Stability of (sub)critical non-local spatial branching processes with and without immigration

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Abstract

We consider the setting of either a general non-local branching particle process or a general non-local superprocess, in both cases, with and without immigration. Under the assumption that the mean semigroup has a Perron-Frobenious type behaviour for the immigrated mass, as well as the existence of second moments, we consider necessary and sufficient conditions that ensure limiting distributional stability. More precisely, our first main contribution pertains to proving the asymptotic Kolmogorov survival probability and Yaglom limit for critical non-local branching particle systems *and* superprocesses under a second moment assumption on the offspring distribution. Our results improve on existing literature by removing the requirement of bounded offspring in the particle setting [21] and generalising [43] to allow for non-local branching mechanisms. Our second main contribution pertains to the stability of both critical and sub-critical non-local branching particle systems and superprocesses with immigration. At criticality, we show that the scaled process converges to a Gamma distribution under a necessary and sufficient integral test. At subcriticality we show stability of the process, also subject to an integral test. In these cases, our results complement classical results for (continuous-time) Galton–Watson processes with immigration and continuous-state branching processes with immigration; see [22, 40, 42, 48, 51], among others. In the setting of superprocesses, the only work we know of at this level of generality is summarised in [34]. The proofs of our results, both with and without immigration, appeal to similar technical approaches and accordingly, we include the results together in this paper.

Key words: Branching Markov process, superprocess, immigration, distributional stability.

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1 Introduction

In this article, we revisit foundational results concerning the stability of branching processes with and without immigration. In essence, our objective is to show that, qualitatively speaking, several of the classical results for Galton–Watson processes (with and without immigration) are universal truths in the setting of general branching Markov processes and superprocessses.

In what follows, we focus on critical or subcritical processes. In the setting of Galton–Watson processes, the notion of criticality is dictated by the mean number of offspring. In the general setting we present in this work, the notion of criticality pertains to the value of an assumed lead eigenvalue for the mean semigroup.

The first main focus of this paper pertains to critical processes without immigration. In this case, we are interested in the extent to which the so-called Kolmogorov and Yaglom limits for discrete-time Galton–Watson processes are still an inherent behaviour at generality. The Kolmogorov limit stipulates that the decay of the survival probability at criticality is inversely proportional to time (interpreted as either real-time or generation number). The Yaglom limit asserts that, conditional on survival, the current population normalised by time, converges to an exponential random variable with rate that is written explicitly in terms of the model parameters. In both the Kolmogorov and Yaglom limits, second moments of the offspring distribution are needed in the classical setting.

Improving on recent work in this domain for general non-local branching Markov processes [21] and superprocesses [43], we prove both of these results under a second moment assumption on the offspring distribution. For non-local branching Markov processes, this builds on [21], where a bound on the number of offspring was required. In the setting of superprocesses, we accommodate for non-local branching mechanisms, where previous works have only allowed local branching [43].

The third and fourth main results of this article concern critical and subcritical processes *with* immigration. Returning to the Galton–Watson setting, let us consider the case where we have i.i.d. immigration in each generation, with each immigrant spawning an independent copy of the underlying Galton–Watson process. If $f(s) = \mathcal{E}[s^N]$, $s \in [0,1]$ is the probability generating function of the offspring distribution of the typical family size, N, for the Galton–Watson dynamics and $g(s) = \tilde{\mathcal{E}}[s^{\tilde{N}}]$, $s \in [0,1]$, is the probability generating function of the distribution of the number of immigrants, $\tilde{N}$, in each generation, then it is known (see [4, 17, 22, 44, 36]) that the total population converges in distribution if, and only if, the process is not supercritical, i.e. $\mathcal{E}[N] \leq 1$, and

$$\int_0^1 \frac{1-g(s)}{f(s)-s}\mathrm{d}s < \infty. \tag{1}$$

In the subcritical setting, the integral (1) is equivalent to the requirement that $\tilde{\mathcal{E}}[\log(1+\tilde{N})] < \infty$. In the critical setting, although the integral (1) fails, it is still possible to demonstrate that the process with immigration when scaled by time converges to a gamma distribution (see [16, 17, 37, 45]).

Again, we develop analogous results in the general framework of non-local branching Markov processes and non-local superprocesses with immigration. In the former case, we believe our results to be the first of their kind at this level of generality. For non-local superprocesses, our results complement those in Chapter 9 of [34]. Indeed, in the setting of independent immigration, at subcriticality, we introduce an integral test which seems not to have been noted previously. At criticality we are able to provide the natural analogue of scaled convergence of the population to a gamma distribution, which also appears to be new for superprocesses. As in the first two results, we also work under a second moment assumption on the offspring distribution.

It turns out that there is a natural reason to consider the results with and without immigration together. Indeed, a fundamental feature of the analysis in both cases pertains to how the asymptotic behaviour of the non-linear semigroup of the underlying branching process behaves in relation to its linear semigroup. In particular, the insistence of second moments throughout leads to the use of a second order Taylor approximation in all cases.

2 Non-local spatial branching processes

Let us spend some time describing the general setting in which we wish to work. Let E be a Lusin space. Throughout, will write $B(E)$ for the Banach space of bounded measurable functions on E with norm $\|\cdot\|$, $B^+(E)$ for the space of non-negative bounded measurable functions on E and $B_1^+(E)$ for the subset of functions in $B^+(E)$ that are uniformly bounded by unity. We are interested in spatial branching processes that are defined in terms of a Markov process and a branching mechanism, whether that be a branching particle system or a superprocess. We characterise Markov processes by a semigroup on E, denoted by $\mathtt{P} = (\mathtt{P}_t, t \geq 0)$. Unless otherwise stated, we do not need $\mathtt{P}$ to have the Feller property, and it is not necessary that $\mathtt{P}$ is conservative. Indeed, in the case where it is non-conservative, we can append a cemetery state $\{\dagger\}$ to E, which is to be treated as an absorbing state, and regard $\mathtt{P}$ as conservative on the extended space $E \cup \{\dagger\}$, which can also be treated as a Lusin space. However, we must then alter the definition of $B(E)$ (and accordingly $B^+(E)$ and $B_1^+(E)$) to ensure that any function $f \in B(E)$ satisfies $f(\dagger) = 0$.

2.1 Non-local Branching Markov Processes

Consider now a spatial branching process in which, given their point of creation, particles evolve independently according to a $\mathtt{P}$-Markov process. In an event, which we refer to as 'branching', particles positioned at x die at rate $\beta(x)$, where $\beta \in B^+(E)$, and instantaneously, new particles are created in E according to a point process. The configurations of these offspring are described by the random counting measure

$$\mathcal{Z}(A) = \sum_{i=1}^{N} \delta_{x_i}(A),$$

for Borel subsets A of E, for which we also assume that $\sup_{x\in E} \mathcal{E}_x[N] < \infty$. The law of the aforementioned point process depends on x, the point of death of the parent, and we denote it by $\mathcal{P}_x$, $x \in E$, with associated expectation operator given by $\mathcal{E}_x$, $x \in E$. This information is captured in the so-called branching mechanism

$$\mathtt{G}[g](x) := \beta(x)\mathcal{E}_x\left[\prod_{i=1}^{N} g(x_i) - g(x)\right], \qquad x \in E,$$

where $g \in B_1^+(E)$. Without loss of generality, we can assume that $\mathcal{P}_x(N = 1) = 0$ for all $x \in E$ by viewing a branching event with one offspring as an extra jump in the motion. On the other hand, we do allow for the possibility that $\mathcal{P}_x(N = 0) > 0$ for some or all $x \in E$.

Henceforth we refer to this spatial branching process as a $(\mathtt{P}, \mathtt{G})$-branching Markov process (or $(\mathtt{P}, \mathtt{G})$-BMP for short). It is well known that if the configuration of particles at time t is denoted by $\{x_1(t), \dots, x_{N_t}(t)\}$, then, on the event that the process has not become extinct or exploded, the branching Markov process can be described as the co-ordinate process $X = (X_t, t \geq 0)$, given by

$$X_t(\cdot) = \sum_{i=1}^{N_t} \delta_{x_i(t)}(\cdot), \qquad t \geq 0,$$

evolving in the space of counting measures on E with finite total mass, which we denote $N(E)$. In particular, X is Markovian in $N(E)$. Its probabilities will be denoted $\mathbb{P} := (\mathbb{P}_\mu, \mu \in N(E))$. Sometimes we will write $X^{(\mu)}$ to signify that we are considering X under $\mathbb{P}_\mu$, that is to say, $X_0^{(\mu)} = \mu$. For convenience, we will write for any measure $\mu \in N(E)$ and function $f \in B^+(E)$,

$$\langle f, \mu\rangle = \int_E f(x)\mu(\mathrm{d}x).$$

In particular,

$$\langle f, X_t\rangle = \sum_{i=1}^{N_t} f(x_i(t)), \qquad f \in B^+(E).$$

With this notation in hand, it is worth noting that the independence that is inherent in the definition of the Markov branching property implies that, if we define,

$$\mathrm{e}^{-\mathtt{v}_t[f](x)} = \mathbb{E}_{\delta_x}\left[\mathrm{e}^{-\langle f, X_t\rangle}\right], \qquad t \geq 0,\ f \in B^+(E),\ x \in E,$$

then for $\mu \in N(E)$, we have

$$\mathbb{E}_\mu\left[\mathrm{e}^{-\langle f, X_t\rangle}\right] = \mathrm{e}^{-\langle \mathtt{v}_t[f], \mu\rangle}, \qquad t \geq 0. \tag{2}$$

Moreover, for $f \in B^+(E)$ and $x \in E$,

$$\mathrm{e}^{-\mathtt{v}_t[f](x)} = \mathtt{P}_t[\mathrm{e}^{-f}](x) + \int_0^t \mathtt{P}_s\left[\mathtt{G}[\mathrm{e}^{-\mathtt{v}_{t-s}[f]}]\right](x)\mathrm{d}s, \qquad t \geq 0. \tag{3}$$

The above equation describes the evolution of the semigroup $\mathtt{v}_t[\cdot]$ in terms of the action of transport and branching. That is, either the initial particle has not branched and undergone a Markov transition (including the possibility of being absorbed) by time t or at some time $s \leq t$, the initial particle has branched, producing offspring according to $\mathtt{G}$. We refer the reader to [25, 20] for a proof.

Branching Markov processes enjoy a very long history in the literature, dating back as far as the late 1950s, [46, 47, 49, 26, 27, 28], with a broad base of literature that is arguably too voluminous to give a fair summary of here. Most literature focuses on the setting of local branching. This corresponds to the setting that all offspring are positioned at their parent's point of death (i.e. $x_i = x$ in the definition of $\mathtt{G}$). In that case, the branching mechanism reduces to

$$\mathtt{G}[s](x) = \beta(x)\left[\sum_{k=0}^{\infty} p_k(x)s^k - s\right], \qquad x \in E,$$

where $s \in [0,1]$ and $(p_k(x), k \geq 0)$ is the offspring distribution when a parent branches at site $x \in E$. The branching mechanism $\mathtt{G}$ may otherwise be seen, in general, as a mixture of local and non-local branching.

We want to introduce a variant of the model that includes immigration, where the new particles can arrive into the system from an external source. These arrival times, at which immigration events occur, are determined by a homogeneous Poisson process with rate α. At each arrival time, a random number of particles, $\tilde{N}$, is added to the system at locations $y_1, \ldots, y_{\tilde{N}}$ in E. The latter can be summarised by another random counting measure

$$\tilde{\mathcal{Z}}(\cdot) = \sum_{i=1}^{\tilde{N}} \delta_{y_i}(\cdot). \tag{4}$$

The corresponding law, $\tilde{\mathcal{P}}$, is independent of the state of the system and its expectation is denoted by $\tilde{\mathcal{E}}$. Similarly to before, this can be succinctly described by the immigration mechanism

$$\mathtt{H}[f] = \alpha\tilde{\mathcal{E}}\left[1 - \mathrm{e}^{-\langle f, \tilde{\mathcal{Z}}\rangle}\right], \qquad f \in B^+(E),$$

where we assume $\tilde{\mathcal{P}}(\tilde{N} = 0) = 0$, i.e. we always have at least one immigrant at the arrival times.

Once immigrants are embedded in the system, they evolve according to the same rules as independent copies of the branching Markov process, initiated from their point of arrival.

Definition 1 (Non-local branching Markov process with immigration)**.** *We say that $Y^{(\mu)} = (Y_t^{(\mu)}, t \geq 0)$ is a $(\mathtt{P},\mathtt{G})$-branching Markov process with $\mathtt{H}$-immigration (or a $(\mathtt{P},\mathtt{G},\mathtt{H})$-BMPI for short) with initial mass $\mu \in N(E)$, if*

$$Y_t^{(\mu)} = X_t^{(\mu)} + \sum_{j=1}^{D_t} X_{t-\tau_j}^{(\tilde{\mathcal{Z}}_j)}, \qquad t \geq 0, \tag{5}$$

where $(D_t, t \geq 0)$ is the homogeneous Poisson process with rate α, τ_j is j-th arrival time and $\{\tilde{\mathcal{Z}}_j, j \in \mathbb{N}\}$ are i.i.d. copies of $\tilde{\mathcal{Z}}$. Moreover, given $(\tilde{\mathcal{Z}}_j, j = 1, \cdots, D_t)$, the processes $X^{(\tilde{\mathcal{Z}}_j)}$ are independent copies of $X^{(\mu)}$ issued from the respective measures $\mu = \tilde{\mathcal{Z}}_j$. The probabilities of $Y^{(\mu)}$ are also denoted by $\mathbb{P}_\mu$, $\mu \in N(E)$.

2.2 Non-local Superprocesses

Superprocesses can be thought of as the high-density limit of a sequence of branching Markov processes, resulting in a new family of measure-valued Markov processes; see e.g. [34, 6, 50, 8, 7]. Just as branching Markov processes are Markovian in $N(E)$, the former are Markovian in $M(E)$, the space of finite Borel measures on E equipped with the topology of weak convergence. There is a broad literature for superprocesses, e.g. [34, 6, 50, 13, 11], with so-called local branching mechanisms, which has been broadened to the more general setting of non-local branching mechanisms in [7, 34]. Let us now introduce these concepts with a self-contained definition of what we mean by a non-local superprocess (although the reader will note that we largely conform to the presentation in [34]).

A Markov process $X := (X_t : t \geq 0)$ with state space $M(E)$ and probabilities $\mathbb{P} := (\mathbb{P}_\mu, \mu \in M(E))$ is called a $(\mathtt{P}, \psi, \phi)$-superprocess (or $(\mathtt{P}, \psi, \phi)$-SP for short) if it has semigroup $(\mathtt{V}_t, t \geq 0)$ on $M(E)$ satisfying

$$\mathbb{E}_\mu\left[\mathrm{e}^{-\langle f, X_t\rangle}\right] = \mathrm{e}^{-\langle \mathtt{V}_t[f], \mu\rangle}, \qquad \mu \in M(E), f \in B^+(E), \tag{6}$$

where $(\mathtt{V}_t, t \geq 0)$ is characterised as the minimal non-negative solution of the evolution equation

$$\mathtt{V}_t[f](x) = \mathtt{P}_t[f](x) - \int_0^t \mathtt{P}_s\left[\psi(\cdot, \mathtt{V}_{t-s}[f](\cdot)) + \phi(\cdot, \mathtt{V}_{t-s}[f])\right](x)\mathrm{d}s. \tag{7}$$

Here ψ denotes the local branching mechanism

$$\psi(x, \lambda) = -b(x)\lambda + c(x)\lambda^2 + \int_0^\infty (\mathrm{e}^{-\lambda y} - 1 + \lambda y)\nu(x, \mathrm{d}y), \quad \lambda \geq 0, \tag{8}$$

where $b \in B(E)$, $c \in B^+(E)$ and $(x \wedge x^2)\nu(x, \mathrm{d}y)$ is a uniformly (for $x \in E$) bounded kernel from E to $(0, \infty)$, and ϕ is the non-local branching mechanism

$$\phi(x, f) = \beta(x)(f(x) - \eta(x, f)),$$

where $\beta \in B^+(E)$ and η has representation

$$\eta(x, f) = \gamma(x, f) + \int_{M(E)^\circ} (1 - \mathrm{e}^{-\langle f, \nu\rangle})\Gamma(x, \mathrm{d}\nu),$$

such that $\gamma(x, f)$ is a uniformly bounded function on $E \times B^+(E)$ and $\langle 1, \nu\rangle\Gamma(x, \mathrm{d}\nu)$ is a uniformly (for $x \in E$) bounded kernel from E to $M(E)^\circ := M(E)\backslash\{0\}$ with

$$\gamma(x, f) + \int_{M(E)^\circ} \langle 1, \nu\rangle\Gamma(x, \mathrm{d}\nu) \leq 1.$$

We refer the reader to [7, 39] for more details regarding the above formulae. Lemma 3.1 in [7] tells us that the functional $\eta(x, f)$ has the following equivalent representation

$$\eta(x, f) = \int_{M_0(E)} \left[\delta_\eta(x, \pi)\langle f, \pi\rangle + \int_0^\infty (1 - \mathrm{e}^{-u\langle f, \pi\rangle})n_\eta(x, \pi, \mathrm{d}u)\right] P_\eta(x, \mathrm{d}\pi), \tag{9}$$

where $M_0(E)$ denotes the set of probability measures on E, $P_\eta(x, \mathrm{d}\pi)$ is a probability kernel from E to $M_0(E)$, $\delta_\eta \geq 0$ is a bounded function on $E \times M_0(E)$, and $un_\eta(x, \pi, \mathrm{d}u)$ is a bounded kernel from $E \times M_0(E)$ to $(0, \infty)$ with

$$\delta_\eta(x, \pi) + \int_0^\infty un_\eta(x, \pi, \mathrm{d}u) \leq 1.$$

The reader will note that we have deliberately used some of the same notation for both branching Markov processes and superprocesses. In the sequel there should be no confusion and the motivation for this choice of repeated notation is that our main results are indifferent to which of the two processes we are talking about.

Let us now define what we mean by a $(\mathtt{P}, \psi, \phi)$-superprocess with immigration. In order to do so, we need to introduce two objects, the first of which is the excursion measure for the $(\mathtt{P}, \psi, \phi)$-superprocess. It is known, see [10] or Chapter 8 of [34], that a measure $\mathbb{Q}_x$ exists on the space $\mathbb{D} = \mathbb{D}([0, \infty) \times \mathcal{M}(E))$ which satisfies

$$\mathbb{Q}_x \left(1 - \mathrm{e}^{-\langle f, X_t \rangle}\right) = \mathtt{V}_t[f](x),$$

for $x \in E$, $t \geq 0$ and $f \in B^+(E)$. The second object is the immigration mechanism, which we define, for $f \in B^+(E)$, via

$$\chi[f] = \langle f, \upsilon \rangle + \int_{M(E)^\circ} (1 - \mathrm{e}^{-\langle f, \nu \rangle}) \Upsilon(\mathrm{d}\nu), \tag{10}$$

where $\upsilon \in M(E)$ and $(1 \wedge \langle 1, \nu \rangle)\Upsilon(\mathrm{d}\nu)$ is a finite measure on $M(E)^\circ$. As above, Lemma 3.1 of [7] enforces the necessity of the decomposition

$$\chi[f] = \int_{M_0(E)} \left[\delta_\chi(\pi)\langle f, \pi \rangle + \int_0^\infty (1 - \mathrm{e}^{-u\langle f, \pi \rangle}) n_\chi(\pi, \mathrm{d}u)\right] P_\chi(\mathrm{d}\pi),$$

where $\delta_\chi \geq 0$ is a bounded function on $M_0(E)$, $un_\chi(\pi, \mathrm{d}u)$ is a bounded kernel from $M_0(E)$ to $(0, \infty)$ and $P_\chi(\mathrm{d}\pi)$ is a probability on $M_0(E)$.

Definition 2 (Non-local superprocess with immigration)**.** *We say that $Y^{(\mu)} = (Y_t^{(\mu)}, t \geq 0)$ is a $(\mathtt{P}, \psi, \phi)$-superprocess with χ-immigration (or a $(\mathtt{P}, \psi, \phi, \chi)$-SPI for short) with initial mass $\mu \in M(E)$, if*

$$Y_t^{(\mu)} = X_t^{(\mu)} + \int_0^t \int_{\mathbb{D}} X_{t-s} \mathtt{N}(\mathrm{d}s, \mathrm{d}X), \qquad t \geq 0, \tag{11}$$

where $\mathtt{N}(\mathrm{d}s, \mathrm{d}X)$ is Poisson random measure on $[0, \infty) \times \mathbb{D}$ with intensity

$$\int_{M_0(E)} P_\chi(\mathrm{d}\pi) \left(\delta_\chi(\pi) \int_E \pi(\mathrm{d}y)\mathbb{Q}_y(\mathrm{d}X) + \int_0^\infty n_\chi(\pi, \mathrm{d}u)\mathbb{P}_{u\pi}(\mathrm{d}X)\right) \mathrm{d}s.$$

We also write $\mathbb{P} = (\mathbb{P}_\mu, \mu \in M(E))$ for the probabilities of $Y^{(\mu)}$.

We note that similar constructions for SPI processes can be found in [32, 33].

3 Assumptions and main results

Before stating our results, we first introduce some assumptions that will be crucial in analysing the models defined above. Unless a specific difference is indicated, the assumptions apply both to the setting that X is either a non-local branching Markov process or a non-local superprocess.

(H1): We assume second moments

$$\sup_{x\in E}\mathcal{E}_x\left[N^2\right]<\infty \qquad \text{and} \qquad \sup_{x\in E}\left(\int_0^\infty y^2\nu(x,\mathrm{d}y)+\int_{M(E)^\circ}\langle 1,\nu\rangle^2\Gamma(x,\mathrm{d}\nu)\right)<\infty.$$

Assumption (H1) allows us to define, for $f\in B^+(E)$

$$\mathbb{V}[f](x)=\beta(x)\mathcal{E}_x\left[\sum_{i,j=1;\,i\neq j}^{N} f(x_i)f(x_j)\right], \qquad x\in E,$$

in the branching Markov process setting or with

$$\mathbb{V}[f](x)=\psi''(x,0+)f(x)^2+\beta(x)\int_{M(E)^\circ}\langle f,\nu\rangle^2\Gamma(x,\mathrm{d}\nu), \tag{12}$$

$$=\left(2c(x)+\int_0^\infty y^2\nu(x,\mathrm{d}y)\right)f(x)^2+\beta(x)\int_{M(E)^\circ}\langle f,\nu\rangle^2\Gamma(x,\mathrm{d}\nu), \tag{13}$$

in the superprocess setting.

(H2): There exist a constant $\lambda\le 0$, a function $\varphi\in B^+(E)$ and finite measure $\tilde\varphi\in M(E)$ such that, for $f\in B^+(E)$,

$$\langle \mathtt{T}_t[\varphi],\mu\rangle=\mathrm{e}^{\lambda t}\langle\varphi,\mu\rangle \text{ and } \langle\mathtt{T}_t[f],\tilde\varphi\rangle=\mathrm{e}^{\lambda t}\langle f,\tilde\varphi\rangle,$$

for all $\mu\in N(E)$ (resp. $M(E)$) if $(X,\mathbb{P})$ is a branching Markov process (resp. a superprocess), where

$$\langle\mathtt{T}_t[f],\mu\rangle=\int_E\mu(\mathrm{d}x)\mathbb{E}_{\delta_x}\left[\langle f,X_t\rangle\right]=\mathbb{E}_\mu\left[\langle f,X_t\rangle\right], \qquad t\ge 0.$$

Further let us define

$$\Delta_t=\sup_{x\in E,\,f\in B_1^+(E)}|\varphi(x)^{-1}\mathrm{e}^{-\lambda t}\mathtt{T}_t\left[f\right](x)-\langle f,\tilde\varphi\rangle|, \qquad t\ge 0. \tag{14}$$

We suppose that $\Delta:=\sup_{t\ge0}\Delta_t<\infty$ and

$$\Delta_t=O(\mathrm{e}^{-\varepsilon t}) \text{ as } t\to\infty \text{ for some } \varepsilon>0.$$

Without loss of generality, we conveniently impose the normalisation $\langle\varphi,\tilde\varphi\rangle=1$.

Remark 1. The non-local spatial branching process is known as critical (resp. subcritical) when $\lambda=0$ (resp. $\lambda<0$). Without restriction on the sign of λ, assumption (H2) has been recently named (see [24]) the Asmussen-Hering class of branching processes, acknowledging foundational results for this class in [1, 2, 3].

(H3): For each $x \in E$

$$\mathbb{P}_{\delta_x}(\zeta < \infty) = 1,$$

where $\zeta = \inf\{t > 0 : \langle 1, X_t \rangle = 0\}$.

(H4): There exist constants $K > 0$ and $M > 0$ such that for all $f \in B^+(E)$,

$$\langle \mathbb{V}_M[f], \tilde{\varphi} \rangle \geq K \langle f, \tilde{\varphi} \rangle^2,$$

where $\mathbb{V}_M$ is defined by

$$\mathbb{V}_M[f](x) = \beta(x) \mathcal{E}_x \left[\sum_{i,j=1;\, i \neq j}^{N} f(x_i) f(x_j) \mathbf{1}_{\{N \leq M\}} \right], \qquad x \in E,$$

for branching Markov processes and by

$$\mathbb{V}_M[f](x) = \left(2c(x) + \int_0^\infty y^2 \mathbf{1}_{\{y \leq M\}} \nu(x, \mathrm{d}y) \right) f(x)^2 + \beta(x) \int_{M(E)^\circ} \langle f, \nu \rangle^2 \mathbf{1}_{\{\langle 1, \nu \rangle \leq M\}} \Gamma(x, \mathrm{d}\nu), \qquad x \in E,$$

for superprocesses.

Notice that in both cases, $\mathbb{V}[f] = \lim_{M \to \infty} \mathbb{V}_M[f]$ by monotone convergence.

We are now ready to state our main results. The reader will note that the results are stated for both branching Markov processes and superprocesses simultaneously. Moreover, as alluded to in the introduction, the proofs of these results, whether with or without immigration, are similar in spirit and methodology. Accordingly, as the reader will see, we therefore opt to give all proofs in the BMP setting.

The first two results pertain to the critical system, i.e. when $\lambda = 0$ in (H2), and when there is no immigration. In particular, we show that the Kolmogorov survival probability asymptotic holds, as well as the Yaglom limit. In essence these results support the notion of universality of the exponential distribution for the asymptotic law of $\langle f, X_t \rangle / t$ conditional on survival as $t \to \infty$.

Theorem 1 (Kolmogorov survival probability at criticality)**.** *Suppose that $(X, \mathbb{P})$ is a $(\mathtt{P}, \mathtt{G})$-BMP (resp. a $(\mathtt{P}, \psi, \phi)$-SP) satisfying (H1)–(H4) with $\lambda = 0$. Then, for all $\mu \in N(E)$ (resp. $\mu \in M(E)$),*

$$\lim_{t \to \infty} t \mathbb{P}_\mu(\zeta > t) = \frac{2 \langle \varphi, \mu \rangle}{\langle \mathbb{V}[\varphi], \tilde{\varphi} \rangle}.$$

Theorem 2 (Yaglom limit at criticality)**.** *Suppose that $(X, \mathbb{P})$ is a $(\mathtt{P}, \mathtt{G})$-BMP (resp. a $(\mathtt{P}, \psi, \phi)$-SP) satisfying (H1)–(H4) with $\lambda = 0$. Then, for all $\mu \in N(E)$ (resp. $\mu \in M(E)$),*

$$\lim_{t \to \infty} \mathbb{E}_\mu \left[\exp\left(-\theta \frac{\langle f, X_t \rangle}{t} \right) \Big| \langle 1, X_t \rangle > 0 \right] = \frac{1}{1 + \theta \frac{1}{2} \langle f, \tilde{\varphi} \rangle \langle \mathbb{V}[\varphi], \tilde{\varphi} \rangle},$$

where $\theta \geq 0$ and $f \in B^+(E)$.

We now state the two results that provide conditions for the stability of the system *with* immigration at both criticality and subcriticality. The reader should note that Theorems 3 and 4 do not assume hypotheses (H3) and (H4). In the spirit of the setting without immigration, Theorem 3 supports the notion of universality of the gamma distribution for the asymptotic law of $\langle f, Y_t \rangle / t$ conditional on survival as $t \to \infty$.

Theorem 3 (Stability at criticality)**.** *Suppose that* $(Y, \mathbb{P})$ *is a* $(\mathtt{P}, \mathtt{G}, \mathtt{H})$*-BMPI, resp. a* $(\mathtt{P}, \psi, \phi, \chi)$*-SPI, satisfying (H1) and (H2) with* $\lambda = 0$*. Then, for every* $f \in B^+(E)$*, the random variable* $\langle f, Y_t \rangle / t$ *converges weakly as* $t \to \infty$ *if and only if* $\mathtt{I}[\varphi] < \infty$*, where*

$$\mathtt{I}[\varphi] = \alpha \tilde{\mathcal{E}}[\langle \varphi, \tilde{\mathcal{Z}} \rangle], \qquad \textit{resp. } \mathtt{I}[\varphi] = \langle \varphi, \upsilon \rangle + \int_{M(E)^\circ} \langle \varphi, \nu \rangle \Upsilon(\mathrm{d}\nu),$$

for the BMPI, resp. SPI setting. In that case, for all $\mu \in N(E)$ *(resp.* $\mu \in M(E)$*) and* $\theta \geq 0$*,*

$$\lim_{t \to \infty} \mathbb{E}_\mu \left[\exp\left(-\theta \frac{\langle f, Y_t \rangle}{t} \right) \right] = \left(1 + \theta \frac{1}{2} \langle f, \tilde{\varphi} \rangle \langle \mathbb{V}[\varphi], \tilde{\varphi} \rangle \right)^{-2\mathtt{I}[\varphi] / \langle \mathbb{V}[\varphi], \tilde{\varphi} \rangle}.$$

Remark 2. *Under (H1) and (H2), we deduce from the above theorem that there is no stationary measure* Y_∞ *on* E *such that* $Y_t \to Y_\infty$ *weakly as* $t \to \infty$*. This is due to the fact that the process* $(Y, \mathbb{P})$ *always explodes at criticality.*

Theorem 4 (Stability at subcriticality)**.** *Suppose that* $(Y, \mathbb{P})$ *is a* $(\mathtt{P}, \mathtt{G}, \mathtt{H})$*-BMPI, resp.* $(\mathtt{P}, \psi, \phi, \chi)$*-SPI, satisfying (H1) and (H2) with* $\lambda < 0$*. Then, for all* $\mu \in N(E)$*, resp.* $\mu \in M(E)$*, there exists a measure* Y_∞ *on* E *given by*

$$\mathbb{E}_\mu \left[\mathrm{e}^{-\langle f, Y_\infty \rangle} \right] = \mathrm{e}^{-\int_0^\infty \mathtt{H}[\mathtt{v}_s[f]] \mathrm{d}s}, \ \textit{resp. } \mathbb{E}_\mu \left[\mathrm{e}^{-\langle f, Y_\infty \rangle} \right] = \mathrm{e}^{-\int_0^\infty \chi(\mathtt{V}_s[f]) \mathrm{d}s},$$

such that $Y_t \to Y_\infty$ *weakly as* $t \to \infty$ *if and only if*

$$\int_0^{z_0} \frac{\mathtt{H}[z\varphi]}{z} \mathrm{d}z < \infty, \qquad \textit{resp. } \int_0^{z_0} \frac{\chi[z\varphi]}{z} \mathrm{d}z < \infty, \qquad \textit{for some } z_0 > 0, \tag{15}$$

if and only if

$$\tilde{\mathcal{E}} \left[\log \left(1 + \langle \varphi, \tilde{\mathcal{Z}} \rangle \right) \right] < \infty, \qquad \textit{resp. } \int_{M(E)^\circ} \log \left(1 + \langle \varphi, \nu \rangle \right) \Upsilon(\mathrm{d}\nu) < \infty. \tag{16}$$

Remark 3. Notice that if $\inf_{x \in E} \varphi(x) > 0$, then the eigenfunction φ can be substituted into conditions (15) and (16) by the constant function 1. This will be the case, for instance, in the continuous-time multi-type Galton–Watson model with immigration, where the eigenfunction is the Perron–Frobenius eigenvector of the offspring mean matrix.

4 Discussion

In this section, we spend some time discussing the consistency of our results with the existing literature. Moreover, we also take the opportunity to discuss assumptions (H1)–(H4) in the setting of some specific spatial processes.

4.1 Consistency with known results

Theorems 1 and 2 are the analogues of the Kolmogorov and Yalgom limit theorems which are so classical that they barely need any introduction. Needless to say, one may find them included in the standard branching process texts Athreya and Ney [4] and Asmussen and Hering [3]. Both Theorems 1 and 2 were recently proved in the setting of non-local branching Markov processes in [21] albeit under the significantly more restrictive assumption that $\langle 1, \mathcal{Z}\rangle$ is bounded above almost surely by a constant. In the setting of superprocesses, the limits in Theorems 1 and 2 are studied [43] for local branching mechanisms but, as far as we are aware, these results are new in the setting of non-local superprocesses.

With regards to Theorem 3, as alluded to above, the analogue of (15) in the form of the classical integral test (1) is well know for subcritical Galton–Watson processes as well as corresponding to log-moments of immigrating mass at each generation. At criticality, the convergence towards a gamma distribution after normalization has also been widely studied; we refer the reader to [4, 17, 22, 23, 37, 44, 45] for discrete time results and [38, 48, 51] for continuous time. For models with continuous mass, the natural analogues of Galton–Watson processes are continuous-state branching process. For this setting, the picture was first described by [40] with further detail given in [30]. See also Chapter 3 of [34], where an integral similar to that of (15) can also be found for the setting of CSBPs.

Finally, regarding Theorem 4, [42, 29] provide results which mirror those of Theorem 4 for subcritical multi-type Galton–Watson processes with immigration. It is worth noting that multi-type branching processes may be considered as one of the simplest examples of a spatial branching process, where the spatial component evolves in a discrete or finite set. In this setting, the mean matrix of types across a single generation codes the notion of criticality through the value of its leading eigenvalue in relation to unity. (The assumption (H2) is a direct generalisation of this concept.) We were unable to find any continuous-time analogues in the setting of multi-type Galton–Watson processes, hence we presume that Theorem 4 is a new result in this setting given that they are special cases of BMPIs as we have defined them.

Otherwise, for general BMPIs, we are unaware of any work on immigration which covers the level of generality addressed in Theorems 3 and 4. For the setting of SPIs, the most comprehensive work to date that we could find is nicely summarised in Section 9.6 of [34]; see also [32, 33] for related material. Nonetheless, we note that e.g. the integral test and the scaled limit to a gamma distribution we provide appear to be new.

On a final note, we mention that the setting for general BMPIs and SPIs has some implicit context through the well understood study of martingale change of measures in a variety of settings, see e.g. [15, 14, 12] among many others. As a rule of thumb, it is known that inherent additive martingales, which typically arise from the leading right-eigenfunction described in assumption (H2), when used as a change of measure on the ambient probability space, invoke a so-called spine decomposition, which is akin to a BMPI/SPI. Although distributional stability of the spatial population is not necessarily of concern in this context (whereas martingale convergence typically is), the notion of controlled growth through logarithmic moment conditions is certainly an important part of the dialogue. For a general perspective of martingale changes of measure and immigration in the context of BMPIs, see the discussion in [24].

4.2 Two examples

We consider two concrete examples which resonate with existing literature.

Branching Brownian motion on a compact domain. We consider a regular branching Brownian motion in which particles branch independently at a constant rate $\beta > 0$ with i.i.d. numbers of offspring distributed like N. Branching is local in the sense that offspring are positioned at the point in space where their parent dies. This process is contained in a regular bounded domain D such that when an individual first touches the boundary of the domain it is killed and sent to a cemetery state. This model was considered by [41], for which it was shown that (H2) holds providing ∂D is Lipschitz. It is also known that for subcritial and critical systems, as defined by (H2), the requirement (H3) holds. In fact, it is necessarily the case that $\tilde{\varphi} = \varphi$. Moreover, as soon as $\mathcal{E}[N^2] < \infty$, we also have that (H1) and (H4) hold. Indeed, for the latter, it is easy to see that

$$\mathbb{V}_M[g](x) = \beta g(x)^2 \mathcal{E}\left[N(N-1)\mathbf{1}_{\{N \le M\}}\right], \qquad x \in E,$$

so that

$$\langle \mathbb{V}_M[g], \tilde{\varphi}\rangle = \beta \mathcal{E}\left[N(N-1)\mathbf{1}_{\{N \le M\}}\right] \langle g^2, \tilde{\varphi}\rangle \ge K \langle g, \tilde{\varphi}\rangle^2$$

by Jensen's inequality, which implies (3) holds.

Multi-type continuous-state branching processes (MCSBP). These processes are the natural analogues of multi-type Galton–Watson processes in the context of continous mass. One may also think of them as super Markov chains. In addition, allowing for immigration, MCSBPs were introduced in [5] and can be represented via their semigroup properties or as solutions to SDEs. In essence, they correspond to the setting that $E = \{1, \dots, n\}$, for some $n \in \mathbb{N}$. In this setting, (H2) is a natural ergodic assumption similar to those discussed in Section 4 of [31], where a simple irreducibility assumption ensures that (H2) will hold. In essence, (H2) is the classical Perron–Frobenius behaviour for the matrix of the mean semigroup. The assumption (H3) does not automatically hold as, in the spirit of CSBP processes, extinction can occur by a slow trickle of mass down to zero. The assumption (H1) is also natural, ensuring finite second moments for the MCSBP mass. Finally, for the assumption (H4),

$$\begin{aligned}\mathbb{V}_M[h](i) = &\left(2c(i) + \int_{(0,\infty)} y^2 \mathbf{1}_{\{y \le M\}} \nu(i, \mathrm{d}y)\right) h(i)^2 \\ &+ \beta(i) \int_{M(\{1,\dots,n\})^\circ} \langle h, \nu\rangle^2 \mathbf{1}_{\{\langle 1,\nu\rangle \le M\}} \Gamma(i, \mathrm{d}\nu), \qquad i \in \{1, \dots, n\},\end{aligned}$$

so again by Jensen's inequality it is easy to see that

$$\langle \mathbb{V}_M[h], \tilde{\varphi}\rangle \ge \min_{i \in \{1,\dots,n\}} \left(2c(i) + \int_{(0,\infty)} y^2 \mathbf{1}_{\{y \le M\}} \nu(i, \mathrm{d}y)\right) \langle h^2, \tilde{\varphi}\rangle \ge K \langle h, \tilde{\varphi}\rangle^2,$$

i.e. (H4) is automatically satisfied.

5 Evolution equations

In this section, we consider several semigroup evolution equations that will be useful for proving our main results. We note that, in formulating them, we don't need to assume as much as (H1)–(H4). We recall that the assumptions on the branching mechanisms $\mathtt{G}$, ψ and ϕ ensure that

$$\sup_{x\in E}\mathcal{E}_x[N]<\infty, \qquad \text{resp.}\ \sup_{x\in E}\left(\int_0^\infty |y|\nu(x,\mathrm{d}y)+\int_{M(E)^\circ}\langle 1,\nu\rangle\Gamma(x,\mathrm{d}\nu)\right)<\infty,$$

for branching particle processes, resp. superprocesses, which is needed for some of the results we cite below.

5.1 Non-local branching Markov processes

In the setting of the $(\mathtt{P},\mathtt{G})$-branching Markov process, the evolution equation for the expectation semigroup $(\mathtt{T}_t, t\geq 0)$ is given by

$$\mathtt{T}_t[f](x)=\mathtt{P}_t[f](x)+\int_0^t \mathtt{P}_s[\beta(\mathtt{m}[\mathtt{T}_{t-s}[f]]-\mathtt{T}_{t-s}[f])](x)\mathrm{d}s,$$

for $t\geq 0$, $x\in E$ and $f\in B^+(E)$, where we have used the notation

$$\mathtt{m}[f](x)=\mathcal{E}_x[\langle f,\mathcal{Z}\rangle].$$

See, for example, Lemma 8.1 of [24].

Our next evolution equation will relate the non-linear semigroup to the linear semigroup, which will enable us to use (H2), for example, to study the limiting behaviour of $\mathtt{v}_t$. For this, we will introduce the following modification to the non-linear semigroup,

$$\mathtt{u}_t[g](x)=\mathbb{E}_{\delta_x}\left[1-\prod_{i=1}^{N_t}g(x_i(t))\right]=1-\mathtt{v}_t[-\log g](x), \qquad g\in B_1^+(E).$$

Recalling that we have assumed first moments of the offspring distribution, one can show that

$$\mathtt{u}_t[g](x)=\mathtt{T}[1-g](x)-\int_0^t \mathtt{T}_s[\mathtt{A}[\mathtt{u}_{t-s}[g]]](x)\mathrm{d}s, \quad t\geq 0, \tag{17}$$

where, for $g\in B_1^+(E)$ and $x\in E$,

$$\mathtt{A}[g](x)=\beta(x)\mathcal{E}_x\left[\prod_{i=1}^{N}(1-g(x_i))-1+\sum_{i=1}^{N}g(x_i)\right]. \tag{18}$$

We refer the reader to Theorem 8.2 of [24] for a more general version of (17), along with a proof.

We now consider the process *with* immigration. Let us consider the transition semigroup $(\mathtt{W}_t, t \geq 0)$ for the $(\mathtt{P}, \mathtt{G}, \mathtt{H})$-BMPI, given by

$$\mathrm{e}^{-\mathtt{W}_t[f](x)} = \mathbb{E}_{\delta_x}\left[\mathrm{e}^{-\langle f, Y_t\rangle}\right], \qquad t \geq 0,\ f \in B^+(E),\ x \in E. \tag{19}$$

Denoting $\tilde{Y}_t = \sum_{j=1}^{D_t} X_{t-\tau_j}^{(\tilde{\mathcal{Z}}_j)}$, from Definition 1 it is clear that $Y_t^{(\delta_x)} = X_t^{(\delta_x)} + \tilde{Y}_t$ and

$$\mathrm{e}^{-\mathtt{W}_t[f](x)} = \mathrm{e}^{-\mathtt{v}_t[f](x)}\mathrm{e}^{-\tilde{\mathtt{W}}_t[f]},$$

where $(\tilde{\mathtt{W}}_t, t \geq 0)$ is the transition semigroup associated to $\tilde{Y}$. From the branching property (2) and the immigration counting measure (4), it is clear that the Laplace functional of $X_t^{(\tilde{\mathcal{Z}})}$ is given by

$$\tilde{\mathcal{E}}\left[\mathrm{e}^{-\langle \mathtt{v}_t[f], \tilde{\mathcal{Z}}\rangle}\right], \qquad f \in B^+(E).$$

Therefore, conditioning on the time of the first immigration event, it is possible to obtain

$$\begin{aligned}
\mathrm{e}^{-\tilde{\mathtt{W}}_t[f]} &= \mathrm{e}^{-\alpha t} + \int_0^t \alpha \mathrm{e}^{-\alpha s}\tilde{\mathcal{E}}\left[\mathrm{e}^{-\langle \mathtt{v}_{t-s}[f], \tilde{\mathcal{Z}}\rangle}\right]\mathrm{e}^{-\tilde{\mathtt{W}}_{t-s}[f]}\mathrm{d}s \\
&= \mathrm{e}^{-\alpha t} + \int_0^t \mathrm{e}^{-\alpha s}\left[\alpha - \mathtt{H}[\mathtt{v}_{t-s}[f]]\right]\mathrm{e}^{-\tilde{\mathtt{W}}_{t-s}[f]}\mathrm{d}s.
\end{aligned}$$

It then follows from Theorem 2.1 in [24] or Lemma 1.2 in Chapter 4 of [9], for example, that

$$\mathrm{e}^{-\tilde{\mathtt{W}}_t[f]} = 1 - \int_0^t \mathtt{H}[\mathtt{v}_{t-s}[f]]\mathrm{e}^{-\tilde{\mathtt{W}}_{t-s}[f]}\mathrm{d}s,$$

from which it is easily deduced that

$$\mathrm{e}^{-\mathtt{W}_t[f](x)} = \exp\left(-\mathtt{v}_t[f](x) - \int_0^t \mathtt{H}[\mathtt{v}_s[f]]\mathrm{d}s\right). \tag{20}$$

We note that similar calculations for the semigroup of processes with immigration are common in other literature, e.g. in Chapters 3 and 9 of [34].

5.2 Non-local superprocesses

In the setting of the $(\mathtt{P}, \psi, \phi)$-superprocess, the evolution equation for the expectation semigroup $(\mathtt{T}_t, t \geq 0)$ is well known and satisfies

$$\mathtt{T}_t\left[f\right](x) = \mathtt{P}_t[f](x) + \int_0^t \mathtt{P}_s\left[\beta(\mathtt{m}[\mathtt{T}_{t-s}[f]] - \mathtt{T}_{t-s}[f]) + b\mathtt{T}_{t-s}[f]\right](x)\mathrm{d}s, \tag{21}$$

for $t \geq 0$, $x \in E$ and $f \in B^+(E)$, where, with a meaningful abuse of our branching Markov process notation, we now define

$$\mathtt{m}[f](x) = \gamma(x, f) + \int_{M(E)^\circ}\langle f, \nu\rangle\Gamma(x, \mathrm{d}\nu). \tag{22}$$

See for example equation (3.24) of [7].

Similarly to the branching Markov process setting, let us re-write an extended version of the non-linear semigroup evolution $(\mathtt{V}_t, t \geq 0)$, defined in (7), i.e. the natural analogue of (3), in terms of the linear semigroup $(\mathtt{T}_t, t \geq 0)$. To this end, define

$$\mathrm{e}^{-\mathtt{V}_t[f](x)} = \mathbb{E}_{\delta_x}\left[\mathrm{e}^{-\langle f, X_t\rangle}\right].$$

From [18], we have the following evolution equation,

$$\mathtt{V}_t[f](x) = \mathtt{T}_t[f](x) - \int_0^t \mathtt{T}_s\left[\mathtt{J}[\mathtt{V}_{t-s}[f]]\right](x)\mathrm{d}s, \quad f \in B^+(E), x \in E, t \geq 0, \tag{23}$$

where, for $h \in B^+(E)$ and $x \in E$, we now define

$$\begin{aligned} \mathtt{J}[h](x) &= \psi(x, h(x)) + \phi(x, h) + \beta(x)(\mathtt{m}[h](x) - h(x)) + b(x)h(x) \\ &= c(x)h(x)^2 + \int_{(0,\infty)} (\mathrm{e}^{-h(x)y} - 1 + h(x)y)\nu(x, \mathrm{d}y) \\ &\quad + \beta(x)\int_{M(E)^\circ} (\mathrm{e}^{-\langle h,\nu\rangle} - 1 + \langle h, \nu\rangle)\Gamma(x, \mathrm{d}\nu). \end{aligned} \tag{24}$$

As before, we now consider the non-local superprocess with immigration. In particular, we are interested in the transition semigroup pair $((\mathtt{W}_t, \mathtt{V}_t), t \geq 0)$ for the $(\mathtt{P}, \psi, \phi, \chi)$-SPI, where

$$\mathrm{e}^{-\mathtt{W}_t[f](x)} = \mathbb{E}_{\delta_x}\left[\mathrm{e}^{-\langle f, Y_t\rangle}\right], \qquad t \geq 0. \tag{25}$$

From the definition (11), with the help of Campbell's formula, we have

$$\begin{aligned} \mathrm{e}^{-\mathtt{W}_t[f](x)} &= \mathrm{e}^{-\mathtt{V}_t[f](x)} \exp\left(-\int_0^t \int_{M_0(E)} P_\chi(\mathrm{d}\pi)\delta_\chi(\pi)\int_E \pi(\mathrm{d}y)\mathbb{Q}_y(1 - \mathrm{e}^{-\langle f, X_{t-s}\rangle})\mathrm{d}s\right) \\ &\qquad \times \exp\left(-\int_0^t \int_{M_0(E)} P_\chi(\mathrm{d}\pi)\int_0^\infty n_\chi(\pi, \mathrm{d}u)\mathbb{E}_{u\pi}(1 - \mathrm{e}^{-\langle f, X_{t-s}\rangle})\mathrm{d}s\right) \\ &= \mathrm{e}^{-\mathtt{V}_t[f](x)} \exp\left(-\int_0^t \left[\langle \mathtt{V}_{t-s}[f], \upsilon\rangle + \int_{M^\circ(E)} (1 - \mathrm{e}^{-\langle \mathtt{V}_{t-s}[f], \nu\rangle})\Upsilon(\mathrm{d}\nu)\right]\mathrm{d}s\right) \\ &= \exp\left(-\mathtt{V}_t[f](x) - \int_0^t \chi[\mathtt{V}_{t-s}[f]]\mathrm{d}s\right). \end{aligned} \tag{26}$$

We note that similar calculations can be found in Chapter 9 of [34].

6 Proof of Theorem 1

6.1 Non-local branching Markov processes

Let us define $\mathtt{u}_t(x) := \mathtt{u}_t[\mathbf{0}](x)$, where $\mathbf{0}$ is the constant 0 function. Then $\mathtt{u}_t(x) = \mathbb{P}_{\delta_x}(\zeta > t)$ and hence $\mathbb{P}_{\delta_x}(\zeta > t)$ is a solution to (17) with $f = \mathbf{0}$. Our aim will be to use this evolution equation to obtain the asymptotic behaviour for $\mathtt{u}_t(x)$. In order to do so, it will be convenient to introduce the following quantities,

$$a_t[g] := \langle \mathtt{u}_t[g], \tilde{\varphi} \rangle, \quad \text{and} \quad a_t := a_t[\mathbf{0}] = \langle \mathtt{u}_t, \tilde{\varphi} \rangle.$$

Integrating (17) with respect to $\tilde{\varphi}$ and using the (H2), we obtain

$$a_t[g] = \langle 1 - g, \tilde{\varphi} \rangle - \int_0^t \langle \mathtt{A}[\mathtt{u}_{t-s}[g]], \tilde{\varphi} \rangle \mathrm{d}s. \tag{27}$$

The strategy of the proof is to first find coarse upper and lower bounds of the order $1/t$ for a_t and $\mathtt{u}_t$, and then refine our estimates to obtain the precise constants. The method of proof follows closely that of [21] but with more precise estimates in our calculations. For the convenience of the reader, we will include the details.

We thus proceed via a series of lemmas, the first of which provides useful lower bounds on $\mathtt{u}_t[g]$ and $a_t[g]$ for general g. For the following lemma, we introduce the following change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}^{\varphi}_{\mu}}{\mathrm{d}\mathbb{P}_{\mu}}\right|_{\mathcal{F}_t} := \frac{\langle \varphi, X_t \rangle}{\langle \varphi, \mu \rangle}, \quad t \geq 0, \mu \in N(E), \tag{28}$$

where it follows from (H2) that $\langle \varphi, X_t \rangle / \langle \varphi, \mu \rangle$ is a martingale.

Lemma 1. *There exists $C \in (0, \infty)$ such that*

$$\mathtt{u}_t[g](x) \geq \frac{C\varphi(x)}{\mathbb{E}^{\varphi}_{\delta_x}[\langle \varphi, X_t \rangle] + \sup_{y \in E} \frac{\varphi(y)}{\log(1/g(y))}} \quad \text{and} \quad a_t[g] \geq \frac{C}{\mathbb{E}^{\varphi}_{\delta_x}[\langle \varphi, X_t \rangle] + \sup_{y \in E} \frac{\varphi(y)}{\log(1/g(y))}}$$

for all $t \geq 0$ and $g \in B_1^+(E)$ such that $\sup_{y \in E} \varphi(y)/\log(1/g(y)) < \infty$.

In particular,

$$\mathtt{u}_t(x) \geq \frac{C\varphi(x)}{t} \quad \text{and} \quad a_t \geq \frac{C}{t}, \qquad t \geq 1.$$

Proof. Recall the change of measure (28). By Jensen's inequality, we have

$$\mathbb{E}_{\delta_x}\left[1 - \prod_{i=1}^{N_t} g(x_i(t))\right] = \varphi(x)\mathbb{E}^{\varphi}_{\delta_x}\left[\frac{1 - \prod_{i=1}^{N_t} g(x_i(t))}{\langle \varphi, X_t \rangle}\right] \geq \frac{\varphi(x)}{\mathbb{E}^{\varphi}_{\delta_x}\left[\frac{\langle \varphi, X_t \rangle}{1 - \prod_{i=1}^{N_t} g(x_i(t))}\right]} \tag{29}$$

where we note that $1 - \mathrm{e}^{-x} \geq \min(x/2, 1/2)$ for $x \geq 0$, so that

$$1 - \prod_{i=1}^{N_t} g(x_i(t)) \geq \tfrac{1}{2} \min(\langle \log(1/g), X_t \rangle, 1),$$

(where this also holds for $g = \mathbf{0}$ with the convention that $\log(1/0) = \infty$). Thus

$$\mathbb{E}_{\delta_x}\left[1 - \prod_{i=1}^{N_t} g(x_i(t))\right] \geq \frac{2\varphi(x)}{\mathbb{E}^{\varphi}_{\delta_x}\left[\max\{\langle \varphi, X_t \rangle, \frac{\langle \varphi, X_t \rangle}{\langle \log(1/g), X_t \rangle}\}\right]} \geq \frac{2\varphi(x)}{\mathbb{E}^{\varphi}_{\delta_x}[\langle \varphi, X_t \rangle] + \sup_{y \in E} \frac{\varphi(y)}{\log(1/g(y))}}.$$

The lower bound for $a_t[g]$ then follows from an integration with $\tilde{\varphi}$, recalling that we have normalised the left and right eigenfunctions so that $\langle \varphi, \tilde{\varphi} \rangle = 1$.

The specific claim when $g = \mathbf{0}$ follows from [18, Theorem 1], since this implies that $\mathbb{E}^{\varphi}_{\delta_x}[\langle \varphi, X_t \rangle] \leq Ct$ for some constant $C \in (0, \infty)$ and for all $t \geq 1$. □

The next lemma shows that the leading order term in $\mathtt{A}$, defined in (18), is governed by the operator $\mathbb{V}$.

Lemma 2 (Properties of $\mathtt{A}$ and $\mathbb{V}$)**.** *Under the assumption (H1) we have the following.*

(a) *We have $\mathtt{A}[g](x) \geq 0$ for all $g \in B_1^+$, $x \in E$.*

(b) *There exists a constant $C \in (0,\infty)$ such that $\|\mathbb{V}[h_1] - \mathbb{V}[h_2]\| \leq C\|h_1 - h_2\|$ for all functions $h_1, h_2 \in \{f \in B^+(E) : \|f\| \leq k\}$ for some $k > 0$.*

(c) *$\mathtt{A}[g](x) \leq \frac{1}{2}\mathbb{V}[g](x)$ for all $g \in B_1^+(E)$, $x \in E$.*

(d) *$\|\mathtt{A}[g] - \frac{1}{2}\mathbb{V}[g]\| = o(\|g\|^2)$ as $\|g\| \to 0$, $g \in B_1^+(E)$.*

Proof. (a) is a consequence of the deterministic inequality

$$0 \leq \prod_{i=1}^{N}(1 - z_i) - 1 + \sum_{i=1}^{N} z_i$$

for all $N \in \mathbb{N}$ and $z_1, \dots, z_N \in [0,1]$.

For (b), we write

$$\|\mathbb{V}[h_1] - \mathbb{V}[h_2]\| \leq \|\beta\| \sup_{x\in E} \mathcal{E}_x\left[\sum_{i,j=1;\, i\neq j}^{N} \big|[h_1(x_i) - h_2(x_i)]h_1(x_j) + h_2(x_i)[h_1(x_j) - h_2(x_j)]\big|\right]$$

and the claim follows thanks to (H1).

Finally, we deduce (c) and (d). Using Taylor's theorem, we have

$$\mathtt{A}[h](x) = \beta(x)\mathcal{E}_x\Bigg[\sum_{\substack{i,j=1,\\ i\neq j}}^{N} h(x_i)h(x_j)\int_0^1 (1-r)\prod_{\substack{k=1,\\ k\neq i,j}}^{N}(1 - rh(x_k))\mathrm{d}r\Bigg]. \tag{30}$$

This immediately implies (c), and we also deduce that

$$\begin{aligned}\|\mathtt{A}[h] - \tfrac{1}{2}\mathbb{V}[h]\| &\leq \sup_{x\in E}\beta(x)\mathcal{E}_x\left[\sum\nolimits_{i\neq j} h(x_i)h(x_j)\int_0^1(1-r)\Big|1 - \prod\nolimits_{k\neq i,j}(1 - rh(x_k)\Big|\,\mathrm{d}r\right]\\ &\leq \|\beta\|\|h\|^2\sup_x \mathcal{E}_x[N(N-1)(N\|h\|\mathbf{1}_{\{N\leq\|h\|^{-1/2}\}} + 2\mathbf{1}_{\{N\geq\|h\|^{-1/2}\}})].\end{aligned}$$

This is $o(\|h\|^2)$ thanks to (H1). □

With this in hand, we proceed to show that $\mathtt{u}_t[g]/\varphi$ and $a_t[g]$ are small when $\mathtt{u}_t[g]$ is small, which will be crucial for the proof of Theorem 1.

Lemma 3. *There exists $C \in (0,\infty)$ such that for all $t > 0$ and all $g \in B_1^+(E)$,*

$$\sup_{x\in E}\left|\frac{\mathtt{u}_t[g](x)}{\varphi(x)} - a_t[g]\right| \leq C\left(\mathrm{e}^{-\varepsilon t}\|1-g\| + \int_0^t \mathrm{e}^{-\varepsilon(t-s)}\|\mathtt{u}_s[g]\|^2\mathrm{d}s\right).$$

Proof. We have

$$\left|\frac{\mathtt{u}_t[g](x)}{\varphi(x)}-a_t[g]\right| \le \left|\frac{\mathtt{T}_t[1-g](x)}{\varphi(x)}-\langle 1-g,\tilde\varphi\rangle\right| + \int_0^t \left|\frac{\mathtt{T}_{t-s}[\mathtt{A}[\mathtt{u}_s[g]]](x)}{\varphi(x)}-\langle \mathtt{A}[\mathtt{u}_s[g]],\tilde\varphi\rangle\right| \mathrm{d}s$$
$$\le C\left(\mathrm{e}^{-\varepsilon t}\|1-g\| + \int_0^t \mathrm{e}^{-\varepsilon(t-s)}\|\mathtt{A}[\mathtt{u}_s[g]]\|\,\mathrm{d}s\right),$$

where the second line follows from (H2). The lemma then follows since $\|\mathtt{A}[h]\| \le \frac{1}{2}\|\mathbb{V}[h]\|$ (Lemma 2c) and $\|\mathbb{V}[\mathtt{u}_s[g]]\| \le \sup_{x\in E}\mathcal{E}_x[N(N-1)]\|\beta\|\|\mathtt{u}_s[g]\|^2$, which is thanks to Assumption (H1). □

Lemma 4. *Under the assumptions of Theorem 1, there exists $t_0 \in (0,\infty)$ and a constant $C>0$ such that for all $t \ge t_0$ and all $g \in B_1^+(E)$,*

$$a_t \le \frac{C}{t} \quad \textit{and} \quad \sup_{x\in E}\mathtt{u}_t(x) \le \frac{C}{t}. \tag{31}$$

Proof. We first observe from (H3) that

$$a_t \to 0 \text{ and } \sup_{x\in E}\mathtt{u}_t(x) \to 0$$

as $t\to\infty$. The convergence of a_t to 0 follows from (H3) and dominated convergence. To prove uniform convergence of $\mathtt{u}_t$ to zero, let us note that $\mathtt{u}_{t+s}(x) = \mathtt{u}_t[1-\mathtt{u}_s](x)$ by the Markov branching property. We therefore have (also using Lemma 2a) that

$$0 \le \mathtt{u}_{t+s}(x) = \mathtt{T}_t[\mathtt{u}_s](x) - \int_0^t \mathtt{T}_l\big[\mathtt{A}[\mathtt{u}_{t+s-\ell}]\big](x)\mathrm{d}\ell \le \mathtt{T}_t[\mathtt{u}_s](x) \tag{32}$$

and so

$$\|\mathtt{u}_{t+s}\| \le \|\mathtt{T}_t[\mathtt{u}_s]\| \le a_s\|\varphi\| + O(\mathrm{e}^{-\varepsilon t}) \tag{33}$$

by (H2). Taking t and then s to infinity gives that $\|\mathtt{u}_t\|\to 0$ as $t\to\infty$.

Now we prove the required upper bound on a_t and $\|\mathtt{u}_t\|$. First note that (27) implies that

$$a_t = a_0 + \int_0^t \langle \mathtt{A}[u_s],\tilde\varphi\rangle \mathrm{d}s$$

where the integrand is bounded due to Lemma 2a. Therefore a is differentiable with $a_t' = -\langle \mathtt{A}[\mathtt{u}_t],\tilde\varphi\rangle$ for $t\ge 0$.

Next, by Taylor's Theorem (30), we deduce that if $\|h\|\le 1/2$, then $\mathtt{A}[h](x) \ge 2^{-M}\mathbb{V}_M[h](x)$ for any $M\in\mathbb{N}$. We therefore obtain that for $t\ge t_0$, with t_0 chosen so that $\sup_{t\ge t_0}\|\mathtt{u}_t\|\le 1/2$,

$$a'(t) = -\langle \mathtt{A}[\mathtt{u}_t],\tilde\varphi\rangle \le -2^{-M}\langle \mathbb{V}_M[\mathtt{u}_t],\tilde\varphi\rangle \le -C\langle \mathtt{u}_t,\tilde\varphi\rangle^2 = -Ca_t^2$$

for some $C\in(0,\infty)$, where in the second inequality we have used (H4) with the values of M and C there.

This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{a_t}\right) \ge C \text{ for } t \ge t_0.$$

Integrating from t_0 to t we obtain the desired upper bound for a_t. The upper bound for $\mathtt{u}_t$ follows from the same argument as that given in (33). □

We are now ready to prove Theorem 1, which now entails showing that the long-term behaviour of $\mathtt{u}_t/\varphi$ and a_t are the same.

Proof of Theorem 1. Applying Lemma 3, we have

$$\sup_{x\in E}\left|\frac{\mathtt{u}_t(x)}{\varphi(x)}-a_t\right|\le C\left(\mathrm{e}^{-\varepsilon t}+\int_0^{t/2}\mathrm{e}^{-\varepsilon(t-s)}\|\mathtt{u}_s\|^2\mathrm{d}s+\int_{t/2}^{t}\mathrm{e}^{-\varepsilon(t-s)}\|\mathtt{u}_s\|^2\mathrm{d}s\right).$$

Bounding $\|\mathtt{u}_s\|$ by 1, we see that $\int_0^{t/2}\mathrm{e}^{-\varepsilon(t-s)}\|\mathtt{u}_s\|^2\mathrm{d}s=O(\mathrm{e}^{-\varepsilon t/2})$, and using the bound given in Lemma 4, we obtain $\int_{t/2}^{t}\mathrm{e}^{-\varepsilon(t-s)}\|\mathtt{u}_s\|^2\mathrm{d}s=O(t^{-2})$. Therefore,

$$\sup_{x\in E}\left|\frac{\mathtt{u}_t(x)}{\varphi(x)}-a_t\right|=O(t^{-2}),\quad t\to\infty.$$

On the other hand, from Lemma 1, we have that $a_t^{-1}=O(t)$. It follows that

$$\sup_{x\in E}\left|\frac{\mathtt{u}_t(x)}{\varphi(x)a_t}-1\right|=O(t^{-1}),\quad t\to\infty. \tag{34}$$

Using Lemma 2b, we deduce that

$$\begin{aligned}\sup_{x\in E}a_t^{-2}\left|\mathbb{V}[\mathtt{u}_t](x)-\mathbb{V}[a_t\varphi](x)\right| &= \sup_{x\in E}\left|\mathbb{V}[\mathtt{u}_t/a_t](x)-\mathbb{V}[\varphi](x)\right|\\ &\le C\sup_{x\in E}\left|\frac{\mathtt{u}_t(x)}{a_t}-\varphi(x)\right|=O(t^{-1}).\end{aligned}\tag{35}$$

Therefore, appealing to basic calculus from (27), for all $t\ge t_0$,

$$\left|\frac{1}{ta_t}-\frac{1}{ta_{t_0}}-\langle\tfrac12\mathbb{V}[\varphi],\tilde\varphi\rangle\right|=\frac1t\left|\int_{t_0}^t\frac{\langle\mathtt{A}[\mathtt{u}_s],\tilde\varphi\rangle}{a_s^2}\mathrm{d}s-\int_0^t\langle\tfrac12\mathbb{V}[\varphi],\tilde\varphi\rangle\mathrm{d}s\right|.$$

Noting that $\mathbb{V}[\varphi a_s]=a_s^2\mathbb{V}[\varphi]$, we can bound the right-hand side above by

$$\frac1t\left|\int_0^{t_0}\langle\tfrac12\mathbb{V}[\varphi],\tilde\varphi\rangle\right|+\frac1t\left|\int_{t_0}^t\frac{\frac12\langle\mathbb{V}[a_s\varphi]-\mathbb{V}[\mathtt{u}_s],\tilde\varphi\rangle}{a_s^2}\mathrm{d}s\right|+\frac1t\left|\int_{t_0}^t\frac{\langle\frac12\mathbb{V}[\mathtt{u}_s]-\mathtt{A}[\mathtt{u}_s],\tilde\varphi\rangle}{a_s^2}\mathrm{d}s\right|.$$

The first term in the expression above clearly converges to 0 as $t\to\infty$, while the second term converges to 0 using (35). The final term converges to 0 again using the lower bound $a_s\ge C/s$ from Lemma 1 and Lemma 2d.

Since $1/(ta_{t_0})\to 0$ as $t\to\infty$, this implies that

$$\frac{1}{ta_t}\to\langle\tfrac12\mathbb{V}[\varphi],\tilde\varphi\rangle,\qquad\text{as } t\to\infty,$$

in other words,

$$a_t\sim\frac{2}{\langle\mathbb{V}[\varphi],\tilde\varphi\rangle t},\qquad\text{as } t\to\infty.$$

The desired asymptotic for $\mathtt{u}_t$ then follows from (34). □

6.2 Non-local Superprocesses

We first note that, for $\theta \in \mathbb{R}$ and $\mu \in M(E)$,

$$\mathrm{e}^{-\langle \mathtt{V}_t, \mu\rangle} := \lim_{\theta\to\infty} \mathrm{e}^{-\langle \mathtt{V}_t[\theta], \mu\rangle} = \lim_{\theta\to\infty} \mathbb{E}_\mu\big[\mathrm{e}^{-\theta\langle 1, X_t\rangle}\big] = \mathbb{P}_\mu(\zeta \le t), \tag{36}$$

and hence

$$\mathbb{P}_\mu(\zeta > t) = 1 - \mathrm{e}^{-\langle \mathtt{V}_t, \mu\rangle}, \qquad \mu \in M(E),\, t \ge 0. \tag{37}$$

Under assumption (H3), we notice that $\langle \mathtt{V}_t, \mu\rangle \to 0$ as $t \to \infty$ and consequently (37) implies that

$$\lim_{t\to\infty} \frac{1}{\langle \mathtt{V}_t, \mu\rangle} \mathbb{P}_\mu(\zeta > t) = 1. \tag{38}$$

Thus, in order to understand the decay of the survival probability, it suffices to study the decay of $\mathtt{V}_t$. For this reason, we will conveniently work with

$$a_t[f] = \langle \mathtt{V}_t[f], \tilde\varphi\rangle = \langle f, \tilde\varphi\rangle - \int_0^t \langle \mathtt{J}[\mathtt{V}_{t-s}[f]], \tilde\varphi\rangle \mathrm{d}s, \quad f \in B^+(E), \tag{39}$$

where the second equality follows from integrating (23) with respect to $\tilde\varphi$ and (H2). It follows that for any $t, t_0 > 0$,

$$a_{t+t_0}[f] = a_{t_0}[f] - \int_0^t \langle \mathtt{J}[\mathtt{V}_s[\mathtt{V}_{t_0}[f]]], \tilde\varphi\rangle \mathrm{d}s. \tag{40}$$

Thus, with $a_t := \lim_{\theta\to\infty} a_t[\theta] = \langle \mathtt{V}_t, \tilde\varphi\rangle$, it follows that

$$a_{t+t_0} = a_{t_0} - \int_0^t \langle \mathtt{J}[\mathtt{V}_s[\mathtt{V}_{t_0}]], \tilde\varphi\rangle \mathrm{d}s. \tag{41}$$

The strategy is thus to prove that a_t and $\mathtt{V}_t$ are asymptotically of order $1/t$, as in the proof methodology of [21].

The proof of Theorem 1 for the superprocess setting is almost verbatim the same as in the previous section. In the interests of brevity, we leave the remainder of the proof of Theorem 1 for non-local superprocesses as an exercise, offering as assistance below the analogue of Lemma 2, which is a key ingredient, and referring for a full proof of Theorem 1 to [35].

Lemma 5 (Properties of $\mathtt{J}$ and $\mathbb{V}$)**.** *Suppose that assumption (H1) holds.*

(a) We have $\mathtt{J}[h](x) \ge 0$ for all $h \in B^+(E)$, $x \in E$.

(b) There exists a constant $C \in (0,\infty)$ such that $\|\mathbb{V}[h_1] - \mathbb{V}[h_2]\| \le C\|h_1 - h_2\|$ for all functions $h_1, h_2 \in \{f \in B^+(E) : \|f\| \le k\}$ for some $k > 0$.

(c) $\mathtt{J}[h](x) \le \frac{1}{2}\mathbb{V}[h](x)$ for all $h \in B^+(E)$, $x \in E$.

(d) $\|\mathtt{J}[h] - \frac{1}{2}\mathbb{V}[h]\| = o(\|h\|^2)$ as $\|h\| \to 0$, $h \in B^+(E)$.

Proof. (a) and (c) follows trivially from the deterministic inequalities

$$0 \le \mathrm{e}^{-z} - 1 + z \le \frac{z^2}{2}, \qquad z \in [0, \infty)].$$

For (b), we write

$$\begin{aligned} \|\mathbb{V}[h_1] - \mathbb{V}[h_2]\| &\le \sup_{x\in E} \left(2c(x) + \int_0^\infty y^2 \nu(x, \mathrm{d}y)\right) \|h_1 + h_2\| \|h_1 - h_2\| \\ &\quad + \sup_{x\in E} \beta(x) \int_{M(E)^\circ} \langle \|h_1 + h_2\|, \nu\rangle \langle \|h_1 - h_2\|, \nu\rangle |\Gamma(x, \mathrm{d}\nu) \end{aligned}$$

and the claim follows from (H1).

For (d), the map $m_\star : z \in [0, \infty) \to 1 - z + \frac{1}{2}z^2 - \mathrm{e}^{-z}$ is a non-negative increasing function bounded above by z^2 and z^3, which allows us to write

$$\begin{aligned} |\mathtt{J}[h](x) - \tfrac{1}{2}\mathbb{V}[h](x)| &\le \int_0^\infty m_\star(\|h\|y)\nu(x, \mathrm{d}y) + \beta(x) \int_{M(E)^\circ} m_\star(\langle \|h\|, \nu\rangle)\Gamma(x, \mathrm{d}\nu) \\ &\le \|h\|^2 \int_0^\infty y^2 (\|h\|^{1/2} \mathbf{1}_{\{y \le \|h\|^{-1/2}\}} + \mathbf{1}_{\{y \ge \|h\|^{-1/2}\}})\nu(x, \mathrm{d}y) \\ &\quad + \|h\|^2 \beta(x) \int_0^\infty \langle 1, \nu\rangle^2 (\|h\|^{1/2} \mathbf{1}_{\{\langle 1,\nu\rangle \le \|h\|^{-1/2}\}} + \mathbf{1}_{\{\langle 1,\nu\rangle \ge \|h\|^{-1/2}\}})\Gamma(x, \mathrm{d}\nu) \end{aligned}$$

and this is $o(\|h\|^2)$ thanks to (H1). □

7 Proof of Theorem 2

We give the proof only for the setting of non-local branching Markov processes, again noting that the proof in the setting of non-local superprocesses is almost verbatim, taking account of the fact that the role of $\mathtt{u}_t$ is played by $\mathtt{V}_t$. The reader is again referred to [35] for a full proof.

We are guided by the argument in the proof of Theorem 1.3 in [21]. Given $f \in B^+(E)$, let us consider $\tilde{f} = f - \langle f, \tilde{\varphi}\rangle\varphi$. The first assertion is that $\langle \tilde{f}, X_t\rangle/t$ converges weakly under $\mathbb{P}_\mu(\cdot \mid \zeta > t)$ to 0 as $t \to \infty$. Indeed, by Markov's inequality,

$$\mathbb{P}_\mu\left(\frac{|\langle \tilde{f}, X_t\rangle|}{t} > \epsilon \,\middle|\, \zeta > t\right) \le \frac{\langle \mathtt{T}_t^{(2)}[\tilde{f}], \mu\rangle}{\epsilon^2 t^2 \mathbb{P}_\mu(\langle 1, X_t\rangle > 0)},$$

where $\mathtt{T}_t^{(2)}[\tilde{f}](x) = \mathbb{E}_{\delta_x}[\langle \tilde{f}, X_t\rangle^2]$, for $t \ge 0$, $x \in E$. Moreover, the asymptotic behaviour of $t\mathbb{P}_\mu(\langle 1, X_t\rangle > 0)$ as $t \to \infty$ is given in Theorem 1. Now, from Theorem 1 of [18],

$$\frac{1}{t}\langle \mathtt{T}_t^{(2)}[\tilde{f}], \mu\rangle \le (\langle \tilde{f}, \tilde{\varphi}\rangle^2 \langle \mathbb{V}[\varphi], \tilde{\varphi}\rangle + \|\tilde{f}\|^2 \Delta_t^{(2)})\langle \varphi, \mu\rangle, \tag{42}$$

where $\Delta_t^{(2)} = \sup_{x\in E,\, f\in B_1^+(E)} |s^{-1}\varphi(x)^{-1}\mathtt{T}_t^{(2)}[f](x) - \langle f, \tilde{\varphi}\rangle^2 \langle \mathbb{V}[\varphi], \tilde{\varphi}\rangle|$, which tends to zero as $t \to \infty$ from the aforementioned theorem. Hence, as $\langle \tilde{f}, \tilde{\varphi}\rangle = 0$, we see that the right-hand side of (42) tends to zero as $t \to \infty$.

Then, applying Slutsky's Theorem, it is enough to show that

$$\lim_{t\to\infty} \mathbb{E}_\mu\left[\exp\left(-\frac{\theta}{t}\langle f,\tilde\varphi\rangle\langle\varphi,X_t\rangle\right)\Big|\zeta>t\right] = \frac{1}{1+\frac{1}{2}\theta\langle f,\tilde\varphi\rangle\langle\mathbb{V}[\varphi],\tilde\varphi\rangle},$$

or equivalently,

$$\lim_{t\to\infty}\frac{\mathbb{E}_\mu\left[1-\exp\left(-\theta\langle f,\tilde\varphi\rangle\langle\varphi,X_t\rangle/t\right)\right]}{\mathbb{P}_\mu\left(\langle 1,X_t\rangle>0\right)} = \frac{\frac{1}{2}\theta\langle f,\tilde\varphi\rangle\langle\mathbb{V}[\varphi],\tilde\varphi\rangle}{1+\frac{1}{2}\theta\langle f,\tilde\varphi\rangle\langle\mathbb{V}[\varphi],\tilde\varphi\rangle}. \tag{43}$$

Fix $\tilde\theta\in(0,\infty)$ and define

$$g_t(x) := \mathrm{e}^{-\frac{\tilde\theta\varphi(x)}{t}}.$$

We note that for $0\le s\le t$

$$\begin{aligned}\mathtt{u}_s[g_t](x) &= \mathbb{E}_{\delta_x}[1-\exp(-\tilde\theta\langle\varphi,X_s\rangle/t)]\\ &= \varphi(x)\mathbb{E}^{\varphi}_{\delta_x}\left[\frac{1-\exp(-\tilde\theta\langle\varphi,X_s\rangle/t)}{\langle\varphi,X_s\rangle}\right]\\ &= \frac{\tilde\theta\varphi(x)}{t}\mathbb{E}^{\varphi}_{\delta_x}\left[\frac{1-\exp(-\tilde\theta\langle\varphi,X_s\rangle/t)}{\tilde\theta\langle\varphi,X_s\rangle/t}\right].\end{aligned}$$

Since $x^{-1}(1-\mathrm{e}^{-x}) = \int_0^1\mathrm{e}^{-ux}\,\mathrm{d}u$ for $x>0$, this yields that

$$\mathtt{u}_s[g_t](x) = \frac{\tilde\theta\varphi(x)}{t}\int_0^1\mathbb{E}^{\varphi}_{\delta_x}[\exp(-u\tilde\theta\langle\varphi,X_s\rangle/t)]\mathrm{d}u \le \frac{\tilde\theta\varphi(x)}{t},\qquad \forall s\in[0,t]. \tag{44}$$

We have

$$\left|\frac{1}{ta_t[g_t]}-\frac{1}{ta_0[g_t]}-\langle\tfrac{1}{2}\mathbb{V}[\varphi],\tilde\varphi\rangle\right| = \frac{1}{t}\left|\int_0^t\frac{\langle\mathtt{A}[\mathtt{u}_s[g_t]],\tilde\varphi\rangle}{a_s[g_t]^2}\mathrm{d}s - \int_0^t\langle\tfrac{1}{2}\mathbb{V}[\varphi],\tilde\varphi\rangle\mathrm{d}s\right| \tag{45}$$

which we can bound above by

$$\frac{1}{t}\left|\int_0^t\frac{\frac{1}{2}\langle\mathbb{V}[a_s[g_t]\varphi]-\mathbb{V}[\mathtt{u}_s[g_t]],\tilde\varphi\rangle}{a_s[g_t]^2}\mathrm{d}s\right| + \frac{1}{t}\left|\int_0^t\frac{\langle\frac{1}{2}\mathbb{V}[\mathtt{u}_s[g_t]]-\mathtt{A}[\mathtt{u}_s[g_t]],\tilde\varphi\rangle}{a_s[g_t]^2}\mathrm{d}s\right|. \tag{46}$$

The second term above can be identified as $t^{-1}\int_0^t o(\|\mathtt{u}_s[g_t]\|^2)(a_s[g_t])^{-2}\mathrm{d}s$ by Lemma 2d, and so converges to 0 as $t\to\infty$, using (44) and Lemma 1. For the first term in (46), we note that

$$\left|\frac{\frac{1}{2}\langle\mathbb{V}[a_s[g_t]\varphi]-\mathbb{V}[\mathtt{u}_s[g_t]],\tilde\varphi\rangle}{a_s[g_t]^2}\right| = \left|\tfrac{1}{2}\langle\mathbb{V}[\varphi]-\mathbb{V}[\mathtt{u}_s[g_t]/a_s[g_t]],\tilde\varphi\rangle\right|$$

which, by Lemma 2b, is bounded above by a finite constant times

$$\left\|\frac{\mathtt{u}_s[g_t]}{a_s[g_t]}-\varphi\right\| \le \frac{C}{a_s[g_t]}\left(\mathrm{e}^{-\varepsilon s}\|1-g_t\| + \int_0^s\mathrm{e}^{-\varepsilon(s-r)}\|\mathtt{u}_r[g_t]\|\mathrm{d}r\right). \tag{47}$$

Finally using the lower bound Lemma 1 on $a_s[g_t]$, the fact that $\|1-g_t\| \leq C/t$ and the upper bound (44), we obtain that

$$\left|\frac{\frac{1}{2}\langle \mathbb{V}[a_s[g_t]\varphi] - \mathbb{V}[\mathtt{u}_s[g_t]], \tilde{\varphi}\rangle}{a_s[g_t]^2}\right| \leq C\left(\mathrm{e}^{-\varepsilon s} + \frac{1}{t}\right)$$

for some $C \in (0,\infty)$ and all $s \in (0,t)$. This means that the second term in (46) also converges to 0 as $t \to \infty$.

From (27) we have

$$\frac{1}{ta_0[g_t]} \to \frac{1}{\tilde{\theta}} \text{ as } t \to \infty,$$

and, hence, we obtain from (45) that

$$\frac{1}{ta_t[g_t]} \to \frac{\langle \mathbb{V}[\varphi], \tilde{\varphi}\rangle}{2} + \frac{1}{\tilde{\theta}} \tag{48}$$

as $t \to \infty$.

But as we see from (47), (44) and (48),

$$\left\|\frac{\mathtt{u}_t[g_t]}{a_t[g_t]} - \varphi\right\| \leq C\left(\mathrm{e}^{-\varepsilon t} + \frac{1}{t}\right)$$

and therefore

$$\lim_{t\to\infty} \frac{t\langle \mathtt{u}_t[g_t], \mu\rangle}{\langle \varphi, \mu\rangle} = \frac{\tilde{\theta}}{1 + \frac{1}{2}\tilde{\theta}\langle \mathbb{V}[\varphi], \tilde{\varphi}\rangle}.$$

This is precisely the proof of (43) when we take $\tilde{\theta} = \theta\langle f, \tilde{\varphi}\rangle$, where we have used also the statement of Theorem 1. □

8 Proof of Theorem 3

Once again, we give only the proof in the setting of non-local branching Markov processes, and leave the setting of non-local superprocesses with the assurance that the proof is almost verbatim and that the full proofs can be found in [35].

The proof in the non-local branching Markov process setting is quite long, and so we break it into several steps.

Step 1: We start by the considering necessary and sufficient condition for the existence of a limiting distribution. Recalling (20), assuming that $\lim_{t\to\infty}\langle \mathtt{v}_t[\theta f/t], \mu\rangle = 0$, it is enough to prove that the limit

$$\lim_{t\to\infty} \int_0^t \mathtt{H}[\mathtt{v}_s[\theta f/t]]\mathrm{d}s \tag{49}$$

converges if and only if

$$\mathtt{I}[\varphi] = \alpha\tilde{\mathcal{E}}[\langle \varphi, \tilde{\mathcal{Z}}\rangle] < \infty.$$

We start by looking for a functional upper bound for $\mathtt{v}_s[\theta f/t]$, for any $s \le t$. To this end, recall Δ_s and $\Delta := \sup_{s\ge 0}\Delta_s$ from (H2). We have

$$\mathtt{v}_s[\theta f/t](x) \le \mathtt{T}_s[\theta f/t](x) \le \left(\frac{\theta}{t}\langle f,\tilde\varphi\rangle + \frac{\theta}{t}\|f\|\Delta_s\right)\varphi(x) \le \left(\langle 1,\tilde\varphi\rangle + \Delta\right)\frac{\theta}{t}\|f\|\varphi(x),$$

where we used Jensen's inequality for the first inequality and (H2) for the second. This tells us in particular that

$$\lim_{t\to\infty}\sup_{s\le t}\langle \mathtt{v}_s[\theta f/t],\mu\rangle = 0. \tag{50}$$

To verify that $\mathtt{I}[\varphi] < \infty$ is a sufficient condition for (49) to hold, notice that $\mathtt{H}[\cdot]$ and $\mathtt{v}_t[\cdot]$ are monotone in the sense that if $f, g \in B^+(E)$ with $f \le g$, then $\mathtt{H}[f] \le \mathtt{H}[g]$ and $\mathtt{v}_t[f] \le \mathtt{v}_t[g]$ for all $t \ge 0$. Thus, by Monotone Convergence Theorem,

$$\begin{aligned}\lim_{t\to\infty}\int_0^t \mathtt{H}[\mathtt{v}_s[\theta f/t]]\mathrm{d}s &\le \lim_{t\to\infty} t\mathtt{H}\left[\left(\langle 1,\tilde\varphi\rangle + \Delta\right)\theta\|f\|\varphi/t\right]\\ &= \lim_{t\to\infty} t\alpha\tilde{\mathcal{E}}\left[1-\exp\Big(-\frac{1}{t}\left(\langle 1,\tilde\varphi\rangle+\Delta\right)\theta\|f\|\langle\varphi,\tilde{\mathcal{Z}}\rangle\Big)\right]\\ &= \left(\langle 1,\tilde\varphi\rangle + \Delta\right)\theta\|f\|\mathtt{I}[\varphi],\end{aligned}$$

so $\mathtt{I}[\varphi] < \infty$ is a sufficient condition for the existence of the limit distribution.

We also claim that $\mathtt{I}[\varphi] < \infty$ is a necessary condition for the convergence of (49). Indeed, for all $s > 0$, $x \in E$ and $g \in B^+(E)$, using $\mathrm{e}^{-y} \le 1 - y + \frac{1}{2}y^2$ if $y \ge 0$, we have

$$\mathrm{e}^{-\mathtt{v}_s[g](x)} \le 1 - \mathtt{T}_s[g](x) + \frac{1}{2}\mathtt{T}^{(2)}_s[g](x), \tag{51}$$

where $\mathtt{T}^{(2)}_s[g](x) = \mathbb{E}_{\delta_x}[\langle g, X_s\rangle^2]$ is well defined due to assumption (H1); see [18]. The asymptotic behaviour at criticality of the first two moments is known due to (H2) and Theorem 1 in [18]. From those results, we can state the following inequalities for the moments,

$$\mathtt{T}_s[g](x) \ge \left(\langle g,\tilde\varphi\rangle - \|g\|\Delta_s\right)\varphi(x), \tag{52}$$

$$\mathtt{T}^{(2)}_s[g](x) \le s\left(\langle g,\tilde\varphi\rangle^2\langle \mathbb{V}[\varphi],\tilde\varphi\rangle + \|g\|^2\Delta^{(2)}_s\right)\varphi(x). \tag{53}$$

where $\Delta^{(2)}_s = \sup_{x\in E,\, f\in B^+_1(E)}\left|s^{-1}\varphi(x)^{-1}\mathtt{T}^{(2)}_s\left[f\right](x) - \langle f,\tilde\varphi\rangle^2\langle\mathbb{V}[\varphi],\tilde\varphi\rangle\right| \to 0$ as $s \to \infty$. Applying the three inequalities to $g = t^{-1}h$ with $t \ge 1$ and $h \in B^+(E)$, we get

$$\mathrm{e}^{-\mathtt{v}_s[t^{-1}h](x)} \le 1 - \frac{1}{t}\left(\langle h,\tilde\varphi\rangle - \|h\|\Delta_s\right)\varphi(x) + \frac{s}{2t^2}\left(\langle h,\tilde\varphi\rangle^2\langle\mathbb{V}[\varphi],\tilde\varphi\rangle + \|h\|^2\Delta^{(2)}_s\right)\varphi(x). \tag{54}$$

For the terms involving Δ_s and $\Delta^{(2)}_s$, recall that $\lim_{s\to\infty}(\|h\|\Delta_s + \frac{1}{2}\|h\|^2\Delta^{(2)}_s) = 0$, so for all $\varepsilon \in (0, \frac{1}{4})$ there exists $t_0 \ge 1$ such that $\|h\|\Delta_s + \frac{1}{2}\|h\|^2\Delta^{(2)}_s \le \varepsilon\langle h,\tilde\varphi\rangle$ if $s \ge t_0$. For the other terms, choose a constant $k \in (0,1)$ small enough such that $k\langle 1, \tilde{\varphi}\rangle \langle \mathbb{V}[\varphi], \tilde{\varphi}\rangle \le 1$. Then it is clear that $\langle h, \tilde{\varphi}\rangle^2 \langle \mathbb{V}[\varphi], \tilde{\varphi}\rangle \le \langle h, \tilde{\varphi}\rangle$ if $\|h\| \le k$. Combining this, we obtain

$$\begin{aligned}\exp(-\mathtt{v}_s[t^{-1}h](x)) &\le 1 + \frac{\left(\frac{1}{2}\langle h, \tilde{\varphi}\rangle^2 \langle \mathbb{V}[\varphi], \tilde{\varphi}\rangle - \langle h, \tilde{\varphi}\rangle\right) + \left(\|h\|\Delta_s + \frac{1}{2}\|h\|^2\Delta_s^{(2)}\right)}{t}\varphi(x) \\ &\le 1 + \frac{-\frac{1}{2}\langle h, \tilde{\varphi}\rangle + \varepsilon\langle h, \tilde{\varphi}\rangle}{t}\varphi(x) \le 1 - \frac{\langle h, \tilde{\varphi}\rangle \varphi(x)}{4t},\end{aligned}$$

for $t \ge s \ge t_0$ and $h \in B^+(E)$ with $\|h\| \le k$. Let us use this final expression. For all $t \ge s \ge t_0$ and $f \in B^+(E)$, we have $t^{-1}(k \wedge \theta f) \le t^{-1}\theta f$ and

$$\exp(-\mathtt{v}_s[t^{-1}\theta f](x)) \le \exp(-\mathtt{v}_s[t^{-1}(k \wedge \theta f)](x)) \le 1 - \frac{\langle k \wedge \theta f, \tilde{\varphi}\rangle \varphi(x)}{4t},$$

which implies

$$\mathtt{v}_s[\theta f/t](x) \ge \frac{\langle k \wedge \theta f, \tilde{\varphi}\rangle \varphi(x)}{4t}.$$

Therefore,

$$\int_0^t \mathtt{H}[\mathtt{v}_s[\theta f/t]]\mathrm{d}s \ge \int_0^{t_0} \mathtt{H}[\mathtt{v}_s[\theta f/t]]\mathrm{d}s + \int_{t_0}^t \mathtt{H}\left[\frac{\langle k \wedge \theta f, \tilde{\varphi}\rangle \varphi}{4t}\right] \mathrm{d}s.$$

The first term on the right-hand side converges to 0 and the second tends to $\langle k \wedge \theta f, \tilde{\varphi}\rangle \mathtt{I}[\varphi]/4$ again by monotone convergence. This shows that $\mathtt{I}[\varphi] < \infty$ is also a necessary condition.

Step 2: Next, we compute the explicit limit distribution assuming $\mathtt{I}[\varphi] < \infty$. Recalling (26), our starting point is the expression

$$\begin{aligned}\mathbb{E}_\mu\left[\exp(-\langle \theta f, Y_t\rangle/t)\right] &= \mathbb{E}_\mu\left[\exp(-\langle \theta f, X_t\rangle/t)\right] \\ &\quad \times \exp\left(-\int_0^t \alpha\tilde{\mathcal{E}}\left[1 - \prod_{i=1}^{\tilde{N}}\left(1 - \mathtt{u}_s[\mathrm{e}^{-\theta f/t}](x_i)\right)\right]\mathrm{d}s\right)\end{aligned}$$

We are interested in the second term on the right hand side since, thanks to (50), $\mathbb{E}_\mu[\mathrm{e}^{-\theta\langle f, X_t\rangle/t}] \to 1$ as $t \to \infty$. Let us fix $f \in B^+(E)$ and $\theta \in (0,\infty)$ and, recalling (27), we choose $t_0 > 0$ large enough such that $0 \le a_s[\mathrm{e}^{-\theta f/t}]\varphi(x) \le \langle 1 - \mathrm{e}^{-\theta f/t}, \tilde{\varphi}\rangle\|\varphi\| \le \theta\langle f, \tilde{\varphi}\rangle\|\varphi\|/t \le 1$ for all $t \ge t_0$ and $s \in [0,t]$. Then

$$\begin{aligned}&\int_0^t \alpha\tilde{\mathcal{E}}\left[1 - \prod_{i=1}^{\tilde{N}}\left(1 - \mathtt{u}_s[\mathrm{e}^{-\theta f/t}](x_i)\right)\right]\mathrm{d}s \\ &\quad = \int_0^t \alpha\tilde{\mathcal{E}}\left[\prod_{i=1}^{\tilde{N}}\left(1 - a_s[\mathrm{e}^{-\theta f/t}]\varphi(x_i)\right) - \prod_{i=1}^{\tilde{N}}\left(1 - \mathtt{u}_s[\mathrm{e}^{-\theta f/t}](x_i)\right)\right]\mathrm{d}s \\ &\quad\quad + \int_0^t \alpha\tilde{\mathcal{E}}\left[1 - \prod_{i=1}^{\tilde{N}}\left(1 - a_s[\mathrm{e}^{-\theta f/t}]\varphi(x_i)\right)\right]\mathrm{d}s. \qquad (55)\end{aligned}$$

The first term on the right hand side of (55) converges to 0 as $t \to \infty$. Indeed, using the deterministic inequality

$$\left|\prod_{i=1}^{n}(1-z_i)-\prod_{i=1}^{n}(1-y_i)\right| \le \sum_{i=1}^{n}|z_i-y_i|, \qquad z_i, y_i \in [0,1],\ n\in\mathbb{N},$$

we see that an upper bound for said term is

$$\int_0^t \alpha\tilde{\mathcal{E}}\left[\langle\left|\mathtt{u}_s[\mathrm{e}^{-\theta f/t}]-a_s[\mathrm{e}^{-\theta f/t}]\varphi\right|, \tilde{\mathcal{Z}}\rangle\right]\mathrm{d}s \le \alpha\tilde{\mathcal{E}}\left[\langle\varphi,\tilde{\mathcal{Z}}\rangle\right]\int_0^t\left\|\frac{\mathtt{u}_s[\mathrm{e}^{-\theta f/t}]}{\varphi}-a_s[\mathrm{e}^{-\theta f/t}]\right\|\mathrm{d}s.$$

The statement follows from

$$\begin{aligned}\left\|\frac{\mathtt{u}_s[\mathrm{e}^{-\theta f/t}]}{\varphi}-a_s[\mathrm{e}^{-\theta f/t}]\right\| &\le C\left(\mathrm{e}^{-\varepsilon s}\|\tfrac{\theta}{t}f\|+\int_0^s\mathrm{e}^{-\varepsilon(s-r)}\|\mathtt{T}_r[\tfrac{\theta}{t}f]\|^2\mathrm{d}r\right)\\ &\le C\left(\mathrm{e}^{-\varepsilon s}\frac{\theta}{t}\|f\|+\frac{\theta^2}{t^2}\|f\|^2\frac{1}{\varepsilon}\right),\end{aligned}\tag{56}$$

where we used Lemma 3 and remind the reader that the constant $C>0$ can change its value in the second inequality.

Now, for the second term on the right-hand side of (55), we will show that

$$\begin{aligned}\lim_{t\to\infty}\int_0^t\alpha\tilde{\mathcal{E}}&\left[1-\prod_{i=1}^{\tilde{N}}\left(1-a_s[\mathrm{e}^{-\theta f/t}]\varphi(x_i)\right)\right]\mathrm{d}s\\ &=\frac{2\mathtt{I}[\varphi]}{\langle\mathbb{V}[\varphi],\tilde{\varphi}\rangle}\log\left(1+\frac{1}{2}\theta\langle f,\tilde{\varphi}\rangle\langle\mathbb{V}[\varphi],\tilde{\varphi}\rangle\right),\end{aligned}\tag{57}$$

which will then conclude the proof.

Step 3: In this step we will convert the desired limit (57) to an easier to handle form. It is not hard to see that

$$\begin{aligned}&\tilde{\mathcal{E}}\left[1-\prod_{i=1}^{\tilde{N}}\left(1-a_s[\mathrm{e}^{-\theta f/t}]\varphi(x_i)\right)\right]\\ &\quad=\tilde{\mathcal{E}}\left[\sum_{i=1}^{\tilde{N}}a_s[\mathrm{e}^{-\theta f/t}]\varphi(x_i)\int_0^1\prod_{\substack{j=1\\ j\neq i}}^{\tilde{N}}\left(1-za_s[\mathrm{e}^{-\theta f/t}]\varphi(x_j)\right)\mathrm{d}z\right]\end{aligned}$$

by Taylor's remainder theorem, so that

$$\begin{aligned}&\tilde{\mathcal{E}}\left[1-\prod_{i=1}^{\tilde{N}}\left(1-a_s[\mathrm{e}^{-\theta f/t}]\varphi(x_i)\right)\right]\\ &=\tilde{\mathcal{E}}\left[\langle a_s[\mathrm{e}^{-\theta f/t}]\varphi,\tilde{\mathcal{Z}}\rangle\right]\\ &\qquad+\tilde{\mathcal{E}}\left[\sum_{i=1}^{\tilde{N}}a_s[\mathrm{e}^{-\theta f/t}]\varphi(x_i)\int_0^1\left(\prod_{\substack{j=1\\ j\neq i}}^{\tilde{N}}\left(1-za_s[\mathrm{e}^{-\theta f/t}]\varphi(x_j)\right)-1\right)\mathrm{d}z\right].\end{aligned}$$

Taking into account the deterministic inequalities

$$0 \le 1 - \prod_{i=1}^{n}(1 - z_i) \le \sum_{i=1}^{n} z_i, \qquad z_i \in [0,1],\ n \in \mathbb{N} \tag{58}$$

and $0 \le a_s[\exp(-\theta f/t)] \le \langle 1 - \exp(-\theta f/t), \tilde{\varphi}\rangle \le \theta\langle f, \tilde{\varphi}\rangle/t$, we get

$$\begin{aligned}
&\tilde{\mathcal{E}}\left[\sum_{i=1}^{\tilde{N}} a_s[\mathrm{e}^{-\theta f/t}]\varphi(x_i)\int_0^1 \left|\prod_{\substack{j=1\\ j\neq i}}^{\tilde{N}}\left(1 - z a_s[\mathrm{e}^{-\theta f/t}]\varphi(x_j)\right) - 1\right| \mathrm{d}z\right] \\
&\le \tilde{\mathcal{E}}\left[\sum_{i=1}^{\tilde{N}} a_s[\mathrm{e}^{-\theta f/t}]\varphi(x_i)\sum_{\substack{j=1\\ j\neq i}}^{\tilde{N}} a_s[\mathrm{e}^{-\theta f/t}]\varphi(x_j)\mathbf{1}_{\{\langle\varphi,\tilde{\mathcal{Z}}\rangle<\sqrt{t}\}}\right] + a_s[\mathrm{e}^{-\theta f/t}]\tilde{\mathcal{E}}\left[\langle\varphi,\tilde{\mathcal{Z}}\rangle\mathbf{1}_{\{\langle\varphi,\tilde{\mathcal{Z}}\rangle\ge\sqrt{t}\}}\right] \\
&\le \frac{\theta^2}{t^2}\langle f,\tilde{\varphi}\rangle^2\sqrt{t}\tilde{\mathcal{E}}\left[\langle\varphi,\tilde{\mathcal{Z}}\rangle\mathbf{1}_{\{\langle\varphi,\tilde{\mathcal{Z}}\rangle<\sqrt{t}\}}\right] + \frac{\theta^2}{t}\langle f,\tilde{\varphi}\rangle\tilde{\mathcal{E}}\left[\langle\varphi,\tilde{\mathcal{Z}}\rangle\mathbf{1}_{\{\langle\varphi,\tilde{\mathcal{Z}}\rangle\ge\sqrt{t}\}}\right]
\end{aligned}$$

Integrating the previous expression over $s \in [0,t]$, we see that it converges to 0. Consequently,

$$\lim_{t\to\infty}\int_0^t \alpha\tilde{\mathcal{E}}\left[1 - \prod_{i=1}^{\tilde{N}}\left(1 - a_s[\mathrm{e}^{-\theta f/t}]\varphi(x_i)\right)\right]\mathrm{d}s = \lim_{t\to\infty}\int_0^t \alpha\tilde{\mathcal{E}}\left[\langle a_s[\mathrm{e}^{-\theta f/t}]\varphi, \tilde{\mathcal{Z}},\rangle\right]\mathrm{d}s,$$

which means that in order to prove (57), it is sufficient to show that

$$\lim_{t\to\infty}\int_0^t a_s[\mathrm{e}^{-\theta f/t}]\mathrm{d}s = \frac{2}{\langle\mathbb{V}[\varphi],\tilde{\varphi}\rangle}\log\left(1 + \frac{1}{2}\theta\langle f,\tilde{\varphi}\rangle\langle\mathbb{V}[\varphi],\tilde{\varphi}\rangle\right). \tag{59}$$

Step 4: To prove (59), let us momentarily fix $f \in B^+(E)$ such that $\inf_{x\in E} f(x) > 0$. From the definition of $a_s[\cdot]$ in (27), in a similar spirit to (45) and (46), we have

$$\frac{1}{a_s[\mathrm{e}^{-\theta f/t}]} - \frac{1}{a_0[\mathrm{e}^{-\theta f/t}]} = \int_0^s \frac{\langle \mathtt{A}[\mathtt{u}_r[\mathrm{e}^{-\theta f/t}]],\tilde{\varphi}\rangle}{a_r[\mathrm{e}^{-\theta f/t}]^2}\mathrm{d}r = \langle\tfrac{1}{2}\mathbb{V}[\varphi],\tilde{\varphi}\rangle s + F_1[f](s,t) + F_2[f](s,t)$$

and using $a_0[\mathrm{e}^{-\theta f/t}] = \langle 1 - \mathrm{e}^{-\theta f/t},\tilde{\varphi}\rangle$ we easily obtain

$$a_s[\mathrm{e}^{-\theta f/t}] = \frac{1}{\langle 1 - \mathrm{e}^{-\theta f/t},\tilde{\varphi}\rangle^{-1} + \langle\frac{1}{2}\mathbb{V}[\varphi],\tilde{\varphi}\rangle s + F_1[f](s,t) + F_2[f](s,t)}, \tag{60}$$

where

$$F_1[f](s,t) = \int_0^s \frac{\langle \mathtt{A}[\mathtt{u}_r[\mathrm{e}^{-\theta f/t}]] - \frac{1}{2}\mathbb{V}[\mathtt{u}_r[\mathrm{e}^{-\theta f/t}]],\tilde{\varphi}\rangle}{a_r[\mathrm{e}^{-\theta f/t}]^2}\mathrm{d}r,$$

$$F_2[f](s,t) = \int_0^s \frac{\frac{1}{2}\langle\mathbb{V}[\mathtt{u}_r[\mathrm{e}^{-\theta f/t}]] - \mathbb{V}[a_r[\mathrm{e}^{-\theta f/t}]\varphi],\tilde{\varphi}\rangle}{a_r[\mathrm{e}^{-\theta f/t}]^2}\mathrm{d}r.$$

Next, we look at controlling $F_1[f](s,t)$ and $F_2[f](s,t)$. In what follows, the constant C may vary in value from line to line and formula to formula, however its value is not important other than it is strictly positive and finite. By Lemma 2b,

$$|F_2[f](s,t)| \le \int_0^s \frac{1}{2}\langle \left\| \mathbb{V}[\mathtt{u}_r[\mathrm{e}^{-\theta f/t}]/a_r[\mathrm{e}^{-\theta f/t}]] - \mathbb{V}[\varphi]\right\|, \tilde{\varphi}\rangle \mathrm{d}r \le C\int_0^s \left\| \frac{\mathtt{u}_r[\mathrm{e}^{-\theta f/t}]}{a_r[\mathrm{e}^{-\theta f/t}]} - \varphi\right\| \mathrm{d}r.$$

From [18], we know that there exists $C_1 > 0$ such that $\mathbb{E}^{\varphi}_{\delta_x}[\langle \varphi, X_r\rangle] \le C_1(1+r)$ (see [18]) for all $r \in [0,t]$ and hence, from Lemma 1

$$a_r[\mathrm{e}^{-\theta f/t}] \ge \frac{C}{C_1(1+r) + \|\varphi\| t(\theta \inf_{y\in E} f(y))^{-1}} \ge \frac{C/t}{1 + (\theta\inf_{y\in E} f(y))^{-1}},$$

for some $C > 0$. Combining this with (56), we see that

$$\left\| \frac{\mathtt{u}_r[\mathrm{e}^{-\theta f/t}]}{a_r[\mathrm{e}^{-\theta f/t}]} - \varphi\right\| \le C\left(1 + \frac{1}{\theta \inf_{y\in E} f(y)}\right)\left(\mathrm{e}^{-\varepsilon r}\theta\|f\| + \frac{\theta^2}{t}\|f\|^2\frac{1}{\varepsilon}\right) \tag{61}$$

and hence

$$\int_0^s \left\| \frac{\mathtt{u}_r[\mathrm{e}^{-\theta f/t}]}{a_r[\mathrm{e}^{-\theta f/t}]} - \varphi\right\| \mathrm{d}r \le \frac{C}{\varepsilon}\|f\|\left(\theta + \frac{1}{\inf_{y\in E} f(y)}\right)(1 + \theta\|f\|), \tag{62}$$

where the constant $C > 0$ may change from line to line. As a consequence we have the bound

$$|F_2[f](s,t)| \le C_{F_2}, \tag{63}$$

where $C_{F_2} > 0$ is a constant.

Applying Lemma 2d, for each $\epsilon > 0$ there exists $t_0 > 1$ large enough such that

$$|F_1[f](s,t)| \le \int_0^s \frac{\langle \|\mathtt{A}[\mathtt{u}_r[\mathrm{e}^{-\theta f/t}]] - \frac{1}{2}\mathbb{V}[\mathtt{u}_r[\mathrm{e}^{-\theta f/t}]]\|, \tilde{\varphi}\rangle}{a_r[\mathrm{e}^{-\theta f/t}]^2}\mathrm{d}r \le \epsilon\int_0^s \left\| \frac{\mathtt{u}_r[\mathrm{e}^{-\theta f/t}]}{a_r[\mathrm{e}^{-\theta f/t}]}\right\|^2 \mathrm{d}r \tag{64}$$

for all $s \in [0,t]$ and $t > t_0$. Using the triangle inequality,

$$\left\| \frac{\mathtt{u}_r[\mathrm{e}^{-\theta f/t}]}{a_r[\mathrm{e}^{-\theta f/t}]}\right\|^2 \le \left\| \frac{\mathtt{u}_r[\mathrm{e}^{-\theta f/t}]}{a_r[\mathrm{e}^{-\theta f/t}]} - \varphi\right\|^2 + \|\varphi\|^2$$

and hence, appealing to (62), we have back in (64) that

$$|F_1[f](s,t)| \le \epsilon(C_{F_1} + s) \tag{65}$$

for some constant $C_{F_1} > 0$.

With (63) and (65) in hand, the integral in (59) is bounded below and above (respectively depending on the $\pm$ sign) by

$$\int_0^t \frac{\mathrm{d}s}{\langle 1 - \mathrm{e}^{-\theta f/t}, \tilde{\varphi}\rangle^{-1} + \langle \frac{1}{2}\mathbb{V}[\varphi], \tilde{\varphi}\rangle s \pm (\epsilon(C_{F_1} + s) + C_{F_2})}$$
$$= \frac{1}{\langle \frac{1}{2}\mathbb{V}[\varphi], \tilde{\varphi}\rangle \pm \epsilon}\log\left(1 + \frac{\left(\langle \frac{1}{2}\mathbb{V}[\varphi], \tilde{\varphi}\rangle \pm \epsilon\right)t}{\langle 1 - \mathrm{e}^{-\theta f/t}, \tilde{\varphi}\rangle^{-1} \pm (\epsilon C_{F_1} + C_{F_2})}\right).$$

Taking limit as $t \to \infty$, we get

$$\frac{\log\left(1+\theta\langle f,\tilde{\varphi}\rangle\left(\langle\frac{1}{2}\mathbb{V}[\varphi],\tilde{\varphi}\rangle\pm\epsilon\right)\right)}{\langle\frac{1}{2}\mathbb{V}[\varphi],\tilde{\varphi}\rangle\pm\epsilon}.$$

Since ϵ is as small as we want, we achieve the desired limit as in (59).

Step 5: To complete the proof of (59), we want to remove the assumption that $f \in B^+(E)$ satisfies $\inf_{x\in E} f(x) > 0$. To do this, we define

$$\mathtt{F}[h] := \lim_{t\to\infty}\int_0^t a_s[\mathrm{e}^{-h/t}]\mathrm{d}s, \qquad h \in B^+(E).$$

The functional $\mathtt{F}$ is well defined because the integrand of the above integral is non-negative and $\int_0^t a_s[\mathrm{e}^{-h/t}]\mathrm{d}s \le \int_0^t\langle 1-\mathrm{e}^{-h/t},\tilde{\varphi}\rangle\mathrm{d}s \le \langle h,\tilde{\varphi}\rangle$. We claim that $\mathtt{F}$ is a continuous functional. Indeed, given $h, g \in B^+(E)$, and taking into account the deterministic inequality $|\mathrm{e}^{-x}-\mathrm{e}^{-y}| \le |x-y|$ for all $x, y \ge 0$, the claim follows from the fact that

$$\begin{aligned}|\mathtt{F}[h]-\mathtt{F}[g]| &\le \lim_{t\to\infty}\int_0^t\langle\mathbb{E}_{\delta_\cdot}\left[|\mathrm{e}^{-\langle h,X_s\rangle/t}-\mathrm{e}^{-\langle g,X_s\rangle/t}|\right],\tilde{\varphi}\rangle\mathrm{d}s\\ &\le \lim_{t\to\infty}\frac{1}{t}\int_0^t\langle\mathtt{T}_s[\|h-g\|],\tilde{\varphi}\rangle\mathrm{d}s = \langle 1,\tilde{\varphi}\rangle\|h-g\|.\end{aligned}$$

Now we show that (59) holds for any function $f \in B^+(E)$ satisfying $\inf_{x\in E} f(x) = 0$. By considering a sequence of functions $f_n = \max(n^{-1}, f)$, $n \in \mathbb{N}$, noting that $\inf_{x\in E} f_n(x) > 0$, $\lim_{n\to\infty}\|f_n - f\| = 0$ and

$$\mathtt{F}[\theta f_n] = \frac{2}{\langle\mathbb{V}[\varphi],\tilde{\varphi}\rangle}\log\left(1+\frac{1}{2}\theta\langle f_n,\tilde{\varphi}\rangle\langle\mathbb{V}[\varphi],\tilde{\varphi}\rangle\right), \qquad n \in \mathbb{N},$$

continuity gives us that $\mathtt{F}[\theta f] = \lim_{n\to\infty}\mathtt{F}[\theta f_n]$ and that (59) holds for $f \in B^+(E)$ with $\inf_{x\in E} f(x) = 0$.

To conclude Steps 2-5, Step 5 ensures that (59) holds, which ensures that (57) holds, which, in turn, from (55) gives us the limiting result

$$\mathbb{E}_\mu\left[\exp(-\theta\langle f,Y_t\rangle/t)\right] = \left(1+\theta\frac{1}{2}\langle f,\tilde{\varphi}\rangle\langle\mathbb{V}[\varphi],\tilde{\varphi}\rangle\right)^{-2\mathtt{I}[\varphi]/\langle\mathbb{V}[\varphi],\tilde{\varphi}\rangle}$$

as required. □

9 Proof of Theorem 4

As usual, we restrict ourselves to the setting of non-local branching Markov processes, noting that the non-local superprocess setting is almost verbatim (with $\mathtt{V}_t$ playing the role of $\mathtt{v}_t$), with a full proof available in [35].

From equations (19) and (20), it is clear that $Y_t \to Y_\infty$ weakly as $t \to \infty$ if, and only if, for all $f \in B^+(E)$, we have $\lim_{t\to\infty} \|\mathtt{v}_t[f]\| < \infty$ and

$$\int_0^\infty \mathtt{H}[\mathtt{v}_s[f]]\mathrm{d}s < \infty. \tag{66}$$

The strategy now is to prove the equivalence between (66) and the integral condition,

$$\int_0^{z_0} \frac{\mathtt{H}[z\varphi]}{z}\mathrm{d}z < \infty \text{ for some } z_0 > 0. \tag{67}$$

We proceed by finding two functions, one that bounds $\mathtt{v}_t[\cdot]$ above and one below. The necessesity and sufficiency of (67) follows, respectively, from the monotonicity of the immigration mechanism $\mathtt{H}[\cdot]$ applied to said bounding functions.

Let us start by showing that (67) is a sufficient condition. For all $t \geq 0$, $f \in B^+(E)$ and $x \in E$, we have

$$\mathtt{v}_t[f](x) \leq \mathtt{T}_t[f](x) \leq (\langle 1, \tilde{\varphi}\rangle + \Delta)\, \|f\|\mathrm{e}^{\lambda t}\varphi(x), \tag{68}$$

where the first inequality is due to Jensen's inequality and the second is thanks to (H2). As a consequence, $\lim_{t\to\infty} \|\mathtt{v}_t[f]\| = 0$ because $\lambda < 0$ by assumption. Moreover, with the change of variables $z = (\langle 1, \tilde{\varphi}\rangle + \Delta)\, \|f\|\mathrm{e}^{\lambda t}$, we get

$$\int_0^\infty \mathtt{H}[\mathtt{v}_t[f]]\mathrm{d}t \leq -\frac{1}{\lambda}\int_0^{(\langle 1,\tilde{\varphi}\rangle+\Delta)\|f\|} \frac{\mathtt{H}[z\varphi]}{z}\mathrm{d}z,$$

that is, (67) is a sufficient condition.

Let us now show that (67) is necessary. The counterpart inequalities of (52) and (53) at subcriticality (see [19]) are

$$\mathtt{T}_t[g](x) \geq (\langle g, \tilde{\varphi}\rangle - \|g\|\Delta_t)\, \mathrm{e}^{\lambda t}\varphi(x), \tag{69}$$

$$\mathtt{T}^{(2)}_t[g](x) \leq \left(L_2(g) + \|g\|^2\Delta^{(2)}_t\right) \mathrm{e}^{\lambda t}\varphi(x), \tag{70}$$

for all $t \geq 0$, $g \in B^+(E)$ and $x \in E$, where $\mathtt{T}^{(2)}_s[g](x) = \mathbb{E}_{\delta_x}[\langle g, X_s\rangle^2]$, Δ_t is defined in (H2) and $\Delta^{(2)}_t = \sup_{x\in E,\, f\in B^+_1(E)} |\mathrm{e}^{-\lambda t}\varphi(x)^{-1}\mathtt{T}^{(2)}_t\, [f]\,(x) - L_2(f)|$ with

$$L_2(g) = \langle g^2, \tilde{\varphi}\rangle + \int_0^\infty \mathrm{e}^{-\lambda s}\langle \mathbb{V}[\mathtt{T}_s[g]], \tilde{\varphi}\rangle\mathrm{d}s. \tag{71}$$

Using the upper bound for the expectation semigroup in (68), we obtain $L_2(g) \leq [\langle 1, \tilde{\varphi}\rangle - \lambda^{-1}(\langle 1, \tilde{\varphi}\rangle + \Delta)^2\langle \mathbb{V}[\varphi], \tilde{\varphi}\rangle]\|g\|^2$. That is, there exists a constant $C > 0$ such that

$$L_2(g) \leq C\|g\|^2, \qquad \text{for all } g \in B^+(E). \tag{72}$$

Combining (51), (69), (70) and (72), we have

$$\mathrm{e}^{-\mathtt{v}_t[g](x)} \leq 1 - (\langle g, \tilde{\varphi}\rangle - \|g\|\Delta_t)\, \mathrm{e}^{\lambda t}\varphi(x) + \frac{1}{2}\left(C + \Delta^{(2)}\right) \|g\|^2\mathrm{e}^{\lambda t}\varphi(x),$$

where $\Delta^{(2)} = \sup_{t\geq 0}\Delta_t^{(2)}$, which is finite under (H1); see [19]. Let us apply this expression to $g = \mathrm{e}^{\lambda t} f \leq f$, with $f \in B^+(E)$, then

$$\mathrm{e}^{-\mathtt{v}_t[f](x)} \leq \mathrm{e}^{-\mathtt{v}_t[\mathrm{e}^{\lambda t} f](x)} \leq 1 - (\langle f, \tilde{\varphi}\rangle - \|f\|\Delta_t)\,\mathrm{e}^{2\lambda t}\varphi(x) + \frac{C + \Delta^{(2)}}{2}\|f\|^2\mathrm{e}^{3\lambda t}\varphi(x).$$

As $\Delta_t \to 0$ as $t \to \infty$, we know that for each $f \in B^+(E)$ there exists $t_0 > 0$ large enough such that for each $t \geq t_0$,

$$\|f\|\Delta_t \leq \frac{1}{3}\langle f, \tilde{\varphi}\rangle \qquad \text{and} \qquad \frac{C + \Delta^{(2)}}{2}\|f\|^2\mathrm{e}^{3\lambda t} \leq \frac{1}{3}\langle f, \tilde{\varphi}\rangle\mathrm{e}^{2\lambda t}.$$

Finally, for all $t \geq t_0$, we achieve our desired lower bound,

$$\mathrm{e}^{-\mathtt{v}_t[f](x)} \leq 1 - \frac{1}{3}\langle f, \tilde{\varphi}\rangle\mathrm{e}^{2\lambda t}\varphi(x) \quad \Longrightarrow \quad \mathtt{v}_t[f](x) \geq \frac{1}{3}\langle f, \tilde{\varphi}\rangle\mathrm{e}^{2\lambda t}\varphi(x).$$

Under the change of variables $z = \frac{1}{3}\langle f, \tilde{\varphi}\rangle\mathrm{e}^{2\lambda t}$, we see that (67) is a necessary condition for (66) due to the fact that

$$\int_{t_0}^{\infty} \mathtt{H}[\mathtt{v}_t[f]]\mathrm{d}t \geq -\frac{1}{2\lambda}\int_0^{\frac{1}{3}\langle f,\tilde{\varphi}\rangle\mathrm{e}^{2\lambda t_0}} \frac{\mathtt{H}[z\varphi]}{z}\mathrm{d}z.$$

The last step is the equivalence between (67) and the log moment condition (16). This follows by Tonelli's theorem,

$$\int_0^{z_0} \frac{\mathtt{H}[z\varphi]}{z}\mathrm{d}z = \alpha\tilde{\mathcal{E}}\left[\int_0^{z_0\langle\varphi,\tilde{\mathcal{Z}}\rangle} \frac{1 - \mathrm{e}^{-y}}{y}\mathrm{d}y\right] < \infty,$$

for some $z_0 > 0$. The latter holds iff $\tilde{\mathcal{E}}[\log(1 + \langle\varphi, \tilde{\mathcal{Z}}\rangle)] < \infty$. □

Remark 4. Roughly speaking, paraphrasing Corollary 9.53 in [34] in the setting of non-local Markov branching processes, it states that, for $\mathtt{v}_t[1](x)$, uniform (in $x \in E$) exponential bounds (in time) are needed to ensure that $\tilde{\mathcal{E}}[\log(1 + \langle\varphi, \tilde{\mathcal{Z}}\rangle)] < \infty$ is a necessary and sufficient condition for a stationary distribution to exist. We note that the proof above essentially verifies such exponential temporal bounds starting at a certain instant $t_0 > 0$, albeit they are not uniform in $x \in E$.

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