



# Backbone decomposition for continuous-state branching processes with immigration

A.E. Kyprianou<sup>a,\*</sup>, Y.-X. Ren<sup>b</sup>

<sup>a</sup> Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, UK

<sup>b</sup> LMAM School of Mathematical Sciences & Center for Statistical Science, Peking University, Beijing 100871, PR China

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## ABSTRACT

In the spirit of Duquesne and Winkel (2007) and Berestycki et al. (2011), we show that supercritical continuous-state branching process with a general branching mechanism and general immigration mechanism is equivalent in law to a continuous-time Galton–Watson process with immigration (with Poissonian dressing). The result also helps to characterise the limiting backbone decomposition which is predictable from the work on consistent growth of Galton–Watson trees with immigration in Cao and Winkel (2010).

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## 1. Introduction

In this article, we are interested in the case that the  $[0, \infty)$ -valued strong Markov process with absorbing state at zero,  $X = \{X_t : t \geq 0\}$ , is a conservative, supercritical continuous-state branching process with general branching mechanism  $\psi$  taking the form

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x < 1\}}) \Pi(dx), \quad \lambda \geq 0,$$

where  $\alpha \in \mathbb{R}$ ,  $\beta \geq 0$  and  $\Pi$  is a measure concentrated on  $(0, \infty)$  which satisfies  $\int_{(0,\infty)} (1 \wedge x^2) \Pi(dx) < \infty$  and a general immigration mechanism  $\varphi$  taking the form

$$\varphi(\lambda) = \delta\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x}) \nu(dx),$$

where  $\delta \geq 0$  and  $\nu$  is a measure concentrated on  $(0, \infty)$  which satisfies  $\int_{(0,\infty)} (1 \wedge x) \nu(dx) < \infty$ . Our requirement that  $X$  is supercritical and conservative means that we necessarily have that  $\psi'(0+) < 0$  and

$$\int_{0+} \frac{1}{|\psi(\xi)|} d\xi = \infty$$

respectively.

\* Corresponding author.

E-mail address: [a.kyprianou@bath.ac.uk](mailto:a.kyprianou@bath.ac.uk) (A.E. Kyprianou).

The process  $X$ , henceforth denoted a  $(\psi, \varphi)$ -CSBP, can be described through its semi-group as follows. Suppose that  $\mathbb{P}_x$  denotes the law of  $X$  on cadlag path space  $D[0, \infty)$  when the process is issued from  $x \geq 0$ . Then the semi-group associated with the  $(\psi, \varphi)$ -CSBP can be described as follows. For all  $x, \lambda \geq 0$  it necessarily follows that

$$\mathbb{E}_x(e^{-\lambda X_t}) = e^{-xu_t(\lambda) - \int_0^t \varphi(u_{t-s}(\lambda)) ds}, \quad t \geq 0,$$

where  $u_t(\lambda)$  uniquely solves the evolution equation

$$u_t(\lambda) + \int_0^t \psi(u_s(\lambda)) ds = \lambda, \tag{1}$$

with initial condition  $u_0(\lambda) = \lambda$ . Note in particular that  $u_t(\lambda)$  describes the semi-group of the  $(\psi, 0)$ -CSBP.

Another process related to the  $(\psi, 0)$ -CSBP is that of the  $(\psi, 0)$ -CSBP conditioned to become extinguished. To understand what this means, let us momentarily recall that for all supercritical continuous-state branching processes (without immigration) the event  $\{\lim_{t \uparrow \infty} X_t = 0\}$  occurs with positive probability. Moreover, for all  $x \geq 0$ ,

$$\mathbb{P}_x \left( \lim_{t \uparrow \infty} X_t = 0 \right) = e^{-\lambda^* x},$$

where  $\lambda^*$  is the unique root on  $(0, \infty)$  of the equation  $\psi(\lambda) = 0$ . Note that  $\psi$  is strictly convex with the property that  $\psi(0) = 0$  and  $\psi(+\infty) = \infty$ , thereby ensuring that the root  $\lambda^* > 0$  exists; see Chapters 8 and 9 of Kyprianou (2006) for further details. It is straightforward to show that the law of  $(X, \mathbb{P}_x)$  conditional on the event  $\{\lim_{t \uparrow \infty} X_t = 0\}$ , say  $\mathbb{P}_x^*$ , agrees with the law of a  $(\psi^*, 0)$ -CSBP, where

$$\psi^*(\lambda) = \psi(\lambda + \lambda^*). \tag{2}$$

See for example Sheu (1997).

In Duquesne and Winkel (2007) and Berestycki et al. (2011) it was shown for the case that  $\varphi \equiv 0$  that the law of process  $X$  can be recovered from a supercritical continuous-time Galton–Watson process (GW), issued with a Poisson number of initial ancestors, and dressed in a Poissonian way using the law of the original process conditioned to become extinguished.

To be more precise, they showed that for each  $x \geq 0$ ,  $(X, \mathbb{P}_x)$  has the same law as the process  $\{\Lambda_t : t \geq 0\}$  which has the following pathwise construction. First sample from a continuous-time Galton–Watson process with branching generator

$$F(r) = q \left( \sum_{n \geq 0} p_n r^n - r \right) = \frac{1}{\lambda^*} \psi(\lambda^*(1 - r)). \tag{3}$$

Note that in the above generator, we have that  $q = \psi'(\lambda^*)$  is the rate at which individuals reproduce and  $\{p_n : n \geq 0\}$  is the offspring distribution. With the particular branching generator given by (3),  $p_0 = p_1 = 0$ , and for  $n \geq 2$ ,  $p_n := p_n[0, \infty)$  where for  $y \geq 0$ ,

$$p_n(dy) = \frac{1}{\lambda^* \psi'(\lambda^*)} \left\{ \beta(\lambda^*)^2 \delta_0(dy) \mathbf{1}_{\{n=2\}} + (\lambda^*)^n \frac{y^n}{n!} e^{-\lambda^* y} \Pi(dy) \right\}.$$

If we denote the aforesaid GW process by  $Z = \{Z_t : t \geq 0\}$  then we shall also insist that  $Z_0$  has a Poisson distribution with parameter  $\lambda^* x$ . Next, dress the life-lengths of  $Z$  in such a way that a  $(\psi^*, 0)$ -CSBP is independently grafted on to each edge of  $Z$  at time  $t$  with rate

$$2\beta d\mathbb{N}^* + \int_0^\infty y e^{-\lambda^* y} \Pi(dy) d\mathbb{P}_y^*.$$

Here the measure  $\mathbb{N}^*$  is the excursion measure on the space  $D[0, \infty)$  which satisfies

$$\mathbb{N}^*(1 - e^{-\lambda X_t}) = u_t^*(\lambda) = -\frac{1}{x} \log \mathbb{E}_x^*(e^{-\lambda X_t})$$

for  $\lambda, t \geq 0$ , where  $u_t^*(\lambda)$  is the unique solution to the integral equation

$$u_t^*(\lambda) + \int_0^t \psi^*(u_s^*(\lambda)) ds = \lambda, \tag{4}$$

with initial condition  $u_0^*(\lambda) = \lambda$ . See El Karoui and Roelly (1991), Le Gall (1999) and Dynkin and Kuznetsov (2004) for further details. Moreover, on the event that an individual dies and branches into  $n \geq 2$  offspring, with probability  $p_n(dx)$ , an additional independent  $(\psi^*, 0)$ -CSBP is grafted on to the branching point with initial mass  $x \geq 0$ . The quantity  $\Lambda_t$  is now understood to be the total dressed mass present at time  $t$  together with the mass present at time  $t$  in an independent  $(\psi^*, 0)$ -CSBP issued at time zero with initial mass  $x$ .

It was also shown in Berestycki et al. (2011) that for each  $t \geq 0$ , the law of  $Z_t$  given  $\Lambda_t$  is that of a Poisson random measure with intensity  $\lambda^* \Lambda_t$ .

Our objective here is to describe a similar decomposition for the  $(\psi, \varphi)$ -CSBP. In the case that we include immigration, it will turn out that the backbone is rather naturally replaced by a continuous-time Galton–Watson process with immigration.

## 2. Backbone decomposition

In order to describe the backbone decomposition for the  $(\psi, \varphi)$ -CSBP, let us first remind ourselves of the basic structure of a continuous-time Galton–Watson process with immigration. Such processes are characterised by the two generators  $(F, G)$  where, as mentioned before,

$$F(r) = q \left( \sum_{n \geq 0} p_n r^n - r \right)$$

encodes the fact that individuals live for an independent and exponentially distributed length of time, after which they give birth to a random number of offspring with distribution  $\{p_n : n \geq 0\}$ , and

$$G(r) = p \sum_{n \geq 0} \pi_n r^n,$$

reflecting the fact that at times of a Poisson arrival process with rate  $p > 0$ , a random number of immigrants with distribution  $\{\pi_n : n \geq 0\}$  issue independent copies of a continuous-time Galton–Watson process with generator  $F$ .

Our forthcoming backbone decomposition will be built from an  $(F, G)$ -GW process with  $F$  given by (3) and

$$G(r) = \varphi(\lambda^*) - \varphi(\lambda^*(1 - r)). \tag{5}$$

It can be seen from the above expression for  $G(r)$  that  $p = \varphi(\lambda^*)$ . To describe the distribution  $\{\pi_n : n \geq 0\}$  let us introduce an associated probability measure, concentrated on  $\{1, 2, \dots\} \times (0, \infty)$ ,

$$\pi_n(dy) = \frac{1}{\varphi(\lambda^*)} \left[ (\delta\lambda^*)\delta_0(dy)\mathbf{1}_{(n=1)} + \frac{(\lambda^*y)^n}{n!} e^{-\lambda^*y} \nu(dy) \right]. \tag{6}$$

It is straightforward to check that, in (5),  $\pi_0 := 0, \pi_n := \pi_n(0, \infty), n \geq 1$  and  $p = \varphi(\lambda^*)$  respectively.

Fix  $x > 0$ . Our backbone decomposition for the process  $(X, \mathbb{P}_x)$  will consist of the bivariate Markov process  $(Z, \Lambda) = \{(Z_t, \Lambda_t) : t \geq 0\}$  valued in  $\{0, 1, 2, \dots\} \times [0, \infty)$ . Here the process,  $Z$ , the backbone, is an  $(F, G)$ -GW process as described above with the additional property that  $Z_0$  is Poisson distributed in number with rate  $\lambda^*x$ . The process of continuous mass,  $\Lambda$ , is described as follows.

- (i) As in Berestycki et al. (2011), along the life length of each individual alive in the process  $Z$ , there is Poissonian dressing with rate

$$2\beta d\mathbb{N}^* + \int_0^\infty y e^{-\lambda^*y} \Pi(dy) d\mathbb{P}_y^*. \tag{7}$$

- (ii) At the branch points of  $Z$ , on the event that there are  $n$  offspring, an additional copy of a  $(\psi^*, 0)$ -CSBP with initial mass  $y \geq 0$  is issued with probability  $p_n(dy)$ .
- (iii) At the same time, along the time-line between each immigration of  $Z$ , there is again Poissonian dressing with rate

$$\delta d\mathbb{N}^* + \int_0^\infty e^{-\lambda^*y} \nu(dy) d\mathbb{P}_y^*. \tag{8}$$

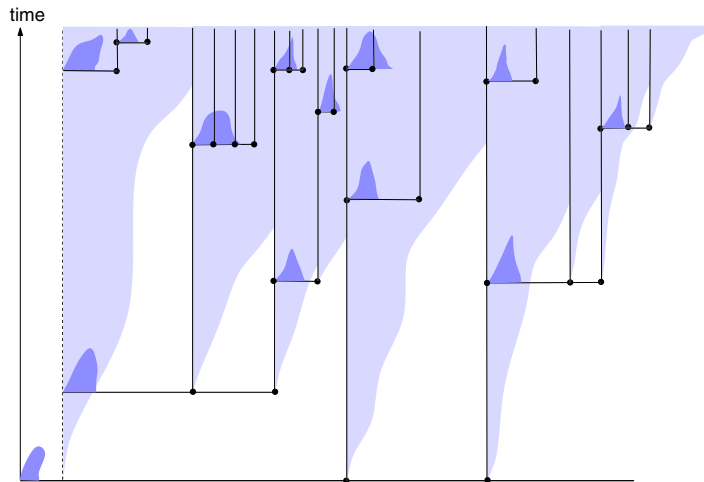
- (iv) Moreover, on the event that there are  $n \geq 1$  immigrants in  $Z$ , an additional copy of a  $(\psi^*, 0)$ -CSBP with initial mass  $y \geq 0$  is issued with probability  $\pi_n(dy)$ .

The quantity  $\Lambda_t$  is now taken to be the total dressed mass present at time  $t$  together with the mass at time  $t$  of an independent  $(\psi^*, 0)$ -CSBP issued at time zero with initial mass  $x$ . Fig. 1 gives a pictorial representation of this decomposition. Henceforth we shall denote the law of the process  $(Z, \Lambda)$  by  $\mathbf{P}_x$ .

**Theorem 2.1** (Backbone Decomposition for  $(\psi, \varphi)$ -CSBP). Fix  $x > 0$ . The law of  $(X, \mathbb{P}_x)$  agrees with that of  $(\Lambda, \mathbf{P}_x)$ . Moreover, for all  $t \geq 0$ , the law of  $Z_t$  given  $\Lambda_t$  is that of a Poisson random variable with law  $\lambda^* \Lambda_t$ .

**Remark 2.2.** The above decomposition complements the recent work of Cao and Winkel (2010). In their paper, it is shown how to consistently grow GW trees with immigration in such a way that, with suitable rescaling, the resulting total mass at each fixed time converges in law to that of a  $(\psi, \varphi)$ -CSBP process. In an appropriate sense, the decomposition in Theorem 2.1 helps to give a description of what the rescaled GW trees with immigration in Cao and Winkel (2010) will converge to.

**Remark 2.3.** Before progressing to the proof, we note that the above theorem can also be cited in the setting of a general superprocess where the motion, taken as a general Borel right Markov process with Lusin state space, is independent of the branching mechanism (now reading  $Z, X$  and  $\Lambda$  as random measures) with minor modification to the forthcoming proof, providing one insists further that  $|\psi'(0+)| < \infty$ . The additional condition is inherited from Berestycki et al. (2011). Whilst this condition is not required in the case that motion is neglected, Berestycki et al. (2011) requires it as soon as spatial considerations come into play.



**Fig. 1.** The diagram above gives a symbolic representation of the backbone decomposition for the  $(\psi, \varphi)$ -CSBP. Working from left to right: An independent copy of a  $(\psi^*, 0)$ -CSBP (shaded dark) is issued at time zero with initial mass  $x$  together with an  $(F, G)$ -GW process which admits a Poisson distributed number of initial individuals with rate  $\lambda^*x$ . Along the (vertical dotted) time-line of the immigration process the dressing (shaded light) has rate  $\delta d\mathbb{N}^* + \int_0^\infty e^{-\lambda^*y} \nu(dy) d\mathbb{P}_y^*$  and additional independent  $(\psi^*, 0)$ -CSBPs (shaded dark) are grafted on at times of immigration of the  $(F, G)$ -GW process such that the probability there are  $n$  simultaneous immigrants with grafted mass of initial size  $y \geq 0$  is  $\pi_n(dy)$ . Along the life length of individuals in the  $(G, F)$ -GW process (vertical black lines) there is dressing (shaded light) at rate  $2\beta d\mathbb{N}^* + \int_0^\infty y e^{-\lambda^*y} \Pi(dy) d\mathbb{P}_y^*$  with additional independent mass (shaded dark) grafted on at branching times such that the probability of there being  $n$  offspring with grafted mass of initial size  $y > 0$  is  $p_n(dy)$ .

**3. Proof of main result**

We first need a result in Berestycki et al. (2011) which was originally stated for superprocesses. We use it here in a reduced form (the spatial movement of particles in their formulation is ignored).

**Lemma 3.1.** Let  $(Z^\theta, \Lambda^\theta)$  be a copy of the backbone decomposition for a  $(\psi, 0)$ -CSBP, where the process  $Z^\theta$ , the backbone, is an  $(F, 0)$ -GW process as described above with the additional property that  $Z_0^\theta = n \in \{0, 1, 2, \dots\}$ , the process of continuous mass,  $\Lambda^\theta$ , is described as above with the additional property that  $\Lambda_0^\theta = y$ . Let  $\mathbf{P}_{(y,n)}^\theta$  be the law of  $(Z^\theta, \Lambda^\theta)$ . Then

$$\mathbf{E}_{(y,n)}^\theta (r^{Z_t^\theta} e^{-\theta \Lambda_t^\theta}) = e^{-y u_t^*(\theta) - n w_t(r, \theta)},$$

where

$$\lambda^* (1 - e^{-w_t(r, \theta)}) = u_t(\theta + \lambda^* (1 - r)) - u_t^*(\theta). \tag{9}$$

**Proof.** According to Theorem 1 in Berestycki et al. (2011),

$$\mathbf{E}_{(y,n)}^\theta (r^{Z_t^\theta} e^{-\theta \Lambda_t^\theta}) = e^{-y u_t^*(\theta) - n w_t(r, \theta)},$$

where  $e^{-w_t(r, \theta)}$  is the unique  $[0, 1]$ -valued solution to the integral equation

$$e^{-w_t(r, \theta)} = r + \frac{1}{\lambda^*} \int_0^t ds [\psi^*(-\lambda^* e^{-w_{t-s}(r, \theta)} + u_{t-s}^*(\theta)) - \psi^*(u_{t-s}^*(\theta))]$$

for  $t \geq 0$ . With the help of (2) and (4), it is straightforward to show that  $u_t^*(\theta) + \lambda^*(1 - e^{-w_t(r, \theta)})$  solves (1) with initial condition  $\lambda = \theta + \lambda^*(1 - r)$ . Therefore we have

$$\lambda^* (1 - e^{-w_t(r, \theta)}) = u_t(\theta + \lambda^* (1 - r)) - u_t^*(\theta)$$

as required.  $\square$

**Proof of Theorem 2.1.** For the first part we need to show that the process  $(\Lambda, \mathbf{P}_x)$  is Markovian and its semi-group agrees with that of  $(X, \mathbb{P}_x)$ . For the first part it suffices to show that for  $r \in [0, 1]$  and  $\theta \geq 0$ ,

$$\mathbf{E}_x(r^{Z_t} e^{-\theta \Lambda_t}) = \mathbf{E}_x(e^{-(\theta + \lambda^*(1-r)) \Lambda_t}). \tag{10}$$

In fact, a little thought shows that both of these facts can be simultaneously established by proving that for all  $x \geq 0$ ,  $r \in [0, 1]$  and  $\theta \geq 0$ ,

$$\mathbf{E}_x(r^{Z_t} e^{-\theta \Lambda_t}) = e^{-x u_t(\theta + \lambda^*(1-r)) - \int_0^t \varphi(u_{t-s}(\theta + \lambda^*(1-r))) ds}. \tag{11}$$

Indeed, note that (11) directly implies (10) and by setting  $r = 1$  in (11) we also see that  $\Lambda$  has the required semi-group.

To this end, let us split the process  $(Z, \Lambda)$  into the independent sum of processes  $(Z^\emptyset, \Lambda^\emptyset)$  and  $(Z^I, \Lambda^I)$  where the first is an independent copy of the backbone decomposition for a  $(\psi, 0)$ -CSBP and  $(Z^I, \Lambda^I)$  is the part of  $Z$  rooted at immigration times together with its dressing. Note immediately by independence we have that

$$\mathbf{E}_x(r^{Z_t} e^{-\theta \Lambda_t}) = \mathbf{E}_x(r^{Z_t^\emptyset} e^{-\theta \Lambda_t^\emptyset}) \mathbf{E}_x(r^{Z_t^I} e^{-\theta \Lambda_t^I}) = e^{-xu_t(\theta + \lambda^*(1-r))} \mathbf{E}_x(r^{Z_t^I} e^{-\theta \Lambda_t^I}),$$

where the second equality follows from the Poissonization that is known to hold for the backbone embedding of  $(\psi, 0)$ -CSBPs as described in Berestycki et al. (2011) (see also the discussion in Section 1).

It therefore suffices to prove that for all  $x \geq 0, s \in [0, 1]$  and  $\theta \geq 0$

$$\mathbf{E}_x(r^{Z_t^I} e^{-\theta \Lambda_t^I}) = e^{-\int_0^t \varphi(u_{t-s}(\theta + \lambda^*(1-r))) ds}.$$

With this as our goal, let us now write for each  $t \geq 0$ ,

$$\Lambda_t^I = \Lambda_t^{I,1} + \Lambda_t^{I,2},$$

where  $\Lambda_t^{I,1}$  is the mass at time  $t$  due to the Poissonian dressing along the time-line between each immigration of  $Z$  and  $\Lambda^{I,2}$  is the mass at time  $t$  due to the dressing at immigration times together with the dressing of the immigrating  $(F, 0)$ -GW processes. First note that with the help of Campbell's Formula,

$$\begin{aligned} \mathbf{E}_x(e^{-\theta \Lambda_t^{I,1}}) &= \exp \left\{ -\int_0^t ds \cdot \delta \mathbb{N}^*(1 - e^{-\theta X_{t-s}}) - \int_{(0,\infty)} e^{-\lambda^* y} \nu(dy) \mathbb{E}_y^*(1 - e^{-\theta X_{t-s}}) \right\} \\ &= \exp \left\{ -\int_0^t ds \cdot \delta u_{t-s}^*(\theta) - \int_{(0,\infty)} (1 - e^{-y u_{t-s}^*(\theta)}) e^{-\lambda^* y} \nu(dy) \right\} \\ &= \exp \left\{ -\int_0^t ds \cdot \varphi^*(u_{t-s}^*(\theta)) \right\}, \end{aligned} \tag{12}$$

where

$$\varphi^*(\lambda) := \varphi(\lambda + \lambda^*) - \varphi(\lambda^*) = \delta \lambda + \int_{(0,\infty)} (1 - e^{-\lambda y}) e^{-\lambda^* y} \nu(dy).$$

Recalling that the immigration of  $Z$  is characterised by  $G$ , by using Lemma 3.1 and applying Campbell's Formula, we have

$$\begin{aligned} \mathbf{E}_x(r^{Z_t^I} e^{-\theta \Lambda_t^{I,2}}) &= \exp \left\{ -\int_0^t ds \cdot \varphi(\lambda^*) \sum_{n \geq 1} \int_{(0,\infty)} \pi_n(dy) (1 - e^{-y u_{t-s}^*(\theta) - n w_{t-s}(r,\theta)}) \right\} \\ &= \exp \left\{ -\int_0^t ds \cdot \left( \delta \lambda^* + \int_{(0,\infty)} (1 - e^{-\lambda^* y}) \nu(dy) \right. \right. \\ &\quad \left. \left. - \int_{(0,\infty)} \sum_{n \geq 1} \frac{(\lambda^* y e^{-w_{t-s}(r,\theta)})^n}{n!} e^{-\lambda^* y} e^{-y u_{t-s}^*(\theta)} \nu(dy) - \delta \lambda^* e^{-w_{t-s}(r,\theta)} \right) \right\} \\ &= \exp \left\{ -\int_0^t ds \cdot \left( \varphi(\lambda^*) - \int_{(0,\infty)} (\exp\{\lambda^* y e^{-w_{t-s}(r,\theta)}\} - 1) e^{-y(\lambda^* + u_{t-s}^*(\theta))} \nu(dy) - \delta \lambda^* e^{-w_{t-s}(r,\theta)} \right) \right\} \\ &= \exp \left\{ -\int_0^t ds \cdot \left( \varphi(\lambda^*) + \varphi_{u_{t-s}^*(\theta)}^*(-\lambda^* e^{-w_{t-s}(r,\theta)}) \right) \right\}, \end{aligned} \tag{13}$$

where for  $u \geq -\lambda^*$ ,

$$\begin{aligned} \varphi_u^*(\lambda) &= \varphi^*(\lambda + u) - \varphi^*(u) = \varphi(\lambda + \lambda^* + u) - \varphi(\lambda^* + u) \\ &= \delta \lambda + \int_{(0,\infty)} (1 - e^{-\lambda y}) e^{-y(\lambda^* + u)} \nu(dy). \end{aligned}$$

Putting the pieces together in (12) and (13) with the help of (9), we see that

$$\begin{aligned} \mathbf{E}_x(r^{Z_t^I} e^{-\theta \Lambda_t^I}) &= \exp \left\{ -\int_0^t ds \cdot \varphi(u_{t-s}^*(\theta) + \lambda^*(1 - e^{-w_{t-s}(r,\theta)})) \right\} \\ &= \exp \left\{ -\int_0^t ds \cdot \varphi(u_{t-s}(\theta + \lambda^*(1 - r))) \right\} \end{aligned}$$

as required.  $\square$

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