Entrance and exit at infinity for stable jump diffusions

Andreas Kyprianou (based on joint work with Leif Döring)

Feller boundary classification for diffusions

- In his seminal work in the 1950s, William Feller classified one-dimensional diffusion processes on $-\infty \leq a < b \leq \infty$
- The four types of boundary points are:
  regular, if it is both accessible and enterable;
  exit, if it is accessible but not enterable;
  entrance, if it is enterable but not accessible;
  natural if it is neither accessible nor enterable.
- Feller's definitions and proofs are purely analytic, using Hille-Yosida theory to generate Feller semigroup of a process $(X_t, t \geq 0)$ from differential operators (diffusion generators)

$$\mathcal{A} := \kappa(x)\frac{d}{dx} + \frac{\sigma(x)^2}{2}\frac{d^2}{dx^2}$$

  taking account of the different boundary conditions.
- A change of space via the so-called scale function (say s which makes $(s(X_t), t \geq 0)$ a martingale)

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  on a new interval $(\tilde{a}, \tilde{b})$.

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The case of an infinite boundary

- In the setting of the entire real line, i.e. $a = -\infty$ and $b = +\infty$, the notion of entrance (in applications also called coming down from infinity) and exit (explosion) becomes interesting
- Depending on the growth of σ at infinity the infinite boundary points can be of an entrance type. Feller's results for this scenario imply that $+\infty$ is an entrance boundary if and only if

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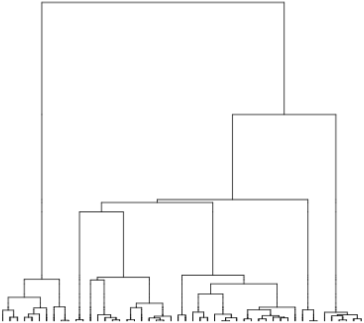
- The notion of coming down from infinity becoming more important in other classes of Feller processes e.g. Kingman's Coalescent



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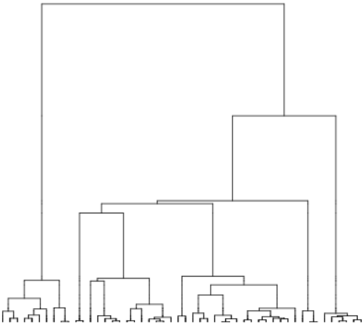

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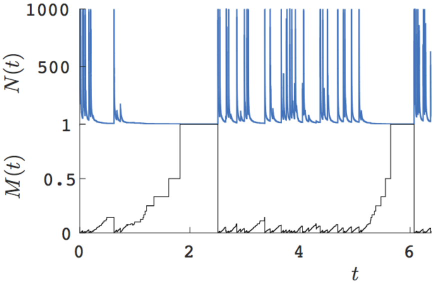
Coming down from infinity: II

- Kingman coalescent dynamics, fragment each block at a constant rate into an infinite number of blocks [cf. K., Pagett, Rogers & Schweinsberg (2017)] - what happens after the first fragmentation event?
- Nothing more than a Markov chain $(N(t) : t \geq 0)$ on $\mathbb{N} \cup \{\infty\}$ specified by the Q-matrix

$$Q_{i,j} = \begin{cases} c\binom{i}{2} & \text{if } j = i-1, \\ \lambda i & \text{if } j = \infty. \end{cases}$$

  Let $\theta := 2\lambda/c$.
- If $0 < \theta < 1$, then $(N(t) : t \geq 0)$ is a recurrent Feller process on $\mathbb{N} \cup \{\infty\}$ such that $\{\infty\}$ is instantaneously regular (that is to say 0 is a not a holding point).
- If $\theta \geq 1$, then $\{\infty\}$ is an absorbing state for $(N(t) : t \geq 0)$.



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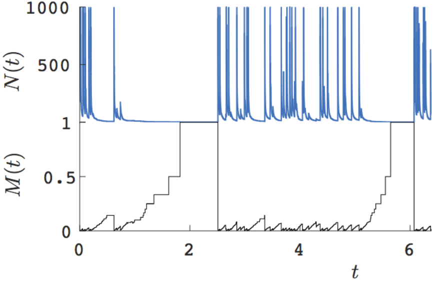

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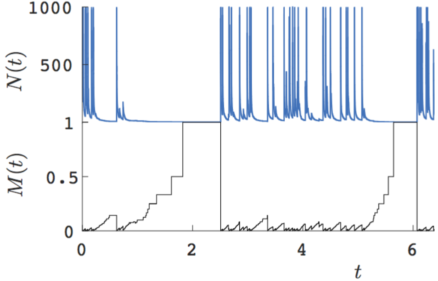

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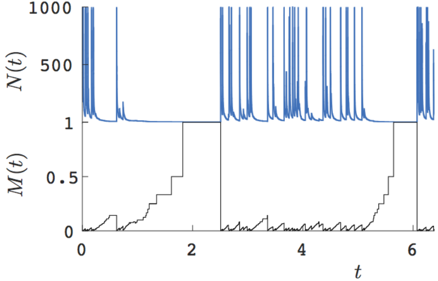

Coming down from infinity: III

- Lambert's logistic Continuous-state branching process

$$\mathrm{d}Z_t = bZ_t\mathrm{d}t + \gamma Z_t\mathrm{d}B_t - cZ_t^2\mathrm{d}t, \qquad t \geq 0.$$

Lambert (2005)

- More generally

$$Z_t = x - a\int_0^t Z_s\mathrm{d}s + \sigma\int_0^t\int_0^{Z_{s-}} W(\mathrm{d}s,\mathrm{d}u) + \int_0^t\int_0^{Z_{s-}}\int_0^\infty r\tilde{N}(\mathrm{d}s,\mathrm{d}v,\mathrm{d}r) - \int_0^t G(Z_s)\mathrm{d}s, \qquad t \geq 0.$$

[Berestycki, Fitipaldi & Fontobona (2018)]

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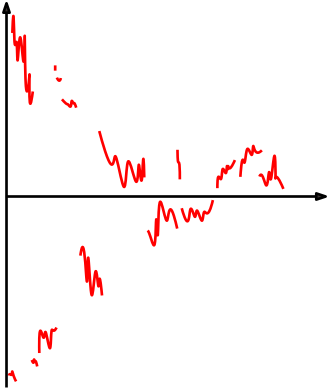
[Berestycki, Fitipaldi & Fontobona (2018)]

STABLE JUMP-DIFFUSIONS

- Focus our study on so-called stable jump diffusions:

$$dZ_t = \sigma(Z_{t-})\,dX_t, \qquad Z_0 = z \in \mathbb{R}, t \geq 0.$$

- Intersted in entrance from $\{+\infty\}$, $\{-\infty\}$ and $\pm\infty := \{+\infty\} \cup \{-\infty\}$



Stable process

- A stable process lies in the intersection of the class of Lévy process (stationary and independent increments) and the class of self-similar Markov processes: for all $c > 0$ and $x \in \mathbb{R}$,

$$(cX_{c^{-\alpha}t}, t \geq 0) \text{ under } \mathbb{P}_x \text{ is equal in law to } (X_t, t \geq 0) \text{ under } \mathbb{P}_{cx},$$

  where $(\mathbb{P}_x, x \in \mathbb{R})$ are the probabilities of X and $\alpha \in (0,2)$.
- Semigroup of X is entirely characterised by $\Psi(z) := -\log \mathbb{E}_0\left[e^{izX_1}\right]$, satisfying

$$\Psi(z) = |z|^\alpha \left(e^{\pi i\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{\{z>0\}} + e^{-\pi i\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{\{z<0\}}\right), \quad z \in \mathbb{R}.$$

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- The Lévy measure associated with Ψ:

$$\frac{\Pi(dx)}{dx} = \Gamma(1+\alpha)\frac{\sin(\pi\alpha\rho)}{\pi}\frac{1}{x^{1+\alpha}}\mathbf{1}_{(x>0)} + \Gamma(1+\alpha)\frac{\sin(\pi\alpha\hat{\rho})}{\pi}\frac{1}{|x|^{1+\alpha}}\mathbf{1}_{(x<0)},$$

  where $\hat{\rho} := 1-\rho$. In the case that $\alpha = 1$, we take $\rho = 1/2$, meaning that X corresponds to the Cauchy process.
  Convention from now on: Anything with a $\hat{}$ is associated to the law of $-X$. E.g. $\hat{\mathbb{P}}_x$ is the law of $-X$ with $X_0 = -x$.
- If X has only upwards (resp. downwards) jumps we say X is spectrally positive (resp. negative). If X has jumps in both directions we say X is two-sided. A spectrally positive (resp. negative) stable process with $\alpha < 1$ is necessarily increasing (resp. decreasing).

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Proposition (Zanzotto (2002), Döring & K. (2018))

Suppose that σ is strictly positive. Then there is a unique (possibly exploding) weak solution Z to the SDE

$$dZ_t = \sigma(Z_{t-})\, dX_t, \qquad Z_0 = z \in \mathbb{R}, t \geq 0.$$

and Z can be expressed as time-change under $\mathbb{P}_z$ via

$$Z_t := X_{\tau_t}, \quad t < T,$$

where

$$\tau_t = \inf\left\{s > 0 : \int_0^s \sigma(X_s)^{-\alpha} \mathrm{d}s > t\right\}$$

and the finite or infinite explosion time is $T = \int_0^\infty \sigma(X_s)^{-\alpha} \mathrm{d}s$.

The law of the unique solution Z will be denoted by P_z, $z \in \mathbb{R}$.

Technical point: when $\alpha \in (1, 2)$, the origin is a recurrent point, hence as $\sigma > 0$, $T = \infty$.

However, when $\alpha \in (1, 2)$, $\mathtt{k} := \inf\{t > 0 : Z_t = 0\}$ is almost surely finite (irrespective of Z_0).

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Definition

We say that $\pm\infty$ is a (continuous) entrance point for a Feller process Y on $\mathbb{R}$ with transition semigroup $\mathcal{P}$ (with probabilities $\mathbb{P}_x$, $x \in \mathbb{R}$) if

(i) the point $\pm\infty$ is not accessible,

(ii) the semigroup $\mathcal{P}$ can be extended to a Feller semigroup $\overline{\mathcal{P}}$ on $C_b(\overline{\underline{\mathbb{R}}})$,

(iii) there is continuous entrance in the sense that

$$\mathbb{P}_{\pm\infty}\left(\lim_{t\downarrow 0}|Y_t| = \infty, \limsup_{t\downarrow 0} Y_t = +\infty, \liminf_{t\downarrow 0} Y_t = -\infty\right) = 1$$

Analogously, we define entrance from $-\infty$ as extension to $C_b(\underline{\mathbb{R}})$ and entrance from $+\infty$ as extension to $C_b(\overline{\mathbb{R}}) = C(\overline{\mathbb{R}})$.

Theorem (Döring & K. (2018))

Suppose that σ is uniformly bounded away from the origin and let

$$I^{\sigma,\alpha}(A) = \int_A \sigma(x)^{-\alpha}|x|^{\alpha-1}\,\mathrm{d}x \quad \text{and} \quad I^{\sigma,1} = \int_{\mathbb{R}} \sigma(x)^{-1}\log|x|\,\mathrm{d}x.$$

Then the following table exhaustively summarizes entrance points at infinity of

$$dZ_t = \sigma(Z_{t-})\,dX_t, \qquad Z_0 = z \in \mathbb{R}, t \geq 0.$$

Necessary and sufficient conditions for entrance from infinite boundary points				
α	Jumps	$+\infty$	$-\infty$	$\pm\infty$
< 1	only $\downarrow$	✗	✗	✗
	only $\uparrow$	✗	✗	✗
	$\uparrow$ & $\downarrow$	✗	✗	✗
$= 1$	$\uparrow$ & $\downarrow$	✗	✗	✓ iff $I^{\sigma,1} < \infty$
> 1	only $\downarrow$	✗	✓ iff $I^{\sigma,\alpha}(\mathbb{R}_-) < \infty$	✗
	only $\uparrow$	✓ iff $I^{\sigma,\alpha}(\mathbb{R}_+) < \infty$	✗	✗
	$\uparrow$ & $\downarrow$	✗	✗	✓ iff $I^{\sigma,\alpha}(\mathbb{R}) < \infty$
$= 2$	none	✓ iff $I^{\sigma,2}(\mathbb{R}_+) < \infty$	✓ iff $I^{\sigma,2}(\mathbb{R}_-) < \infty$	✗

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Henceforth concentrate on the case of two-sided jumps.

Riesz–Bogdan–Żak transform

Convention from now on: Anything with a $\hat{}$ is associated to the law of $-X$. E.g. $\hat{\mathbb{P}}_x$ is the law of $-X$ with $X_0 = -x$.

Theorem (Bogdan & Żak (2010), K. (2016))

Suppose that X is a stable process with two-sided jumps. Define

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} \mathrm{d}u > t\}, \qquad t \geq 0.$$

Then, for all $x \in \mathbb{R}\backslash\{0\}$,

$$\frac{1}{X_{\eta(t)}}, \qquad t \geq 0$$

under $\hat{\mathbb{P}}_x$ a self-similar Markov process equal in law to $(X, \mathbb{P}^\circ_{1/x})$, where

$$\left.\frac{\mathrm{d}\mathbb{P}^\circ_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{h(X_t)}{h(x)} \mathbf{1}_{(t<\tau^{\{0\}})}$$

$$h(z) = (\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\mathrm{sgn}(z))\,|z|^{\alpha-1}$$

and $\mathcal{F}_t := \sigma(X_s : s \leq t)$, $t \geq 0$.

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Stable conditioned to avoid the origin

- Recalling that $\alpha \in (1,2)$, $|x|^{\alpha-1}$ as a Doob h-function, rewards paths that are far from the origin ($|x| \gg 1$) and punishes paths that stray too close to the origin ($|x| \ll 1$).
- In fact it has been shown [Chaumont, Panti & Rivero (2013), Kuznetsov, K., Pardo, Watson (2014)] that $(X, \mathbb{P}^\circ_y)$, $y \neq 0$, can be identified by the limit

$$\mathbb{P}^\circ_y(A) = \lim_{s\to\infty} \mathbb{P}_y(A \,|\, T_0 > t+s),$$

for $A \in \mathcal{F}_t$ and $T_0 = \inf\{t > 0 : X_t = 0\}$.
- (WARNING! Ultra specialist information): As X is a point recurrent process, there exists an excursion measure $n(\cdot)$ for the Poisson random field of excursions from the origin, from which one can construct (up to a constant)

$$\mathbb{P}^\circ_0(X^\circ_t \in \mathrm{d}z) := h(z) n(X_t \in \mathrm{d}z,\ t < \zeta)$$

consistently with $\mathbb{P}^\circ_y$, $y \neq 0$, where ζ is the excursion lifetime and

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- (Executive summary of last point): The limit

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Time change and Riesz-Bogdan-Żak

Remember there is a unique weak solution Z to the SDE

$$dZ_t = \sigma(Z_{t-})\,dX_t, \qquad Z_0 = z \in \mathbb{R}, t \geq 0.$$

and Z can be expressed as time-change under $\mathbb{P}_z$ via $Z_t := X_{\tau_t}$, $t < T$, where

$$\tau_t = \inf\left\{s > 0 : \int_0^s \sigma(X_s)^{-\alpha} \mathrm{d}s > t\right\}$$

Proposition (Döring & K. (2018))

Set

$$\beta(x) = \sigma(1/x)^{-\alpha}|x|^{-2\alpha}, \qquad x \in \mathbb{R}\backslash\{0\}.$$

Define the time-space transformation

$$Z_t^{\dagger} = \frac{1}{\hat{X}^{\circ}_{\theta_t}}, \qquad t < \int_0^{\infty} \beta(\hat{X}^{\circ}_u)\,du,$$

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- We want to show that $\int_{\mathbb{R}} \sigma(x)^{-\alpha}|x|^{\alpha-1}\,\mathrm{d}x < \infty$ implies that $\pm\infty$ is an entrance point for
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SUFFICIENCY (HEURISTIC)

Writing $G_{\hat{X}^\circ}(x, \mathrm{d}y)$ for the resolvent of $\hat{X}^\circ$ and $G_{\hat{X}^\dagger}(x, \mathrm{d}y)$ for the resolvent of X killed on first hitting the origin,

$$
\begin{aligned}
&\hat{\mathbb{E}}_x^\circ\left[\int_0^\infty \beta(\hat{X}_u^\circ)du\right]\\
&= \int_{\mathbb{R}} G_{\hat{X}^\circ}(x, \mathrm{d}y)\sigma(1/y)^{-\alpha}|y|^{-2\alpha}\\
&= \int_{\mathbb{R}} G_{\hat{X}^\dagger}(x, \mathrm{d}y)\frac{\hat{h}(y)}{\hat{h}(x)}\sigma(1/y)^{-\alpha}|y|^{-2\alpha}\\
&\approx \int_{\mathbb{R}} \left(|y|^{\alpha-1}s(y) - |y-x|^{\alpha-1}s(y-x) + |x|^{\alpha-1}s(-x)\right)\frac{|y|^{\alpha-1}}{|x|^{\alpha-1}}\sigma(1/y)^{-\alpha}|y|^{-2\alpha},
\end{aligned}
$$

which is finite if

$$\int_{\mathbb{R}} \sigma(x)^{-\alpha}|x|^{\alpha-1}\,\mathrm{d}x < \infty.$$

Note, for a Markov process Y, with probabilities P_x, $x \in E$,

$$G_Y(x, \mathrm{d}y) = \int_0^\infty \mathrm{P}_x(Y_t \in \mathrm{d}y)\mathrm{d}t, \qquad x, y \in E.$$

Proposition (Döring & K. (2018))

Suppose that $\hat{X}^\circ$ has probabilities $\hat{\mathbb{P}}_x^\circ$, $x \in \mathbb{R}$. Define $\hat{Z}_t^\circ = \hat{X}_{\iota_t}^\circ$, $t \geq 0$, where the time-change ι is given by

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Recall that Z has the law of the unique weak solution to the SDE and $Z^\dagger$ is the same process killed on first hitting 0.

If $\pm\infty$ is an entrance point for Z, then the time reversed process $Z^\dagger_{(\mathtt{k}-t)-}$, $t \leq \mathtt{k}$, under $\mathrm{P}_{\pm\infty}$ is a time-homogenous Markov process with transition semigroup which agrees with that of $\hat{Z}^\circ$, where $\mathtt{k}$ is any almost surely finite last passage time for $Z^\dagger$ (e.g. $\mathtt{k} = \inf\{t > 0 : Z_t^\dagger = 0\}$).

Remark on proof: Important step is to prove weak duality:

$$p_{Z^\dagger}(t, y, \mathrm{d}z)\mu(\mathrm{d}y) = p_{\hat{Z}^\circ}(t, z, \mathrm{d}y)\mu(\mathrm{d}z)$$

where

$$\mu(dy) = \int_{\mathbb{R}} \nu(\mathrm{d}x) G_{\hat{Z}^\circ}(x, \mathrm{d}y) = \sigma(x)^{-\alpha} h(x) \mathrm{d}x$$

and $G_{\hat{Z}^\circ}$ is the resolvent of $\hat{Z}^\circ$

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If $\pm\infty$ is an entrance point for Z, then the time reversed process $Z^\dagger_{(\mathtt{k}-t)-}$, $t \leq \mathtt{k}$, under $\mathrm{P}_{\pm\infty}$ is a time-homogenous Markov process with transition semigroup which agrees with that of $\hat{Z}^\circ$, where $\mathtt{k}$ is any almost surely finite last passage time for $Z^\dagger$ (e.g. $\mathtt{k} = \inf\{t > 0 : Z^\dagger_t = 0\}$).

Remark on proof: Important step is to prove weak duality:

$$p_{Z^\dagger}(t, y, \mathrm{d}z)\mu(\mathrm{d}y) = p_{\hat{Z}^\circ}(t, z, \mathrm{d}y)\mu(\mathrm{d}z)$$

where

$$\mu(dy) = \int_{\mathbb{R}} \nu(\mathrm{d}x) G_{\hat{Z}^\circ}(x, \mathrm{d}y) = \sigma(x)^{-\alpha} h(x)\mathrm{d}x$$

and $G_{\hat{Z}^\circ}$ is the resolvent of $\hat{Z}^\circ$

HUNT-NAGASAWA DUALITY

Proposition (Döring & K. (2018))

Suppose that $\hat{X}^\circ$ has probabilities $\hat{\mathbb{P}}^\circ_x$, $x \in \mathbb{R}$. Define $\hat{Z}^\circ_t = \hat{X}^\circ_{\iota_t}$, $t \geq 0$, where the time-change ι is given by

$$\iota_t = \inf\left\{s > 0 : \int_0^s \sigma(\hat{X}^\circ_s)^{-\alpha} ds > t\right\}, \qquad t < \int_0^\infty \sigma(\hat{X}^\circ_s)^{-\alpha} ds.$$

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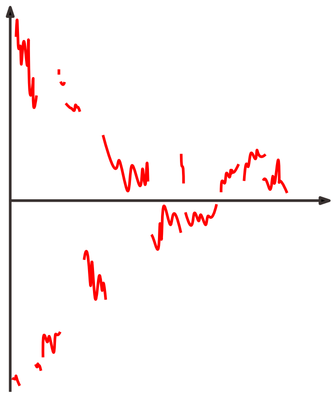

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NECESSITY (HEURISTIC)

- We want to show that if $\pm\infty$ is an entrance point for

$$dZ_t = \sigma(Z_{t-})\, dX_t, \qquad Z_0 = z \in \mathbb{R}, t \geq 0,$$

then necessarily $\int_{\mathbb{R}} \sigma(x)^{-\alpha} |x|^{\alpha-1}\, \mathrm{d}x < \infty$.

- If $\pm\infty$ is an entrance point, then Z can be seen as a Feller process on the compact space $\overline{\mathbb{R}}$.
- Getoor's equivalent definitions of transience:
  - On the one hand, last exit from any compact set is a.s. finite
  - On the other hand the resolvent of any compact set is finite
- As $\overline{\mathbb{R}}$ is compact itself,

$$G_{Z^\dagger}(\pm\infty, \overline{\mathbb{R}}) < \infty$$

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$$\infty > G_{\hat{Z}^\circ}(0, \overline{\mathbb{R}}) \approx G_{\hat{Z}^\circ}(x, \mathbb{R}) = \int_{\mathbb{R}} G_{\hat{X}^\dagger}(x, \mathrm{d}y) \frac{\hat{h}(y)}{\hat{h}(x)} \sigma(1/y)^{-\alpha} |y|^{-2\alpha} \approx \int_{\mathbb{R}} \sigma(x)^{-\alpha} |x|^{\alpha-1}\, \mathrm{d}x,$$

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Difficulties in other regimes

- Two sided jumps
  - $\alpha \leq 1$ Cannot hit the origin, so cannot time reverse from the origin or condition to avoid the origin
  - $\alpha = 1$ Can time reverse from first entry into strip $(-1, 1)$
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EXPLOSION (EXIT AT INFINITY)

Theorem (Döring & K. (2018))

Suppose that $\sigma > 0$ and let

$$I^{\sigma,\alpha}(A) = \int_A \sigma(x)^{-\alpha}|x|^{\alpha-1}dx.$$

Then the following table exhaustively summarises finite time explosion for

$$dZ_t = \sigma(Z_{t-})\,dX_t, \qquad Z_0 = z \in \mathbb{R}, t \geq 0.$$

Necessary and sufficient conditions for exit at infinite boundary points				
α	Jumps	$+\infty$	$-\infty$	$\pm\infty$
< 1	only $\downarrow$	✗	✓ iff $I^{\sigma,\alpha}(\mathbb{R}_-) < \infty$	✗
	only $\uparrow$	✓ iff $I^{\sigma,\alpha}(\mathbb{R}_+) < \infty$	✗	✗
	$\uparrow$ & $\downarrow$	✗	✗	✓ iff $I^{\sigma,\alpha}(\mathbb{R}) < \infty$
$= 1$	$\uparrow$ & $\downarrow$	✗	✗	✗
> 1	only $\downarrow$	✗	✗	✗
	only $\uparrow$	✗	✗	✗
	$\uparrow$ & $\downarrow$	✗	✗	✗
$= 2$	none	✗	✗	✗

Thank you!