r, \upsilon)}g(r + \upsilon t, \upsilon)\mathbf{1}_{(t < \kappa_{r,\upsilon}^D)}$$

(9.3)
$$+ \int_0^{t \wedge \kappa_{r,\upsilon}^D} \alpha(r + \upsilon s, \upsilon)\mathrm{e}^{-\lambda_* s + \int_0^s \beta(r + \upsilon\ell, \upsilon) - \alpha(r + \upsilon\ell, \upsilon)\,\mathrm{d}\ell}\frac{\varphi(r + \upsilon s, \upsilon)}{\varphi(r, \upsilon)}$$

$$\times \int_V \psi_{t-s}^{\varphi}[g](r + \upsilon s, \upsilon')\frac{\varphi(r + \upsilon s, \upsilon')}{\varphi(r + \upsilon s, \upsilon)}\pi(r + \upsilon s, \upsilon, \upsilon')\,\mathrm{d}\upsilon'\,\mathrm{d}s.$$

Now appealing to (8.4), then using the standard trick of replacing the role of an additive potential by the role of a multiplicative potential in such semigroup evolutions, see, for example, Lemma 1.2, Chapter 4 in [16], and noting (9.2) we get

(9.4)
$$\psi_t^{\varphi}[g](r, \upsilon) = \mathrm{U}_t[g](r, \upsilon)$$

$$+ \int_0^t \mathrm{U}_s\big[(\mathrm{L}_{\varphi} + \alpha^{\varphi}\mathrm{I})\psi_{t-s}^{\varphi}[g] - (\varphi^{-1}(\mathrm{S} + \mathrm{F})\varphi + (\beta - \alpha)\mathrm{I})\psi_{t-s}^{\varphi}[g]\big](r, \upsilon)\,\mathrm{d}s,$$

where

$$(9.5) \qquad \mathrm{L}_{\varphi}f(r, \upsilon) = \alpha^{\varphi}(r, \upsilon)\int_V [f(r, \upsilon') - f(r, \upsilon)]\pi^{\varphi}(r, \upsilon, \upsilon')\,\mathrm{d}\upsilon',$$

for $f \in L_{\infty}^+(D \times V)$. Referring to (2.6), (2.7) and (4.1), we note that

$$\alpha^{\varphi}(r, \upsilon) - \frac{(\mathrm{S} + \mathrm{F})\varphi(r, \upsilon)}{\varphi(r, \upsilon)} - \beta(r, \upsilon) + \alpha(r, \upsilon)$$

$$= \int_V \alpha(r, \upsilon)\frac{\varphi(r, \upsilon')}{\varphi(r, \upsilon)}\pi(r, \upsilon, \upsilon')\,\mathrm{d}\upsilon'$$

$$- \int_V \frac{\varphi(r, \upsilon')}{\varphi(r, \upsilon)}\big(\sigma_{\mathrm{s}}(r, \upsilon)\pi_{\mathrm{s}}(r, \upsilon, \upsilon') + \sigma_{\mathrm{f}}(r, \upsilon)\pi_{\mathrm{f}}(r, \upsilon, \upsilon')\big)\,\mathrm{d}\upsilon' - \sigma_{\mathrm{s}}(r, \upsilon) - \sigma_{\mathrm{f}}(r, \upsilon)$$

$$- \sigma_{\mathrm{f}}(r, \upsilon)\left(\int_V \pi_{\mathrm{f}}(r, \upsilon, \upsilon')\,\mathrm{d}\upsilon' - 1\right) + \sigma_{\mathrm{s}}(r, \upsilon) + \sigma_{\mathrm{f}}(r, \upsilon)\int_V \pi_{\mathrm{f}}(r, \upsilon, \upsilon')\,\mathrm{d}\upsilon'$$

$$= 0.$$

Hence (9.4) reduces to the somewhat simpler recurrence equation

$$\psi_t^\varphi[g](r, \upsilon) = \mathrm{U}_t[g](r, \upsilon) + \int_0^t \mathrm{U}_s[\mathrm{L}_\varphi \psi_{t-s}^\varphi[g]](r, \upsilon)\, \mathrm{d}s, \quad t \geq 0,$$

where we recall that L_φ was defined in (9.5). This is nothing more than the mild equation for the semigroup evolution $\mathbf{E}_{(r,\upsilon)}^\varphi[g(R_t, \Upsilon_t)]$, $t \geq 0$, which has a unique bounded solution from the usual Grönwall arguments. Note that when $g = 1$, we see the solution is 1. This, together with the Markov property implies that the right-hand side of (6.5) is a martingale. Moreover, it follows that the martingale change of measure in (6.5) describes law of the $\alpha^\varphi \pi^\varphi$-NRW.

The fact that $\psi_t^\varphi[1](r, \upsilon) = 1$ for all $r \in D$, $\upsilon \in V$, implies that $((R, \Upsilon), \mathbf{P}^\varphi)$ is conservative. Moreover,

$$\mathbf{P}_{(r,\upsilon)}^\varphi[g(R_t, \Upsilon_t)] = \psi_t^\varphi[g](r, \upsilon) = \mathrm{e}^{-\lambda_* t} \frac{\psi_t[g\varphi](r, \upsilon)}{\varphi(r, \upsilon)}, \quad r \in D, \upsilon \in V,$$

where $g \in L_\infty^+(D \times V)$; cf. (9.2). Hence $\lim_{t \to \infty} \mathbf{P}_{(r,\upsilon)}^\varphi[g(R_t, \Upsilon_t)] = \langle g, \tilde{\varphi}\varphi \rangle$ for all $g \in L_\infty^+(D \times V)$. In other words, $\tilde{\varphi}\varphi$ is the density of the stationary distribution of (R, Υ) under $\mathbb{P}^\varphi$.

10. Proof of Theorem 5.1.

The proof we offer here is a variant of a standard one, which has been used to analyse the convergence of many analogous martingales in the setting of different spatial branching processes. We mention [3, 31, 39] and [17] to name but a few of the contexts with similar results.

In the case that $\lambda_* < 0$ and $\lambda_* > 0$, we need (H3) to ensure that the NBP can undergo fission. In the setting $\lambda_* = 0$ we need the stricter condition (H3)* for technical reasons in the proof to ensure a minimal rate of reproduction.

A standard measure theoretic result (cf. page 242 of [15]) tells us that the martingale change of measure in (6.1) is uniformly integrable if and only if

$$\mathbb{P}_{\delta_{(r,\upsilon)}}^\varphi \left(\limsup_{t \to \infty} W_t < \infty \right) = 1.$$

In the case that $\limsup_{t \to \infty} W_t = \infty$, $\mathbb{P}^\varphi$ almost surely, we have $\mathbb{P}_{\delta_{(r,\upsilon)}}(W_\infty = 0) = 1$.

(i) Let us first deal with the case that $\lambda_* > 0$. To this end, let us define the sigma algebra $\mathcal{S} = \sigma(T_i, \mathcal{Z}_i : i \geq 0)$, where T_i, $i \geq 1$, are the times at which the spine undergoes fission and $\mathcal{Z}_i$, $i \geq 1$, are point processes on V that describe the velocities of fission offspring (i.e., whose law is given by the family (2.2) under the change of measure (6.3)). For convenience we will write $T_0 = 0$.

Appealing to the pathwise spine decomposition in Theorem 6.1, we can write

$$\mathbb{E}_{\delta_{(r,\upsilon)}}^\varphi[W_t] = \mathbb{E}_{\delta_{(r,\upsilon)}}^\varphi \left[\mathrm{e}^{-\lambda_* t} \frac{\varphi(R_t, \Upsilon_t)}{\varphi(r, \upsilon)} \right]$$

$$+ \mathbb{E}_{\delta_{(r,\upsilon)}}^\varphi \left[\mathbb{E}_{\delta_{(r,\upsilon)}}^\varphi \left[\sum_{j=1}^\infty \mathrm{e}^{-\lambda_* T_j} \mathbf{1}_{(T_j \leq t)} \sum_{i=1}^{N^j} \varphi(R_{T_j}, \upsilon_i) W_{t-T_j}^j(R_{T_j}, \upsilon_i) \Big| \mathcal{S} \right] \right]$$

(10.1)

$$= \mathbb{E}_{\delta_{(r,\upsilon)}}^\varphi \left[\mathrm{e}^{-\lambda_* t} \frac{\varphi(R_t, \Upsilon_t)}{\varphi(r, \upsilon)} + \sum_{j=1}^\infty \mathrm{e}^{-\lambda_* T_j} \mathbf{1}_{(T_j \leq t)} \mathcal{E}_{(R_{T_j}, \Upsilon_{T_{j-1}})}^\varphi[\langle \varphi, \mathcal{Z}_j \rangle] \right]$$

$$= \mathbb{E}_{\delta_{(r,\upsilon)}}^\varphi \left[\mathrm{e}^{-\lambda_* t} \frac{\varphi(R_t, \Upsilon_t)}{\varphi(r, \upsilon)} + \sum_{j=1}^\infty \mathrm{e}^{-\lambda_* T_j} \mathbf{1}_{(T_j \leq t)} \frac{\mathcal{E}_{(R_{T_j}, \Upsilon_{T_{j-1}})}[\langle \varphi, \mathcal{Z} \rangle^2]}{\mathcal{E}_{(R_{T_j}, \Upsilon_{T_{j-1}})}[\langle \varphi, \mathcal{Z} \rangle]} \right],$$

where, for a given $(r, \upsilon) \in D \times V$, the process $W_s^j(r, \upsilon)$ is an independent copy of $(W_s, s \geq 0)$, under $\mathbb{P}_{\delta_{(r,\upsilon)}}$ (and consequently has unit mean, which is also used above). Our objective

is to prove that the sum on the right-hand side of (10.1) is $\mathbb{P}^{\varphi}_{\delta_{(r,\upsilon)}}$-almost surely finite. In that case, it will follow with the help of Fatou's lemma that

$$(10.2) \qquad \infty > \limsup_{t\to\infty} \mathbb{E}^{\varphi}_{\delta_{(r,\upsilon)}}[W_t] \geq \mathbb{E}^{\varphi}_{\delta_{(r,\upsilon)}}\Big[\liminf_{t\to\infty} W_t\Big].$$

Recalling that W is a nonnegative $\mathbb{P}$-martingale, it holds that $1/W$ is a nonnegative $\mathbb{P}^{\varphi}$-supermartingale and thus its limit exists. The conditional expectation in (10.2) ensures that $\liminf_{t\to\infty} W_t$ (and hence from the immediately preceding remarks $\limsup_{t\to\infty} W_t$) is $\mathbb{P}^{\varphi}_{\delta_{(r,\upsilon)}}$-almost surely finite.

We must thus show that the upper bound on the right-hand side of (10.1) is $\mathbb{P}^{\varphi}_{\delta_{(r,\upsilon)}}$-almost surely finite. To do so, we again recall the description of the pathwise spine decomposition in Theorem 6.1 and note that fission along the spine occurs at the accelerated rate $\varphi^{-1}(\mathrm{F} + \sigma_{\mathrm{f}}\mathrm{I})\varphi$. Hence (recalling the generic point process $\mathcal{Z}$ defined in (2.2))

$$\mathbb{E}^{\varphi}_{\delta_{(r,\upsilon)}}\left[\sum_{j=1}^{\infty} e^{-\lambda_* T_j}\mathbf{1}_{\{T_j \leq t\}}\frac{\mathcal{E}_{(R_{T_j},\Upsilon_{T_{j-1}})}[\langle\varphi,\mathcal{Z}\rangle^2]}{\mathcal{E}_{(R_{T_j},\Upsilon_{T_{j-1}})}[\langle\varphi,\mathcal{Z}\rangle]}\right]$$

$$\leq (\|\varphi\|_{\infty}n_{\max})^2 \mathbb{E}^{\varphi}_{\delta_{(r,\upsilon)}}\left[\sum_{j=1}^{\infty} e^{-\lambda_* T_j}\frac{1}{\mathcal{E}_{(R_{T_j},\Upsilon_{T_{j-1}})}[\langle\varphi,\mathcal{Z}\rangle]}\right]$$

$$= (\|\varphi\|_{\infty}n_{\max})^2$$

$$(10.3) \qquad \times \mathbf{E}^{\varphi}_{(r,\upsilon)}\left[\int_0^{\infty} e^{-\lambda_* t}\frac{\sigma_{\mathrm{f}}(R_t,\Upsilon_t)}{\mathcal{E}_{(R_t,\Upsilon_t)}[\langle\varphi,\mathcal{Z}\rangle]}\int_V \frac{\varphi(R_t,\upsilon')}{\varphi(R_t,\Upsilon_t)}\pi_{\mathrm{f}}(R_t,\Upsilon_t,\upsilon')\,\mathrm{d}\upsilon'\,\mathrm{d}t\right]$$

$$\leq \bar{\sigma}_{\mathrm{f}}(\|\varphi\|_{\infty}n_{\max})^2\mathbf{E}^{\varphi}_{(r,\upsilon)}\left[\int_0^{\infty}\frac{e^{-\lambda_* t}}{\varphi(R_t,\Upsilon_t)}\,\mathrm{d}t\right]$$

$$\leq \bar{\sigma}_{\mathrm{f}}\frac{(\|\varphi\|_{\infty}n_{\max})^2}{\varphi(r,\upsilon)}\mathbf{E}_{(r,\upsilon)}\left[\int_0^{\infty}e^{-2\lambda_* t+\int_0^t\beta(R_s,\Upsilon_s)\,\mathrm{d}s}\mathbf{1}_{\{t<\tau^D\}}\,\mathrm{d}t\right]$$

$$\leq \bar{\sigma}_{\mathrm{f}}(\|\varphi\|_{\infty}n_{\max})^2\int_0^{\infty}e^{-2\lambda_* t}\frac{\psi_t[1](r,\upsilon)}{\varphi(r,\upsilon)}\,\mathrm{d}t,$$

where we have used (H4) in the first inequality, features of the spine decomposition for the first equality, (8.8) and (H1) in the second inequality, the change of measure (6.5) in the third inequality and the semigroup (4.2) for the final line. Finally, note that since φ is uniformly bounded above, the contribution from the spine term in (10.1) is zero in the limit $t\to\infty$. Now using Theorem 3.1, it follows that

$$\limsup_{t\to\infty}\mathbb{E}^{\varphi}_{\delta_{(r,\upsilon)}}[W_t] < \infty$$

as required.

(ii) Next, for the case $\lambda_* < 0$, it is easy to see that, on the event $\{T_j \leq t < T_{j+1}\}$,

$$W_t \geq e^{-\lambda_* t}\varphi(R_t,\Upsilon_{T_{j-1}})$$

which ensures that $\mathbb{P}^{\varphi}_{\delta_{(r,\upsilon)}}(\limsup_{t\to\infty} W_t = \infty) = 1$ for all $(r,\upsilon)\in D\times V$ and hence $\mathbb{P}_{\delta_{(r,\upsilon)}}(W_{\infty} = 0) = 1$.

(iii) Finally, for the case $\lambda_* = 0$, our aim is to show that, for each $r\in D$, $\upsilon\in V$,

$$\mathbb{P}^{\varphi}_{\delta_{(r,\upsilon)}}\Big(\limsup_{t\to\infty} W_t = \infty\Big) = 1.$$

We do this by constructing a random sequence of times $(s_n : n \geq 0)$ such that

$$\limsup_{n \to \infty} W_{s_n} = \infty$$

almost surely with respect to $\mathbb{P}^\varphi$.

Lemma 6.1 tells us that $\tilde{\varphi}\varphi$ is the density of the stationary distribution of (R, Υ) under $\mathbf{P}^\varphi$. Moreover, thanks to Theorem 3.1, the density $\tilde{\varphi}\varphi$ is a.e. uniformly bounded away from 0 on each $\Omega \subset\subset D \times V$. It follows that $\langle \mathbf{1}_\Omega, \varphi\tilde{\varphi} \rangle > 0$ for all $\Omega \subseteq D_\varepsilon \times V$ and that the spine R visits Ω infinitely often under $\mathbf{P}^\varphi$.

Fix $k \in \mathbb{N}$. We want to show that there is an $\Omega \subseteq D_\varepsilon \times V$ such that

$$(10.4) \qquad \inf_{(r', \upsilon') \in \Omega} \mathbb{P}^\varphi_{\delta_{(r', \upsilon')}} \left(X_\iota(D_\varepsilon \times V) \geq k \right) > 0,$$

where $\iota = 2\operatorname{diam}(D)/\mathrm{v}_{\min}$ (note that ι is twice the time it would take a neutron to cross the equivalent of the diameter of D, when moving at minimal speed). To this end, fix $r \in D$, $\upsilon \in V$ and choose $\varepsilon > 0$ sufficiently small such that both $r \in D_\varepsilon := \{r \in D : \inf_{y \in \partial D} |r - y| > \varepsilon\mathrm{v}_{\max}\}$ and B (introduced in the assumption (H3)*) is in D_ε. Then, define

$$\Omega = \{(r, \upsilon) \in D_\varepsilon \times V : \{r + \upsilon s : s \geq 0\} \cap B \neq \varnothing\}.$$

Write m for the smallest natural number such that $m(n_{\max} - 1) + 1 \geq k$. Recalling from Theorem 3.1 that $\inf_{r \in D, \upsilon, \upsilon' \in V} \alpha(r, \upsilon)\pi(r, \upsilon, \upsilon') > 0$, and taking account of the positivity properties of φ, we can lower bound the probability that, from any $(r, \upsilon) \in \Omega$, the spine can enter B. Moreover, on this event, due to (H3)*, we can also lower bound the probability that the spine immigrates $n_{\max} - 1$ particles on m (evenly spaced in time) separate occasions, all of which are still inside of B by time ι. The strategy for doing so is to head into B from the given point of issue in Ω by travelling in a straight line within a small cone of possible velocities (which would be guaranteed to happen within $\iota/2$ units of time), and then for neutrons to cycle around the perimeter of B in an annulus by scattering within a narrow cone of velocities each time; see Figure 2. As such we can provide the lower bound desired in (10.4). The technical details are tedious and left to the reader.

With (10.4) in hand, we can construct the sequence $(t_n : n \geq 0)$ by defining $t_0 = 0$ and subsequently, for $n \geq 1$,

$$t_n = \inf\{s > t_{n-1} + (10m \times \iota) : (R_s, \Upsilon_s) \in \Omega\}.$$

Note that since (R, Υ) visits Ω infinitely often under $\mathbf{P}^\varphi$ we have that $t_n < \infty$, $\mathbf{P}^\varphi$-almost surely for $n \geq 0$, and $t_n \to \infty$, $\mathbf{P}^\varphi$-almost surely. By applying the strong Markov property at
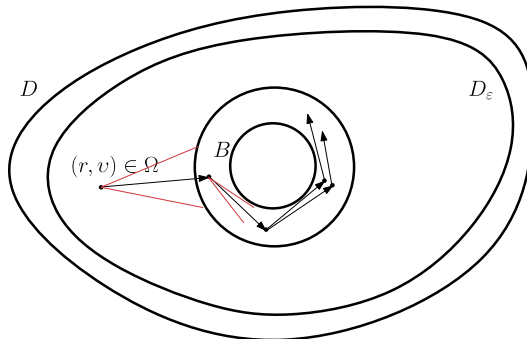

FIG. 2. *There is a uniform lower bound on the probability that the spine issued from $(r, \upsilon) \in \Omega$ heads directly into the annulus contained in B and subsequently immigrates $n_{\max} - 1$ neutrons on each of m separate occasions, which then cycle around the annulus, and all this is completed over the time horizon $\iota = 2\operatorname{diam}(D)/\mathrm{v}_{\min}$ elapses.*

the sequence of times $(t_n, n \geq 0)$, it now follows from (10.4) that, in the spirit of a sequence of independent Bernoulli trials, $\limsup_{n\to\infty} X_{s_n}(D_\varepsilon) \geq k$ almost surely with respect to $\mathbb{P}^\varphi$, where $s_n = t_n + (m \times \iota)$. Since the integer k can be chosen arbitrarily large, we also have that $\limsup_{n\to\infty} X_{s_n}(D_\varepsilon) = \infty$ almost surely with respect to $\mathbb{P}^\varphi$.

As φ is uniformly bounded below away from 0 on $D_\varepsilon \times V$ (see Theorem 3.1), it follows that

$$W_t \geq cX_t(D_\varepsilon \times V), \quad t \geq 0,$$

for some constant $c > 0$. The analysis above, thus shows that $\mathbb{P}^\varphi_{\delta_{(r,v)}}(\limsup_{t\to\infty} W_t = \infty) > 0$, as required, for each $r \in D$, $v \in V$.

11. Proof of Theorem 5.2.   We need to show that in all three cases, the event $\{\zeta < \infty\}$ agrees almost surely with $\{W_\infty = 0\}$. To this end, first note that $\{\zeta < \infty\} \subseteq \{W_\infty = 0\}$ and hence

$$(11.1) \qquad \mathbb{P}_{\delta_{(r,v)}}(\zeta < \infty) \leq \mathbb{P}_{\delta_{(r,v)}}(W_\infty = 0),$$

for all $r \in D$, $v \in V$. It thus suffices to show that (11.1) is in fact an equality.

We will give two preparatory technical lemmas before coming to the main part of the proof of Theorem 5.2.

LEMMA 11.1.   *For all $r \in D$ and $v \in V$, we have $\mathbb{P}_{\delta_{(r,v)}}$-almost surely that*

$$\lim_{n\to\infty} \mathbb{P}_{X_n}(\zeta \leq t_0) = \mathbf{1}_{\{\zeta < \infty\}}.$$

PROOF.   On the event $\{\zeta < \infty\}$, it is immediate that, for all $r \in D$ and $v \in V$,

$$(11.2) \qquad \lim_{t\to\infty} \mathbb{P}_{X_t}(\zeta \leq t_0) = 1$$

$\mathbb{P}_{\delta_{(r,v)}}$-almost surely. Let $(T_n)_{n\in\mathbb{N}}$ be any increasing sequence of stopping times. Using the strong Markov property and (11.2), we have that, for all $n \in \mathbb{N}$,

$$\mathbb{P}_{\delta_{(r,v)}}(\zeta < \infty) = \mathbb{E}_{\delta_{(r,v)}}\big[\mathbb{P}_{X_{T_n}}(\zeta < \infty)\big] \geq \mathbb{E}_{\delta_{(r,v)}}\big[\mathbb{P}_{X_{T_n}}(\zeta \leq t_0)\big].$$

Using this inequality and Fatou's lemma, we deduce that

$$\begin{aligned}
\mathbb{P}_{\delta_{(r,v)}}(\zeta < \infty) &\geq \liminf_{n\to\infty} \mathbb{E}_{\delta_{(r,v)}}\big[\mathbb{P}_{X_{T_n}}(\zeta \leq t_0)\big]\\
&\geq \mathbb{E}_{\delta_{(r,v)}}\Big[\liminf_{n\to\infty} \mathbb{P}_{X_{T_n}}(\zeta \leq t_0)\Big]\\
&\geq \mathbb{E}_{\delta_{(r,v)}}[\mathbf{1}_{\{\zeta<\infty\}}] + \delta\mathbb{P}_{\delta_{(r,v)}}\Big(\zeta = \infty \text{ and } \liminf_{n\to\infty}\mathbb{P}_{X_{T_n}}(\zeta \leq t_0) \geq \delta\Big).
\end{aligned}$$

It follows that, for all $\delta \in (0, 1]$, we have $\mathbb{P}_{\delta_{(r,v)}}(\zeta = \infty$ and $\liminf_{n\to\infty} \mathbb{P}_{X_{T_n}}(\zeta \leq t_0) \geq \delta) = 0$. This implies that, on $\{\zeta = \infty\}$, $\liminf_{n\to\infty} \mathbb{P}_{X_{T_n}}(\zeta \leq t_0) = 0$. Since this is true for any sequence of increasing stopping times, we deduce that, on $\{\zeta = \infty\}$, $\limsup_{n\to\infty} \mathbb{P}_{X_n}(\zeta \leq t_0) = 0$. Together with (11.2), this gives us

$$\lim_{n\to\infty} \mathbb{P}_{X_n}(\zeta \leq t_0) = \mathbf{1}_{\{\zeta<\infty\}}$$

$\mathbb{P}_{\delta_{(r,v)}}$-almost surely, as required.   □

LEMMA 11.2. *For all $r \in D$ and $\upsilon \in V$, on $\{\zeta = \infty\}$, we have $\mathbb{P}_{\delta_{(r,\upsilon)}}$-almost surely that*

$$\lim_{t \to \infty} \langle \varphi, X_t \rangle = \infty. \tag{11.3}$$

PROOF. Recall that ζ is the time of extinction of the NBP. For any $r \in D$, $\upsilon \in V$ and $t \geq 0$, we have

$$\mathbb{P}_{\delta_{(r,\upsilon)}}(t < \zeta) \leq \mathbb{E}_{\delta_{(r,\upsilon)}}[N_t] = \psi_t[\mathbf{1}_{\{D \times V\}}](r, \upsilon).$$

Using (3.5), we deduce that there exists a $t_0 > 0$ such that, for all $r \in D$, $\upsilon \in V$,

$$\mathbb{P}_{\delta_{(r,\upsilon)}}(t_0 < \zeta) \leq \psi_{t_0}[\mathbf{1}_{\{D \times V\}}](r, \upsilon) \leq 2e^{\lambda_* t_0}\varphi(r, \upsilon).$$

It is straightforward to show that $\sup_{x \in E} \mathbb{P}_x(t_0 < \zeta) < 1$. Hence, there exists a constant $c_0 \in (0, 1)$ such that, uniformly for all $r \in D$ and $\upsilon \in V$,

$$\mathbb{P}_{\delta_{(r,\upsilon)}}(t_0 < \zeta) \leq c_0 \wedge [2e^{\lambda_* t_0}\varphi(r, \upsilon)].$$

Using the branching property, we deduce that, for all $\mu = \sum_{i=1}^{n} \delta_{(r_i, \upsilon_i)} \in \mathcal{M}(D \times V)$,

$$\mathbb{P}_{\mu}(\zeta \leq t_0) \geq \prod_{i=1}^{n}(1 - c_0 \wedge [2e^{\lambda_* t_0}\varphi(r_i, \upsilon_i)]).$$

Now, using the Lemma 11.1 we have, taking the limit in $n \in \mathbb{N}$, we obtain

$$\mathbf{1}_{\{\zeta < \infty\}} = \limsup_{n \to \infty} \mathbb{P}_{X_n}(\zeta \leq t_0) \geq \limsup_{n \to \infty} \prod_{i=1}^{N_n}(1 - c_0 \wedge [2e^{\lambda_0 t_0}\varphi(r_i(n), \upsilon_i(n))]), \tag{11.4}$$

where we have used the notation from (2.1). Since $c_0 < 1$, taking logarithms in (11.4) and using $\log x \leq x - 1$, we deduce that, on $\{\zeta = \infty\}$, again taking limits on $\mathbb{N}$,

$$\lim_{n \to \infty} \sum_{i=1}^{N_n}(c_0 \wedge [2e^{\lambda_0 t_0}\varphi(r_i(n), \upsilon_i(n))]) = \infty$$

and hence the statement of the lemma follows. $\square$

Let us now return to the proof of Theorem 5.2. First we consider the setting that $\lambda_* \leq 0$. Noting that, up to a normalising constant $W_t = e^{-\lambda_* t}\langle \varphi, X_t \rangle \geq \langle \varphi, X_t \rangle$, as W is almost surely convergent, the conclusion of Lemma 11.2 forces us to deduce that $\{\zeta < \infty\}$ almost surely in order to avoid a contradiction. Hence, from (11.1), we have that $\{\zeta < \infty\} = \{W_\infty = 0\}$ and both occur with probability one (irrespective of the starting configuration of X).

Next we consider the setting that $\lambda_* > 0$. Due to our assumptions and the boundedness of φ, we have uniformly, for all $r \in D$, $\upsilon \in D$ and all times t such that there is a discontinuity in W, $|W_t - W_{t-}|$ is uniformly bounded by some constant $M > 0$, $\mathbb{P}_{\delta_{(r,\upsilon)}}$ almost surely. Defining the stopping time $T_1 = \inf\{t \geq 0, W_t \geq 1\}$, using the fact that W is a nonnegative, L^1, and hence uniformly integrable martingale, and using Doob's optional stopping theorem, we deduce that, for all $r \in D$, $\upsilon \in V$,

$$1 = \mathbb{E}_{\delta_{(r,\upsilon)}}[W_{T_1}] = \mathbb{E}_{\delta_{(r,\upsilon)}}[W_{T_1}\mathbf{1}_{\{W_\infty > 0\}}] \leq \frac{1}{\varphi(r, \upsilon)}\mathbb{E}_{\delta_{(r,\upsilon)}}((1 + M)\mathbf{1}_{\{W_\infty > 0\}}).$$

It follows that, for all $r \in D$ and $\upsilon \in V$, $\mathbb{P}_{\delta_{(r,\upsilon)}}(W_\infty > 0) \geq \varphi(r,\upsilon)/M$. Hence that there exists $c_1 > 0$ such that

$$\mathbb{P}_{\delta_{(r,\upsilon)}}(W_\infty = 0) \leq 1 - c_1\varphi(r,\upsilon), \quad r \in D, \upsilon \in V.$$

Now, using the branching property, we obtain for all $\mu = \sum_{i=1}^n \delta_{(r_i,\upsilon_i)} \in \mathcal{M}(D \times V)$,

$$\log \mathbb{P}_\mu(W_\infty = 0) \leq \sum_{i=1}^n \log(1 - c_1\varphi(x_i)) \leq -c_1 \sum_{i=1}^n \varphi(x_i).$$

Due to the conclusion of Lemma 11.2 we have, on $\{\zeta = \infty\}$,

$$\limsup_{n \to \infty} \log \mathbb{P}_{X_n}(W_\infty = 0) \leq -c_1 \lim_{n \to \infty} \langle \varphi, X_n \rangle = -\infty.$$

With the upper bound of any probability being unity, we can thus write

$$\limsup_{n \to \infty} \mathbb{P}_{X_n}(W_\infty = 0) \leq \mathbf{1}_{\{\zeta < \infty\}}.$$

Markov's property now entails, for $r \in D$, $\upsilon \in V$,

$$\mathbb{P}_{\delta_{(r,\upsilon)}}(W_\infty = 0) = \mathbb{E}_{\delta_{(r,\upsilon)}}\big[\mathbb{P}_{X_n}(W_\infty = 0)\big] \quad \forall n \in \mathbb{N},$$

so that, by the reverse Fatou's lemma,

$$\mathbb{P}_{\delta_{(r,\upsilon)}}(W_\infty = 0) = \limsup_{n \to \infty} \mathbb{E}_{\delta_{(r,\upsilon)}}\big[\mathbb{P}_{X_n}(W_\infty = 0)\big]$$

$$\leq \mathbb{E}_{\delta_{(r,\upsilon)}}\Big[\limsup_{n \to \infty} \mathbb{P}_{X_n}(W_\infty = 0)\Big]$$

$$\leq \mathbb{E}_{\delta_{(r,\upsilon)}}[\mathbf{1}_{\{\zeta < \infty\}}]$$

$$= \mathbb{P}_{\delta_{(r,\upsilon)}}(\zeta < \infty).$$

Together with (11.1), this completes the proof of the theorem.

12. Proof of Corollary 5.4.   Doob's martingale inequality ensures that, for $\mu \in \mathcal{M}(D \times V)$

$$\mathbb{E}_\mu\Big[\Big(\sup_{t \geq 0} W_t\Big)^2\Big] \leq \liminf_{s \to \infty} 4\mathbb{E}_\mu[(W_s)^2].$$

Showing that the right-hand side above is finite is sufficient to obtain $L_2(\mathbb{P})$ convergence. Note, however, that $\mathbb{E}_\mu[(W_s)^2] = \mathbb{E}_\mu^\varphi[W_t]$, $t \geq 0$, and hence, from (10.2), the desired upper bound is proved.

Glossary of some commonly used notation
(Th. = Th., a. = above, b. = below)

Notation	Description	Introduced
$(\psi_t, t \geq 0)$	Solution to mild NTE/NBP expectation semigroup	(2.4), (2.8)
D and V	Physical and velocity domain	Section 1
$\sigma_{\mathrm{s}}, \sigma_{\mathrm{f}}$ and σ	Scatter, fission and total cross-sections	b. (1.1)
π_{s} and π_{f}	Scatter and fission kernels	b. (1.1)
S and F	Scatter and fission operators	(2.6), (2.7)
$\mathcal{P}_{(r,\upsilon)}$	Offspring law of X when parent at $(r, \upsilon) \in D \times V$	(2.3)
$((r_i, \upsilon_i), i = 1, \ldots, N)$	Position and number of offspring positions of a family in X	a. (2.2)
$(X, \mathbb{P}_\mu)$	NBP when issued from μ	Section 2.2
U_t	Linear advection semigroup	b. (2.5), (2.8)
G_{f}	Branching generator of $(X, \mathbb{P}_\mu)$	(8.9)
$(u_t, t \geq 0)$	Nonlinear semigroup of X	(8.5)
$\hat{\mathrm{U}}_t$	Nonlinear advection semigroup	(8.11)
$n_{\max}$	Maximum number of neutrons in a fission event	b. (2.3), (H4)
λ_*, φ and $\tilde{\varphi}$	Leading eigenvalue, right- and left-eigenfunctions	Th. 3.1
$(W_t, t \geq 0)$	Additive martingale	(5.1)
ζ	Extinction time	Th. 5.1
$((R, \Upsilon), \mathbf{P})$	Many-to-one NRW	Lemma 4.1
α and π	Scatter rate and kernel for many-to-one NRW	(3.2), (3.3)
τ^D	First exit time of spatial component of $\alpha\pi$-NRW from D	Lemma 4.1
$((R, \Upsilon), \mathbf{P}^\dagger)$	Killed $\alpha\pi$-NRW	(4.3)
$\mathrm{P}^\dagger$	Semigroup of killed $\alpha\pi$-NRW	(4.3)
β	Many-to-one potential	(4.1), (4.2)
k	Killing time of $\alpha\pi$-NRW	(4.4)
$(J_k, k \geq 1)$	Ordered jump times of killed $\alpha\pi$-NRW	b. (7.5)
$(X, \mathbb{P}_\mu^\varphi)$	NBP after change of measure with W when issued from μ	(6.1)
G_{f}^φ	Branching generator of $(X, \mathbb{P}_\mu^\varphi)$	(8.7), (8.13)
$(u_t^\varphi, t \geq 0)$	Nonlinear semigroup of $(X, \mathbb{P}_\mu^\varphi)$	(8.1)
$(\psi_t^\varphi, t \geq 0)$	Linear semigroup of $(X, \mathbb{P}_\mu^\varphi)$	(9.2)
$(X^\varphi, \tilde{\mathbb{P}}_\mu^\varphi)$	Dressed spine when issued from configuration μ	(6.2)
$(\tilde{u}_t^\varphi, t \geq 0)$	Nonlinear semigroup of $(X^\varphi, \tilde{\mathbb{P}}_\mu^\varphi)$	(8.14)
$((R^\varphi, \Upsilon^\varphi), \tilde{\mathbf{P}}^\varphi)$	Marginal of $\tilde{\mathbb{P}}^\varphi$ giving law of spine NRW	Th. 6.1, (6.5)
α^φ and π^φ	Scatter rate and kernel of auxiliary NRW	(6.4)
$\mathcal{P}_{(r,\upsilon)}^\varphi$	Scattering of velocities along the spine	(6.3)
$\mathbf{P}^\varphi$	Law of $\alpha^\varphi\pi^\varphi$-NRW that agrees with $\tilde{\mathbf{P}}^\varphi$	a. (6.4)

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