General path integrals and stable SDEs[1]

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(based on joint work with Leif Döring & Sam Baguley)

[1] Based on a paper of the same name to appear in JEMS

A classical SDE

- When does there exist a non-trivial solution to the SDE:

$$\mathrm{d}\, X_t = \sigma(X_{t-})\, \mathrm{d}\, Y_t$$

where Y is a one-dimensional α-stable Lévy process, $X_0 \in \mathbb{R}$ and σ is measurable?

α-STABLE PROCESS

- A stable process lies in the intersection of the class of Lévy process (stationary and independent increments) and the class of self-similar Markov processes: for all $c > 0$ and $x \in \mathbb{R}$,

$$(cX_{c^{-\alpha}t}, t \geq 0) \text{ under } \mathbb{P}_x \text{ is equal in law to } (X_t, t \geq 0) \text{ under } \mathbb{P}_{cx},$$

  where $(\mathbb{P}_x, x \in \mathbb{R})$ are the probabilities of X and $\alpha \in (0, 2)$.

- Semigroup of X is entirely characterised by $\Psi(z) := -\log \mathbb{E}_0\left[\mathrm{e}^{\mathrm{i}zX_1}\right]$, satisfying

$$\Psi(z) = |z|^\alpha \left(e^{\pi \mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{\{z>0\}} + e^{-\pi \mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{\{z<0\}}\right), \quad z \in \mathbb{R}.$$

  where $\rho = \mathbb{P}(X_1 > 0)$.

- The Lévy measure associated with Ψ:

$$\frac{\Pi(\mathrm{d}x)}{\mathrm{d}x} = \Gamma(1+\alpha)\frac{\sin(\pi\alpha\rho)}{\pi}\frac{1}{x^{1+\alpha}}\mathbf{1}_{(x>0)} + \Gamma(1+\alpha)\frac{\sin(\pi\alpha\hat{\rho})}{\pi}\frac{1}{|x|^{1+\alpha}}\mathbf{1}_{(x<0)},$$

  where $\hat{\rho} := 1 - \rho$. In the case that $\alpha = 1$, we take $\rho = 1/2$, meaning that X corresponds to the Cauchy process.
  %pause

α-STABLE LÉVY PROCESS

index	jumps	path	asymptotic behaviour
$\alpha \in (0,1)$			transient
$\rho = 0$	$-$	monotone decreasing	$\mathbb{P}_x(\tau^{\{0\}} < \infty) = 0, x \in \mathbb{R}, \lim_{t\to\infty} X_t = -\infty$
$\rho = 1$	$+$	monotone increasing	$\mathbb{P}_x(\tau^{\{0\}} < \infty) = 0, x \in \mathbb{R}, \lim_{t\to\infty} X_t = \infty$
$\rho \in (0,1)$	$+, -$	bounded variation	$\mathbb{P}_x(\tau^{\{0\}} < \infty) = 0, x \in \mathbb{R}, \lim_{t\to\infty} \|X_t\| = \infty$
$\alpha = 1$			recurrent
$\rho = \frac{1}{2}$	$+, -$	unbounded variation	$\mathbb{P}_x(\tau^{\{0\}} < \infty) = 0, x \in \mathbb{R}$, $\limsup_{t\to\infty} \|X_t\| = \infty, \liminf_{t\to\infty} \|X_t\| = 0$
$\alpha \in (1,2)$			recurrent
$\alpha\rho = 1$	$-$	unbounded variation	$\mathbb{P}_x(\tau^{\{0\}} < \infty) = 1, x \in \mathbb{R}$, $\liminf_{t\to\infty} X_t = -\infty, \limsup_{t\to\infty} X_t = \infty$
$\alpha\rho = \alpha - 1$	$+$	unbounded variation	$\mathbb{P}_x(\tau^{\{0\}} < \infty) = 1, x \in \mathbb{R}$, $\liminf_{t\to\infty} X_t = -\infty \limsup_{t\to\infty} X_t = \infty$
$\alpha\rho \in (\alpha - 1, 1)$	$+, -$	unbounded variation	$\mathbb{P}_x(\tau^{\{0\}} < \infty) = 1, x \in \mathbb{R}$, $\liminf_{t\to\infty} X_t = -\infty, \limsup_{t\to\infty} X_t = \infty$
$\alpha = 2$			recurrent
$\rho = \frac{1}{2}$	no jumps	unbounded variation	$\mathbb{P}_x(\tau^{\{0\}} < \infty) = 1, x \in \mathbb{R}$, $\liminf_{t\to\infty} X_t = -\infty, \limsup_{t\to\infty} X_t = \infty$

Path integrals

$$\mathrm{d}\, X_t = \sigma(X_{t-})\, \mathrm{d}\, Y_t$$

- Every non-trivial solution can be written in the form

$$X_t = Y_{\tau_t} \quad \text{where} \quad \tau_t = \inf\{s > 0 : \int_0^s \sigma(Y_u)^{-2}\, \mathrm{d}\, u > t\}$$

  providing $\int_0^s \sigma(Y_u)^{-\alpha}\, \mathrm{d}\, u < \infty$ for all $s > 0$ (up to some $s_0 > 0$).

- Hence this is really about the question of understanding when path integrals of the form

$$\int_0^s \sigma(Y_u)^{-\alpha}\, \mathrm{d}\, u$$

  are finite.
- In fact we will end up addressing finiteness of such path integrals for **any** 'standard' Markov process on a general state space E (locally compact Hausdorff space with a countable bas)

Engelbert-Schmidt type results

- $\alpha = 2$: Engelbert-Schmidt (1981).
  Classic results of ES81 tell us that when $Y_t = B_t$ (standard Brownian motion)
  Theorem:

$$\exists t > 0 : \int_0^t \sigma(Y_s)^{-2} \, \mathrm{d}\, s < \infty \iff \int_{-\epsilon}^{\epsilon} \sigma(x)^{-2} \, \mathrm{d}\, x < \infty \text{ for some } \epsilon > 0.$$

- $\alpha \in (1, 2)$: Zanzotto (2002)
  Extends ES81 to the setting that Y_t is an α-stable Lévy process.
  Theorem: The exact same statement as ES81 holds except σ^{-2} replaced by $\sigma^{-\alpha}$.
- Rough idea of proof for $\alpha \in (1, 2]$:
  Can appeal to the existence of local time at each $x \in \mathbb{R}$, say $L_t^{(x)}$ and write

$$\int_0^t \sigma(Y_s)^{-\alpha} \, \mathrm{d}\, s = \int_{\mathbb{R}} \sigma(x)^{-\alpha} L_t^{(x)} \, \mathrm{d}\, x$$

  and then argue away the $L_t^{(x)}$ in the integral.
- $\alpha \in (0, 1)$: existence of local time at a point fails $\longrightarrow$ dealt with in this talk
- $\alpha = 1$: existence of local time at a point fails $\longrightarrow$ no clue what is going on

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- Is it true for α-stable Y that the natural criteria for

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to have a solution under $\mathbb{P}$ is always

$$\int_{-\epsilon}^{\epsilon} \sigma(x)^{-\alpha}\,\mathrm{d}\, x < \infty?$$

- Sanity check: Consider

$$\mathrm{d}\, X_t = |X_{t-}|^{\beta}\,\mathrm{d}\, Y_t, \qquad \beta > 0.$$

- Engelbert-Schmidt-Zanzotto integral tests predicts **for** $\alpha \in (0, 2]$ that a non-trivial solution occurs when

$$\int_{-\epsilon}^{\epsilon} |x|^{-\alpha\beta}\,\mathrm{d}\, x < \infty \Longrightarrow \beta \leq 1/\alpha$$

- It is known from (e.g. Bass-Burdzy-Chen 2004) that when $\alpha \in (0, 1)$, a sharp condition in this setting is that $\beta < 1$. In which case the Engelbert-Schmidt-Zanzotto cannot be right for $\alpha \in (0, 1)$.

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- Döring-K 2016 (following results of multiple authors e.g. including Koshnevisan, Salminen and Yor) have the following result:

Theorem: Suppose that Y is a Lévy process with local times at a point, $Y_t \to \infty$ and $f \geq 0$ is locally bounded, then

$$\mathbb{P}\left(\int_0^\infty f(Y_s)\,\mathrm{d}s < \infty\right) = 0 \text{ or } 1.$$

and

$$\int_0^\infty f(Y_s)\,\mathrm{d}s < \infty \text{ a.s.} \iff \int_{\mathbb{R}} f(x)\,\mathrm{d}x < \infty.$$

Proof: Write $U(\mathrm{d}x) = \int_0^\infty \mathbb{P}(Y_s \in \mathrm{d}x)\,\mathrm{d}s$ for the potential measure of Y. The existence of local times is equivalent to the existence of a density $U(\mathrm{d}x) = u(x)\,\mathrm{d}x$, so

$$\mathbb{E}\left[\int_0^\infty f(Y_s)\,\mathrm{d}s\right] = \int_{\mathbb{R}} f(x)u(x)\,\mathrm{d}x.$$

Then argue away the $u(x)$.

Note: we always start the Lévy process from 0 WLOG because of stationary and independents.

Perpetual integrals for Lévy processes

- Kolb-Savov 2020 have the following result (stated in terms of its added value):

  Theorem: Suppose that Y is a Lévy process without local time such that $Y_t \to \infty$, $f \geq 0$ is continuous or ultimately non-increasing (locally bounded) function

  $$\int_0^\infty f(Y_s)\,\mathrm{d}s < \infty \text{ a.s.} \iff \int_{\mathbb{R}\setminus B} f(x)U(\mathrm{d}x) < \infty$$

  for some transient set[2] B

  Proof: $\Longleftarrow$: easy to see that

  $$\int_0^\infty f(Y_s)\,\mathrm{d}s < \infty \Leftrightarrow \int_0^\infty f(Y_s)\mathbf{1}_{(Y_s\in\mathbb{R}\setminus B)}\,\mathrm{d}s < \infty \Leftarrow \int_{\mathbb{R}\setminus B} f(x)U(\mathrm{d}x) < \infty.$$

  $\Longrightarrow$: this is where the meat is.

- Main problem with these and previous results is that they require locally bounded functions.

[2] B is a transient set if the process eventually leaves it and never revisits the set. Formally: a Borel set with $\mathbb{P}(\sup\{t > 0 : Y_t \in B\} < \infty) = 1$.

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General Markov processes

- Lets consider a general standard[3] Markov process X on a general[4] state space E with a possible cemetery state (and lifetime denoted by ζ). Denote its probabilities by $\mathbb{P} = (\mathbb{P}_z, z \in E)$.

- A Borel set $B \in E$ is called $\mathbb{P}_z$-avoidable if $\mathbb{P}_z(D_B < \zeta) < 1$, where $D_B = \inf\{t \geq 0 : X_t \in D\}$.

- If B is avoidable then its complement $M = E \backslash B$ is called $\mathbb{P}_z$-supportive, and satisfies $\mathbb{P}_z(X_t \in M \text{ for all } t \in [0, \zeta)) > 0$.

- In general it is not at all clear what form avoidable or supportive sets should take but a vast literature exists for special processes. Examples include:

  - If X is a recurrent Markov process then only polar sets are avoidable.
  - If X is a symmetric stable process on $\mathbb{R}$ of index $\alpha \in (1, 2]$ then only the empty set is avoidable.
  - If X is a symmetric stable process on $\mathbb{R}$ of index $\alpha \in (0, 1)$ then any compact set not containing z is $\mathbb{P}_z$-avoidable.

[3] We mean cadlag, quasi leftcontinous, strong Markov property

[4] locally compact Hausdorff space with a countable base

Path integrals for general Markov processes

- Let us introduce the potential of X given by $U(x, \mathrm{d}\, z) = \mathbb{E}_x \left[\int_0^{\zeta} \mathbf{1}_{(X_t \in \mathrm{d}\, z)} \, \mathrm{d}\, t \right]$
  Note: as a general Markov process, we now index by the point of issue x.

- **Theorem:** Let $f : E \to [0, +\infty]$ be measurable and X a standard Markov process with (possibly infinite) life time ζ. Let $z \in E$. Then the following are equivalent.
  - $\mathbb{P}_z\big(\int_0^{\infty} f(X_s) \, \mathrm{d}\, s < \infty \big) > 0$;
  - The integral test $\int_{E \setminus B} f(x) \, U(z, \mathrm{d}\, x) < \infty$ holds for a $\mathbb{P}_z$-avoidable set B.

- **Proof:** $\Longleftarrow$. This is the easy direction as $\int_{E \setminus B} f(x) \, U(z, \mathrm{d}\, x) < \infty$ implies that $\int_0^{\infty} f(X_s) \mathbf{1}_{(X_s \in E \setminus B)} \, \mathrm{d}\, s < \infty$ w.p.p. which implies $\int_0^{\infty} f(X_s) \, \mathrm{d}\, s < \infty$ w.p.p. as $E \setminus B$ is a supporting set.

- $\Longrightarrow$ The crux of the argument is to show that if, for some $n \in \mathbb{N}, p \in (0, 1)$,

$$M_{n,p} = \{ y \in E : \mathbb{P}_y\Big(\int_0^{\infty} f(X_s) \, \mathrm{d}\, s \leq n \Big) > p \}$$

  is non-empty then $\int_M f(x) \, U(z, \mathrm{d}\, x) < \infty$.
  Moreover, $\mathbb{P}_z\big(\int_0^{\infty} f(X_s) \, \mathrm{d}\, s < \infty \big) > 0$ implies that $M_{n,p}$ is non-empty and a $\mathbb{P}_z$-supportive set.

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Various corollaries

- **Corollary:** Let X be a standard Markov process on state space E and $f : E \to [0, \infty]$ measurable. Then the following are equivalent.
  - $\mathbb{P}_z\big(\int_0^\infty f(X_s)\,\mathrm{d}\,s < \infty \big) = 1$;
  - For every $\varepsilon > 0$ there exists a $\mathbb{P}_z$-supportive set M such that $\int_M f(x)\,U(z, \mathrm{d}\,x) < \infty$ and X stays in M with probability at least $1 - \varepsilon$.

- **Corollary:** Let X be a standard Markov process on $\mathbb{R}^d$ with trivial tail σ-algebra when issued from $z \in \mathbb{R}^d$, and let $f : \mathbb{R}^d \to [0, \infty)$ be measurable and bounded on compact sets. Suppose in addition that the process is conservative ($\mathbb{P}_z(\zeta = \infty) = 1$). Then the following are equivalent:
  - $\mathbb{P}_z\big(\int_0^\infty f(X_s)\,\mathrm{d}\,s < \infty \big) > 0$;
  - $\mathbb{P}_z\big(\int_0^\infty f(X_s)\,\mathrm{d}\,s < \infty \big) = 1$;
  - There exists a transient set[5] B such that integral test $\int_{\mathbb{R}^d \setminus B} f(x)\,U(z, \mathrm{d}\,x) < \infty$ holds.

[5] a Borel set with $\mathbb{P}_z(\sup\{t > 0 : X_t \in B\} < \infty) = 1$

VARIOUS COROLLARIES

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Back to the SDE driven by an α-stable process with $\alpha \in (0,1)$

- When $\alpha \in (0,1)$, we know $U(x, \mathrm{d}y) = |x-y|^{\alpha-1}\,\mathrm{d}y$

- For finite time integrals, the role of avoidable sets is replaced by sets which are avoided for a positive amount of time, so-called thin sets.

- **Theorem:** If $N(\sigma)$ denotes the zero-set of σ and

$$\mathcal{O}(\sigma,\alpha) = \left\{ x \in \mathbb{R} : \int_{\mathbb{R}\setminus B} \sigma(y)^{-\alpha}|x-y|^{\alpha-1}\,\mathrm{d}y = \infty \text{ for all } \mathbb{P}_x\text{-thin sets } B \right\}$$

denotes the set of irregular points, then the following statements hold.

1. For fixed $z \in \mathbb{R}$ there exists a non-trivial (i.e. non-constant) local weak solution if and only if $z \notin \mathcal{O}(\sigma,\alpha)$.
2. A global weak solution exists for all initial conditions $z \in \mathbb{R}$ if and only if $\mathcal{O}(\sigma,\alpha) \subseteq N(\sigma)$.
3. A non-trivial global weak solution exists for all $z \in \mathbb{R}$ if and only if $\mathcal{O}(\sigma,\alpha) = \emptyset$.
4. There exists a global weak solution for all $z \in \mathbb{R}$, each of which is unique in law, if and only if $\mathcal{O}(\sigma,\alpha) = N(\sigma)$.

- If σ has only isolated monotone zeros (e.g. $\sigma(x) = |x|^\beta$) then we can work with the cleaner integral tests

$$\mathcal{O}(\sigma,\alpha) = \left\{ x \in \mathbb{R} : \int_{x-\varepsilon}^{x+\varepsilon} \sigma(y)^{-\alpha}|x-y|^{\alpha-1}\,\mathrm{d}y = \infty \text{ for all } \varepsilon > 0 \right\}$$

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Sanity check $\sigma(y) = |y|^\beta$, $\alpha \in (0,1)$

- For fixed $z \in \mathbb{R}$ there exists a non-trivial (i.e. non-constant) local weak solution if and only if $z \notin \mathcal{O}(\sigma, \alpha)$
- Need
$$\int_{x-\varepsilon}^{x+\varepsilon} \sigma(y)^{-\alpha} |y|^{\alpha-1} \, \mathrm{d}\, y = \int_{x-\varepsilon}^{x+\varepsilon} |y|^{\alpha(1-\beta)-1} \, \mathrm{d}\, y < \infty$$
for all $x \in \mathbb{R}$, $\varepsilon > 0$.
- Suffices that $\beta < 1$, consistent with Bass-Burdzy-Chen 2004.

Thank you!