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Asymptotic Radial Speed of the Support of Supercritical Branching Brownian Motion and Super-Brownian Motion in R^{d}

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Abstract. It has long been known that the left-most or right-most particle in a one dimensional dyadic branching Brownian motion with constant branching rate $\beta > 0$ has almost sure asymptotic speed $\sqrt{2\beta}$, (cf. [18]). Recently similar results for higher dimensional branching Brownian motion and super-Brownian motion have also been established but in the weaker sense of convergence in probability; see [20] and [8]. In this short note we confirm the 'folklore' for higher dimensions and establish an asymptotic radial speed of the support of the latter two processes in the almost sure sense. The proofs rely on *Pinsky's local extinction criterion*, martingale convergence, projections of branching processes from higher to one dimensional spaces together with simple geometrical considerations.

KEYWORDS: spatial branching processes, super-Brownian motion, branching Brownian motion, local extinction

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1. Introduction

In this short note we shall consider X as either a $(\triangle/2, \beta; \mathbf{R}^d)$ *binary* branching Brownian motion, with β a positive constant, or a $(\triangle/2, \beta, \alpha; \mathbf{R}^d)$ super-Brownian motion, with β and α positive constants, together with probabilities $\{\mathsf{P}_\mu : \mu \in \mathcal{M}_c(\mathbf{R}^d)\}$. Here $\mathcal{M}_c(\mathbf{R}^d)$ denotes the space of Borel measures on $\mathbf{R}^d$ which are finite and compactly supported in $\mathbf{R}^d$.

The branching Brownian motion $(\triangle/2, \beta; \mathbf{R}^d)$ under measure P_μ with $\mu \in \mathcal{M}_c(\mathbf{R}^d)$ is constructed as follows. Start with an initial (finite) configuration of points represented by μ in $\mathbf{R}^d$. From each point an independent Brownian motion is initiated. Each particle diffuses until an independent exponentially distributed time with mean $1/\beta$ at which point it undergoes fission producing two particles. The two particles move and reproduce independently starting from their point of creation in a way that is stochastically identical to that of their parent and so on. This process has an almost surely finite number of particles at any given time; that is to say it is $\mathcal{M}_c(\mathbf{R}^d)$-valued. A simple generalization of this process in one dimension that we shall also work with is the $(\triangle/2 - \gamma d/dx, \beta; \mathbf{R})$ branching process for $\gamma \in \mathbf{R}$ and $\beta > 0$. This process has virtually the same definition but for the fact that particles diffuse as a standard Brownian motion with drift $-\gamma t$. For any of the aforementioned particle processes, since each particle has exactly two offspring, they are supercritical (in the traditional sense of Galton−Watson processes) and survive with probability one.

The super-Brownian motion $(\triangle/2, \beta, \alpha; \mathbf{R}^d)$ under P_μ with $\mu \in \mathcal{M}_c(\mathbf{R}^d)$ arises as the weak limit of an appropriately rescaled, branching Brownian motion in which particles have a random number of offspring with finite variance and mean which is greater than 1. Pinsky [21] gives specific details of the limiting procedure. It suffices to note for our purposes that the resulting process (X, P_μ) is valued in $\mathcal{M}_c(\mathbf{R}^d)$ with $X_0 = \mu$ and has the property that for each g belonging to the cone of non-negative, continuous, bounded functions, $\mathsf{E}_\mu(\exp\langle -g, X_t\rangle) = \exp\langle -u(\cdot, t), \mu\rangle$ where $\langle \cdot, \cdot \rangle$ is the usual inner product and u is the minimal non-negative solution to the PDE

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\triangle u + \beta u - \alpha u^2 \quad \text{for } (x,t) \in \mathbf{R}^d \times \mathbf{R}^+, \\ u(x,0) &= g(x) \quad \text{for } x \in \mathbf{R}^d. \end{aligned} \tag{1.1}$$

In one dimension we shall also talk of a $(\triangle/2 - \gamma d/dx, \beta, \alpha; \mathbf{R}^d)$ superprocess for $\beta > 0$ and $\gamma \in \mathbf{R}$. This process satisfies dynamics similar to (1.1) with the exception that $\triangle/2$ is replaced by $\triangle/2 - \gamma d/dx$. In any of the aforementioned superprocesses, the fact that $\beta > 0$ qualifies this process as supercritical since, contrary to the case $\beta \leq 0$, the process may survive with positive probability. In contrast, the definition of supercritical super-Brownian motion we have given here has binary splitting and hence this ensures there is survival with probability one. Given $\mu \in \mathcal{M}_c(\mathbf{R}^d)$, we have for super-Brownian motion

$$\mathsf{P}_\mu(X \text{ survives}) = 1 - \exp\{-\beta\|\mu\|/\alpha\} \tag{1.2}$$

where $\|\mu\| = \langle 1, \mu\rangle$ is the total mass of μ; see [9] for details.

In the sequel, when speaking of initial state $\mu \in \mathcal{M}_c(\mathbf{R}^d)$ for either class of processes, it shall be understood that we are excluding the case $\mu = 0$.

2. Asymptotic radial speed

This paper concerns the question how fast the support of these processes spreads in time. In order to describe how this will happen, we need the following definitions.

Definition 2.1. Let B_r be the closed ball of radius $r > 0$ centered at the origin. We define the *radius of the support* $M_t = \inf\{r > 0 : X_t(B_r^c) = 0\}$.

Definition 2.2. For one dimensional processes, we shall also talk about the right most extreme of the support, $R_t = \inf\{y \in \mathbf{R} : X_t(y, \infty) = 0\}$.

It is well known that when $d = 1$ and X is a branching Brownian motion, there exists an almost sure asymptotic speed for extreme particles (cf. [3,4,18]). This is captured in the following proposition.

Proposition 2.1. *If X is a supercritical branching Brownian motion in one dimension and $\mu \in \mathcal{M}_c(\mathbf{R})$, then $R_t/t \to \sqrt{2\beta}$ P_μ-almost surely.*

A generalized version of this result for both types of process in any Euclidian dimension reads as follows.

Theorem 2.1. *Suppose that X is either the $(\triangle/2, \beta; \mathbf{R}^d)$ branching Brownian motion or the $(\triangle/2, \beta, \alpha; \mathbf{R}^d)$ super-Brownian motion. Assume that $\mu \in \mathcal{M}_c(\mathbf{R}^d)$, then $M_t/t \to \sqrt{2\beta}$ P_μ-almost surely on the survival set of X.*

Note that in the previous statement and in forthcoming statements, the qualifier "on the survival set of X" is superfluous for the case of branching Brownian motion but is included anyway in order to avoid making two separate statements.

Although Theorem 2.1 would seem to be an intuitively obvious and natural extension to Proposition 2.1, the existence in the literature of this *strong law of large numbers* for the radius of the support is essentially folklore. Recently weaker versions of this result have appeared in related problems. Specifically in [8] for branching Brownian motion where the convergence occurs in probability and in [20] for super-Brownian motion where again convergence in probability is proved.

The aim of this note is, in a certain sense, to round off these existing results by proving Theorem 2.1. For the most part, the proof is based on four principle ideas. The first is *Pinsky's local extinction criterion* (proved for superdiffusions in Theorem 6 in [21] and for branching particle diffusions in Theorem 3 in [10]). The second is that a natural martingale associated with the exponential function contains information about the asymptotic behaviour of the processes in question. The third is the fact that one-dimensional marginal measures of branching processes correspond again to those of one-dimensional branching processes. Finally the fourth is a fundamental concept from Euclidian geometry concerning the approximation of hyperspheres by polytopes.

We end this section by presenting the first two of the aforementioned principle ideas.

Theorem 2.2 (Pinsky's local extinction criterion). *Suppose* $\mu \in \mathcal{M}_c(\mathbf{R})$. *Let* $\gamma \in \mathbf{R}$ *and take* X^γ *as either the* $(\triangle/2 - \gamma d/dx, \beta, \mathbf{R})$ *branching process or the* $(\triangle/2 - \gamma d/dx, \beta, \alpha, \mathbf{R})$ *superprocess. Denote the generalized principle eigenvalue*

$$\begin{aligned}\lambda_c &= \lambda_c(\triangle/2 - \gamma d/dx + \beta)\\ &= \inf\left\{\lambda \in \mathbf{R} : (\triangle/2 - \gamma d/dx + \beta - \lambda)h = 0 \text{ for some } h > 0 \text{ in } C^2(\mathbf{R})\right\}.\end{aligned}$$

Let $I \subset \mathbf{R}$ *be any bounded interval of positive length. There exists a* P_μ*-almost surely finite* $T = T(I,\mu)$ *such that*

$$\mathsf{P}_\mu\left(X^\gamma_t(I) = 0 \text{ for all } t \geq T\right) = 1$$

if and only if $\lambda_c \leq 0$.

Remark 2.1. Since all positive eigenfunctions of the operator $(\triangle/2 - \gamma d/dx + \beta)$ on $\mathbf{R}$ are exponential, it follows from a straightforward calculation that

$$\lambda_c(\triangle/2 - \gamma d/dx + \beta) = \beta - \frac{\gamma^2}{2}$$

and hence the criterion $\lambda_c \leq 0$ in Theorem 2.2 is equivalent to $|\gamma| \geq \sqrt{2\beta}$.

Theorem 2.3. *We have that*

(i) *the process*

$$\left\{W_t(\gamma) := \exp\{-(\beta + \gamma^2/2)t\}\langle \exp\{\gamma\cdot\}, X\rangle : t \geq 0\right\} \tag{2.1}$$

is a P_μ*-martingale with respect to the natural filtration* $\mathcal{F}_t = \sigma(X_u : u \leq t)$,

(ii) *when* $0 < |\gamma| < \sqrt{2\beta}$ *its limit, say* $W(\gamma)$, *is such that*

$$\mathsf{P}_\mu(W(\gamma) > 0 \mid X \text{ survives}) = 1. \tag{2.2}$$

Remark 2.2. The last theorem is well known for branching Brownian motion, even in a stronger form. Namely that, in addition to the above, $W(\gamma)$ is an $L^p(\mathsf{P}_\mu)$ limit (for some $p \in [1,2]$) if and only if $|\gamma| < \sqrt{(2\beta/p)}$ and otherwise $W(\gamma)$ is identically equal to zero; see [19]. For super-Brownian motion, the results in Theorem 2.3, although clearly justifiable in an intuitive way given their validity for branching Brownian motion, are absent from the literature. In principle, the stronger results stated above for branching Brownian motion should also be true for super-Brownian motion. We leave this issue for another occasion preferring to remain focused on our main objective. Also in this spirit, we give the proof of Theorem 2.3 for the case of super-Brownian motion in the Appendix.

3. Proof of Theorem 2.1

In order to prove Theorem 2.1 we shall first address what happens in one dimension and then re-cycle some of this information for the proof of higher dimensions. Our proofs will be generic for both branching Brownian motion and super-Brownian motion (although in one dimension, as the experienced reader may already suspect, the proof for branching Brownian motion is of course a simple extension of Proposition 2.1 and could be performed in several different easier ways anyway). Recall then that in the sequel, for any $\gamma \in \mathbf{R}$ we shall understand X^γ to be either the $(\triangle/2 - \gamma d/dx, \beta; \mathbf{R})$ branching Brownian motion or the $(\triangle/2 - \gamma d/dx, \beta, \alpha; \mathbf{R})$ super-Brownian motion. Also as before we shall always take $\mu \in \mathcal{M}_c(\mathbf{R}^d)$.

Proof of Theorem 2.1 *for* $d = 1$. For any $\gamma \geq 0$ and any bounded interval of positive length $I \subset \mathbf{R}$, we have

$$\begin{aligned}&\mathsf{P}_\mu\left(X_t(I + \gamma t) = 0 \text{ for all sufficiently large } t\right)\\ &\qquad = \mathsf{P}_\mu\left(X_t^\gamma(I) = 0 \text{ for all sufficiently large } t\right).\end{aligned}$$

From Theorem 2.2 and Remark 2.1 it follows that when $|\gamma| \geq \sqrt{2\beta}$,

$$\mathsf{P}_\mu\left(X_t(I + \gamma t) = 0 \text{ for all sufficiently large } t\right) = 1. \tag{3.1}$$

This leads us to conclude that

$$\mathsf{P}_\mu\left(\limsup_{t\uparrow\infty} \frac{M_t}{t} \leq \sqrt{2\beta} \text{ or } \limsup_{t\uparrow\infty} \frac{M_t}{t} = \infty \mid X \text{ survives}\right) = 1,$$

that is to say, the radius of the support travels no faster than linearly with maximum speed $\sqrt{2\beta}$ or superlinearly. We can upgrade this last equality to

$$\mathsf{P}_\mu\left(\limsup_{t\uparrow\infty} \frac{M_t}{t} \leq \sqrt{2\beta}\right) = 1 \tag{3.2}$$

as follows.

We proceed by contradiction. Suppose that $\mathsf{P}_\mu(\limsup_{t\uparrow\infty} M_t/t = \infty) > 0$. Suppose further that

$$\mathsf{P}_\mu\left(\liminf_{t\uparrow\infty} \frac{M_t}{t} \leq c, \limsup_{t\uparrow\infty} \frac{M_t}{t} = \infty\right) > 0$$

for some constant $c > 0$. The latter two assumptions (together with the fact that $R_t \leq M_t$) imply that there exists a random sequence of times $0 < t_1 < T_1 < t_2 < T_2 < \ldots$ such that for all $n \geq 1$

$$R_{t_n} \leq \left(c \vee \sqrt{2\beta}\right)t_n \quad \text{and} \quad R_{T_n} \geq \left(\left(c \vee \sqrt{2\beta}\right) + 1\right)T_n$$

with positive P_μ-probability. Since our class of branching processes have (versions with) continuous paths, this in turn implies that for any bounded interval of positive length $I \subset \mathbf{R}$, $\mathsf{P}_\mu(X_t^{c\vee\sqrt{2\beta}}(I) > 0 \text{ i.o.}) > 0$. However, this contradicts the joint conclusion of Theorem 2.2 and Remark 2.1.

In conclusion, if we assume that $\mathsf{P}_\mu(\limsup_{t\uparrow\infty} M_t/t = \infty) > 0$ then we are also forced to assume further that $\mathsf{P}_\mu(\liminf_{t\uparrow\infty} M_t/t = \infty) > 0$ and hence $\mathsf{P}_\mu(\lim_{t\uparrow\infty} M_t/t = \infty) > 0$. However this would contradict [20] for super-Brownian motion and [8] for branching Brownian motion, both of which concluded that M_t/t converges in probability to $\sqrt{2\beta}$ on the survival set. Therefore the original assumption that $\mathsf{P}_\mu(\limsup_{t\uparrow\infty} M_t/t = \infty) > 0$ is false and (3.2) is proved.

Now let us deduce that

$$\liminf_{t\uparrow\infty} \frac{M_t}{t} \geq \sqrt{2\beta} \tag{3.3}$$

P_μ-almost surely on the survival set of X. We make use of the following observation which is essentially taken from [13]: for $0 < \varepsilon < \sqrt{2\beta}/2$ and $\gamma = \sqrt{2\beta} - \varepsilon$

$$\limsup_{t\uparrow\infty} \exp\{-(\beta + \gamma^2/2)t\} \left\langle \exp\{\gamma\cdot\}\mathbf{1}\{(\cdot \leq (\gamma - \varepsilon)t)\}, X_t \right\rangle = 0 \tag{3.4}$$

and hence

$$\lim_{t\uparrow\infty} \exp\{-(\beta + \gamma^2/2)t\} \left\langle \exp\{\gamma\cdot\}\mathbf{1}\{(\cdot > (\gamma - \varepsilon)t)\}, X_t \right\rangle = W(\gamma) \tag{3.5}$$

P_μ-almost surely on the survival set of X. The proof of (3.4) follows from the fact that

$$e^{\gamma x}\mathbf{1}\{(x \leq (\gamma - \varepsilon)t)\} \leq e^{(\gamma-\varepsilon)x} e^{\varepsilon(\gamma-\varepsilon)t},$$

so that

$$\exp\{-(\beta + \gamma^2/2)t\} \left\langle \exp\{\gamma\cdot\}\mathbf{1}\{(\cdot \leq (\gamma - \varepsilon)t)\}, X_t \right\rangle \leq \exp\{-\varepsilon^2 t/2\}\, W_t(\gamma - \varepsilon),$$

together with the fact that $W_t(\gamma - \varepsilon)$ converges almost surely. Since ε may be taken arbitrarily close to zero in (3.5), we may conclude from (2.2) that

$$\liminf_{t\uparrow\infty} \frac{R_t}{t} \geq \sqrt{2\beta} \tag{3.6}$$

P_μ-almost surely on the survival set of X and hence that (3.3) is true.

Now combining (3.2) with (3.3) we have that $\lim_{t\uparrow\infty} M_t/t = \sqrt{2\beta}$, P_μ-almost surely on the survival set of X. □

Remark 3.1. Reconsideration of the above arguments shows that in fact for all $\gamma \geq \sqrt{2\beta}$, there exists a P_μ-almost surely finite random time T^γ such that for any $x > 0$,

$$\mathsf{P}_\mu\left(X(x + t\gamma, \infty) = 0 \text{ for all } t \geq T^\gamma\right) = 1.$$

Proof of Theorem 2.1 *for* $d \geq 2$. As usual X denotes either a supercritical branching Brownian motion or a supercritical super-Brownian motion. Let $\mathbf{e}$ be any vector on the sphere of unit radius, $S_1 = \{\mathbf{e} \in \mathbf{R}^d : |\mathbf{e}| = 1\}$, and define $X^{\mathbf{e}}$ to be the projection of X onto the one dimensional space in the direction of $\mathbf{e}$. That is to say, letting $O^{\mathbf{e}}$ be the $\mathbf{R}^{d-1}$ space orthogonal to $\mathbf{e}$, for any Borel $A \in \mathbf{R}$,

$$X_t^{\mathbf{e}}(A) = X_t(A \times O^{\mathbf{e}}).$$

The law of $X^{\mathbf{e}}$ under P_μ is identical to that of a one dimensional branching Brownian motion or super-Brownian motion (respectively with the type of process that X is) under $\mathsf{P}_{\mu^{\mathbf{e}}}$ where $\mu^{\mathbf{e}} \in \mathcal{M}_c(\mathbf{R})$ satisfies $\mu^{\mathbf{e}}(A) = \mu(A \times O^{\mathbf{e}})$ for each Borel $A \in \mathbf{R}$. Note that $X^{\mathbf{e}}$ survives if and only if X survives.

Now define for each $\mathbf{e} \in S_1$,

$$M_t^{\mathbf{e}} = \inf\big\{y \in \mathbf{R} : X_t^{\mathbf{e}}[(-y, y)^c] = 0\big\}.$$

It is immediate that $M_t \geq M_t^{\mathbf{e}}$ for each $\mathbf{e} \in S_1$ and hence from (3.3) applied to $X^{\mathbf{e}}$ we have

$$\liminf_{t\uparrow\infty} \frac{M_t}{t} \geq \sqrt{2\beta}$$

P_μ-almost surely on the survival set of X.

Now let us complete the proof by establishing that

$$\limsup_{t\uparrow\infty} \frac{M_t}{t} \leq \sqrt{2\beta} \tag{3.7}$$

P_μ-almost surely. To this end, we may reconsider Remark 3.1 and deduce that, for each $x > 0$, there exists a P_μ-almost surely finite time $T^{\mathbf{e}}$ such that

$$\mathsf{P}_\mu\left(X_t^{\mathbf{e}}(x + \sqrt{2\beta}t, \infty) = 0 \text{ for all } t \geq T^{\mathbf{e}}\right) = 1.$$

Next, we shall need some standard geometrical results which trace back to Euclid's works, *Elements* (cf. [14], Book XII, Proposition 17, p. 425). For any constant $c > 0$ define S_c the sphere with radius $c > 0$. Let x be any positive number as before. Then for each $\varepsilon > 0$ there exists a regular polytope $\mathcal{P}$ centred at the origin which contains S_x and is contained in $S_{x+\varepsilon}$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the outward normal unit vectors to each surface of $\mathcal{P}$. We know from the previous paragraph that $T^{\mathbf{e}_i} < \infty$, P_μ-almost surely for each $i = 1, \ldots, n$.

Now define $\mathcal{P}(t)$ to be the radially scaled version of $\mathcal{P}$, with scaling factor $(\sqrt{2\beta}t + x)/x$, which contains $S_{\sqrt{2\beta}t+x}$ and is contained in $S_{(1+\varepsilon/x)(\sqrt{2\beta}t+x)}$. Since $\sup_{i=1,\ldots,n} T^{\mathbf{e}_i} < \infty$ P_μ-almost surely, it follows that all mass is eventually contained in the inflating polytope $\{\mathcal{P}(t) : t > 0\}$ P_μ-almost surely. This in turn implies that all mass is eventually contained in the inflating sphere $\{S_{(1+\varepsilon/x)(\sqrt{2\beta}t+x)} : t > 0\}$ P_μ-almost surely. That is to say,

$$\limsup_{t\uparrow\infty} \frac{M_t}{t} \leq (1 + \varepsilon)\sqrt{2\beta}.$$

Since ε can be chosen arbitrarily close to zero, we have shown that

$$\limsup_{t\uparrow\infty}\frac{M_t}{t}\leq\sqrt{2\beta}$$

P_μ-almost surely and the proof is complete. □

Remark 3.2. The proof of Theorem 2.1 shows something a little stronger than the statement of the theorem. Namely that the (space-time) exterior of the inflating sphere $\{S_{x+\gamma t}: t>0\}$, for arbitrary $x>0$, is charged for arbitrarily large t when X survives P_μ-almost surely when $\gamma\in[0,\sqrt{2\beta})$ and not charged for all sufficiently large t P_μ-almost surely when $\gamma>\sqrt{2\beta}$. As usual it is understood that $\mu\in\mathcal{M}_c(\mathbf{R}^d)$.

Remark 3.3. In principle it is possible to extract the result stated in Theorem 2.1 for branching Brownian motion from the results by Biggins [2] which concern general branching walks.

Another study which also needs mentioning in this respect is the recent paper by Fleischmann and Swart [11]. Here it is shown that the support of a branching particle system can be 'embedded' within the support of an associated superprocess defined on the same probability space for all sufficiently large times. This would suggest that, with careful checking, one could derive for example (3.3) for super-Brownian motion as a consequence of the same statement being true for branching Brownian motion. With some further analysis, it is likely one could also demonstrate that, even though the super-Brownian motion is a 'bigger' process than the branching Brownian motion (in the sense of the embedding), the radial support of the former will not stray much further beyond the radial support of the latter. In essence however, the proofs would not become significantly easier and we refrain from developing them here.

Appendix. Proof of Theorem 2.3 for super-Brownian motion

Proof. We shall keep the proof brief, leaving some details for the reader to fill in as, to some extent, it re-uses many of the ideas concerning the relationship between $W(\gamma)$ and the KPP travelling wave equation for branching Brownian motion given in [12, 19] and [17].

One particular fact we shall make heavy use of below is that the KPP travelling wave equation

$$\frac{1}{2}\Phi''-c\Phi'+\beta(\Phi-\Phi^2)=0\quad\text{on }\mathbf{R}$$

may take up to three solutions bounded in $[0,1]$ depending on the value of c. If $|c|\leq\sqrt{2\beta}$, then the only such solutions are $\Phi=0$ and $\Phi=1$. Otherwise there is an additional solution valued in $(0,1)$ which has the properties that it is monotone with $\Phi(-\infty)=1$ and $\Phi(\infty)=0$. By defining $\Psi=\Phi\beta/\alpha$ the above travelling wave equation transforms uniquely to the equation

$$\frac{1}{2}\Psi''-c\Psi'+\beta\Psi-\alpha\Psi^2=0 \quad \text{on } \mathbf{R} \tag{A.1}$$

which is the form in which we shall frequently encounter it below. The three possible solutions (depending on the value of c) bounded in $[0,\beta/\alpha]$ are now either $\Psi=0$, $\Psi=\beta/\alpha$ or monotone, valued in $(0,\beta/\alpha)$ having $\Psi(-\infty)=\beta/\alpha$ and $\Psi(\infty)=0$.

(i) Take $\gamma\in\mathbf{R}$ and $\mu\in\mathcal{M}_c(\mathbf{R})$. It is standard to derive from the log-Laplace equation (1.1) that $v(x,t):=E_{\delta_x}\langle e^{\gamma\cdot},X_t\rangle$ is the minimal non-negative solution to the heat equation

$$\frac{\partial v}{\partial t}=\frac{1}{2}\frac{\partial^2 v}{\partial x^2}+\beta v$$

with initial condition $v(x,0)=\exp\{\gamma x\}$, which itself is solved uniquely by $\exp\{\gamma x+(\beta+\gamma^2/2)t\}$. Applying the Markov and Continuous Branching properties (cf. [15] and [5], Theorem I.1.3) we have then that for $0\le s\le t$,

$$\begin{aligned}&\mathsf{E}_\mu\left(\exp\{-(\beta+\gamma^2/2)t\}\langle\exp\{\gamma\cdot\},X_t\rangle\mid\mathcal{F}_s\right)\\ &\qquad=\exp\{-(\beta+\gamma^2/2)t\}\,\mathsf{E}_{X_s}\langle\exp\{\gamma\cdot\},X_{t-s}\rangle\\ &\qquad=\int\mathsf{E}_{\delta_x}\langle\exp\{\gamma\cdot\},X_{t-s}\rangle X_s(dx)=\exp\{-(\beta+\gamma^2/2)s\}\langle\exp\{\gamma\cdot\},X_s\rangle\end{aligned}$$

showing that $W_t(\gamma)$ is a martingale. Since $W_t(\gamma)\ge 0$ for all t, it converges almost surely to a limit, namely $W(\gamma)$.

(ii) For reasons of symmetry, it suffices to prove the result for $\gamma\in(0,\sqrt{2\beta})$. To this end, for each $z\in\mathbf{R}$, define the space-time line $\Gamma_{-z}=\{(y,s)\in\mathbf{R}\times\mathbf{R}^+ : y=-z\}$. Associated with these space-time barriers are the exit measures $X^{c_\gamma}_{\Gamma_{-z}}$ embedded within the process X^{c_γ} where $c_\gamma:=\beta/\gamma+\gamma/2$. See [6] for the precise definition of an exit measure. Since it has been proved that M_t and hence R_t grows no faster than linearly with speed $\sqrt{2\beta}$ (before the facts we are proving about the martingale were used), it follows that the exit measures $X^{c_\gamma}_{\Gamma_{-z}}$ have compact support when $\gamma\in(0,\sqrt{2\beta})$ (and hence $c_\gamma>\sqrt{2\beta}$).

Next fix $\gamma\in(0,\sqrt{2\beta})$, $\mu\in\mathcal{M}_c(\mathbf{R})$, $z\in\mathbf{R}$ and define for each $x>-z$, $t\in\mathbf{R}^+$ and continuous, non-negative and bounded g

$$u(z,x,t)=-\log\mathsf{E}_{\delta_{(x,t)}}\left(\exp\{-\langle g,X^{c_\gamma}_{\Gamma_{-z}}\rangle\}\right)\in[0,\infty) \tag{A.2}$$

where $\delta_{(x,t)}$ should be understood as the measure corresponding to a unit mass placed at space-time position (x,t) (in which case for notational consistency we should note that $\delta_{(x,0)}=\delta_x$). Theorem II.3.1 in [5] now tells us that u is the minimal non-negative solution to

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}-c_\gamma\frac{\partial u}{\partial x}+\beta u-\alpha u^2=0 \quad \text{on } (x,t)\in(-z,\infty)\times\mathbf{R}^+$$

with boundary condition $u = g$ on $x = -z$. When $c_\gamma > \sqrt{2\beta}$ by choosing $g = \Psi$, the travelling wave solution to the KPP equation (A.1) at speed c_γ, it becomes clear that $u = \Psi$. Now applying the Special Markov and Continuous Branching properties for exit measures, together with the fact that each Γ_{-z} is compactly supported, it follows that for $\zeta > 0$ and $z > z_0 := \inf\{y \in \mathbf{R} : (-\infty, -y) \cap \operatorname{supp}\mu = \emptyset\}$

$$\mathsf{E}_\mu\left(\exp\{-\langle \Psi, X^{c_\gamma}_{\Gamma_{-(\zeta+z)}}\rangle\} \mid \mathcal{F}_{\Gamma_{-z}}\right) = \exp\{-\langle \Psi, X^{c_\gamma}_{\Gamma_{-z}}\rangle\}$$

where $\mathcal{F}_{\Gamma_{-z}} = \sigma(X^{c_\gamma}_{\Gamma_{-x}} : x \leq z)$ and hence $\{\exp\{-\langle \Psi, X^{c_\gamma}_{\Gamma_{-z}}\rangle\} : z \geq z_0\}$ is a P_μ-martingale. Indeed, it is a positive, bounded martingale and hence converges in $L^1(\mathsf{P}_\mu)$. It follows then that $\langle \Psi, X^{c_\gamma}_{\Gamma_{-z}}\rangle$ has a limit as z tends to infinity and further

$$\mathsf{P}_\mu\left(\lim_{z\uparrow\infty}\langle \Psi, X^{c_\gamma}_{\Gamma_{-z}}\rangle > 0\right) = \mathsf{P}_\mu\left(\lim_{z\uparrow\infty}\Psi(-z)\|X^{c_\gamma}_{\Gamma_{-z}}\| > 0\right) > 0$$

where $\|X^{c_\gamma}_{\Gamma_{-z}}\| = \langle 1, X^{c_\gamma}_{\Gamma_{-z}}\rangle$. Again referring to classical analysis of travelling waves to the KPP equation (cf. [12] for a modern account) it is well known that $\Psi(-z) \sim k \times \exp\{-\gamma z\}$, for some positive and finite constant k, as z tends to infinity. We thus conclude that

$$\mathsf{P}_\mu\left(\lim_{z\uparrow\infty} e^{-\gamma z}\|X^{c_\gamma}_{\Gamma_{-z}}\| > 0\right) > 0. \tag{A.3}$$

In a similar vein to the discussion at the begining of part (i), we can deduce from (A.2) that $\mathsf{E}_{\delta_{(x,t)}}\langle \exp\{\gamma\cdot\}, X^{c_\gamma}_{\Gamma_{-z}}\rangle$ is the minimal non-negative solution to

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - c_\gamma \frac{\partial u}{\partial x} + \beta u = 0 \quad \text{on } (x,t) \in (-z,\infty) \times \mathbf{R}^+$$

with $u(x,t) = \exp\{\gamma x\}$ on $x = -z$. The latter is solved by $u(x,t) = \exp\{\gamma x\}$. Taking conditional expectations with respect to the filtration $\{\mathcal{F}_{\Gamma_{-z}} : z \geq z_0\}$ and again using the Special Markov, Continuous Branching and compact support properties for the exit measures we are considering, we may deduce, in a similar way to the reasoning in part (i), that

$$\left\{\langle \exp\{\gamma\cdot\}, X^{c_\gamma}_{\Gamma_{-z}}\rangle = \exp\{-\gamma z\}\|X^{c_\gamma}_{\Gamma_{-z}}\| : z \geq z_0\right\}$$

is a P_μ-martingale. From (A.3), its limit, say $\Delta(\gamma)$, is a non-negative random variable which is not identically zero with probability one. Fatou's Lemma implies that for any $x \in \mathbf{R}$,

$$\mathsf{E}_{\delta_x}\left(\Delta(\gamma)\right) \leq e^{\gamma x}$$

and hence again an application of the Strong Markov and Continuous Branching properties shows that

$$\mathsf{E}_\mu\left(\Delta(\gamma) \mid \mathcal{F}_t\right) = \mathsf{E}_{X^{c_\gamma}_t}\left(\Delta(\gamma)\right) \leq \langle \exp\{\gamma\cdot\}, X^{c_\gamma}_t\rangle \overset{d}{=} W_t(\gamma) \tag{A.4}$$

where the last equality is in distribution. We are now forced to conclude that $\mathsf{P}_\mu(W(\gamma) > 0) > 0$ as otherwise, taking limits in (A.4) one would have that $\mathsf{P}_\mu(\Delta(\gamma) = 0) = 1$.

Finally we need to prove that

$$\mathsf{P}_\mu\left(W(\gamma) > 0 \mid X \text{ survives}\right) = 1. \tag{A.5}$$

To start with, note that $\{X$ eventually becomes extinct $\} \subseteq \{W(\gamma) = 0\}$. Let $\eta(x) := -\log \mathsf{P}_{\delta_x}(W(\gamma) = 0)$. Our previous observation together with the fact that $\mathsf{P}_\mu(W(\gamma) > 0) > 0$ and (1.2) shows that $0 < \eta \leq \beta/\alpha$. Our aim is to show that $\eta = \beta/\alpha$ in which case (A.5) is proved. To this end, apply the Markov and Continuous Branching properties to get

$$\mathsf{P}_\mu\left(W(\gamma) = 0 \mid \mathcal{F}_t\right) = \mathsf{P}_{X_t}\left(W(\gamma) = 0\right) = \exp\{-\langle \eta, X_t\rangle\},$$

which in turn shows that $\{\exp\{-\langle \eta, X_t\rangle\} : t \geq 0\}$ is a martingale. It follows from an application of Theorem II.3.1 in [5] that

$$\frac{1}{2}\eta'' + \beta\eta - \alpha\eta^2 = 0.$$

As discussed at the beginning of this Appendix, the only solution to the above equation which is valued in $(0, \beta/\alpha]$ is $\eta = \beta/\alpha$ and hence (A.5) is proved. □

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