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General path integrals and stable SDEs

Abstract. The theory of one-dimensional stochastic differential equations driven by Brownian motion is classical and has been largely understood for several decades. For stochastic differential equations with jumps the picture is still incomplete, and even some of the most basic questions are only partially understood. In the present article we study existence and uniqueness of weak solutions to

$$\mathrm{d}Z_t = \sigma(Z_{t-})\,\mathrm{d}X_t$$

driven by a two-sided α-stable Lévy process, in the spirit of the classical Engelbert–Schmidt time-change approach. Extending and completing results of Zanzotto we derive a complete characterisation for existence and uniqueness of weak solutions for $\alpha \in (0, 1)$. Our approach is not based on classical stochastic calculus arguments but on the general theory of Markov processes. We prove integral tests for finiteness of path integrals under minimal assumptions.

Keywords. Stochastic differential equations, stable processes, Markov processes, perpetuities, time change

1. Introduction

Itô diffusions are solutions of stochastic differential equations driven by a Brownian motion, and their study as stochastic processes in their own right has a rich history dating back to the foundational works of Feller [14, 15] in the 1950s. Itô diffusions continue to motivate theoretical research, due in some part to their many applications, and this sustained interest has also given rise to various generalisations. The present article is concerned with solutions of stochastic differential equations driven not by a Brownian motion but by a Lévy process. Our objective is to study solutions to the one-dimensional driftless stochastic differential equation

$$\mathrm{d}Z_t = \sigma(Z_{t-})\,\mathrm{d}X_t, \quad Z_0 = z \in \mathbb{R}, \tag{1}$$

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where X is a Lévy process and σ is a non-negative, measurable function on $\mathbb{R}$. Solution processes Z are distinguished in contrast to continuous Itô diffusions by the presence of discontinuities and are therefore known as *jump diffusions*. In the present article we always assume the driving Lévy process X is *strictly stable*, that is, the scaling property

$$(c X_{c^{-\alpha}t}, t \geq 0) \text{ under } \mathbb{P}_x \text{ is equal in law to } (X_t, t \geq 0) \text{ under } \mathbb{P}_{cx}$$

holds for some $\alpha \in (0, 2]$. The strictly stable process of index $\alpha = 2$ is simply the Brownian motion.

In 1981 Engelbert and Schmidt [13] proved a zero-one law for finiteness of so-called finite-time path integrals $\int_0^t f(B_s)\, ds$ of the Brownian motion. In the same paper they made use of their zero-one law and a time-change representation of Brownian SDEs to comprehensively address the question of weak existence and uniqueness of solutions to (1) – see [13, Theorem 4] or Karatzas and Shreve [22, Theorems 3.4.2 and 3.4.6]. Almost two decades later Zanzotto [34] extended Engelbert and Schmidt's results to stable processes of index $\alpha \in (1, 2]$. Proving the time-change representation for stable SDEs is significantly harder than for Brownian SDEs, because it is no longer possible to argue using quadratic variation. Despite this difficulty, Zanzotto's result is nearly identical in form to that for the Brownian motion: see Theorem 2.32 of his paper. A few years later Zanzotto generalised his result further – see [36, Theorems 2.2 and 2.6], but was still restricted to $\alpha \in (1, 2]$ for precise results. Harder results for SDEs concern strong existence and pathwise uniqueness. Following seminal work of Yamada and Watanabe [33] for the Brownian case, strong existence and pathwise uniqueness have been studied in many variations. The classical setup gives positive results for Lipschitz continuous drift and $\frac{1}{2}$-Hölder continuous noise coefficient. Extensions to stable SDEs are surprisingly recent, mostly motivated by the study of continuous state branching processes. For spectrally positive stable processes (only positive jumps) and $\alpha \in (1, 2)$ the Yamada/Watanabe argument was generalised by Li and Mytnik [26] leading to Lipschitz continuous drift and $(1 - 1/\alpha)$-Hölder continuous noise coefficient. The symmetric case was dealt with by Bass [4] leading to $1/\alpha$-Hölder continuous noise coefficient. Since $1/\alpha > 1$ for $\alpha \in (0, 1)$ it comes as no surprise that results must be structurally different for small α. Partial results for $\alpha \in (0, 1)$ on the pathwise uniqueness problem appeared in the past two decades (see for instance Bass–Burdzy–Chen [5] for a counterexample). To the best of our knowledge proofs for sharp conditions are still unknown. The aim of the present article is not to solve the strong existence/uniqueness problem, but to give a complete solution to the weak existence/uniqueness problem in the spirit of Engelbert and Schmidt's results.

The works of Engelbert–Schmidt and Zanzotto together answer questions for weak solutions to (1) in the case of $\alpha \in (1, 2]$. In all of them the time change method is used to reduce the SDE to a time change of the driving process X using the time change $\int_0^t \sigma(X_s)^{-\alpha}\, ds$, which is usually called a *path integral* (or *perpetual integral*). The crucial tool for analysing those path integrals is the occupation time formula and local time for X. This is the barrier to results for $\alpha \in (0, 1]$: while the time-change representations of Zanzotto continue to hold, there is no local time to work with. The challenge we tackle

in the present article is the study of finite- and infinite-time path integrals of the form

$$I_t^f = \int_0^t f(X_s)\, \mathrm{d}s, \quad t \in (0, \infty],$$

where f is a measurable function taking values in $[0, \infty]$. The value ∞ for f is crucial as it corresponds to the zeros of σ. What we seek is a collection of 'integral tests', which are statements tying the law of a path integral to finiteness of a related deterministic integral. Path integrals have attracted some interest in recent years, mostly for restricted classes of integrands (such as f being locally integrable or continuous) and processes (such as Lévy processes or diffusion processes) leading to very clean results. For X a Brownian motion with positive drift, Salminen and Yor [29, 30] obtained results via the Ray–Knight theorem, and similar results for spectrally negative Lévy processes can be found in Koshnevisan, Salminen, and Yor [23]. The recent article of Kolb and Savov [24] has the most up-to-date results for Lévy processes. The technique we use only requires minimal assumptions, i.e. f measurable and X a standard Markov process. Allowing f to be infinite changes the picture completely as known zero-one laws for path integrals (see for instance Kyprianou and Döring [11]) fail immediately for transient X (take f to be an infinite indicator on a set which X visits with probability in $(0, 1)$). Hence, it is clear that for a general theorem sets which are hit with probability less than 1 must appear in the statements of results. Such sets are called *avoidable*; we call their complements *supportive*, and these sets will play a fundamental role in this work. Assuming only that X is a standard Markov process on a general state space E with potential measure U we will prove the following theorem on path integrals.

Theorem. *Let $f : E \to [0, \infty]$ be measurable and X a standard Markov process on E with (possibly infinite) lifetime ζ. Let $z \in E$. Then the following are equivalent:*

(i) $\mathbb{P}_z(\int_0^\zeta f(X_s)\, \mathrm{d}s < \infty) > 0.$

(ii) *The integral test $\int_{E \setminus B} f(x)\, U(z, \mathrm{d}x) < \infty$ holds for a $\mathbb{P}_z$-avoidable set B.*

We remind the reader that the integral test in (ii) says nothing but

$$\mathbb{E}_z\left[\int_0^\zeta \mathbf{1}_{B^C}(X_s) f(X_s)\, \mathrm{d}s \right] < \infty.$$

Thus, the path integral is finite with positive probability if and only if it has finite mean away from an avoidable set. In several situations the potential measure U is explicit and avoidable sets can be described analytically to turn the integral tests into analytic statements.

The general form of the path integral theorem allows us to deduce a couple of consequences on finite-time path integrals which we will need to study the SDE (1) via time-change techniques. For finite-time path integrals the role of avoidable sets is replaced by sets which are avoided for a positive amount of time, called *thin sets*. Let us summarise the main findings for the SDE (1) driven by a two-sided stable Lévy process with $\alpha \in (0, 1)$; see Section 5 for the theorems.

Theorem. *If $N(\sigma)$ denotes the zero-set of σ and*

$$\mathcal{O}(\sigma, \alpha) = \left\{ x \in \mathbb{R} : \int_{\mathbb{R} \setminus B} \sigma(y)^{-\alpha} |x - y|^{\alpha - 1} \, dy = \infty \text{ for all } \mathbb{P}_x\text{-thin sets } B \right\} \quad (2)$$

denotes the set of irregular points, then the following statements hold:

(i) *For fixed $z \in \mathbb{R}$ there exists a non-trivial (i.e. non-constant) local weak solution if and only if $z \notin \mathcal{O}(\sigma, \alpha)$.*

(ii) *A global weak solution exists for all initial conditions $z \in \mathbb{R}$ if and only if $\mathcal{O}(\sigma, \alpha) \subseteq N(\sigma)$.*

(iii) *A non-trivial global weak solution exists for all $z \in \mathbb{R}$ if and only if $\mathcal{O}(\sigma, \alpha) = \emptyset$.*

(iv) *There exists a global weak solution for all $z \in \mathbb{R}$, each of which is unique in law, if and only if $\mathcal{O}(\sigma, \alpha) = N(\sigma)$.*

Since thin sets of stable processes have analytic descriptions through capacities, the theorem gives a full analytic description of weak solutions for the stable SDE (1). The analytic description allows for the $\mathbb{P}_x$-thin sets to be removed in many examples, such as when σ is monotone on either side of a zero at x.

Statements (i)–(iv) are identical to the Engelbert–Schmidt–Zanzotto theorems for Brownian SDEs and stable SDEs for $\alpha \in (1, 2)$, but differ in the set $\mathcal{O}(\sigma, \alpha)$ of irregular points, which in those cases is given by

$$\mathcal{O}(\sigma, \alpha) = \left\{ x \in \mathbb{R} : \int_x \sigma(y)^{-\alpha} \, dy = \infty \right\}.$$

This set is different from (2) in two respects. First, since complements of $\mathbb{P}_x$-thin sets contain balls around x, for $\alpha \in (1, 2)$ those sets do not appear. Second, the polynomial factor in the integral tests is not present; for $\alpha \in (0, 1)$ this factor has a pole, which puts a tighter restriction on the function σ. Let us quickly give two examples that highlight the two differences.

Integral test: To get an idea for the additional factor $|y|^{\alpha - 1}$ it is instructive to check the simple example

$$dZ_t = |Z_t|^\beta \, dX_t, \quad Z_0 = 0. \quad (3)$$

Since the trivial solution $Z \equiv 0$ always exists, using (i) the Engelbert–Schmidt–Zanzotto criterion implies that weak uniqueness holds for $\alpha \in (1, 2]$ if and only if $\int_0 \sigma^{-\alpha}(y) \, dy = \int_0 |y|^{-\beta\alpha} \, dy = +\infty$ around 0, which is equivalent to $\beta > 1/\alpha$ and thus coincides with the Yamada–Watanabe–Bass criterion for pathwise uniqueness. Since $\sigma(x) = |x|^\beta$ is Lipschitz continuous for $\beta \geq 1$ the Engelbert–Schmidt–Zanzotto integral test must be wrong for $\alpha \in (0, 1)$. Let us check our modified integral test with the additional factor $|y|^{\alpha-1}$ but ignoring the thin sets (which we justify later in Corollary 5.7 as σ has a monotone zero at 0):

$$\int_0 \sigma(y)^{-\alpha} |y|^{\alpha-1} \, dy = \int_0 |y|^{-\beta\alpha} |y|^{\alpha-1} \, dy = \int_0 |y|^{\alpha(1-\beta)-1} \, dy = +\infty \iff \beta \geq 1.$$

Hence, (i) implies that weak uniqueness for the SDE (3) holds if and only if $\beta \geq 1$, the Lipschitz case. In contrast to $\alpha \in (1, 2)$ the statement is independent of α which bears some similarity to the results of Bass–Burdzy–Chen [5]. Of course, this is an artifact of σ being a polynomial, in the general setting our integral test is not independent of α.

Thin sets: The next example is helpful for understanding how the path behaviour of the driving stable process forces the thin sets to appear in the integral test. For a two-sided stable process with $\alpha \in (0, 1)$ we can choose a $\mathbb{P}_0$-thin set A with positive potential (for example a union of well-chosen small disjoint intervals accumulating at 0, similarly to Example 2.14). If we define $\sigma = \mathbf{1}_{A^C}$ we can construct a local weak solution by running the stable process until it hits A. Since A is thin at 0, the solution is non-trivial. Without the thin set $B = A$ in the integral test, statement (i) would fail as $\sigma^{-\alpha}$ equals $+\infty$ on A and A has positive potential. The situation is simpler for $\alpha \in (1, 2]$ because in that case a set is thin at 0 if and only if its complement contains a ball around 0.

The present article only deals with SDEs for two-sided stable processes with $\alpha \in (0, 1)$, exploiting crucially the transience. We were only able to derive partial results for the Cauchy process ($\alpha = 1$) which we defer to future research.

Organisation of the article

The article is organised as follows. The core of the work is carried out in Section 2, where we present the general integral test for infinite-time-horizon path integrals $\int_0^\infty f(X_s) \, \mathrm{d}s$ for general standard processes. By killing these processes in various ways we deduce integral tests for finite-time-horizon integral tests $\int_0^t f(X_s) \, \mathrm{d}s$ in Sections 3 and 4. In Section 5 the results from Sections 2–4 are translated into SDE theorems using Zanzotto's time-change techniques.

2. Infinite-time path integral tests

2.1. Setting

Before we begin it is worthwhile precisely establishing the setting in which our theorems are proved, as several aspects of the proofs rely upon it. We do not assume anything out of the ordinary, and the reader familiar with the potential theory of Markov processes can skip ahead to Section 2.2.

Take E to be a locally compact Hausdorff space with a countable base, and let Δ be adjoined to E as the point at infinity if E is non-compact, and as an isolated point if E is compact. Let $\mathcal{E}$ be the Borel σ-algebra on E, and $\mathcal{E}_\Delta$ the Borel σ-algebra on $E_\Delta := E \cup \{\Delta\}$. Let D be the space of paths $w : [0, \infty] \to E_\Delta$ such that $w(\infty) = \Delta$, and if $w(t) = \Delta$ then $w(s) = \Delta$ for all $s \geq t$. Let $(X_t, t \in [0, \infty])$ be the family of coordinate maps $X_t : D \to E_\Delta$, i.e. $X_t(w) = w_t$ for all $t \in [0, \infty]$, and denote by $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$ and $\mathcal{F} = \sigma(X_s, s \in [0, \infty])$ the canonical filtration of X. Let $(\theta_t, t \in [0, \infty])$ be the family of translation maps $\theta_t : D \to D : w \mapsto (w_{t+s}, s \geq 0)$, and

let $(\mathbb{P}_x,\ x \in E_\Delta)$ be a family of probability measures on $(D, \mathcal{F})$ satisfying the following conditions:

(i) (*Regularity*) For each measurable $Y : D \to E$ and each $B \in \mathcal{E}$, the map $x \mapsto \mathbb{P}_x(Y \in B)$ is $\mathcal{E}_\Delta$-measurable.

(ii) (*Normality*) $\mathbb{P}_x(X_0 = x) = 1$ for all $x \in E_\Delta$.

(iii) (*Càdlàg paths*) The path functions $t \to X_t(w)$ are right-continuous on $[0, \infty)$ and have left limits on $[0, \zeta)$ $\mathbb{P}_x$-almost surely for all $x \in E$, where the random time $\zeta(w) := \inf\{t > 0 : X_t(w) = \Delta\}$ is called the *lifetime* of X.

(iv) (*Quasi-left-continuity*) For any sequence T_n of $\mathcal{F}_t$-stopping times with limit T, we have $X_{T_n} \to X_T$ $\mathbb{P}_x$-almost surely on $\{T < \zeta\}$, for all $x \in E$.

(v) (*Strong Markov property*) For all $x \in E_\Delta$, stopping times T, $s \in [0, \infty]$, and bounded measurable f the strong Markov property holds:

$$\mathbb{E}_x[f(X_{T+s})|\mathcal{F}_T] = \mathbb{E}_{X_T}[f(X_s)] \quad \mathbb{P}_x\text{-almost surely.}$$

The family $(\mathbb{P}_x,\ x \in E_\Delta)$ is called a *standard Markov process* on the state space $(E, \mathcal{E})$, with *cemetary state* Δ and *lifetime* ζ. It is often more convenient to refer to X (with associated laws $\mathbb{P}_x$) as the process. The assumption that E is Hausdorff ensures that compact subsets of E are closed, and therefore $\mathcal{E}$-measurable. In the proofs that follow we shall also work with a strong Markov process (a time-changed Markov process) which may not satisfy either (ii) or (iv).

For future use we define, for a Borel set $B \in \mathcal{E}_\Delta$, the random times

$$D_B := \inf\{t \geq 0 : X_t \in B\}, \quad T_B := \inf\{t > 0 : X_t \in B\}, \quad L_B := \sup\{t \geq 0 : X_t \in B\}.$$

Both D_B and T_B are stopping times, and are called the *first entry time* and *first hitting time* of B respectively. L_B is not in general a stopping time, and is called the *last exit time* of B. By convention we set $\inf \emptyset = \infty$ and $\sup \emptyset = 0$.

The (*infinite-time-horizon*) *path integral* over a Markov process X on the state space $(E, \mathcal{E})$ and for a non-negative $\mathcal{E}_\Delta$-measurable function $f : E_\Delta \to [0, \infty]$ is defined to be

$$I_\infty^f = \int_0^\infty f(X_s)\,\mathrm{d}s.$$

If X has semigroup $(P_t)_{0 < t < \infty}$ and lifetime ζ, then the *potential measure* of X is defined as

$$U(z, B) = \int_0^\infty P_t(z, B)\,\mathrm{d}t = \mathbb{E}_z\left[\int_0^\zeta \mathbf{1}_{(X_t \in B)}\,\mathrm{d}t\right], \quad \text{where } P_t(z, B) = \mathbb{P}_z(X_t \in B),$$

for $x \in E$ and measurable B, with the corresponding *potential operator*

$$Uf(z) := \int_E f(x)\,U(z, \mathrm{d}x) = \mathbb{E}_z\left[\int_0^\zeta f(X_s)\,\mathrm{d}s\right]$$

for $B \in \mathcal{E}_\Delta$ and f measurable. In the Markov process literature, functions f on E are commonly extended over E_Δ by setting $f(\Delta) = 0$, and this conveniently allows us to replace ζ in the definition of U by ∞. In what follows we will study how finiteness of I_∞^f (almost surely and with positive probability) is related to the law of the potential operator Uf.

Several complementary concepts of transience exist for Markov processes. The classical definition is that a standard Markov process on the state space $(E, \mathcal{E})$ is *transient* if there exists a strictly positive, universally measurable function h such that Uh is finite everywhere. This concept is not strong enough for our purposes, and we instead define a *transient standard Markov process* as one which simultaneously satisfies the following two conditions:

 (i) $U(\cdot, K)$ is bounded for all compact K;

(ii) $\mathbb{P}_z(L_K < \infty) = 1$ for all compact K and all $z \in E$.

Chung and Walsh [10, §3.7] show that both conditions above imply classical transience, and in Corollary 2.3 of his seminal paper Getoor [16] demonstrated that, under a regularity condition on the excessive functions of X, they are equivalent to it. If further

(iii) $\mathbb{P}_z(L_K < \zeta) = 1$ for all compact K and all $z \in E$,

we say that X is *strongly transient*. Clearly any transient process with an almost surely infinite lifetime is also strongly transient. It is worth mentioning that for Lévy processes all the definitions of transience discussed above are automatically equivalent – see for example Sato [31, Theorem 35.4] – and in addition are equivalent to the condition that

(iv) $\lim_{t \to \infty} |X_t| = \infty$ almost surely.

Our analysis of path integrals uses the trick of viewing a path integral as the explosion time of a time-changed process. To define the time change we use the finite-time path integrals

$$I_t^f = \int_0^t f(X_s) \, \mathrm{d}s, \quad t \in [0, \infty).$$

Note that the convention $f(\Delta) = 0$ allows us to write either $t \in [0, \infty)$ or $t \in [0, \zeta)$. The right-continuous inverse of $(I_t^f, t \geq 0)$ will be denoted by φ^f. Each φ_t^f is a stopping time for X. When f is unambiguous we shall drop it from the notation. The time-changed process $(Y_t, t \in [0, \infty])$ of interest is defined as

$$Y_t = X_{\varphi_t} \quad \text{for } t \in [0, \infty), \quad Y_\infty = \Delta, \tag{4}$$

which moves on the same state space E as X. It is not immediately clear which of the properties of X are inherited by X_φ. Volkonskiĭ [32] proved that X_φ is a strong Markov process if

$$t \mapsto I_t \text{ is continuous on the whole of } [0, \infty), \tag{5}$$

which is equivalent to assuming $I_\infty = \bar{I} := \sup_{t \in [0,\infty): I_t < \infty} I_t$. Note that

$$\varphi_{\bar{I}} = \varphi_\infty := \lim_{t \to \infty} \varphi_t. \tag{6}$$

A sufficient condition for (5) to hold almost surely, though not necessary, is that

$$I_t < \infty \quad \text{for all } t \in [0, \infty) \text{ almost surely,} \tag{7}$$

which is what we are going to check when we use the strong Markov property of the time change. This implies (indeed is equivalent to) $\varphi_\infty = \infty$ almost surely, and we get

$$\int_0^\infty \sigma(X_{\varphi_s}) \, \mathrm{d}s = \int_0^\infty \sigma(X_t) f(X_t) \, \mathrm{d}t \quad \text{almost surely.} \tag{8}$$

This will be of particular use to us later in this section.

As alluded to earlier, in contrast to existing theory of stable SDEs for $\alpha \in (1, 2]$, our analysis is based on general Markov process theory. For this reason we now introduce an object which commonly appears in potential theory of Markov processes, called the *fine topology*. If M is a Borel set then $x \in E$ is called a *regular point* for M if $\mathbb{P}_x(T_M > 0) = 0$. We denote the set of regular points by M^r, and note that $X_{T_M} \in M^r \cup M$ $\mathbb{P}_y$-almost surely for all $y \in E$. A measurable set is called *finely closed* if $M^r \subseteq M$ and *finely open* if the complement is finely closed. The collection of finely open sets with compact closure in E forms the base of a topology called the *fine topology*. It is immediately seen that right-continuity of paths of X implies that all open sets are finely open, and therefore closed sets are finely closed.

We will also make use of a result on the regularity of q-excessive functions, for which we refer to Blumenthal and Getoor [9, Theorem VI(4.9)] or Chung and Walsh [10, Theorem 13.80]. If X is a standard Markov process on $(E, \mathcal{E})$ with strong Feller resolvents (and dual resolvents, which is automatically fulfilled for Lévy processes) then the following are equivalent:

(H) If $K \subseteq E$ is a non-polar set then some point of K must be regular for K.

(R) If f is a locally integrable q-excessive function for some $q > 0$, then $t \mapsto f(X_t)$ is continuous whenever $t \mapsto X_t$ is continuous on $[0, \zeta)$.

(R)′ If f is a locally integrable q-excessive function for some $q > 0$, and T_n is an increasing sequence of stopping times with limit T, then $f(X_{T_n}) \to f(X_T)$ on $\{T < \zeta\}$.

The assumption that X has strong Feller resolvents is fairly strong for general standard Markov X, but Hawkes [18] showed that for Lévy processes it is equivalent to existence of a density u^q for the q-potential measure, which is known to hold for a broad class of Lévy processes, including stable processes (see Sato [31, Theorem 41.15, Remark 41.20]). Property (H) is commonly known as *Hunt's condition*, and equivalent formulations can be found in Chung and Walsh [10, Chapter 13]. Hunt's condition is satisfied by all non-trivial stable processes on $\mathbb{R}^d$ (see Kanda [20, 21]), but for general Lévy processes the question of whether or not it holds remains an open problem (for an example of a Lévy process not satisfying (H) see Hawkes [18, Theorem 4.1], and for further discussion see Bertoin [6, Chapter II.7]). Property (R) is called *regularity of f*, and it is interesting to compare it with the fact that for every standard Markov process X, $t \mapsto f(X_t)$ is right-continuous and has left-hand limits on $[0, \infty)$ for any q-excessive function f.

2.2. *Positive probability case*

The classification of finiteness and the computation of distributions of path integrals I_∞^f and I_t^f have a rich history. Integral tests were derived for different classes of stochastic processes under different assumptions on f. Applications can be found for instance in SDEs, branching processes, or random walks in random environments. The closest article in its general form was recently published by Kolb and Savov [22]. For a Lévy process drifting to $+\infty$ a complete integral-test characterisation of finiteness was proved under rather strong assumptions on f – most importantly, f was assumed finite:

Theorem 2.1 (Kolb–Savov [22]). *If the measurable function $f : \mathbb{R} \to [0, \infty)$ is either continuous or ultimately non-increasing and X is a transient Lévy process with limit $+\infty$, the following are equivalent:*

(i) $\mathbb{P}_z(\int_0^\infty f(X_s)\, \mathrm{d}s = \infty) > 0.$

(ii) $\mathbb{P}_z(\int_0^\infty f(X_s)\, \mathrm{d}s = \infty) = 1.$

(iii) *The integral test $\int_{\mathbb{R}\setminus B} f(x)\, U(z, \mathrm{d}x) = \infty$ holds for all Borel sets B such that $\mathbb{P}_z(L_B < \infty) = 1$.*

The sets B appearing in (iii) are called $\mathbb{P}_z$-*transient* sets. The occurrence of transient sets is clearly necessary as the finite-time occupation of a transient set does not influence finiteness of the path integral due to the assumptions on f. In Kolb and Savov's theorem either of the two assumptions on f, in combination with the fact that X is a Lévy process, is enough to ensure that the zero-one law $\mathbb{P}_z(I_\infty^f < \infty) \in \{0, 1\}$ holds. It might be unclear how to use the theorem due to the occurrence of the stochastically defined transient sets. Kolb and Savov provide several examples of Lévy processes for which the sets can be characterised. For example, if X has local time at points, the theorem works with $B = (-\infty, a]$, $a \in \mathbb{R}$, and recovers the integral test at infinity of [11].

The present article extends the Kolb–Savov theorem in two directions. First, we shall allow extended measurable functions $f : \mathbb{R} \to [0, \infty]$ without any further assumptions, and second, we shall consider X to be a general standard Markov process. The generalisation to extended measurable functions is crucial in order to study SDEs with singular coefficients, such as $\sigma(x) = |x|^\beta$, since path integrals for $f = \sigma^{-\alpha}$ need to be analysed as part of the representation of solutions via a time change. The generalisation to standard Markov processes is useful for instance to understand finite-time-horizon path integrals by applying the infinite-time-horizon theorem to killed processes.

The main difficulty arising from our weaker assumptions on f is a breakdown of the zero-one law for path integrals, even for a Lévy process. As an example, let X be a symmetric α-stable process with $\alpha \in (0, 1)$. Then it is well-known that $\mathbb{P}_0(T_{[1,2]} < \infty) \in (0, 1)$, which implies that $\mathbb{P}_0\left(\int_0^\infty +\infty \mathbf{1}_{[1,2]}(X_s)\, \mathrm{d}s < \infty\right) \in (0, 1)$. Similarly to transient sets in the setting of Kolb and Savov, avoidable sets must play a special role in the setting of path integrals for which zero-one laws fail. As a consequence we will prove different theorems for the positive probability case and the probability 1 case.

In this section we provide a complete classification of finiteness with positive probability of infinite-time-horizon path integrals for standard Markov processes. The almost sure case and the general form of the Kolb–Savov theorem will be treated in the next section.

Definition 2.2. Let X be a standard Markov process on E with lifetime ζ. A Borel set $B \in \mathcal{E}$ is called $\mathbb{P}_z$-*avoidable* if $\mathbb{P}_z(D_B < \zeta) < 1$. If B is $\mathbb{P}_z$-avoidable then its complement $M = E \setminus B$ is called $\mathbb{P}_z$-*supportive*, and satisfies

$$\mathbb{P}_z(X_t \in M \text{ for all } t \in [0, \zeta)) > 0.$$

In general it is not at all clear what form avoidable or supportive sets should take but a vast literature exists for special processes. Here are the most relevant examples for the present article:

- If X is a recurrent Markov process then only polar sets are avoidable.
- If X is a two-sided stable process on $\mathbb{R}$ of index $\alpha \in (1, 2]$ then only the empty set is avoidable.
- If X is a two-sided stable process on $\mathbb{R}$ of index $\alpha \in (0, 1)$ then any compact set not containing z is $\mathbb{P}_z$-avoidable.

We can now formulate our general theorem on path integrals, an integral test which is not a zero-one law, and which requires minimal assumptions on f and X.

Theorem 2.3. *Let* $f : E \to [0, \infty]$ *be measurable and* X *a standard Markov process with* (*possibly infinite*) *lifetime* ζ. *Let* $z \in E$. *Then the following are equivalent:*

(i) $\mathbb{P}_z(\int_0^\infty f(X_s) \, \mathrm{d}s < \infty) > 0$.

(ii) *The integral test* $\int_{E \setminus B} f(x) \, U(z, \mathrm{d}x) < \infty$ *holds for a* $\mathbb{P}_z$-*avoidable set* B.

Just as the occurrence of transient sets is clearly needed for the Kolb–Savov theorem, the occurrence of avoidable (or supportive) sets is needed here since f can be anything in avoidable sets without harming the positivity of the probability of finite path integral. As for the Kolb–Savov theorem, extra knowledge on avoidable sets for particular processes (such as the Wiener test for stable processes) can turn the integral test into a purely analytic statement.

An equivalent formulation of Theorem 2.3 using supportive sets of Definition 2.2 is that the following are equivalent:

(i) $\mathbb{P}_z(\int_0^\infty f(X_s) \, \mathrm{d}s = \infty) = 1$.

(ii) The integral test $\int_M f(x) \, U(z, \mathrm{d}x) = \infty$ holds for all $\mathbb{P}_z$-supportive sets M.

Supportive sets will appear most often in our proofs as an explicitly constructed set of 'safe points', in the sense that those are the issuing points of the state space from which the path integral remains finite with positive probability. The idea of the proof of Theorem 2.3 is to use the observation that $I_\infty^f < \infty$ is equivalent to finite-time explosion of the time-changed strong Markov process X_φ from (4) if the time change is well-behaved in the sense of (7). The latter leads to the introduction of an additional indicator over a so-called

super-finite set for the pair (X, f), in the forthcoming Lemma 2.6 and Proposition 2.9. For the analysis of explosion we exploit an old argument from the general study of transience due to Getoor [16]; see Proposition 2.7.

In all of this section, X is a standard Markov process on a state space $(E, \mathcal{E})$, and $f : E \to [0, \infty]$ is a measurable function.

Definition 2.4. A *super-finite* set for (X, f) is defined by

$$M = \left\{ y \in E : \mathbb{P}_y \left(\int_0^\infty f(X_s) \, ds \le n \right) > c \right\}.$$

for some $n \in \mathbb{N}$ and $c \in (0, 1)$.

The regularity condition on X allows the pullback of (c, ∞) by $x \mapsto \mathbb{P}_x(I_\infty^f \le n)$ to be a set in $\mathcal{E}_\Delta$, and it is trivial to remove the Δ point and see that $M \in \mathcal{E}$.

Our first lemma shows that if there exists some super-finite set for (X, f) then it is possible to choose another measurable $g : E \to [0, \infty]$ such that the whole state space E is super-finite for (X, g). The constant c^2 is not optimal, but it suffices for our use of this lemma.

Lemma 2.5. *Suppose* $M = \{ y \in E : \mathbb{P}_y(\int_0^\infty f(X_s) \, ds \le n) > c \}$ *is a non-empty super-finite set for* (X, f) *and let* $g = \mathbf{1}_M \cdot f$. *Then*

$$\mathbb{P}_y \left(\int_0^\infty g(X_s) \, ds \le 2n \right) > c^2, \quad \forall y \in E.$$

That is, the entire state space E *is super-finite for* (X, g).

Proof. Since $g \le f$, also $I_t^g \le I_t^f$ for all $t \in [0, \infty]$. Then the result holds immediately for $y \in M$, since by definition of that set,

$$\mathbb{P}_y(I_\infty^g \le 2n) \ge \mathbb{P}_y(I_\infty^f \le 2n) \ge \mathbb{P}_y(I_\infty^f \le n) > c > c^2.$$

We now prove the result for regular points $y \in M^r$. Lemma I.10.19 of [9] gives a nested increasing sequence of compact sets $K_m \subseteq M$, $m \in \mathbb{N}$, such that, $\mathbb{P}_y$-almost surely, $T_{K_m} \downarrow T_M$ as $m \to \infty$. Since $\mathbb{P}_y(T_M = 0) = 1$ it then follows that

$$
\begin{aligned}
\mathbb{P}_y & \left(\int_0^{T_{K_1}} g(X_s) \, ds \le n \right) \\
&= \mathbb{P}_y \left(\bigcap_{m=1}^\infty \left\{ \int_{T_{K_m}}^{T_{K_1}} g(X_s) \, ds \le n \right\} \right) = \lim_{m \to \infty} \mathbb{P}_y \left(\int_{T_{K_m}}^{T_{K_1}} g(X_s) \, ds \le n \right) \\
&\ge \lim_{m \to \infty} \mathbb{P}_y \left(\int_{T_{K_m}}^{T_{K_1}} g(X_s) \, ds \le n; \, T_{K_m} < \infty \right) \\
&= \lim_{m \to \infty} \int_{K_m} \mathbb{P}_a \left(\int_0^{T_{K_1}} g(X_s) \, ds \le n \right) \mathbb{P}_y(X_{T_{K_m}} \in da; \, T_{K_m} < \infty) \\
&\ge \lim_{m \to \infty} \int_{K_m} \mathbb{P}_a \left(\int_0^\infty f(X_s) \, ds \le n \right) \mathbb{P}_y(X_{T_{K_m}} \in da; \, T_{K_m} < \infty).
\end{aligned}
$$

From [9, Lemma I.11.4] we know that $X_{T_{K_m}} \in K_m \subseteq M$ almost surely, and thus we obtain the lower bound

$$\mathbb{P}_y \left(\int_0^{T_{K_1}} g(X_s) \, ds \leq n \right) > c \cdot \lim_{m \to \infty} \mathbb{P}_y(T_{K_m} < \infty) = c \cdot \mathbb{P}_y(T_M < \infty) = c.$$

In addition, since T_{K_m} decreases to $T_M = 0$ $\mathbb{P}_y$-almost surely, it follows that we have $\mathbb{P}_y(A; T_{K_m} < \infty) \uparrow \mathbb{P}_y(A)$ for any $A \in \mathcal{F}$. This, in combination with the inequality above, shows that there exists a choice of $j \in \mathbb{N}$ such that

$$\mathbb{P}_y \left(\int_0^{T_{K_1}} g(X_s) \, ds \leq n; \, T_{K_j} < \infty \right) > c.$$

Monotonicity of the sequence of sets $(K_m, m \in \mathbb{N})$ yields $T_{K_1} \geq T_{K_j}$, and thus

$$\mathbb{P}_y \left(\int_0^{T_{K_j}} g(X_s) \, ds \leq n; \, T_{K_j} < \infty \right) > c. \tag{9}$$

Fix this j. Next, using the strong Markov property, we get

$$\mathbb{P}_y \left(\int_0^\infty g(X_s) \, ds \leq 2n; \, T_{K_j} < \infty \right)$$

$$\geq \mathbb{P}_y \left(\int_0^{T_{K_j}} g(X_s) \, ds \leq n; \, \int_{T_{K_j}}^\infty g(X_s) \, ds \leq n; \, T_{K_j} < \infty \right)$$

$$= \mathbb{E}_y \left[\mathbf{1}_{(\int_0^{T_{K_j}} g(X_s) \, ds \leq n)} \mathbf{1}_{(T_{K_j} < \infty)} \mathbb{E}_y \left[\mathbf{1}_{(\int_{T_{K_j}}^\infty g(X_s) \, ds \leq n)} \mid \mathcal{F}_{T_{K_j}} \right] \right]$$

$$= \mathbb{E}_y \left[\mathbf{1}_{(\int_0^{T_{K_j}} g(X_s) \, ds \leq n)} \mathbf{1}_{(T_{K_j} < \infty)} \mathbb{P}_{X_{T_{K_j}}} \left(\int_0^\infty g(X_s) \, ds \leq n \right) \right],$$

and now since $I_\infty^g \leq I_\infty^f$,

$$\geq \mathbb{E}_y \left[\mathbf{1}_{(\int_0^{T_{K_j}} g(X_s) \, ds \leq n)} \mathbf{1}_{(T_{K_j} < \infty)} \mathbb{P}_{X_{T_{K_j}}} \left(\int_0^\infty f(X_s) \, ds \leq n \right) \right].$$

Since $K_j \subseteq M$ the inner probability is bounded below by c. This holds even in the extreme case $K_j = M$, because then M is compact and $X_{T_{K_j}} \in M$ almost surely. The remaining expectation can also be bounded from below by c using (9). In total this leads to

$$\mathbb{P}_y \left(\int_0^\infty g(X_s) \, ds \leq 2n \right) \geq \mathbb{P}_y \left(\int_0^\infty g(X_s) \, ds \leq 2n; \, T_{K_j} < \infty \right) > c^2.$$

Thus the lemma is proved for regular points $y \in M^r$. What now remains is to extend it to all $y \in E$. In this case,

$$\mathbb{P}_y \left(\int_0^\infty g(X_s) \, ds \leq 2n \right)$$

$$= \mathbb{P}_y \left(\int_0^\infty g(X_s) \, ds \leq 2n; \, T_M < \infty \right) + \mathbb{P}_y \left(\int_0^\infty g(X_s) \, ds \leq 2n; \, T_M = \infty \right)$$

$$= \mathbb{P}_y\left(\int_{T_M}^\infty g(X_s)\,\mathrm{d}s \le 2n;\ T_M < \infty\right) + \mathbb{P}_y(T_M = \infty),$$

$$= \int_E \mathbb{P}_a\left(\int_0^\infty g(X_s)\,\mathrm{d}s \le 2n\right)\mathbb{P}_y(X_{T_M} \in \mathrm{d}a;\ T_M < \infty) + \mathbb{P}_y(T_M = \infty).$$

Lemma I.11.4 of [9] tells us that the integrating measure is concentrated on $M \cup M^r$, and thus we can use what we have already proved for elements of $M \cup M^r$ to conclude that

$$\mathbb{P}_y\left(\int_0^\infty g(X_s)\,\mathrm{d}s \le 2n\right) > c^2 \mathbb{P}_y(T_M < \infty) + \mathbb{P}_y(T_M = \infty)$$

$$\ge c^2(\mathbb{P}_y(T_M < \infty) + \mathbb{P}_y(T_M = \infty)) = c^2.$$

The proof of the lemma is now complete. ∎

The next lemma serves as preparation for the proposition that follows it. It shows that restricting f to a super-finite set ensures that the path integral does not explode in finite time, which is exactly condition (7), and ensures that the time change of X with the inverse of the path integral is strong Markov.

Lemma 2.6. *Suppose* $M = \{y \in E : \mathbb{P}_y(\int_0^\infty f(X_s)\,\mathrm{d}s \le n) > c\}$ *is a non-empty super-finite set for* (X, f). *Let* $g = \mathbf{1}_M \cdot f$. *Then*

$$\mathbb{P}_y\left(\exists t \in (0, \infty) : \int_0^t g(X_s)\,\mathrm{d}s = \infty\right) = 0, \quad \forall y \in E.$$

Proof. Fix $y \in E$ and recall the stopping times

$$\varphi_n^g = \inf\{t > 0 : I_t^g > n\},\ n \in \mathbb{N}, \quad \varphi_\infty^g = \lim_{n\to\infty} \varphi_n^g.$$

Since

$$\varphi_n^g < \infty \iff \int_0^\infty g(X_s)\,\mathrm{d}s > n \iff \exists t \in (0, \infty) : \int_0^t g(X_s)\,\mathrm{d}s > n, \quad (10)$$

we deduce from Lemma 2.5 that for all $y \in E$,

$$\mathbb{P}_y(\varphi_{2n}^g < \infty) \le 1 - c^2. \quad (11)$$

We then see for $k, n \in \mathbb{N}$ that

$$\mathbb{P}_y(\varphi_{2kn}^g < \infty) = \mathbb{P}_y\left(\exists t > 0 : \int_0^t g(X_s)\,\mathrm{d}s > 2kn\right)$$

$$= \mathbb{P}_y\left(\varphi_{2n(k-1)}^g < \infty;\ \exists t' > 0 : \int_{\varphi_{2n(k-1)}^g}^{\varphi_{2n(k-1)}^g + t'} g(X_s)\,\mathrm{d}s > 2n\right)$$

$$= \mathbb{E}_y\left[\mathbf{1}_{(\varphi_{2n(k-1)}^g < \infty)}\mathbb{E}_y\left[\mathbf{1}_{(\exists t' > 0 : \int_{\varphi_{2n(k-1)}^g}^{\varphi_{2n(k-1)}^g + t'} g(X_s)\,\mathrm{d}s > 2n)}\ \Big|\ \mathcal{F}_{\varphi_{2n(k-1)}^g}\right]\right]$$

$$= \mathbb{E}_y\left[\mathbf{1}_{(\varphi_{2n(k-1)}^g < \infty)}\mathbb{P}_{X_{\varphi_{2n(k-1)}^g}}\left(\exists t' > 0 : \int_0^{t'} g(X_s)\,\mathrm{d}s > 2n\right)\right]$$

$$= \mathbb{E}_y\left[\mathbf{1}_{(\varphi_{2n(k-1)}^g < \infty)}\mathbb{P}_{X_{\varphi_{2n(k-1)}^g}}(\varphi_{2n}^g < \infty)\right],$$

and by (11),

$$\leq (1 - c^2) \mathbb{P}_y(\varphi^g_{2n(k-1)} < \infty)$$

$$\vdots$$

$$\leq (1 - c^2)^k \to 0 \quad \text{as } k \to \infty.$$

Continuity of measures yields $\mathbb{P}_y(\varphi^g_\infty < \infty) = 0$, which via (10) implies the claim. ∎

The next proposition is partially motivated by ideas from Getoor [16], in particular the proof of Lemma (3.1) there, albeit used in a different fashion. It proves that super-finite sets for (X, f) have finite f-potential.

Proposition 2.7. *Suppose that* $M = \{y \in E : \mathbb{P}_y(\int_0^\infty f(X_s)\, ds \leq n/2) > p\}$ *for some* $n \in \mathbb{N}$ *and* $p \in (0, 1)$ *is a non-empty super-finite set for* (X, f). *Then*

$$\int_M f(x)\, U(y, dx) \leq \frac{n}{p^2} \quad \text{for all } y \in E.$$

Proof. With $g = \mathbf{1}_M \cdot f$ we introduce the time-changed process $Y_t = X_{\varphi^g_t}$ from (4), where

$$\varphi^g_t = \inf \left\{ s > 0 : \int_0^s g(X_u)\, du > t \right\}, \quad t \in [0, \infty).$$

Lemma 2.6 tells us that (7) and thus (5) are fulfilled almost surely and therefore, as per the discussion of the work of Volkonskiĭ [32] in Section 2.1, Y is a strong Markov process. For this proof we will use the notation

$$h(x) = \mathbb{P}_x(I^g_\infty \leq n), \quad x \in E. \tag{12}$$

By Lemma 2.5, h is bounded below by p^2 on E.

We denote by (P_t) the transition operator of Y and by U_Y the corresponding potential operator, to distinguish it from U_X, the potential operator of X. The function h from (12) is extended to E_Δ by setting $h(\Delta) = 0$ as usual. We then see that

$$P_n \mathbf{1}_E(x) = \mathbb{P}_x(X_{\varphi^g_n} \in E) = \mathbb{P}_x(\varphi^g_n < \zeta_X)$$

$$= \mathbb{P}_x \left(\exists s < \zeta_X : \int_0^s g(X_u)\, du > n \right)$$

$$\leq \mathbb{P}_x \left(\int_0^{\zeta_X} g(X_u)\, du > n \right) = \mathbf{1}_E(x) - h(x)$$

for all $x \in E_\Delta$. The main part of the proof is showing that $U_Y h$ is bounded above. Using the definition of the potential, the estimate of $P_n \mathbf{1}_E$ above, and the semigroup property yields

$$U_Y h(x) = \lim_{t \to \infty} \int_0^t P_s h(x)\, ds$$

$$\leq \lim_{t \to \infty} \int_0^t (P_s \mathbf{1}_E(x) - P_s P_n \mathbf{1}_E(x))\, ds$$

$$= \lim_{t \to \infty} \left(\int_0^t P_s \mathbf{1}_E(x) \, ds - \int_0^t P_{s+n} \mathbf{1}_E(x) \, ds \right)$$

$$= \lim_{t \to \infty} \left(\int_0^t P_s \mathbf{1}_E(x) \, ds - \int_n^{t+n} P_s \mathbf{1}_E(x) \, ds \right)$$

$$= \lim_{t \to \infty} \left(\int_0^n P_s \mathbf{1}_E(x) \, ds - \int_t^{t+n} P_s \mathbf{1}_E(x) \, ds \right)$$

$$\leq \int_0^n P_s \mathbf{1}_E(x) \, ds \leq n \tag{13}$$

for all $x \in E_\Delta$. It has already been noted that h is bounded below by $p^2 > 0$ on E. Then by monotonicity of the potential operator,

$$U_Y \mathbf{1}_E(y) \leq \frac{1}{p^2} U_Y h(y) \quad \text{for all } y \in E. \tag{14}$$

Combining (13) and (14) implies $U_Y \mathbf{1}_E(y) \leq \frac{1}{p^2} U_Y h(y) \leq \frac{n}{p^2}$ for all $y \in E$. Bearing in mind the following identity for bounded measurable functions b, obtained via change of variables and the definition $g = \mathbf{1}_M \cdot f$:

$$U_Y b(y) = \mathbb{E}_y \left[\int_0^\infty b(X_{\varphi_s^g}) \, ds \right] = \mathbb{E}_y \left[\int_0^\infty b(X_t) g(X_t) \, dt \right] = \int_M b(x) f(x) \, U_X(y, dx),$$

we obtain the desired estimate

$$\int_M f(x) \, U_X(y, dx) = U_Y \mathbf{1}_E(y) < \frac{n}{p^2} \quad \text{for all } y \in E.$$

The proof is now complete. ∎

Proposition 2.7 is a strong result for super-finite sets. All that remains to be done in order to prove Theorem 2.3 is to prove that non-empty super-finite sets are also supportive. Before doing so, another technical lemma is needed.

Lemma 2.8. *For $n \in \mathbb{N}$ and $c \in (0, 1)$, the set*

$$B = \left\{ x \in E : \mathbb{P}_x \left(\int_0^\infty f(X_s) \, ds \leq n \right) \leq c \right\}$$

contains its regular points, and is therefore finely closed.

Proof. The definition of B is exactly that of the complement in E of some super-finite set for (X, f). Let $K \subseteq B$ be a compact set, so that, by right-continuity of paths, $X_{T_K} \in K$ almost surely. Then for any $a \in E$ we obtain

$$\mathbb{P}_a(I_\infty^f \leq n) = \mathbb{P}_a(I_\infty^f \leq n; T_K < \zeta) + \mathbb{P}_a(I_\infty^f \leq n; T_K \geq \zeta)$$

$$\leq \mathbb{P}_a \left(\int_{T_K}^\infty f(X_s) \, ds \leq n; T_K < \zeta \right) + \mathbb{P}_a(I_\infty^f \leq n; T_K \geq \zeta)$$

$$= \int \mathbb{P}_y(I_\infty^f \leq n) \mathbb{P}_a(X_{T_K} \in dy; T_K < \zeta) + \mathbb{P}_a(I_\infty^f \leq n; T_K \geq \zeta)$$

$$\leq c + \mathbb{P}_a(T_K \geq \zeta), \tag{15}$$

as $K \subseteq B$. Since (15) is true for all compact sets $K \subseteq B$, we find for all $a \in E$ that

$$\mathbb{P}_a\left(\int_0^\infty f(X_s)\,ds \leq n\right) \leq c + \inf_{K \subseteq B} \mathbb{P}_a(T_K \geq \zeta). \tag{16}$$

We now argue that the second summand of the right-hand side of (16) equals 0 if a is regular for B. From [9, Lemma I.10.19] it follows that there exists an increasing sequence (K_n) of compact subsets of B such that $\mathbb{P}_a(T_{K_n} \downarrow T_B) = 1$. Monotonicity of measures implies $\inf_{K \subseteq B} \mathbb{P}_a(T_K \geq \zeta) = \lim_{n \to \infty} \mathbb{P}_a(T_{K_n} \geq \zeta) = \mathbb{P}_a(T_B \geq \zeta) = 0$. But then (16) implies $a \in B$. Thus $B^r \subseteq B$. Equivalence of B being finely closed and containing its regular points is Exercise II.4.9 of [9]. ∎

We have built enough structure around super-finite sets to now prove the final proposition of this section, which establishes that super-finite sets are supportive when the process is issued from them.

Proposition 2.9. *For $n \in \mathbb{N}$ and $c \in (0, 1)$ the super-finite set*

$$M_{n,c} = \left\{y \in E : \mathbb{P}_y\left(\int_0^\infty f(X_s)\,ds \leq n\right) > c\right\}$$

is $\mathbb{P}_z$-supportive if and only if $z \in M_{n,c}$. In particular, if $\mathbb{P}_z(I_\infty^f < \infty) > 0$ for some fixed $z \in E$ then there exists a super-finite set for (X, f) which is also $\mathbb{P}_z$-supportive.

Proof. For ease of notation let $h_n(x) = \mathbb{P}_x(\int_0^\infty f(X_s)\,ds \leq n)$ and define $B_{n,c} = \{x \in E : h_n(x) \leq c\} = E \setminus M_{n,c}$ as in Lemma 2.8.

Necessity of $z \in M_{n,c}$ for $M_{n,c}$ to be supportive is clear, and we shall prove sufficiency by proving the contrapositive. Let us suppose for the moment that $M_{n,c}$ is not $\mathbb{P}_z$-supportive, that is,

$$\mathbb{P}_z(D_{B_{n,c}} < \zeta) = \mathbb{P}_z(D_{E \setminus M_{n,c}} < \zeta) = 1.$$

From this we obtain

$$\begin{aligned}
h_n(z) &= \mathbb{P}_z\left(\int_0^\infty f(X_s)\,ds \leq n;\ D_{B_{n,c}} < \zeta\right) \\
&\leq \mathbb{P}_z\left(\int_{D_{B_{n,c}}}^\infty f(X_s)\,ds \leq n;\ D_{B_{n,c}} < \zeta\right) \\
&= \int_E \mathbb{P}_a\left(\int_0^\infty f(X_s)\,ds \leq n\right)\mathbb{P}_z(X_{D_{B_{n,c}}} \in da;\ D_{B_{n,c}} < \zeta). \tag{17}
\end{aligned}$$

Due to Lemma 2.8 the regular points of $B_{n,c}$ belong to $B_{n,c}$, and therefore Lemma I.11.4 of [9] tells us that $\mathbb{P}_y(X_{D_{B_{n,c}}} \in da;\ D_{B_{n,c}} < \zeta)$ is concentrated on $B_{n,c}$. Then returning to (17) we can deduce from the definition of $B_{n,c}$ that

$$h_n(z) \leq \int_E \mathbb{P}_a\left(\int_0^\infty f(X_s)\,ds \leq n\right)\mathbb{P}_z(X_{D_{B_{n,c}}} \in da;\ D_{B_{n,c}} < \zeta) \leq c.$$

To recap, we have proved that

$$M_{n,c} \text{ is not } \mathbb{P}_z\text{-supportive} \implies h_n(z) \leq c,$$

or equivalently

$$h_n(z) > c \implies M_{n,c} \text{ is } \mathbb{P}_z\text{-supportive.} \tag{18}$$

Since $z \in M_{n,c}$ if and only if $h_n(z) > c$, the first claim is proved. For the second claim, note that, by continuity of measures, $\mathbb{P}_z(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty) > 0$ implies

$$h_{n_0}(z) = \mathbb{P}_z\left(\int_0^\infty f(X_s)\,\mathrm{d}s \leq n_0\right) > 0$$

for some $n_0 \in \mathbb{N}$. Hence in this case (18) implies the existence of some c_0 such that M_{n_0,c_0} is $\mathbb{P}_z$-supportive. ∎

Proof of Theorem 2.3. Path integral proofs typically have a simple and a complicated direction. The simple direction "$\Longleftarrow$" uses the potential expression to deduce finiteness of the expectation. In our setting the argument goes as follows. Suppose that B is a $\mathbb{P}_z$-avoidable set with complement $M = E \setminus B$ satisfying

$$\mathbb{E}_z\left[\int_0^\infty \mathbf{1}_M(X_s)f(X_s)\,\mathrm{d}s\right] = \int_M f(x)\,U(z,\mathrm{d}x) < \infty.$$

Then $\mathbb{P}_z(\int_0^\infty \mathbf{1}_M(X_s)f(X_s)\,\mathrm{d}s < \infty) = 1$, and thus

$$\mathbb{P}_z\left(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty\right) \geq \mathbb{P}_z\left(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty;\ X_s \in M\ \forall s < \zeta\right)$$

$$= \mathbb{P}_z\left(\int_0^\infty \mathbf{1}_M(X_s)f(X_s)\,\mathrm{d}s < \infty;\ X_s \in M\ \forall s < \zeta\right)$$

$$= \mathbb{P}_z(X_s \in M \text{ for all } s < \zeta) > 0.$$

This shows the "$\Longleftarrow$" direction of Theorem 2.3.

To prove the "$\Longrightarrow$" direction we take the super-finite supportive set from Proposition 2.9 and obtain the integral test from Proposition 2.7. ∎

2.3. Almost sure case

Theorem 2.3 was formulated and proved in great generality. In the following sections we derive a couple of corollaries by adding or changing the assumptions. A first variant of our proof technique provides a complete theorem in the probability 1 setting. The unpleasant supportive sets do not vanish in the integral test $\int_M f(x)\,U(z,\mathrm{d}x) < \infty$, but can be made large.

Theorem 2.10. *Let X be a standard Markov process on the state space E, and let $f : E \to [0,\infty]$ be measurable. Then the following are equivalent:*

(i) $\mathbb{P}_z(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty) = 1$.

(ii) *For every $\varepsilon > 0$ there exists a $\mathbb{P}_z$-supportive set M such that $\int_M f(x)\,U(z,\mathrm{d}x) < \infty$ and X stays in M with probability at least $1 - \varepsilon$.*

Before going into the proof let us recall again that the theorem can be formulated equivalently using the more familiar notion of avoidable sets, the complements of supportive sets. We stick to the notion of supportive sets as those are constructed in the proof.

Proof of Theorem 2.10. (i)⇒(ii) Assuming (i) it follows that for every $\varepsilon > 0$, and in particular for any choice of $\varepsilon \in (0, 1/2)$, there exists an $n \in \mathbb{N}$ such that

$$\mathbb{P}_z\left(\int_0^\infty f(X_s)\,\mathrm{d}s \le n\right) > 1 - \varepsilon. \tag{19}$$

Fix these n, ε and define the super-finite set $M = \{y \in E : \mathbb{P}_y(\int_0^\infty f(X_s)\,\mathrm{d}s \le n) > \varepsilon\}$, which by (19) contains z, and thus by Propsition 2.9 is $\mathbb{P}_z$-supportive. Moreover,

$$\mathbb{P}_z\left(\int_0^\infty f(X_s)\,\mathrm{d}s \le n\right)$$
$$= \mathbb{P}_z\left(\int_0^\infty f(X_s)\,\mathrm{d}s \le n;\ D_{E \setminus M} = \infty\right) + \mathbb{P}_z\left(\int_0^\infty f(X_s)\,\mathrm{d}s \le n;\ D_{E \setminus M} < \infty\right)$$
$$\le \mathbb{P}_z(D_{E \setminus M} = \infty) + \mathbb{P}_z\left(\int_{D_{E \setminus M}}^\infty f(X_s)\,\mathrm{d}s \le n;\ D_{E \setminus M} < \infty\right).$$

Applying the strong Markov property at $D_{E \setminus M}$ and recalling from Lemma 2.8 that $E \setminus M$ contains its regular points, which implies that $X_{D_{E \setminus M}} \in E \setminus M$ almost surely, gives the upper bound $\mathbb{P}_z(D_{E \setminus M} = \infty) + \varepsilon$. In combination with (19) this yields $1 - 2\varepsilon < \mathbb{P}_z(D_{E \setminus M} = \infty)$. It remains only to note via Proposition 2.7 that $\int_M f(x)\,U(z, \mathrm{d}x) < \infty$.

(i)⇐(ii) Take $\varepsilon > 0$ small and let M be such that $\mathbb{P}_z(D_{E \setminus M} < \infty) \le \varepsilon$ and

$$\mathbb{E}_z\left[\int_0^\infty f(X_s)\mathbf{1}_M(X_s)\,\mathrm{d}s\right] = \int_M f(x)\,U(z, \mathrm{d}x) < \infty.$$

Then $\mathbb{P}_z(\int_0^\infty f(X_s)\mathbf{1}_M(X_s)\,\mathrm{d}s < \infty) = 1$ and, in particular,

$$\mathbb{P}_z\left(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty\right) \ge \mathbb{P}_z\left(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty;\ D_{E \setminus M} = \infty\right)$$
$$= \mathbb{P}_z\left(\int_0^\infty f(X_s)\mathbf{1}_M(X_s)\,\mathrm{d}s < \infty;\ D_{E \setminus M} = \infty\right)$$
$$= \mathbb{P}_z(D_{E \setminus M} = \infty) > 1 - \varepsilon.$$

This holds for arbitrarily small $\varepsilon > 0$, and thus $\mathbb{P}_z(\int_0^\infty f(X_s)\,\mathrm{d}s < \infty) = 1$. ∎

Example 2.13 below shows what the large supportive sets might be, and why they cannot be omitted. In that example f has a pole, and the large supportive sets can be chosen to be all sets that do not contain a small interval around the pole of f.

We can immediately derive an interesting consequence of Theorem 2.10 when f is locally bounded and X is transient. The main point here is that the local boundedness of f implies that finiteness of path integrals is not affected by the behaviour of X for a finite amount of time. Hence, speaking about sets in which X spends no time (avoidable) or finite time (transient) should be the same. The first result makes this idea precise, and

the following theorem specialises to processes with trivial tail σ-algebra (such as Lévy processes) for which transience of sets are zero-one events.

Corollary 2.11. *Let X be a transient standard Markov process on $\mathbb{R}^d$, and let $f : \mathbb{R}^d \to [0, \infty)$ be measurable and bounded on compact sets. Then the following are equivalent:*

(i) $\mathbb{P}_z(\int_0^\infty f(X_s) \, ds < \infty) = 1$.

(ii) *For every $\varepsilon > 0$ there exists a set $B \in \mathcal{B}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d \setminus B} f(x) \, U(z, dx) < \infty$ and $\mathbb{P}_z(L_B < \zeta) \geq 1 - \varepsilon$.*

Proof. For any Borel set B the last exit time L_B is clearly zero on the event that the first entry time D_B is infinite. Therefore, if we assume (i), Theorem 2.10 gives a set $B := \mathbb{R}^d \setminus M$ such that $\mathbb{P}_z(L_B < \zeta) \geq 1 - \varepsilon$ and $\int_{\mathbb{R}^d \setminus B} f(x) \, U(z, dx) < \infty$. So (ii) is proven.

Now suppose (ii) holds. We first make use of the following argument from [9, proof of Proposition I(9.3)]. Let (K_n) be an increasing sequence of compact sets with $K_n \subseteq K_{n+1}^\circ$ for all n and $\bigcup K_n = \mathbb{R}^d$. Let $T := \lim_{n \to \infty} T_{\mathbb{R}^d \setminus K_n}$. By quasi-left-continuity, $X_{T_{\mathbb{R}^d \setminus K_n}} \to X_T$ almost surely on $\{T < \zeta\}$, and by right-continuity, $X_{T_{\mathbb{R}^d \setminus K_{n+1}}} \notin K_n$ for all n. Therefore $X_T = \Delta$ almost surely on $\{T < \zeta\}$; clearly this is also true on $\{T \geq \zeta\}$, and thus it holds almost surely. Since the lifetime ζ is the first time that X reaches the cemetary state, we conclude that $T \geq \zeta$ almost surely. Now fix some $\varepsilon > 0$, and let B be the set satisfying $\mathbb{P}_z(L_B < \zeta) \geq 1 - \varepsilon$ and $\int_{\mathbb{R}^d \setminus B} f(x) \, U(z, dx) < \infty$. Due to continuity of measure,

$$\mathbb{P}_z(L_B \geq T_{\mathbb{R}^d \setminus K_n}) \downarrow \mathbb{P}_z(L_B \geq \zeta) < \varepsilon.$$

Then we can find a compact set $K \subseteq \mathbb{R}^d$ such that $\mathbb{P}_z(L_B < T_{\mathbb{R}^d \setminus K}) > 1 - 2\varepsilon$. Define $M = K \cup (\mathbb{R}^d \setminus B)$. This definition implies that on the event $L_B < T_{\mathbb{R}^d \setminus K}$, the process X never leaves M. That is,

$$\mathbb{P}_z(D_{\mathbb{R}^d \setminus M} = \infty) \geq \mathbb{P}_z(L_B < T_{\mathbb{R}^d \setminus K}) > 1 - 2\varepsilon,$$

so X stays in M with probability at least $1 - 2\varepsilon$. In addition

$$\int_M f(x) \, U(z, dx) \leq \int_{\mathbb{R}^d \setminus B} f(x) \, U(z, dx) + U(z, K) \sup_{x \in K} f(x).$$

This together with the integral test of (ii), boundedness of f on compact sets, and $U(z, K) < \infty$ implies that $\int_M f(x) \, U(z, dx) < \infty$. Our choice of $\varepsilon > 0$ was arbitrary, and so the result follows from Theorem 2.10. ∎

In the case of a trivial tail σ-algebra, Corollary 2.11 simplifies and for the special case of a Lévy process matches exactly the main theorem of Kolb and Savov [24], which we recalled in Theorem 2.1. For a Markov process X with infinite lifetime, we say X has a *trivial tail σ-algebra when issued from $z \in E$* if

$$A \in \bigcap_{s \geq 0} \sigma(X_t, t \geq s) \implies \mathbb{P}_z(A) \in \{0, 1\}.$$

As an example, a Lévy process on $\mathbb{R}^d$ has a trivial tail σ-algebra when issued from every

$z \in \mathbb{R}^d$. Since the events $\{L_B < \infty\}$ are in the tail σ-algebra, the previous theorem can be reformulated using transient sets, i.e. the sets with $\mathbb{P}_z(L_B < \infty) = 1$.

Theorem 2.12 (Generalised Kolb–Savov zero-one law for perpetual integrals). *Let X be a standard Markov process on $\mathbb{R}^d$ with trivial tail σ-algebra when issued from $z \in \mathbb{R}^d$, and let $f : \mathbb{R}^d \to [0, \infty)$ be measurable and bounded on compact sets. Suppose in addition that $\mathbb{P}_z(\zeta = \infty) = 1$. Then the following are equivalent:*

(i) $\mathbb{P}_z(\int_0^\infty f(X_s)\, ds < \infty) > 0.$

(ii) $\mathbb{P}_z(\int_0^\infty f(X_s)\, ds < \infty) = 1.$

(iii) *There exists a transient set B (that is, a Borel set with $\mathbb{P}_z(L_B < \infty) = 1$) such that the integral test $\int_{\mathbb{R}^d \setminus B} f(x)\, U(z, dx) < \infty$ holds.*

2.4. Examples for stable Lévy processes

In this section we give two examples that go beyond the drastic example $f = +\infty \mathbf{1}_{[1,2]}$ from Section 2.2. Both examples are for transient symmetric stable processes on $\mathbb{R}$, i.e. $\alpha \in (0, 1)$. In that setting the potential measure is absolutely continuous with (ignoring constants)

$$U(z, dx) = |z - x|^{\alpha-1}\, dx, \quad z, x \in \mathbb{R}.$$

The first example shows that supportive sets also cannot be omitted in the integral test for finite (or bounded) f. Nonetheless, the supportive sets can be chosen large as proved in Theorem 2.10.

Example 2.13. Assume $z = 0$, fix a point $y \in \mathbb{R}$ not equal to zero, and let $\varepsilon > 0$ be such that $|y| > \varepsilon$. Suppose that f has support on $\mathbf{B}_\varepsilon(y) = \{x : |x - y| < \varepsilon\}$. Since the density $u(x) = |x|^{\alpha-1}$ of $U(0, dx)$ is bounded on the support of f we find that

$$\int_{\mathbb{R}} f(x)\, U(0, dx) < \infty \iff \int_{\mathbb{R}} f(x)\, dx < \infty.$$

Hence, if we define such an f with a non-integrable pole at y, for instance $f(x) = (x - y)^{-2}\mathbf{1}_{\mathbf{B}_\varepsilon(y)}(x)$, we have $\int_{\mathbb{R}} f(x)\, U(0, dx) = \infty$. Now recall that X stays a positive distance away from the pole y almost surely.[1] It follows that

$$\xi := \sup_{t \in [0,\infty)} f(X_t) < \infty \quad \text{almost surely.}$$

In addition, $\mathbb{P}_0(L_{\mathbf{B}_\varepsilon(y)} < \infty) = 1$. Therefore, the infinite-time path integral is finite almost surely, that is,

$$\int_0^\infty f(X_s)\, ds \le \xi L_{\mathbf{B}_\varepsilon(y)} < \infty,$$

but the integral test fails without a supportive set. The supportive set can be chosen large. Any set not containing a small ball around the pole of f at y is possible. This is exactly the phenomenon that we found in Theorem 2.10.

---

[1]This well-known fact can be proven using the density of the point of closest reach of X (see [25]).

The next example gives a bounded function f for which the supportive set in the integral test can be given explicitly.

Example 2.14. We construct an avoidable set A with potential $U(z, A) = \infty$ for symmetric α-stable processes with $\alpha \in (2/3, 1)$. Then the function $f := \mathbf{1}_A$ is a further (bounded) counterexample for which $\int_0^\infty f(X_s)\,\mathrm{d}s$ is finite with positive probability (it is actually finite almost surely as the tail σ-algebra of a Lévy process is trivial) but $\int f(x)\,U(z, \mathrm{d}x) = \infty$. The set A is a disjoint union of shrinking intervals drifting off to infinity. The sizes of the intervals are shrinking sufficiently fast to be avoidable, but not fast enough to have finite potential.

For transient isotropic stable processes a result called Wiener's criterion gives an analytic description of sets which are thin at a point; see [8, Corollary V.4.17] for a classical presentation. That result can be extended to an analytic description of avoidable sets via [28, Corollary A.4], and is stated as follows. With so-called shells $S_n := \{x \in \mathbb{R} : \lambda^{n-1} < |x - z| \leq \lambda^n\}$ for arbitrary fixed $\lambda \in (1, \infty)$,

$$B \text{ is } \mathbb{P}_0\text{-avoidable} \iff \sum_{n=-\infty}^\infty \lambda^{n(\alpha-1)} C(B \cap S_n) < \infty,$$

where $C(B \cap S_n)$ is the capacity of $B \cap S_n$. The capacity of a stable process can be computed or estimated using results from probabilistic potential theory. Define the sets $A, A_1, \ldots \in \mathcal{B}(\mathbb{R})$ by

$$A_n := [2^n - 2^{(n-1)/3}, 2^n) \quad \text{and} \quad A := \bigcup_{n=1}^\infty A_n.$$

Because $2^n - 2^{(n-1)/3} > 2^n - 2^{(n-1)} = 2^{n-1}$, we can take $\lambda = 2$ in the definition of S_n and have $A_n \subseteq S_n$ for each $n \in \mathbb{N}$, and in particular $A \cap S_n = A_n$. We now show that A is $\mathbb{P}_0$-avoidable for all $\alpha \in (0, 1)$.

Let $\mathbf{B}_\varepsilon = \{x \in \mathbb{R} : |x| < \varepsilon\}$. Notice that each $A \cap S_n$ is just a translation of the ball of radius $2^{-1}2^{(n-1)/3} = 2^{(n-4)/3}$. Then Proposition 42.12 of Sato [31] yields $C(A \cap S_n) = C(\mathbf{B}_{2^{(n-4)/3}}) = C(2^{(n-4)/3}\mathbf{B}_1)$. From the scaling property $C(aB) = a^{1-\alpha}C(B)$ for capacities of stable processes it follows that

$$C(A \cap S_n) = 2^{(n-4)(1-\alpha)/3} C(\mathbf{B}_1).$$

Now the summation test of Wiener with $\lambda = 2$ is

$$\sum_{n=-\infty}^\infty 2^{n(\alpha-1)} C(A \cap S_n) = \sum_{n=1}^\infty 2^{n(\alpha-1)} C(A \cap S_n)$$

$$= C(\mathbf{B}_1) \sum_{n=1}^\infty 2^{n(\alpha-1)} 2^{(n-4)(1-\alpha)/3}$$

$$= 2^{4(\alpha-1)/3} C(\mathbf{B}_1) \sum_{n=1}^\infty 2^{(2n/3)(\alpha-1)}.$$

Since $C(\mathbf{B}_1) = \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{\alpha}{2})\Gamma((\frac{1-\alpha}{2})+1)}$ is a finite constant (see Bliedtner and Hansen [8, Example V.4.16(2)]), the quantity above is finite if and only if $\sum_{n=1}^{\infty} 2^{(2n/3)(\alpha-1)}$ is finite, and this is equivalent to $2^{(2/3)(\alpha-1)} < 1$, that is, $\alpha < 1$. Hence, A is an avoidable set for all stable processes with $\alpha \in (0,1)$.

With the explicit form of the potential we can compute the potential $U(0, A)$:

$$
\begin{aligned}
U(0, A) &= \int_A |x|^{\alpha-1} \, dx = \sum_{n=1}^{\infty} \int_{2^n - 2^{(n-1)/3}}^{2^n} x^{\alpha-1} \, dx \\
&\geq \sum_{n=1}^{\infty} 2^{n(\alpha-1)} (2^n - (2^n - 2^{(n-1)/3})) \quad \text{(monotonicity of } x^{\alpha-1}) \\
&= 2^{-1/3} \sum_{n=1}^{\infty} 2^{n(\alpha-2/3)}.
\end{aligned}
$$

This sum is infinite if $\alpha > 2/3$. Combining both computations we found an avoidable set A with infinite potential for symmetric stable processes of index $\alpha \in (2/3, 1)$. Setting $f = \mathbf{1}_A$ we have an example function for which $\int_0^{\infty} f(X_s) \, ds = 0 < \infty$ with positive probability and $\int_{\mathbb{R}} f(x) U(0, dx) = \infty$. Hence, the supportive set $M = \mathbb{R} \backslash A$ in $\int_M f(x) U(0, dx)$ is needed also for bounded f. The example of course allows modifications for which f has support on A and f is smooth or vanishes (slowly enough) at infinity.

3. Finite-time path integral tests

The results of Section 2 for infinite-time-horizon path integrals will now be used to study finite-time-horizon path integrals simultaneously for all finite times. We profit here from the very general assumptions of Theorem 2.3 which allows us to use path integrals in different setups, such as for killed processes. We derive conditions under which

$$
\int_0^t f(X_s) \, ds < \infty \quad \text{for every } t < \zeta
$$

holds with positive probability or probability 1. These questions were raised and answered for the Brownian motion by Engelbert and Schmidt [13] in 1981, in the form of the following zero-one law. For $f : \mathbb{R} \to [0, \infty]$ measurable they proved the equivalence of

(i) $\mathbb{P}_0(\int_0^t f(W_s) \, ds < \infty$ for every $t \geq 0) > 0$;

(ii) $\mathbb{P}_0(\int_0^t f(W_s) \, ds < \infty$ for every $t \geq 0) = 1$;

(iii) $\int_K f(y) \, dy < \infty$ for all compact $K \subseteq \mathbb{R}$.

The proof is a direct consequence of the occupation time formula

$$
\int_0^t f(W_s) \, ds = \int_{\mathbb{R}} f(x) L_t(x) \, dx
$$

and the joint continuity of the local time in both coordinates. In one dimension the Brownian motion shares many properties with stable processes of index $\alpha \in (1, 2)$, and Zanzotto [34] extended Engelbert and Schmidt's equivalence of (i)–(iii) to this class. Zanzotto's proof is identical to the Brownian case as the local time for the stable processes is also jointly continuous. With the same proof, the equivalence of (i)–(iii) holds equally for all Lévy processes with jointly continuous local time. As in the previous section the same result cannot hold for $\alpha \in (0, 1)$ due to the transient nature of those stable processes, which allows regions with large f values to be avoided, so that avoidable (or supportive) sets must occur in the theorems. The results presented in this section are Engelbert–Schmidt-type results for general transient Markov processes. In Section 3.1 we prove theorems which are close to the Engelbert–Schmidt theorem with the exception of the appearance of supportive sets. In Section 3.2 we use our methods to extend the Engelbert–Schmidt–Zanzotto theorem to Lévy processes with local time which are not necessarily jointly continuous.

3.1. Transient Markov processes

For general transient Markov processes the Engelbert–Schmidt zero-one law mentioned above fails, since, with positive probability, X can avoid regions in which f can be arbitrarily large. This in particular implies that a general result should not be a zero-one law at all. Here we present two theorems that together provide a counterpart to Engelbert and Schmidt's theorem for standard transient Markov process. The first concerns the positive probability case, and the theorem for almost sure finiteness is given below.

Theorem 3.1. *Let X be a strongly transient standard Markov process and $f : E \to [0, \infty]$ measurable. For $z \in E$, the following are equivalent:*

(i) $\mathbb{P}_z(\int_0^t f(X_s)\, ds < \infty$ *for every* $t < \zeta) > 0$.

(ii) *There exists a $\mathbb{P}_z$-supportive set M such that $\int_{M \cap K} f(y)\, U(z, dy) < \infty$ for all compact $K \subseteq E$.*

Proof. (i)$\Rightarrow$(ii) Let $(K_n, n \geq 1)$ be a sequence of increasing compact sets satisfying $\bigcup_{n \geq 1} K_n = E$, and define $f_n = f\mathbf{1}_{K_n}$. Since X is strongly transient, we have $\mathbb{P}_z(L_{K_n} < \zeta) = 1$ for all n, and thus

$$\mathbb{P}_z\left(\int_0^t f(X_s)\, ds < \infty \text{ for every } t < \zeta\right) > 0,$$

implying that

$$\mathbb{P}_z\left(\int_0^{L_{K_n}} f(X_s)\, ds < \infty \text{ for every } n \in \mathbb{N}\right) > 0$$

and so

$$\exists p > 0 : \mathbb{P}_z\left(\int_0^\infty f_n(X_s)\, ds < \infty \text{ for every } n \in \mathbb{N}\right) > p. \tag{20}$$

It follows that there exists a constant $C_1 > 0$ such that

$$\mathbb{P}_z\left(\int_0^\infty f_1(X_s)\,\mathrm{d}s < C_1, \int_0^\infty f_n(X_s)\,\mathrm{d}s < \infty \text{ for every } n \geq 2\right) > p.$$

Repeating this process inductively for every n gives a sequence of positive, non-decreasing $(C_n, n \geq 1)$ such that $\mathbb{P}_z(\bigcap_{n=1}^k B_n) > p$ for all $k \geq 1$, where $B_n = \{\int_0^\infty f_n(X_s)\,\mathrm{d}s < C_n\}$. Now letting $B := \bigcap_{n=1}^\infty B_n$ we see from continuity of measures that

$$\mathbb{P}_z(B) = \lim_{k\to\infty} \mathbb{P}_z\left(\bigcap_{n=1}^k B_n\right) \geq p. \tag{21}$$

This motivates us to define the Borel set $M := \{y \in E : \mathbb{P}_y(B) > p/2\}$, which (21) tells us contains the point z, and the remainder of the proof will be showing that M is $\mathbb{P}_z$-supportive and satisfies the integral test.

   We begin by showing that M is $\mathbb{P}_z$-supportive, which requires first showing that $M^c = E \setminus M$ contains its regular points, a property known as being finely closed. We argue in the spirit of the reasoning used in the proof of Lemma 2.8. Let y be an arbitrary point which is regular for M^c, and so satisfies $\mathbb{P}_y(T_{M^c} = 0) = 1$. Let $K \subseteq M^c$ be a compact set, so that, by right-continuity of paths, $X_{T_K} \in K$ almost surely. Then

$$\begin{aligned}
\mathbb{P}_y(B) &= \mathbb{P}_y\left(\int_0^\infty f_1(X_s)\,\mathrm{d}s < C_1, \ldots; T_K < \zeta\right) \\
&\quad + \mathbb{P}_y\left(\int_0^\infty f_1(X_s)\,\mathrm{d}s < C_1, \ldots; T_K \geq \zeta\right) \\
&\leq \mathbb{P}_y\left(\int_{T_K}^\infty f_1(X_s)\,\mathrm{d}s < C_1, \int_{T_K}^\infty f_1(X_s)\,\mathrm{d}s < C_2, \ldots; T_K < \zeta\right) + \mathbb{P}_y(T_K \geq \zeta) \\
&= \int \mathbb{P}_a\left(\int_0^\infty f_1(X_s)\,\mathrm{d}s < C_1, \ldots\right)\mathbb{P}_y(X_{T_K} \in \mathrm{d}a; T_K < \zeta) + \mathbb{P}_y(T_K \geq \zeta) \\
&= \int \mathbb{P}_a(B)\mathbb{P}_y(X_{T_K} \in \mathrm{d}a; T_K < \zeta) + \mathbb{P}_y(T_K \geq \zeta) \\
&\leq \frac{p}{2} + \mathbb{P}_y(T_K \geq \zeta)
\end{aligned}$$

since $K \subseteq M^c$. It follows exactly as in the proof of Lemma 2.8 that $\mathbb{P}_y(T_K \geq \zeta) = 0$ and therefore $y \in M^c$. Thus M^c contains its regular points.

   Now we can show that M is supportive for any point contained in it. Let $x \in E$ be arbitrary and suppose first that M is *not* $\mathbb{P}_x$-supportive, that is, $\mathbb{P}_x(D_{M^c} < \zeta) = 1$. From this it follows that

$$\begin{aligned}
\mathbb{P}_x(B) &= \mathbb{P}_x\left(\int_0^\infty f_1(X_s)\,\mathrm{d}s < C_1, \ldots; D_{M^c} < \zeta\right) \\
&\leq \mathbb{P}_x\left(\int_{D_{M^c}}^\infty f_1(X_s)\,\mathrm{d}s < C_1, \ldots; D_{M^c} < \zeta\right) \\
&= \int_E \mathbb{P}_a(B)\mathbb{P}_x(X_{D_{M^c}} \in \mathrm{d}a; D_{M^c} < \zeta). \tag{22}
\end{aligned}$$

We demonstrated above that M^c contains its regular points, and it follows from Blumenthal and Getoor [9, Theorem I(11.4)] that $\mathbb{P}_x(X_{D_{M^c}} \in \mathrm{d}a; \ D_{M^c} < \zeta)$ is concentrated on M^c. In particular, it follows from (22) that

$$\mathbb{P}_x(B) \leq \int_E \mathbb{P}_a(B) \mathbb{P}_x(X_{D_{M^c}} \in \mathrm{d}a; \ D_{M^c} < \zeta) \leq \frac{p}{2}.$$

What we have thus proven is that for arbitrary $x \in E$, if M is not $\mathbb{P}_x$-supportive then $\mathbb{P}_x(B) \leq p/2$. Equivalently,

$$\mathbb{P}_x(B) > \frac{p}{2} \implies M \text{ is } \mathbb{P}_x\text{-supportive.}$$

We saw earlier in the proof that our particular choice of $z \in E$ is contained in M, that is, $\mathbb{P}_z(B) > p/2$, and thus it follows that M is $\mathbb{P}_z$-supportive.

Finally, we show that M satisfies the integral test of (ii). It is clear that for any $n \geq 1$,

$$M \subseteq M_n := \{y \in E : \mathbb{P}_y(B_n) > p/2\} = \left\{ y \in E : \mathbb{P}_y \left(\int_0^\infty f_n(X_s) \, \mathrm{d}s < C_n \right) > p/2 \right\}.$$

Since M_n is super-finite for (X, f_n), Proposition 2.7 yields $\int_{M_n} f_n(x) \, U(y, \mathrm{d}x) < \infty$ for all $y \in E$. We therefore see that

$$\int_{M \cap K_n} f(x) \, U(z, \mathrm{d}x) \leq \int_{M_n} f_n(x) \, U(z, \mathrm{d}x) < \infty \quad \text{for all } n \geq 1.$$

Because $K_n \uparrow E$, any compact set will be covered by K_n for some $n \in \mathbb{N}$. Hence, we obtain the integral test $\int_{M \cap K} f(x) \, U(z, \mathrm{d}x) < \infty$ for all compact $K \subseteq E$.

(i)$\Leftarrow$(ii) Suppose the existence of a $\mathbb{P}_z$-supportive M such that for all compact K,

$$\mathbb{E}_z \left[\int_0^\infty f(X_s) \mathbf{1}_{M \cap K}(X_s) \, \mathrm{d}s \right] = \int_{M \cap K} f(y) \, U(z, \mathrm{d}y) < \infty.$$

Since M is supportive, there exists a positive $c > 0$ such that $\mathbb{P}_z(T_{M^c} \geq \zeta) > c$. Let $K_1, K_2, \ldots$ be an increasing sequence of compact sets with limit E. Then for all $n \in \mathbb{N}$,

$$\mathbb{E}_z \left[\int_0^\infty f(X_s) \mathbf{1}_{M \cap K_n}(X_s) \, \mathrm{d}s \right] < \infty \implies \mathbb{P}_z \left(\int_0^\infty f(X_s) \mathbf{1}_{M \cap K_n}(X_s) \, \mathrm{d}s < \infty \right) = 1$$

$$\implies \mathbb{P}_z \left(\int_0^\infty f(X_s) \mathbf{1}_{K_n}(X_s) \, \mathrm{d}s < \infty \right) > c$$

$$\implies \mathbb{P}_z \left(\int_0^{T_{E \setminus K_n}} f(X_s) \, \mathrm{d}s < \infty \right) > c$$

so that

$$c \leq \lim_{n \to \infty} \mathbb{P}_z \left(\int_0^{T_{E \setminus K_n}} f(X_s) \, \mathrm{d}s < \infty \right)$$

$$= \mathbb{P}_z \left(\int_0^{T_{E \setminus K_n}} f(X_s) \, \mathrm{d}s < \infty \text{ for all } n \in \mathbb{N} \right). \tag{23}$$

Since X is strongly transient, $\lim_{n\to\infty} T_{E\setminus K_n} = \zeta$, and thus

$$\mathbb{P}_z\left(\int_0^t f(X_s)\,\mathrm{d}s < \infty \text{ for all } t < \zeta\right) \geq c > 0.$$

The proof is complete. ∎

In general almost nothing is known about the supportive – or equivalently avoidable – set in the statement of Theorem 3.1, but under some tight restrictions on X, some results exist. For example, in two recent papers Mimica and Vondraček studied unavoidable unions of balls in $\mathbb{R}^d$, for censored stable processes in [27] and for isotropic Lévy processes satisfying a particular scaling condition – which generalises the scaling of stable processes – in [28]. In a similar vein, Grzywny and Kwaśnicki [17] give an explicit form of the hitting probability of a ball by a unimodal isotropic Lévy process. However, a general characterisation of avoidable sets for general Lévy processes remains a challenging problem. The final example of Hawkes' seminal paper [18] gives a nice demonstration of how the problem differs from that of polar sets. Nevertheless we will later discuss stronger assumptions on f, with X a stable process, under which the supportive sets can be omitted entirely from Theorem 3.1.

The following theorem addresses the same problem as Theorem 3.1 but in the case where the infinite-time path integral is finite almost surely. As in Theorem 2.10, the supportive sets do not disappear, but they can be made large.

Theorem 3.2. *Let X be a strongly transient standard Markov process and $f : E \to [0,\infty]$ measurable. For $z \in E$, the following are equivalent:*

(i) $\mathbb{P}_z(\int_0^t f(X_s)\,\mathrm{d}s < \infty$ *for every* $t < \zeta) = 1$.

(ii) *For every $\varepsilon \in (0,1)$ there is a $\mathbb{P}_z$-supportive set M such that $\int_{M\cap K} f(y)\,U(z,\mathrm{d}y) < \infty$ for all compact $K \subseteq E$ and X stays in M with probability at least $1 - \varepsilon$.*

Proof. (i)⇒(ii) Let $K_1 \subseteq K_2 \subseteq \cdots$ be an increasing sequence of compact sets with limit E. For $n \in \mathbb{N}$ let L_{K_n} be the last exit time from K_n. Then, since X is strongly transient,

$$\mathbb{P}_z\left(\int_0^t f(X_s)\,\mathrm{d}s < \infty \text{ for every } t < \zeta\right) = 1,$$

$$\text{so} \quad \mathbb{P}_z\left(\int_0^{L_{K_n}} f(X_s)\,\mathrm{d}s < \infty \text{ for every } n \in \mathbb{N}\right) = 1,$$

$$\text{so} \quad \mathbb{P}_z\left(\int_0^{L_{K_n}} f(X_s)\,\mathrm{d}s < \infty\right) = 1 \text{ for every } n \in \mathbb{N},$$

$$\text{so} \quad \mathbb{P}_z\left(\int_0^\infty f(X_s)\mathbf{1}_{K_n}(X_s)\,\mathrm{d}s < \infty\right) = 1 \text{ for every } n \in \mathbb{N}. \tag{24}$$

Now fix $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$. We can choose a constant N_n such that

$$\mathbb{P}_z\left(\int_0^\infty f(X_s)\mathbf{1}_{K_n}(X_s)\,\mathrm{d}s \leq N_n\right) > 1 - 2^{-n}\varepsilon.$$

Then write $f_n = f \mathbf{1}_{K_n}$ and define a super-finite set for each (X, f_n)

$$M_n^\varepsilon := \left\{ y \in E : \mathbb{P}_y\left(\int_0^\infty f_n(X_s)\, \mathrm{d}s \leq N_n \right) > 2^{-n}\varepsilon \right\}, \quad n \in \mathbb{N}.$$

By Proposition 2.9 each M_n^ε is $\mathbb{P}_z$-supportive. Moreover, since by Lemma 2.8, $E \setminus M_n^\varepsilon$ contains its regular points,

$$
\begin{aligned}
1 - 2^{-n}\varepsilon &< \mathbb{P}_z\left(\int_0^\infty f_n(X_s)\, \mathrm{d}s \leq N_n \right) \\
&= \mathbb{P}_z\left(\int_0^\infty f_n(X_s)\, \mathrm{d}s \leq N_n;\ T_{E \setminus M_n^\varepsilon} < \infty \right) \\
&\quad + \mathbb{P}_z\left(\int_0^\infty f_n(X_s)\, \mathrm{d}s \leq N_n;\ T_{E \setminus M_n^\varepsilon} = \infty \right) \\
&\leq \mathbb{P}_z\left(\int_{T_{E \setminus M_n^\varepsilon}}^\infty f_n(X_s)\, \mathrm{d}s \leq N_n;\ T_{E \setminus M_n^\varepsilon} < \infty \right) + \mathbb{P}_z(T_{E \setminus M_n^\varepsilon} = \infty) \\
&= \int_{E \setminus M_n^\varepsilon} \mathbb{P}_a\left(\int_0^\infty f_n(X_s)\, \mathrm{d}s \leq N_n \right) \mathrm{d}\mathbb{P}_z\left(X_{T_{E \setminus M_n^\varepsilon}} \in \mathrm{d}a;\ T_{E \setminus M_n^\varepsilon} < \infty \right) \\
&\quad + \mathbb{P}_z(T_{E \setminus M_n^\varepsilon} = \infty) \\
&\leq 2^{-n}\varepsilon + \mathbb{P}_z(T_{E \setminus M_n^\varepsilon} = \infty),
\end{aligned}
$$

and therefore not only is each M_n^ε $\mathbb{P}_z$-supportive, but the probability of remaining in each is bounded away from 0 by $\mathbb{P}_z(T_{E \setminus M_n^\varepsilon} = \infty) > 1 - 2^{1-n}\varepsilon$. Now let $M^\varepsilon := \bigcap_n M_n^\varepsilon$. Then by subadditivity of measure,

$$\mathbb{P}_z(T_{E \setminus M^\varepsilon} = \infty) \geq 1 - \sum_{n=1}^\infty \mathbb{P}_z(T_{E \setminus M_n^\varepsilon} < \infty) > 1 - \varepsilon \sum_{n=1}^\infty 2^{1-n} = 1 - 2\varepsilon.$$

Because $K_n \uparrow E$, any compact set will be covered by K_n for some $n \in \mathbb{N}$. Therefore

$$\int_{M^\varepsilon \cap K} f(y)\, U(z, \mathrm{d}y) \leq \int_{M_n^\varepsilon \cap K_n} f(y)\, U(z, \mathrm{d}y) = \int_{M_n^\varepsilon} f_n(y)\, U(z, \mathrm{d}y).$$

The right-hand side is finite due to Proposition 2.7 and the definition of M_n^ε.

(i)$\Leftarrow$(ii) Suppose that for any $\varepsilon > 0$ there exists a $\mathbb{P}_z$-supportive set M^ε such that X stays in M^ε with probability at least $1 - \varepsilon$ and $\int_{M^\varepsilon \cap K} f(y)\, U(z, \mathrm{d}y) < \infty$ for all compact $K \subseteq E$. Let $K_1 \subseteq K_2 \subseteq \cdots$ be a nested sequence of compact sets with limit E. Then for all $n \in \mathbb{N}$,

$$
\begin{aligned}
\mathbb{E}_z\left[\int_0^\infty f(X_s) \mathbf{1}_{M^\varepsilon \cap K_n}(X_s)\, \mathrm{d}s \right] < \infty &\implies \mathbb{P}_z\left(\int_0^\infty f(X_s) \mathbf{1}_{M^\varepsilon \cap K_n}(X_s)\, \mathrm{d}s < \infty \right) = 1 \\
&\implies \mathbb{P}_z\left(\int_0^\infty f(X_s) \mathbf{1}_{K_n}(X_s)\, \mathrm{d}s < \infty \right) > 1 - \varepsilon \\
&\implies \mathbb{P}_z\left(\int_0^{T_{E \setminus K_n}} f(X_s)\, \mathrm{d}s < \infty \right) > 1 - \varepsilon.
\end{aligned}
$$

Thus, since X is strongly transient, we have $\lim_{n \to \infty} T_{E \setminus K_n} = \zeta$ and

$$1 - \varepsilon < \lim_{n \to \infty} \mathbb{P}_z \left(\int_0^{T_{E \setminus K_n}} f(X_s) \, ds < \infty \right)$$

$$= \mathbb{P}_z \left(\int_0^{T_{E \setminus K_n}} f(X_s) \, ds < \infty \text{ for all } n \in \mathbb{N} \right)$$

$$= \mathbb{P}_z \left(\int_0^t f(X_s) \, ds < \infty \text{ for all } t < \zeta \right). \tag{25}$$

Since $\varepsilon > 0$ is arbitrary, we see that $\mathbb{P}_z(\int_0^t f(X_s) \, ds < \infty \text{ for all } t < \zeta) = 1$. ∎

3.2. Lévy processes with local time

One can ask whether, for the special case of Lévy processes with jointly continuous local time, the additional supportive set compared to the Engelbert–Schmidt [13] and Zanzotto [34] theorems discussed at the beginning of Section 3 can be removed. Let us quickly recall from Bertoin [6] some important results on local time for Lévy processes. If Ψ denotes the characteristic exponent of a Lévy process X, then X has local time if and only if

$$\int_{-\infty}^\infty \mathcal{R} \left(\frac{1}{1 + \Psi(\xi)} \right) d\xi < \infty$$

(see [6, V.1 Theorem 1]). A deep result due to Barlow [2] and Barlow and Hawkes [3] gives a necessary and sufficient condition for the existence of a jointly continuous version of the local time process $(t, x) \mapsto L_t^x$; see [6, Chapter V].

We now provide a proof of the Engelbert–Schmidt–Zanzotto result without appealing to local time, instead working only with densities of the potential measures of the killed Lévy process. Our method also works for Lévy processes for which no jointly continuous version of the local time exists.

Theorem 3.3. *Let X be a Lévy process on $\mathbb{R}$ which has local time, and $f : \mathbb{R} \to [0, \infty]$ be measurable. The following are equivalent:*

(i) $\mathbb{P}_0(\int_0^t f(X_s) \, ds < \infty \text{ for every } t \geq 0) = 1$.

(ii) $\int_K f(y) \, dy < \infty$ for all compact $K \subseteq \mathbb{R}$.

Proof. The proof makes use of the killed process X^q, that is, the Lévy process killed at an independent exponentially distributed time τ^q with parameter $q > 0$. In the literature such a process is sometimes called a *q-subprocess*; see for example Blumenthal and Getoor [9, Example III(3.17)]. The killed process is a standard Markov process on $\mathbb{R}$ with cemetary state Δ, and has transition semigroup

$$P_t^q(x, A) = \mathbb{P}_x(X_t \in A; \, t < \tau^q) = e^{-qt} P_t(x, A) \tag{26}$$

for a Borel set A. The potential operator of X^q is U^q, the q-potential operator of X. By [6, Theorems II.5 and II.16], U^q has a bounded density u^q. We denote by $\hat{u}^q$ the dual potential density, that is, the potential density of the dual Lévy process $\hat{X} = -X$. Since

τ^q is almost surely finite, any killed Lévy process is transient. Since X has lifetime $\zeta = \infty$ almost surely, X^q is not strongly transient.

(i)$\Rightarrow$(ii) Fix $q > 0$, and let X^q be the killed process above. Since τ^q has support $(0, \infty)$, our assumption implies that X^q satisfies condition (i) of Theorem 2.10. Fix some $\varepsilon > 0$; by Theorem 2.10, there exists a $\mathbb{P}_0$-supportive set M^ε for X^q such that $\mathbb{P}_0(X_s \in M^\varepsilon$ for all $s < \tau^q) = \mathbb{P}_0(X_s^q \in M^\varepsilon$ for all $s < \zeta) > 1 - \varepsilon$ and

$$\int_{M^\varepsilon \cap K} f(y)\, U^q(0, \mathrm{d}y) \leq \int_{M^\varepsilon} f(y)\, U^q(0, \mathrm{d}y) < \infty$$

for any compact $K \subseteq \mathbb{R}$. Set $B^\varepsilon = \mathbb{R} \setminus M^\varepsilon$ and fix a compact $K \subseteq \mathbb{R}$. Then

$$\mathbb{P}_0(X_s \in B^\varepsilon \cap K \text{ for some } s < \tau^q) = \mathbb{P}_0(X_s^q \in B^\varepsilon \cap K \text{ for some } s < \zeta)$$
$$\leq \mathbb{P}_0(X_s^q \in B^\varepsilon \text{ for some } s < \zeta) < \varepsilon.$$

Now suppose $B^\varepsilon \cap K$ is non-empty, so there exists some $x \in B^\varepsilon \cap K$. With $T_{\{x\}} = \inf\{s > 0 : X_s = x\}$ we find that

$$\mathbb{P}_0(X_s \in B^\varepsilon \cap K \text{ for some } s < \tau^q) \geq \mathbb{P}_0(X_s = x \text{ for some } s < \tau^q)$$
$$= \mathbb{E}_0[e^{-qT_{\{x\}}}] = \hat{u}^q(x) C^q,$$

for a positive constant C^q (using [31, Theorem 43.3]). Because X hits points, $\hat{u}^q$ is bounded below on the compact set K (because u^q is lower-semicontinuous as a q-excessive function, and pointwise positive). Hence, there is a $c > 0$ with $\mathbb{P}_0(X_s \in B^\varepsilon \cap K$ for some $s < \tau^q) > c$, and this constant c is independent of ε. Now we have shown that $c < \varepsilon$, but our choice of ε was arbitrary, and the resolution of this apparent contradiction is that for $\varepsilon \leq c$ there does not exist any point $x \in B^\varepsilon \cap K$, that is, $K \subseteq M^\varepsilon$. Thus for such ε,

$$\infty > \int_{M^\varepsilon \cap K} f(y)\, U^q(0, \mathrm{d}y) = \int_K f(y) u^q(y)\, \mathrm{d}y.$$

Again because u^q is bounded below on compacts, u^q can be omitted in the integral test and the claim follows.

(i)$\Leftarrow$(ii) Fix $q > 0$. Since X has local time, Bertoin [6, Theorem II.16] shows that u^q is bounded, and thus for arbitrary compact K,

$$\mathbb{E}_0\left[\int_0^\infty f(X_s^q)\mathbf{1}_K(X_s^q)\, \mathrm{d}s\right] = \int_K f(y) u^q(y)\, \mathrm{d}y \leq \sup_{x \in \mathbb{R}} u^q(x) \int_K f(y)\, \mathrm{d}y < \infty.$$

Now $\mathbb{E}_0[\int_0^\infty f(X_s^q)\mathbf{1}_K(X_s^q)\, \mathrm{d}s] < \infty$ implies $\mathbb{P}_0(\int_0^\infty f(X_s^q)\mathbf{1}_K(X_s^q)\, \mathrm{d}s < \infty) = 1$ for all compact K. Thus by continuity of measure and monotone convergence,

$$\mathbb{P}_0\left(\int_0^{\tau^q} f(X_s)\, \mathrm{d}s < \infty\right) = \mathbb{P}_0\left(\int_0^\infty f(X_s^q)\, \mathrm{d}s < \infty\right) = 1.$$

Since τ^q is independent of X and has support $(0, \infty)$, this implies

$$\mathbb{P}_0\left(\int_0^t f(X_s)\, \mathrm{d}s < \infty \text{ for every } t < \infty\right) = 1,$$

and (ii) has been proven. ∎

4. Small-time path integral tests

We now come to local versions of the path integral tests. Integral tests for path integrals up to finite random times are known by the work of Engelbert and Schmidt for the Brownian case and Zanzotto for the stable case with $\alpha \in (1, 2)$, and take the form of the following equivalent statements:

(i) $\mathbb{P}_z(\exists t > 0 : \int_0^t f(X_s)\,\mathrm{d}s < \infty) > 0.$

(ii) $\mathbb{P}_z(\exists t > 0 : \int_0^t f(X_s)\,\mathrm{d}s < \infty) = 1.$

(iii) There exists an $\varepsilon > 0$ such that the integral test $\int_{z-\varepsilon}^{z+\varepsilon} f(y)\,\mathrm{d}y < \infty$ holds.

Those authors' arguments, as in the case of finite-time-horizon path integrals, rely upon the occupation time formula and the joint continuity of local time. For our study of stable SDEs for $\alpha \in (0, 1)$ we need a version for transient Lévy processes. A more general version for transient Markov processes can be found in the PhD thesis of Baguley [1].

Theorem 4.1. *Let X be a transient Lévy process on $\mathbb{R}^d$ which does not hit points, and let $f : \mathbb{R}^d \to [0, \infty]$ be measurable. For $z \in \mathbb{R}^d$, the following are equivalent:*

(i) $\mathbb{P}_z(\exists t > 0 : \int_0^t f(X_s)\,\mathrm{d}s < \infty) = 1.$

(ii) *There exists a $\mathbb{P}_z$-thin set B such that $\int_{\mathbb{R}^d \setminus B} f(y)\,U(z, \mathrm{d}y) < \infty.$*

Recall that a measurable set B is called $\mathbb{P}_z$-thin if $\mathbb{P}_z(T_B > 0) = 1.$

The reader should keep in mind the following relation of avoidable (or supportive) sets and thin sets. Since an avoidable set is never hit with positive probability, the first hitting time is trivially non-zero with positive probability. Since this event obeys a zero-one law (Blumenthal's zero-one law), the first hitting time is non-zero with probability 1. Hence, any $\mathbb{P}_z$-avoidable set is also $\mathbb{P}_z$-thin. The converse is obviously wrong.

Proof of Theorem 4.1. (i)$\Rightarrow$(ii) We begin by using the fact that X does not hit points, i.e. $\mathbb{P}_z(L_{\{z\}} = 0) = 1$, to show that the last exit times of balls $\mathbf{B}_\varepsilon(z)$ are arbitrarily small as $\varepsilon \downarrow 0$. First suppose that $l := \lim_{\varepsilon \downarrow 0} L_{\mathbf{B}_\varepsilon(z)} > 0$. Then there exists a time $t \in (0, l]$ such that $X_t \in \mathbf{B}_\varepsilon(z)$ for all $\varepsilon > 0$, that is, $X_t = z$. Therefore $L_{\{z\}} \geq t > 0$, which is prohibited by the fact that X does not hit points. So it follows that $\lim_{\varepsilon \downarrow 0} L_{\mathbf{B}_\varepsilon(z)} = 0$ almost surely.

In particular, under the assumption of (i), there exists some open set $G \ni z$ such that $\mathbb{P}_z(\exists t > L_G : \int_0^t f(X_s)\,\mathrm{d}s < \infty) > 0$. Thus

$$\mathbb{P}_z\left(\int_0^{L_G} f(X_s)\,\mathrm{d}s < \infty\right) \geq \mathbb{P}_z\left(\exists t > L_G : \int_0^t f(X_s)\,\mathrm{d}s < \infty\right) > 0.$$

It therefore follows that

$$\mathbb{P}_z\left(\int_0^\infty f(X_s)\mathbf{1}_G(X_s)\,\mathrm{d}s < \infty\right) > 0. \tag{27}$$

Then Theorem 2.3 yields a $\mathbb{P}_z$-supportive set M such that

$$\int_{M \cap G} f(x)\,U(z, \mathrm{d}x) < \infty.$$

Since M is $\mathbb{P}_z$-supportive, the complement of $\mathbb{R}^d \setminus M$ is avoidable so that $\mathbb{P}_z(T_{\mathbb{R}^d \setminus M} > 0)$ $\geq \mathbb{P}_z(T_{\mathbb{R}^d \setminus M} = \infty) > 0$. Hence, Blumenthal's zero-one law implies that $\mathbb{P}_z(T_{\mathbb{R}^d \setminus M} > 0)$ $= 1$, that is, $\mathbb{R}^d \setminus M$ is $\mathbb{P}_z$-thin. Since G is open and contains z, $\mathbb{R}^d \setminus G$ is also $\mathbb{P}_z$-thin. The union of finitely many $\mathbb{P}_z$-thin sets is again $\mathbb{P}_z$-thin, and thus we define $B = (\mathbb{R}^d \setminus M)$ $\cup (\mathbb{R}^d \setminus G) = \mathbb{R}^d \setminus (M \cap G)$, which is $\mathbb{P}_z$-thin and satisfies

$$\int_{\mathbb{R}^d \setminus B} f(x)\, U(z, \mathrm{d}x) = \int_{M \cap G} f(x)\, U(z, \mathrm{d}x) < \infty.$$

(i)$\Leftarrow$(ii) Let B be the $\mathbb{P}_z$-thin set from (ii). Proposition II(4.3) of Blumenthal and Getoor [9] says that there is a compact set $K \subseteq \mathbb{R}^d \setminus B$ such that $z \in K$ and $\mathbb{R}^d \setminus K$ is again $\mathbb{P}_z$-thin. We shall work with the hitting time $\tau = T_{\mathbb{R}^d \setminus K}$. Since $\mathbb{R}^d \setminus K$ is $\mathbb{P}_z$-thin, τ is $\mathbb{P}_z$-almost surely positive. In addition, by transience of X, $\tau \leq L_K < \infty$ $\mathbb{P}_z$-almost surely, and so $\mathbb{P}_z(0 < \tau < \infty) = 1$. We have assumed that

$$\mathbb{E}_z\left[\int_0^\infty f(X_s)\mathbf{1}_{\mathbb{R}^d \setminus B}(X_s)\, \mathrm{d}s\right] = \int_{\mathbb{R}^d \setminus B} f(x)\, U(z, \mathrm{d}x) < \infty.$$

This implies that $\mathbb{P}_z(\int_0^\infty f(X_s)\mathbf{1}_{\mathbb{R}^d \setminus B}(X_s)\, \mathrm{d}s < \infty) = 1$. From $K \subseteq \mathbb{R}^d \setminus B$ it follows that $\tau \leq T_B$. Therefore

$$\mathbb{P}_z\left(\int_0^\tau f(X_s)\, \mathrm{d}s < \infty\right) \geq \mathbb{P}_z\left(\int_0^{T_B} f(X_s)\, \mathrm{d}s < \infty\right)$$
$$\geq \mathbb{P}_z\left(\int_0^\infty f(X_s)\mathbf{1}_{\mathbb{R}^d \setminus B}(X_s)\, \mathrm{d}s < \infty\right) = 1.$$

Then (i) follows from the fact that τ is almost surely positive. ∎

4.1. The case of stable Lévy processes

It is generally impossible to remove the thin sets from Theorem 4.1. The following theorem is a version of Theorem 4.1 in a particular situation in which the thin set can be removed by capacity comparisons. Its proof relies on a remarkably precise analytic description of $\mathbb{P}_z$-thin sets for stable processes. The theorem will be applied later with $f = \sigma^{-\alpha}$ to study the SDE (1), for instance with $\sigma(x) = |x|^\beta$.

Theorem 4.2. *Let X be a symmetric stable process on $\mathbb{R}$ with index $\alpha \in (0, 1)$, and let $f : \mathbb{R} \to [0, \infty]$ be measurable. Suppose that f has an isolated monotone pole at z, in the sense that there exists $\delta > 0$ such that f is increasing on $(z - \delta, z)$ and decreasing on $(z, z + \delta)$. Then the following are equivalent:*

(i) $\mathbb{P}_z(\exists t > 0 : \int_0^t f(X_s)\, \mathrm{d}s < \infty) = 1$.

(ii) *There exists an $\varepsilon > 0$ such that $\int_{z-\varepsilon}^{z+\varepsilon} f(y)|z - y|^{\alpha-1}\, \mathrm{d}y < \infty$.*

Theorem 4.2 above can be directly extended to 'almost monotone' measurable functions g, in the sense that there exists a $C < \infty$ such that for all $|x - z| < \delta$,

$|g(x) - f(x)| \leq C$ for some measurable f which has an isolated monotone pole at z, by virtue of the fact that in this case, for $\varepsilon < \delta$,

$$\left| \int_{z-\varepsilon}^{z+\varepsilon} g(y)|y|^{\alpha-1}\,\mathrm{d}y - \int_{z-\varepsilon}^{z+\varepsilon} f(y)|y|^{\alpha-1}\,\mathrm{d}y \right| \leq \frac{2C\varepsilon^{\alpha}}{\alpha}$$

and

$$\left| \int_0^t g(X_s)\,\mathrm{d}s - \int_0^t f(X_s)\,\mathrm{d}y \right| \leq Ct.$$

Proof of Theorem 4.2. Since X is a Lévy process, we can without loss of generality set $z = 0$. Recall that $U(0, \mathrm{d}y) = |y|^{\alpha-1}\,\mathrm{d}y$, up to a constant factor which we freely ignore. The implication (ii)$\Rightarrow$(i) follows from Theorem 4.1 and the fact that $\mathbb{R} \setminus (-\varepsilon, \varepsilon)$ is $\mathbb{P}_0$-thin for the stable process.

Now suppose (i). Again from Theorem 4.1 there exists a $\mathbb{P}_0$-thin set B such that

$$\int_{\mathbb{R}\setminus B} f(y)\,U(0, \mathrm{d}y) = \int_{\mathbb{R}\setminus B} f(y)|y|^{\alpha-1}\,\mathrm{d}y < \infty.$$

Take $\varepsilon \in (0, \delta)$ and let the map g be defined by $g(y) = f(y)|y|^{\alpha-1}\mathbf{1}_{(-\varepsilon, \varepsilon)}(y)$. It is immediately seen that g shares the same monotonicity property as f. The intuition to have in mind is that the monotone nature of g will allow its behaviour on B to be determined by its behaviour on $\mathbb{R} \setminus B$.

According to Wiener's criterion for thin sets of stable processes (see for instance Bliedtner and Hansen [8, Corollary V.4.17]), the $\mathbb{P}_0$-thin set B satisfies

$$\sum_{k=1}^{\infty} 2^{k(1-\alpha)} C(B \cap S_k) < \infty, \tag{28}$$

where $C(\cdot)$ is capacity and $S_k = \{x \in \mathbb{R} : 2^{-(k+1)} < |x| \leq 2^{-k}\}$ defines a sequence of decreasing shells of Lebesgue measure $2(2^{-k} - 2^{-(k+1)}) = 2^{-k}$. The isoperimetric inequality of Betsakos [7, Theorem 1] states that the capacity of any compact set is no smaller than that of the ball of the same Lebesgue measure. Example 42.17 of Sato [31] then shows that for any compact K,

$$C(K) \geq C(\mathbf{B}_{\frac{1}{2}\lambda(K)}) = C\left(\frac{\lambda(K)}{2}\mathbf{B}_1 \right) = \left(\frac{\lambda(K)}{2} \right)^{1-\alpha} C(\mathbf{B}_1) = C_0\lambda(K)^{1-\alpha},$$

where $\mathbf{B}_r$ is the ball about 0 of radius r, $C_0 := C(\mathbf{B}_1)2^{\alpha-1}$, and λ represents the Lebesgue measure. We can extend this result to a wider class of Borel sets as follows. Let $G \subset \mathbb{R}$ be a bounded open set. Then there exists an increasing sequence of compact sets $K_n \subseteq G$ such that $G = \bigcup_{n=1}^{\infty} K_n$, and thus from Sato [31, Propositions 42.10 and 42.12] it follows that $C(K_n) \uparrow C(G)$. The isoperimetric inequality applied to each K_n then yields

$$C(G) = \lim_{n\to\infty} C(K_n) \geq \lim_{n\to\infty} C_0\lambda(K_n)^{1-\alpha} = C_0\lambda(G)^{1-\alpha}.$$

Now for a general bounded Borel set $A \subset \mathbb{R}$,

$$
\begin{aligned}
C(A) &= \inf \{C(G) : G \text{ open and } A \subseteq G\} \\
&\geq C_0 \inf \{\lambda(G)^{1-\alpha} : G \text{ open, bounded and } A \subseteq G\} = C_0 \lambda(A)^{1-\alpha}.
\end{aligned}
$$

The first equality is due to C being a Choquet capacity, and details can be found in [9, Theorem I(10.5)] and the discussion following it. This in particular holds for the bounded sets $B \cap S_k$, and so from (28) we can deduce that

$$
\sum_{k=1}^{\infty} 2^{k(1-\alpha)} C_0 \lambda(B \cap S_k)^{1-\alpha} = C_0 \sum_{k=1}^{\infty} (2^k \lambda(B \cap S_k))^{1-\alpha} < \infty.
$$

Since $C_0 = C(\mathbf{B}_1) 2^{\alpha-1} < \infty$ and $\lambda(S_k) = 2^{-k}$, this implies

$$
\sum_{k=1}^{\infty} \left(\frac{\lambda(B \cap S_k)}{\lambda(S_k)} \right)^{1-\alpha} < \infty.
$$

From this it follows that for any fixed $c \in (0,1)$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\lambda(B \cap S_n) \leq c \lambda(S_n) = c 2^{-n}$, and therefore

$$
\lambda(B^c \cap S_n) \geq (1-c) \lambda(S_n) = (1-c) 2^{-n}. \tag{29}
$$

We will use this relationship to bound the integral of g over B.

For $n \geq N$ we shall now consider the two pieces of S_n separately, using the notation $S_n^+ = S_n \cap (0, \infty)$, $S_n^- = S_n \cap (-\infty, 0)$. Taking advantage of the monotonicity of g, with the notation $\bar{g}_n = \sup_{S_n^+} g$ and $\underline{g}_n = \inf_{S_n^+} g$, we see for $n \geq N$ that

$$
\int_{B \cap S_n^+} g(x) \, dx \leq \bar{g}_n \lambda(B \cap S_n^+) \leq \bar{g}_n c \lambda(S_n^+) = \bar{g}_n 2c \lambda(S_{n+1}^+).
$$

Using (29) and the fact that $\bar{g}_n \leq \underline{g}_{n+1}$, we continue the chain of inequalities as

$$
\leq \frac{2c}{1-c} \underline{g}_{n+1} \lambda(B^c \cap S_{n+1}^+) \leq \frac{2c}{1-c} \int_{B^c \cap S_{n+1}^+} g(x) \, dx.
$$

Exactly the same procedure works for S_n^-, and adding the two pieces gives

$$
\int_{B \cap S_n} g(x) \, dx \leq \frac{2c}{1-c} \int_{B^c \cap S_{n+1}} g(x) \, dx.
$$

Summing over $n \geq N$ tells us that

$$
\int_{B \cap \mathbf{B}_{2^{-N}}} g(x) \, dx \leq \frac{2c}{1-c} \int_{B^c} g(x) \, dx < \infty.
$$

Let $\tilde{\varepsilon} = \varepsilon \wedge 2^{-N}$. Summing the integrals over B and B^c then yields $\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} f(x)|y|^{\alpha-1} \, dx < \infty$. ∎

5. Stochastic differential equations

In this final section we translate the foregoing results on different path integrals into results for stable stochastic differential equations. The (driftless) *stable SDE equation* with issuing point $z \in \mathbb{R}$ is defined to be the equation

$$\mathrm{d}Z_t = \sigma(Z_{t-})\,\mathrm{d}X_t, \quad Z_0 = z. \tag{30}$$

More precisely, let X be a two-sided stable process on $\mathbb{R}$ and probability space $\mathscr{P} = (\Omega, \mathcal{F}, \mathbb{P})$, and Z an $\mathbb{R}$-valued stochastic process on the same probability space satisfying $\mathbb{P}(Z_0 = z) = 1$ for some $z \in \mathbb{R}$. Let $\sigma : \mathbb{R} \to [0, \infty)$ be a measurable function, extended as usual via $\sigma(\Delta) = 0$. For $A \subseteq \mathbb{R}$ the collection $(X, Z, \mathscr{P})$ is called a *weak solution* to (30) on the measurable set A if

$$Z_t - z = \int_0^t \sigma(Z_{s-})\,\mathrm{d}X_s \quad \text{for all } t < T_{A^c} \tag{31}$$

and $T_{A^c} > 0$. The latter condition only excludes trivial cases. If $A = \mathbb{R}$ then $(X, Z, \mathscr{P})$ is called a *global weak solution*, and if $A \subsetneq \mathbb{R}$ then $(X, Z, \mathscr{P})$ is called a *local weak solution*. We say a solution is *trivial* if it is constant.

5.1. Existence and uniqueness of (local) solutions

Following the classical Engelbert–Schmidt approach to Brownian SDEs, in a sequence of articles Zanzotto succeeded in reformulating weak solutions of $\mathrm{d}Z_t = \sigma(Z_{t-})\,\mathrm{d}X_t$ into time changes of the driving process with path integrals $\int_0^t \sigma(X_s)^{-\alpha}\,\mathrm{d}s$. Using quadratic variations this is straightforward for the Brownian motion (see for instance Karatzas and Shreve [22, Chapter 5]), but the arguments for the stable case are more involved. The following reciprocal connection of SDE solutions and time change is a combination of three of Zanzotto's results ([34, Lemma 2.26], [35, Theorem 2], and [36, Theorem 2.2]) and a time change due to Kallenberg [19, Theorem 4.1].

Theorem 5.1 (Zanzotto–Kallenberg time change). (i) *Let X be a two-sided stable process of index $\alpha \in (0, 2]$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P}_z)$, and let $\sigma : \mathbb{R} \to [0, \infty)$ be measurable. Define*

$$I_t = \int_0^t \sigma(X_s)^{-\alpha}\,\mathrm{d}s, \quad \varphi_t = \inf\left\{s > 0 : \int_0^s \sigma(X_u)^{-\alpha}\,\mathrm{d}u > t\right\}, \quad t \geq 0,$$

and let X_φ be the time-changed process with the definitions from (4) and below. Then there exists a two-sided stable process Y of index α on an extension $\bar{\mathscr{P}} = (\bar{\Omega}, \tilde{\mathcal{F}}, \bar{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathbb{P}_z)$ such that

$$X_{\varphi_t} - z = \int_0^t \sigma(X_{\varphi_s})\,\mathrm{d}Y_s, \quad t \in [0, \bar{I}). \tag{32}$$

(ii) *Let $(X, Z, \mathscr{P})$ be a global weak solution of* (30) *with initial condition z. Then there exists a two-sided stable process Y of index $\alpha \in (0, 2]$ on an extension $\bar{\bar{\mathscr{P}}} = (\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}})$ of $(\Omega, \mathscr{F}, \mathbb{P})$ such that*

$$Z_t = Y_{\tilde{I}_t} \quad \text{for all } t \in [0, \infty) \ \bar{\mathbb{P}}\text{-a.s.,} \tag{33}$$

where $\tilde{I}_t = \int_0^t \sigma(Z_s)^\alpha \, ds$.

Zanzotto's abstract time-change representation in (32) was proved for all $\alpha \in (0, 2)$, whereas the translation into SDEs (i.e. showing the running time is a first hitting time of a known set A) and the precise analysis of situations in which the time-change representation is non-trivial was restricted to $\alpha \in (1, 2)$. This was caused by the lack of understanding of finiteness of path integrals which for $\alpha \in (0, 1)$ we fully provided in the sections above. The usefulness of our precise results lies not only in providing necessary and sufficient properties of σ for the existence and uniqueness of solutions, but also in giving information about properties of those solutions, for example whether they are global, and whether they explode in finite time. In what follows we shall present the complete picture for stable SDEs with $\alpha \in (0, 1)$, utilising the integral tests from Sections 2–4.

This first proposition is a more general version of Lemma 2.3 of Zanzotto [36]. It is more involved than the original because of the complexity of finely open sets for general stable processes, compared to the simpler case for $\alpha \in (1, 2]$. The lemma connects the statement (i) of Theorem 5.1 to the notion of local weak solutions, by demonstrating that the running time $\bar{I}$ of X_φ in (32) is indeed a first hitting time of a measurable set.

Proposition 5.2. *Let X be a Markov process on E with strong Feller resolvent satisfying Hunt's condition (see* (H) *at the end of Section* 2.1*), and let f be non-negative and measurable. Define the path integrals and inverses as before:*

$$I_t = \int_0^t f(X_s) \, ds, \quad \varphi_t = \inf\left\{s > 0 : \int_0^s f(X_u) \, du > t\right\}, \quad t \in [0, \infty).$$

Then $T_{\mathcal{O}} = \varphi_\infty \ \mathbb{P}_y$-almost surely for all $y \in E$, where

$$\mathcal{O} = \left\{x \in \mathbb{R}^d : \mathbb{P}_x\left(\forall t > 0 : \int_0^t f(X_s) \, ds = \infty\right) = 1\right\}.$$

If f is strictly positive, then $I(T_{\mathcal{O}}) = \bar{I} = \inf\{s > 0 : X_{\varphi_s} \in \mathcal{O}\}$.

Note that this statement does not hold for all Markov processes. It fails for instance for X a deterministic positive drift on $\mathbb{R}$ with $f(x) = |x|^{-1}\mathbf{1}_{(-\infty, 0)}(x)$, because in that case $\mathcal{O} = \emptyset$ but $\varphi_\infty = T_{\{0\}} < \infty$ under $\mathbb{P}_x$ if the initial value x is negative. However, the proposition does apply for two-sided stable Lévy processes.

Proof of Proposition 5.2. Fix a $q > 0$ and let X^q be the process killed at an independent exponential time τ^q, so that

$$\int_0^\infty f(X_s^q) \, ds = \int_0^{\tau^q} f(X_s) \, ds \quad \text{almost surely.}$$

For $c \in (0, 1)$ we define the sets $M_c := \{x \in \mathbb{R} : \mathbb{P}_x(\int_0^\infty f(X_s^q)\, ds \leq 1/c) > c\}$ and $A_c := \mathbb{R} \setminus M_c$. The sets M_c are super-finite for (X^q, f), and increasing as $c \downarrow 0$. Their complements A_c decrease as $c \downarrow 0$ to $A := \{x \in \mathbb{R} : \mathbb{P}_x(\int_0^\infty f(X_s^q)\, ds < \infty) = 0\}$. It is clear that $\mathcal{O} \subseteq A$. We can also note that if $x \in A$ then

$$\mathbb{P}_x\left(\int_0^t f(X_s)\, ds = \infty \text{ for all } t \geq 0\right) = 1,$$

since τ^q is independent and has support $(0, \infty)$. Therefore $x \in \mathcal{O}$, and so $A = \mathcal{O}$. For our purposes it is easier to work with A than $\mathcal{O}$ for the rest of the proof.

We saw in Lemma 2.6 that for any choice of c, and for all $y \in \mathbb{R}$,

$$\mathbb{P}_y\left(\int_0^t f(X_s^q)\mathbf{1}_{M_c}(X_s^q)\, ds < \infty \text{ for all } t \geq 0\right) = 1. \tag{34}$$

Since X^q stays in M_c before T_{A_c}, this implies that $\mathbb{P}_y$-almost surely $\int_0^t f(X_s^q)\, ds < \infty$ for all $t \leq T_{A_c}$, where T_{A_c} is the first exit time of M_c by the unkilled process X. Since the sets A_c are decreasing, the times T_{A_c} are almost surely increasing. In particular, since (34) holds for arbitrary $c \in (0, 1)$, it follows that, for any $y \in \mathbb{R}$,

$$\mathbb{P}_y\left(\int_0^t f(X_s^q)\, ds < \infty \text{ for all } t < \lim_{c \downarrow 0} T_{A_c}\right) = 1.$$

Again, since τ^q is independent of X and has support $(0, \infty)$, this implies that

$$\mathbb{P}_y\left(\int_0^t f(X_s)\, ds < \infty \text{ for all } t < \lim_{c \downarrow 0} T_{A_c}\right) = 1.$$

Therefore $\varphi_\infty \geq \lim_{c \downarrow 0} T_{A_c}$ $\mathbb{P}_y$-almost surely. If we can show the first equality of

$$T := \lim_{c \downarrow 0} T_{A_c} = T_A = T_{\mathcal{O}} \tag{35}$$

$\mathbb{P}_y$-almost surely for any $y \in \mathbb{R}$ then we will have proven the first inequality $\varphi_\infty \geq T_{\mathcal{O}}$.

Recall that for a Borel set A and $q > 0$ the function $\Phi_A^q(x) = \mathbb{E}_y[e^{-qT_A}; T_A < \infty] = \mathbb{E}_y[e^{-qT_A}]$ is q-excessive. Since our process X satisfies condition (H) and Φ_A^q is bounded, the discussion at the end of Section 2.1 tells us that Φ_A^q is regular (in the sense of (R) or (R)$'$), and thus in particular

$$\Phi_A^q(X_{T_{A_c}}) \to \Phi_A^q(X_T) \quad \text{almost surely on } \{T < \infty\} \text{ as } c \downarrow 0.$$

If we fix $A = A_{c_0}$ for some $c_0 > 0$ then $\Phi_A^q(X_{T_{A_c}}) = 1$ for all $c \leq c_0$, since the sets A_c are decreasing and contain their regular points by Lemma 2.8 so that $X_{T_{A_c}}$ is contained in $A_c \subseteq A_{c_0}$ on the event $\{T < \infty\}$ for any $y \in \mathbb{R}$. Thus $\Phi_A^q(X_T) = 1$ on $\{T < \infty\}$, that is, $\mathbb{P}_{X_T(w)}(T_{A_{c_0}} = 0) = 1$ for $\mathbb{P}_y$-almost every w such that $T(w) < \infty$. Since A_{c_0} contains its regular points, this implies that $X_T \in A_{c_0}$ on $\{T < \infty\}$. Then because our choice of $c_0 > 0$ was arbitrary, it follows that

$$X_T \in A = \bigcap_{c > 0} A_c \quad \text{almost surely on } \{T < \infty\}.$$

This implies (35) on $\{T < \infty\}$. On the event $\{T = \infty\}$, (35) is trivial, and so it holds almost surely, and thus $\varphi_\infty \geq T_A = T_\mathcal{O}$ almost surely.

The inequality $\varphi_\infty \leq T_\mathcal{O}$ comes from the fact that for any $u > 0$ and any $y \in \mathbb{R}$,

$$\mathbb{P}_y(\varphi_\infty \leq T_\mathcal{O} + u; T_\mathcal{O} < \infty) = \mathbb{P}_y\left(\int_0^{T_\mathcal{O}+u} f(X_s)\, \mathrm{d}s = \infty; T_\mathcal{O} < \infty\right)$$
$$= \mathbb{E}_y\left[\mathbb{P}_{X_{T_\mathcal{O}}}\left(\int_0^u f(X_s)\, \mathrm{d}s = \infty\right); T_\mathcal{O} < \infty\right]$$
$$= \mathbb{P}_y(T_\mathcal{O} < \infty),$$

using the fact that $\mathcal{O} = A = \bigcap_{c>0} A_c$ and all A_c (and thus the intersection) are finely closed by Lemma 2.8 so that $X_{T_\mathcal{O}} \in \mathcal{O}$. This implies that $\mathbb{P}_y(\varphi_\infty \leq T_\mathcal{O} + u) = \mathbb{P}_y(\varphi_\infty \leq T_\mathcal{O} + u < \infty) + \mathbb{P}_y(T_\mathcal{O} = \infty) = 1$. Therefore $\varphi_\infty \leq T_\mathcal{O} + u$ almost surely for all $u > 0$, and so $\varphi_\infty \leq T_\mathcal{O}$ almost surely.

Finally, we note that if $f > 0$ then the integrals $t \mapsto I_t$ are strictly increasing (up to a possibly finite explosion time). Hence, the generalised inverse φ does not jump to $+\infty$, that is, $\varphi_\infty = \bar{\varphi} = \varphi_{\bar{I}}$. Now since $T_\mathcal{O} = \varphi_\infty$ it follows that $\bar{I} = \inf\{s > 0 : X_{\varphi_s} \in \mathcal{O}\}$. It remains to note that since I is left-continuous, $\bar{I} = I(\varphi_\infty) = I(T_\mathcal{O})$. ∎

Although the setting of Proposition 5.2 is quite general, we will always use it for stable SDEs with $f = \sigma^{-\alpha}$. Since the presence of the set $\mathcal{O}$ is crucial (local solutions will live on the complement), and it depends on α and σ, we shall give it a name:

Definition 5.3. For $\sigma : \mathbb{R} \to [0, \infty)$ measurable and X a two-sided stable Lévy process we denote the set of *irregular points* by

$$\mathcal{O}(\sigma, \alpha) := \left\{x \in \mathbb{R} : \mathbb{P}_x\left(\int_0^t \sigma(X_s)^{-\alpha}\, \mathrm{d}s = \infty\right) = 1 \text{ for all } t > 0\right\}$$

and the *null-set* of σ by $N(\sigma) := \{x \in \mathbb{R} : \sigma(x) = 0\}$.

Note that the definition of $\mathcal{O}(\sigma, \alpha)$ is purely stochastic and as such is not very useful in SDE theorems. It is the results on path integrals which will make the following time-change results for SDEs useful.

Before turning to the main theorem let us first translate the first part of Theorem 5.1 into a statement on (local) weak solutions on the complement of the irregular points.

Proposition 5.4. *If $z \in \mathbb{R} \backslash \mathcal{O}(\sigma, \alpha)$, then $(Y, X_\varphi, \bar{\mathscr{P}})$ from Theorem 5.1 (i) is a local weak solution to the SDE (30) on $A = \mathbb{R} \setminus \mathcal{O}(\sigma, \alpha)$. In addition, if either*

(a) $\mathbb{P}_z(\sigma(X_{\varphi_\infty}) = 0) = 1$, *or*

(b) $\mathbb{P}_z(\bar{I} = \infty) = 1$,

then the solution is global.

Proof. Since $\sigma^{-\alpha} > 0$, Proposition 5.2 tells us that $\bar{I} = \inf\{s > 0 : X_{\varphi_s} \in \mathcal{O}(\sigma, \alpha)\}$. Thus, Zanzotto's time change implies that with $Z := X_\varphi$ the triple $(Y, Z, \bar{\mathscr{P}})$ is a local solution

to (30) on the complement of $\mathcal{O}(\sigma, \alpha)$. Assuming either (a) or (b) implies that the integral equation (32) holds for all $t \in [0, \infty)$, and therefore the solution is global. ∎

Conversely, we use the second part of Theorem 5.1 to derive necessary conditions for the existence of solutions.

Proposition 5.5. *If there exists a non-trivial local weak solution to* (30) *on a set A with issuing point z, then $z \in \mathbb{R} \setminus \mathcal{O}(\sigma, \alpha)$. This in particular holds when $A = \mathbb{R}$, that is, in the case of global weak solutions.*

Proof. To make use of Kallenberg's time change let us turn the local into a global solution by defining $\sigma^\dagger := \sigma \cdot (1 - \mathbf{1}_{A^c \cup (A^c)^r})$. Then the process $Z_t^\dagger := Z_{t \wedge T_{A^c}}$ satisfies

$$Z_t^\dagger - z = \int_0^t \sigma^\dagger(Z_{s-}^\dagger) \, dX_s \quad \text{for all } t \geq 0.$$

Thus Kallenberg's time change tells us that there exists a two-sided stable process Y such that $Z_t^\dagger = Y_{\tilde{I}_t^\dagger}$ for all $t \geq 0$ almost surely, where $\tilde{I}_t^\dagger = \int_0^t \sigma^\dagger(Z_s^\dagger) \, ds$, and thus $Z_t = Y_{\tilde{I}_t}$ for all $t \leq T_{A^c}$ almost surely, using $\tilde{I}_t = \int_0^t \sigma(Z_s) \, ds$ and $\tilde{\varphi}_t = \inf\{s > 0 : \tilde{I}_s > t\}$. Since Z is not constant, it follows from the SDE equation that $\sigma(Z_s) > 0$ for some Lebesgue-positive set of times $s < T_{A^c}$, and thus $\tilde{I}$ is also not constantly zero, and $\tilde{\varphi}$ does not jump instantaneously to ∞. Combining this with

$$\tilde{\varphi}_t \geq \int_0^{\tilde{\varphi}_t} \mathbf{1}_{(\sigma(Z_s) > 0)} \, ds = \int_0^{\tilde{\varphi}_t} \sigma(Z_u)^{-\alpha} \sigma(Z_u)^\alpha \, du = \int_0^t \sigma(Z_{\tilde{\varphi}_s})^{-\alpha} \, ds$$

$$= \int_0^t \sigma(Y_s)^{-\alpha} \, ds =: I_t$$

we have shown that there is almost surely some $t < T_{A^c}$ such that $I_t \leq \tilde{\varphi}_t < \infty$. It therefore follows by definition of $\mathcal{O}(\sigma, \alpha)$ that the issuing point z of Y under $\bar{\mathbb{P}}$ is an element of $\mathbb{R} \setminus \mathcal{O}(\sigma, \alpha)$. ∎

Finally, we can use our results on path integrals to turn the abstract formulations into analytic results. For $\alpha \in (1, 2]$, Zanzotto provided the analytic expression

$$\mathcal{O}(\sigma, \alpha) = \left\{ x \in \mathbb{R} : \int_{x-\varepsilon}^{x+\varepsilon} \sigma(y)^{-\alpha} \, dy = \infty \text{ for all } \varepsilon > 0 \right\}. \tag{36}$$

For $\alpha \in (0, 1)$ according to Theorem 4.1 we have

$$\mathcal{O}(\sigma, \alpha) = \left\{ x \in \mathbb{R} : \int_{\mathbb{R} \setminus B} \sigma(y)^{-\alpha} |x - y|^{\alpha - 1} \, dy = \infty \text{ for all } \mathbb{P}_x\text{-thin sets } B \right\} \tag{37}$$

since $U(x, dy) = |x - y|^{\alpha - 1} \, dy$ modulo some normalising constant. Since Wiener's criterion (see [8, Corollary V.4.17]) gives an analytic test for thinness in terms of capacities, the test is also analytic. If in addition σ has only isolated monotone zeros (e.g. $\sigma(x) = |x|^\beta$) then Theorem 4.2 implies for $\alpha \in (0, 1)$ the clean integral tests

$$\mathcal{O}(\sigma, \alpha) = \left\{ x \in \mathbb{R} : \int_{x-\varepsilon}^{x+\varepsilon} \sigma(y)^{-\alpha} |x - y|^{\alpha - 1} \, dy = \infty \text{ for all } \varepsilon > 0 \right\},$$

which is precisely Zanzotto's integral test modulo an additional polynomial factor. Recall from the introduction that complements of all $\mathbb{P}_z$-thin sets for a stable process with $\alpha > 1$ contain a ball around z. Hence, on a structural level the difference between the integral tests for $\alpha \in (1, 2]$ and $\alpha \in (0, 1)$ is the appearance of the new polynomial factor.

We are now in a position to formulate a set of statements which, for the Brownian motion, are known under the name of Engelbert–Schmidt theorems. The main theorem of this article extends the Engelbert–Schmidt theorems to stable SDEs with $\alpha \in (0, 1)$. The same result was proved for $\alpha \in (1, 2)$ by Zanzotto [36], and our proofs follow his closely, with the important distinction being that our Proposition 5.2 is a more general version of his Lemma 2.3 that no longer depends on local times.

Theorem 5.6. *Suppose X is a two-sided stable process with $\alpha \in (0, 1)$ and $\sigma : \mathbb{R} \to [0, \infty)$ is measurable. Then the following statements hold with the set $\mathcal{O}(\sigma, \alpha)$ of irregular points from* (37):

 (i) *For fixed $z \in \mathbb{R}$ there exists a non-trivial local weak solution to the SDE* (30) *if and only if $z \in \mathbb{R} \backslash \mathcal{O}(\sigma, \alpha)$.*

 (ii) *A global weak solution to* (30) *exists for all $z \in \mathbb{R}$ if and only if $\mathcal{O}(\sigma, \alpha) \subseteq N(\sigma)$.*

 (iii) *A non-trivial global weak solution to* (30) *exists for all $z \in \mathbb{R}$ if and only if $\mathcal{O}(\sigma, \alpha) = \emptyset$.*

 (iv) *There exists a global weak solution to* (30) *for all $z \in \mathbb{R}$, each of which is unique in law, if and only if $\mathcal{O}(\sigma, \alpha) = N(\sigma)$. In that case the solution process Z satisfies $Z = Y_\varphi$, where Y is a two-sided stable process on $\mathbb{R}$ of index α and $\varphi_t = \inf \{s > 0 : \int_0^s \sigma(Y_u)^{-\alpha} \, \mathrm{d}u > t\}$ for $t \geq 0$.*

Let us compare again with the case $\alpha \in (1, 2]$. In this case $\mathbb{R} \backslash \mathcal{O}(\sigma, \alpha)$ is open, so in the situation of (i) one can always consider local solutions on small intervals $[u, v]$ around the starting value z. Without further assumptions on σ this is generally false for $\alpha \in (0, 1)$ using examples where σ vanishes on very small disjoint intervals accumulating at z.

Proof of Theorem 5.6. (i) First suppose that $z \in \mathbb{R} \setminus \mathcal{O}(\sigma, \alpha)$. From Proposition 5.4 it follows that there exists a local weak solution, with solution process $Z = X_\varphi$. Further $T_{\mathcal{O}(\sigma, \alpha)} > 0$ almost surely, because $\mathcal{O}(\sigma, \alpha)$ is finely closed for X (see the end of the proof of Proposition 5.2). Otherwise z would be a regular point of $\mathcal{O}(\sigma, \alpha)$ and as such in $\mathcal{O}(\sigma, \alpha)$. This implies that $\varphi_\infty = T_{\mathcal{O}(\sigma, \alpha)} > 0$, and thus $t \mapsto \int_0^t \sigma^\alpha(X_s) \, \mathrm{d}s$ does not jump to $+\infty$ immediately. This implies that the time change does not explode instantaneously (i.e. $\bar\varphi > 0$) almost surely, and so the solution $Z = X_\varphi$ is not trivial.

For the reverse implication, from Proposition 5.5 it follows that if there exists a non-trivial local weak solution then $z \in \mathbb{R} \setminus \mathcal{O}(\sigma, \alpha)$.

(ii) Since $\mathcal{O}(\sigma, \alpha)$ is finely closed for X (see the end of the proof of Proposition 5.2), we see that $X_{T_{\mathcal{O}(\sigma, \alpha)}} \in \mathcal{O}(\sigma, \alpha)$ on the event that the hitting time is finite. Hence if we suppose that $\mathcal{O}(\sigma, \alpha) \subseteq N(\sigma)$, it follows that $\sigma(X_{\varphi_\infty}) = \sigma(X_{T_{\mathcal{O}(\sigma, \alpha)}}) = 0$ $\mathbb{P}_z$-almost surely for any $z \in \mathbb{R}$. Thus Proposition 5.4 gives existence of a global solution, with solution process $Z = X_\varphi$.

Now suppose that a global solution exists for every issuing point $z \in \mathbb{R}$. Proposition 5.5 tells us that if $z \in \mathcal{O}(\sigma, \alpha)$ then there is no non-trivial local weak solution to (30) with issuing point z. If we then assume that there exists a weak solution for all issuing points, it follows that the solution for $z \in \mathcal{O}(\sigma, \alpha)$ is trivial, and therefore $\sigma(z)$ must be zero for $z \in \mathcal{O}(\sigma, \alpha)$.

(iii) First suppose that $\mathcal{O}(\sigma, \alpha)$ is empty. Then since $\sigma^{-\alpha} > 0$, Proposition 5.2 yields $\bar{I} = \inf\{s > 0 : X_{\varphi_s} \in \mathcal{O}\} = \infty$ almost surely under any $\mathbb{P}_z$. Therefore for any $z \in \mathbb{R}$, Proposition 5.4 gives existence of a global weak solution, with solution process $Z = X_\varphi$.

Now we prove the reverse implication. Proposition 5.5 tells us that if $z \in \mathcal{O}(\sigma, \alpha)$ then there is no non-trivial solution to (30) with issuing point z. If we then assume that there exists a non-trivial weak solution for all issuing points, it follows that $\mathcal{O}(\sigma, \alpha)$ is empty.

(iv) "$\Rightarrow$" Suppose that for every $z \in \mathbb{R}$ there exists a global weak solution to (30) and that each of those solutions is unique in law. Then part (ii) of this theorem implies that $\mathcal{O}(\sigma, \alpha) \subseteq N(\sigma)$. Now suppose for contradiction that there is a point $z \in N(\sigma) \cap (\mathbb{R} \setminus \mathcal{O}(\sigma, \alpha))$. Since $z \in N(\sigma)$, the trivial solution is a solution. Since $z \in \mathbb{R} \setminus \mathcal{O}(\sigma, \alpha)$, (i) tells us that there exists a non-trivial weak solution with solution process $Z = X_\varphi$. Then we have two weak solutions issued from z which are not equal in law, which contradicts uniqueness. Hence, $N(\sigma) \subseteq \mathcal{O}(\sigma, \alpha)$, and the equality has been proved.

"$\Leftarrow$" Now suppose that $\mathcal{O}(\sigma, \alpha) = N(\sigma)$. For this part of the proof we will closely follow the proof of Theorem 2.6 in Zanzotto [36], the crucial difference being that our Proposition 5.2 generalises his Lemma 2.3. By (ii), a global weak solution exists for all $z \in \mathbb{R}$. We now use the representation of Kallenberg in Theorem 5.1 (ii) to prove the time-change representation for this solution, and thus to deduce uniqueness. If we fix one of these solutions $(X, Z, \mathscr{P})$, then Kallenberg's time-change representation of (33) shows that there is a two-sided stable process Y such that $Y_{\tilde{I}_t} = Z_t$, $t \in [0, \infty)$, defined in general on an extension of $\mathscr{P}$, where $\tilde{I}_t := \int_0^t \sigma(Z_s)^\alpha \, ds$. Since $\tilde{I}_{\tilde{\varphi}_t} = t \wedge \tilde{I}_\infty$ we obtain $Z_{\tilde{\varphi}_s} = Y_s$ for $s \in [0, \tilde{I}_\infty)$, where $\tilde{\varphi}_s := \inf\{t > 0 : \tilde{I}_t > s\}$. Further,

$$\tilde{\varphi}_t \geq \int_0^{\tilde{\varphi}_t} \mathbf{1}_{\{\sigma(Z_u) > 0\}} \, du = \int_0^{t \wedge \tilde{I}_\infty} \sigma(Z_{\tilde{\varphi}_s})^{-\alpha} \, ds = \int_0^{t \wedge \tilde{I}_\infty} \sigma(Y_s)^{-\alpha} \, ds, \quad t \geq 0. \quad (38)$$

We shall now show that (38) holds as an equality. Let T be the first hitting time of $\mathcal{O}(\sigma, \alpha)$ by Y, and suppose first that $\tilde{I}_\infty \leq T$. We know that the relation $Y_t = Z_{\tilde{\varphi}_t}$ holds for all times $t < \tilde{I}_\infty$, and therefore it follows from the fact that $\tilde{\varphi}$ is strictly increasing that the first hitting time of $\mathcal{O}(\sigma, \alpha)$ by Z is greater than or equal to $\tilde{\varphi}_{\tilde{I}_\infty}$. In particular, since $\mathcal{O}(\sigma, \alpha) = N(\sigma)$, the indicator in the left-hand integral of (38) is identically equal 1 for all $t \geq 0$, and thus the formula is an equality. Now suppose that $\tilde{I}_\infty \geq T$. From the relation $Y_t = Z_{\tilde{\varphi}_t}$ we see that $\tilde{\varphi}_T$ is the first hitting time of $\mathcal{O}(\sigma, \alpha)$ by Z, and as above we note that up to this time equality holds in (38) because the indicator on the left is equal to 1. After time T the right-hand side of (38) is equal to $+\infty$ (this is the main result of Proposition 5.2, where we showed that $T = \varphi_\infty$) and so we have shown in any case that

$$\tilde{\varphi}_t = \int_0^{t \wedge \tilde{I}_\infty} \sigma(Y_s)^{-\alpha} \, ds, \quad t \geq 0. \quad (39)$$

But in fact now we can show something more. By definition either $\tilde{I}_\infty = +\infty$ or $\tilde{\varphi}_{\tilde{I}_\infty} = +\infty$. Since T is (again by Proposition 5.2) less than or equal to any time t for which the integral $I_t = \int_0^t \sigma(Y_s)^{-\alpha}\,\mathrm{d}s$ is infinite, it follows that in either case $\tilde{I}_\infty \geq T$. But in addition, since $\tilde{\varphi}$ is by definition strictly increasing, it follows from (39) that $\int_0^t \sigma(Y_s)^{-\alpha}\,\mathrm{d}s$ must be finite for all times less than $\tilde{I}_\infty$, and so $\tilde{I}_\infty \leq T$. We can deduce that the two times are almost surely equal. It directly follows that

$$\tilde{\varphi}_t = \int_0^t \sigma(Y_s)^{-\alpha}\,\mathrm{d}s, \quad t \geq 0, \tag{40}$$

with both sides being equal to $+\infty$ after time $\tilde{I}_\infty = T$. Plugging this into the relation $Y_{\tilde{I}_t} = Z_t$ for $t \geq 0$ shows that Z has the $\overline{\mathbb{P}}$-almost sure representation $Z_t = Y_{\varphi_t}$ for $t \geq 0$. If $z \in \mathcal{O}(\sigma, \alpha)$ the time change is identically 0 and the time-changed solution and the trivial solution coincide, so the solution is unique. If $z \in \mathbb{R} \setminus N(\sigma)$ then the trivial solution does not exist, so the non-trivial time-changed solution is unique. Since $\mathcal{O}(\sigma, \alpha) = N(\sigma)$, we have proven uniqueness for all $z \in \mathbb{R}$. ∎

We can specialise the theorem a bit more if X is symmetric and σ has a monotone zero. In that case the thin sets disappear from the integral test. The situation might look artificial but occurs in many examples.

Corollary 5.7. *Suppose that X is symmetric and σ has an isolated monotone zero at $z \in \mathbb{R}$. Then there exists a non-trivial local weak solution to the SDE* (30) *started at z if and only if $\int_{z-\varepsilon}^{z+\varepsilon} \sigma(y)^{-\alpha}|z - y|^{\alpha-1}\,\mathrm{d}y < \infty$ for some $\varepsilon > 0$.*

Proof. Follows from Theorem 5.6 (i), where Theorem 4.2 gives the path integral over $f = \sigma^{-\alpha}$. ∎

Let us consider the previous corollary for the particular situation

$$\mathrm{d}Z_t = |Z_{t-}|^\beta\,\mathrm{d}X_t, \quad Z_0 = 0.$$

According to the corollary there is a non-trivial solution if and only if

$$\int_{-\varepsilon}^{\varepsilon} \sigma^{-\alpha}(y)|y|^{\alpha-1}\,\mathrm{d}y = \int_{-\varepsilon}^{\varepsilon} |y|^{-\alpha\beta+\alpha-1}\,\mathrm{d}y < \infty$$

for some $\varepsilon > 0$, which holds if and only if $-\alpha\beta + \alpha - 1 > -1$, that is, $\beta < 1$. Since the trivial solution $Z \equiv 0$ is also a solution, our integral test combined with the known results for $\alpha \in (1, 2]$ applied to this situation gives the failure of uniqueness for the simple polynomial stable SDE if and only if $\beta < (1/\alpha) \wedge 1$. This corresponds nicely to the counterexamples of Bass, Burdzy, and Chen [5] to pathwise uniqueness for stable SDEs.

5.2. Properties of solutions

In this final section we will explore properties of solutions to the stable SDE

$$\mathrm{d}Z_t = \sigma(Z_{t-})\,\mathrm{d}X_t, \quad Z_0 = z,$$

using the integral tests we developed earlier in this paper. Let us assume $\mathcal{O}(\sigma, \alpha) \subseteq N(\sigma)$, which ensures by Theorem 5.6 that there exists a global solution with the solution process having time-change representation $Z = Y_\varphi$. Since σ takes values in $[0, \infty)$, it follows that the path integral $I_t = \int_0^t \sigma(Y_s)^{-\alpha} \, \mathrm{d}s$ is continuous in t, which, as we noted around (5), ensures that Z is a strong Markov process. Note that the solution can explode in finite time, in which case Z is sent to the cemetery state Δ.

Explosion. Let $\zeta := \inf\{t > 0 : Z_t = \Delta\}$ denote the lifetime of Z. Since the driving stable process Y has infinite lifetime, the time-change representation $Z = Y_\varphi$ implies that

$$\zeta = \int_0^\infty \sigma(Y_s)^{-\alpha} \, \mathrm{d}s \quad \text{almost surely.}$$

We say that Z *explodes* if $\zeta < \infty$. If I_t is finite for all $t \geq 0$ almost surely then the event $\{\zeta < \infty\}$ is in the tail σ-algebra of the stable process Y, and so in this case explosion becomes a zero-one law. In general, however, Z can explode with probability in $(0, 1)$. Note that the recurrence of the two-sided stable processes excludes the possibility of explosion for all $\alpha \in (1, 2)$. We can now fully characterise explosion through integral tests via our earlier results: Theorem 2.3 gives a necessary and sufficient condition for explosion of Z with positive probability, and Theorem 2.10 does the same but for explosion with probability 1.

   Here is an example. If we assume that σ is bounded away from zero on compact sets then a unique global weak solution exists for all initial conditions, and according to Theorem 2.12 explosion of Z is a zero-one law, and Z explodes almost surely if and only if there exists a transient set B such that

$$\int_{\mathbb{R}\setminus B} \sigma(x)^{-\alpha} |x - z|^{\alpha - 1} \, \mathrm{d}x < \infty.$$

It would be nice to remove B from the integral test, but Example 2.14 shows that in general this is not possible. Under strong regularity assumptions this question has been addressed for all starting conditions simultaneously in Döring and Kyprianou [12] by appealing to duality theory for Markov processes.

Freezing. We say that Z is *frozen* if there exists a time $t \in [0, \infty)$ such that $Z_s = Z_t$ for all $s \geq t$. It follows from the time-change representation $Z = Y_\varphi$ that Z is frozen if and only if $\varphi_\infty < \infty$, which from the definition of φ occurs if and only if there exists a $t \in [0, \infty)$ such that

$$\int_0^t \sigma(Y_s)^{-\alpha} \, \mathrm{d}s = \infty.$$

Thus it is clear that freezing and explosion preclude each other. If $\alpha \in (1, 2)$ then Y is point recurrent, and Zanzotto's zero-one law shows that freezing is a zero-one law, and gives a sufficient and necessary condition for freezing to occur. If $\alpha \in (0, 1)$ then Y is transient, and Theorem 3.2 gives a sufficient and necessary condition for Z to be frozen with positive probability, while Theorem 3.1 gives a sufficient and necessary condition for freezing to occur almost surely.

References

[1] Baguley, S.: Path integrals of standard Markov processes. PhD thesis, Univ. Mannheim (2020)

[2] Barlow, M. T.: Necessary and sufficient conditions for the continuity of local time of Lévy processes. Ann. Probab. **16**, 1389–1427 (1988) Zbl 0666.60072 MR 958195

[3] Barlow, M. T., Hawkes, J.: Application de l'entropie métrique à la continuité des temps locaux des processus de Lévy. C. R. Acad. Sci. Paris Sér. I Math. **301**, 237–239 (1985) Zbl 0584.60087 MR 801971

[4] Bass, R. F.: Stochastic differential equations driven by symmetric stable processes. In: Séminaire de Probabilités, XXXVI, Lecture Notes in Math. 1801, Springer, Berlin, 302–313 (2003) Zbl 1039.60056 MR 1971592

[5] Bass, R. F., Burdzy, K., Chen, Z.-Q.: Stochastic differential equations driven by stable processes for which pathwise uniqueness fails. Stochastic Process. Appl. **111**, 1–15 (2004) Zbl 1111.60038 MR 2049566

[6] Bertoin, J.: Lévy Processes. Cambridge Tracts in Math. 121, Cambridge Univ. Press, Cambridge (1996) Zbl 0861.60003 MR 1406564

[7] Betsakos, D.: Symmetrization, symmetric stable processes, and Riesz capacities. Trans. Amer. Math. Soc. **356**, 735–755 (2004) Zbl 1034.31002 MR 2022718

[8] Bliedtner, J., Hansen, W.: Potential Theory. Universitext, Springer, Berlin (1986) Zbl 0706.31001 MR 850715

[9] Blumenthal, R. M., Getoor, R. K.: Markov Processes and Potential Theory. Pure Appl. Math. 29, Academic Press, New York (1968) Zbl 0169.49204 MR 0264757

[10] Chung, K. L., Walsh, J. B.: Markov Processes, Brownian Motion, and Time Symmetry. 2nd ed., Grundlehren Math. Wiss. 249, Springer, New York (2005) Zbl 1082.60001 MR 2152573

[11] Döring, L., Kyprianou, A. E.: Perpetual integrals for Lévy processes. J. Theoret. Probab. **29**, 1192–1198 (2016) Zbl 1356.60073 MR 3540494

[12] Döring, L., Kyprianou, A. E.: Entrance and exit at infinity for stable jump diffusions. Ann. Probab. **48**, 1220–1265 (2020) Zbl 1469.60211 MR 4112713

[13] Engelbert, H. J., Schmidt, W.: On the behaviour of certain functionals of the Wiener process and applications to stochastic differential equations. In: Stochastic Differential Systems (Visegrád, 1980), Lecture Notes in Control and Information Sci. 36, Springer, Berlin, 47–55 (1981) Zbl 0468.60077 MR 653645

[14] Feller, W.: The parabolic differential equations and the associated semi-groups of transformations. Ann. of Math. (2) **55**, 468–519 (1952) Zbl 0047.09303 MR 47886

[15] Feller, W.: The general diffusion operator and positivity preserving semi-groups in one dimension. Ann. of Math. (2) **60**, 417–436 (1954) Zbl 0057.09805 MR 65809

[16] Getoor, R. K.: Transience and recurrence of Markov processes. In: Séminaire de Probabilités, XIV (Paris, 1978/1979), Lecture Notes in Math. 784, Springer, Berlin, 397–409 (1980) Zbl 0431.60067 MR 580144

[17] Grzywny, T., Kwaśnicki, M.: Potential kernels, probabilities of hitting a ball, harmonic functions and the boundary Harnack inequality for unimodal Lévy processes. Stochastic Process. Appl. **128**, 1–38 (2018) Zbl 1386.60265 MR 3729529

[18] Hawkes, J.: Potential theory of Lévy processes. Proc. London Math. Soc. (3) **38**, 335–352 (1979) Zbl 0401.60069 MR 531166

[19] Kallenberg, O.: Some time change representations of stable integrals, via predictable transformations of local martingales. Stochastic Process. Appl. **40**, 199–223 (1992) Zbl 0754.60044 MR 1158024

[20] Kanda, M.: Two theorems on capacity for Markov processes with stationary independent increments. Z. Wahrsch. Verw. Gebiete **35**, 159–165 (1976) Zbl 0316.60048 MR 405594

[21] Kanda, M.: Characterization of semipolar sets for processes with stationary independent increments. Z. Wahrsch. Verw. Gebiete **42**, 141–154 (1978) Zbl 0362.60078 MR 483039

[22] Karatzas, I., Shreve, S. E.: Brownian Motion and Stochastic Calculus. 2nd ed., Grad. Texts in Math. 113, Springer, New York (1991) Zbl 0638.60065 MR 1121940

[23] Khoshnevisan, D., Salminen, P., Yor, M.: A note on a.s. finiteness of perpetual integral functionals of diffusions. Electron. Comm. Probab. **11**, 108–117 (2006) Zbl 1111.60061 MR 2231738

[24] Kolb, M., Savov, M.: A characterization of the finiteness of perpetual integrals of Lévy processes. Bernoulli **26**, 1453–1472 (2020) Zbl 1466.60096 MR 4058374

[25] Kyprianou, A. E., Rivero, V., Satitkanitkul, W.: Deep factorisation of the stable process III: the view from radial excursion theory and the point of closest reach. Potential Anal. **53**, 1347–1375 (2020) Zbl 1460.60024 MR 4159383

[26] Li, Z., Mytnik, L.: Strong solutions for stochastic differential equations with jumps. Ann. Inst. H. Poincaré Probab. Statist. **47**, 1055–1067 (2011) Zbl 1273.60070 MR 2884224

[27] Mimica, A., Vondraček, Z.: Unavoidable collections of balls for censored stable processes. J. Math. Anal. Appl. **419**, 938–958 (2014) MR 3225414

[28] Mimica, A., Vondraček, Z.: Unavoidable collections of balls for isotropic Lévy processes. Stochastic Process. Appl. **124**, 1303–1334 (2014) Zbl 1296.60208 MR 3148015

[29] Salminen, P., Yor, M.: Perpetual integral functionals as hitting and occupation times. Electron. J. Probab. **10**, 371–419 (2005) Zbl 1110.60078 MR 2147313

[30] Salminen, P., Yor, M.: Properties of perpetual integral functionals of Brownian motion with drift. Ann. Inst. H. Poincaré Probab. Statist. **41**, 335–347 (2005) Zbl 1082.60073 MR 2139023

[31] Sato, K.-i.: Lévy Processes and Infinitely Divisible Distributions. Cambridge Stud. Adv. Math. 68, Cambridge Univ. Press, Cambridge (2013) Zbl 1287.60003 MR 3185174

[32] Volkonskiĭ, V. A.: Random substitution of time in strong Markov processes. Teor. Veroyatnost. i Primenen. **3**, 332–350 (1958) (in Russian) Zbl 0102.14004 MR 0100919

[33] Yamada, T., Watanabe, S.: On the uniqueness of solutions of stochastic differential equations. J. Math. Kyoto Univ. **11**, 155–167 (1971) Zbl 0236.60037 MR 278420

[34] Zanzotto, P. A.: On solutions of one-dimensional stochastic differential equations driven by stable Lévy motion. Stochastic Process. Appl. **68**, 209–228 (1997) Zbl 0911.60037 MR 1454833

[35] Zanzotto, P. A.: Representation of a class of semimartingales as stable integrals. Teor. Veroyatnost. i Primenen. **43**, 808–818 (1998) Zbl 0956.60055 MR 1692452

[36] Zanzotto, P. A.: On stochastic differential equations driven by a Cauchy process and other stable Lévy motions. Ann. Probab. **30**, 802–825 (2002) Zbl 1017.60058 MR 1905857