Lévy processes and self-similar Markov processes

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J. Appl. Prob. **43**, 967–983 (2006)
Printed in Israel

CONDITIONED STABLE LÉVY PROCESSES AND THE LAMPERTI REPRESENTATION

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Abstract

By variously killing a stable Lévy process when it leaves the positive half-line, conditioning it to stay positive, and conditioning it to hit 0 continuously, we obtain three different, positive, self-similar Markov processes which illustrate the three classes described by Lamperti (1972). For each of these processes, we explicitly compute the infinitesimal generator and from this deduce the characteristics of the underlying Lévy process in the Lamperti representation. The proof of this result bears on the behaviour at time 0 of stable Lévy processes before their first passage time across level 0, which we describe here. As an application, for a certain class of Lévy processes we give the law of the minimum before an independent exponential time. This provides the explicit form of the spatial Wiener–Hopf factor at a particular point and the value of the ruin probability for this class of Lévy processes.

Keywords: Positive, self-similar Markov process; Lamperti representation; infinitesimal generator; stable Lévy process conditioned to stay positive; stable Lévy process; conditioning to hit 0 continuously

2000 Mathematics Subject Classification: Primary 60G18; 60G51; 60B52

1. Introduction and preliminary results

The stochastic processes which are considered in this work take their values in the Skorokhod space $\mathcal{D}$ of càdlàg trajectories (those that are continuous from the right with left limits). We define this set as follows: $\Delta := \infty$ being the cemetery point, a function $\omega\colon [0,\infty) \to \mathbb{R} \cup \Delta$ belongs to $\mathcal{D}$ if and only if

- for all $t \geq \zeta(\omega)$, $\omega_t = \Delta$, where $\zeta(\omega) := \inf\{t \colon \omega_t = \Delta\}$ is the lifetime of $\omega \in \mathcal{D}$ and $\inf \varnothing = \infty$;
- for all $t \geq 0$, $\lim_{s \downarrow t} \omega_s = \omega_t$; and
- for all $t \in (0, \zeta(\omega))$, $\lim_{s \uparrow t} \omega_s := w_{t-}$ is a finite real value.

The space $\mathcal{D}$ is endowed with the Skorokhod's J_1 topology. We denote by $X\colon \mathcal{D} \to \mathcal{D}$ the canonical process of the coordinates and by $(\mathcal{F}_t)$ the natural Borel filtration generated by X, i.e. $\mathcal{F}_t = \sigma(X_s,\ s \leq t)$. A probability measure P_x on $\mathcal{D}$ is the law of a Lévy process if (X, P_x) starts from x, i.e. $\mathrm{P}_x(X_0 = x) = 1$, and has independent and homogeneous increments. Note that $(X, \mathrm{P}_x) = (x + X, \mathrm{P}_0)$ and that the lifetime of (X, P_x) is either almost surely (a.s.) infinite

Received 30 March 2006; revision received 31 August 2006.
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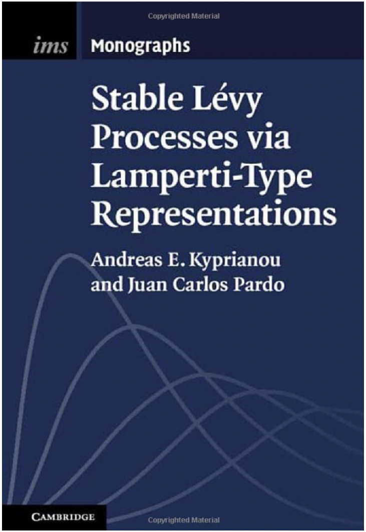

Self-similar Markov processes (ssMp)

Definition

A regular strong Markov process $(Z_t : t \geq 0)$ on $\mathbb{R}^d$, with probabilities $\mathbb{P}_x$, $x \in \mathbb{R}^d$, is a rssMp if there exists an index $\alpha \in (0, \infty)$ such that:

for all $c > 0$ and $x \in \mathbb{R}^d$

$(cZ_{tc^{-\alpha}} : t \geq 0)$ under $\mathbb{P}_x$

is equal in law to

$(Z_t : t \geq 0)$ under $\mathbb{P}_{cx}$.

Some of your best friends are ssMp

- Write $\mathcal{N}_d(\mathbf{0}, \mathbf{\Sigma})$ for the Normal distribution with mean $\mathbf{0} \in \mathbb{R}^d$ and correlation (matrix) $\mathbf{\Sigma}$. The moment generating function of $X_t \sim \mathcal{N}_d(\mathbf{0}, \mathbf{\Sigma}t)$ satisfies, for $\theta \in \mathbb{R}^d$,

$$E[\mathrm{e}^{\theta\cdot X_t}] = \mathrm{e}^{t\theta^{\mathsf{T}}\mathbf{\Sigma}\theta/2} = \mathrm{e}^{(c^{-2}t)(c\theta)^{\mathsf{T}}\mathbf{\Sigma}(c\theta)/2} = E[\mathrm{e}^{\theta\cdot cX_{c^{-2}t}}].$$

- Thinking about the stationary and independent increments of Brownian motion, this can be used to show that $\mathbb{R}^d$-**Brownian motion:** is a ssMp with $\alpha = 2$.

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Suppose that $(X_t : t \geq 0)$ is an $\mathbb{R}$-Brownian motion:

- Write $\underline{X}_t := \inf_{s \leq t} X_s$. Then $(X_t, \underline{X}_t)$, $t \geq 0$ is a Markov process.
- For $c > 0$ and $\alpha = 2$,

$$\begin{pmatrix} c\underline{X}_{c^{-\alpha}t} \\ cX_{c^{-\alpha}t} \end{pmatrix} = \begin{pmatrix} c \inf_{s \leq c^{-\alpha}t} X_s \\ cX_{c^{-\alpha}t} \end{pmatrix} = \begin{pmatrix} \inf_{u \leq t} cX_{c^{-\alpha}u} \\ cX_{c^{-\alpha}t} \end{pmatrix}, \qquad t \geq 0,$$

  and the latter is equal in law to $(X, \underline{X})$, because of the scaling property of X.
- $\Rightarrow$ Markov process $Z_t := X_t - (-x \wedge \underline{X}_t)$, $t \geq 0$ is also a ssMp on $[0, \infty)$ issued from $x > 0$ with index 2.
- $\Rightarrow Z_t := X_t \mathbf{1}_{(\underline{X}_t > 0)}$, $t \geq 0$ is also a ssMp, again on $[0, \infty)$.

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Suppose that $(X_t : t \geq 0)$ is an $\mathbb{R}^d$-Brownian motion:

- Consider $Z_t := |X_t|$, $t \geq 0$. Because of rotational invariance, it is a Markov process. Again the self-similarity (index 2) of Brownian motion, transfers to the case of $|X|$. Note again, this is a ssMp on $[0, \infty)$
- Note that $|X_t|$, $t \geq 0$ is a Bessel-d process. It turns out that all Bessel processes, *and* all squared Bessel processes are self-similar on $[0, \infty)$. Once can check this by e.g. considering scaling properties of their transition semi-groups.

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Suppose that $(X_t : t \geq 0)$ is an $\mathbb{R}^d$-Brownian motion:

- Note when $d = 3$, $|X_t|$, $t \geq 0$ is also equal in law to a Brownian motion conditioned to stay positive: i.e if we define, for a 1-d Brownian motion $(B_t : t \geq 0)$,

$$\mathbb{P}_x^{\uparrow}(A) = \lim_{s\to\infty} \mathbb{P}_x(A | \underline{B}_{t+s} > 0) = \mathbb{E}_x \left[\frac{B_t}{x} \mathbf{1}_{(\underline{B}_t > 0)} \mathbf{1}_{(A)} \right]$$

where $A \in \sigma\{B_t : u \leq t\}$, then

$$(|X_t|, t \geq 0) \text{ with } |X_0| = x \text{ is equal in law to } (B, \mathbb{P}_x^{\uparrow}).$$

More examples?

- All of the previous examples have in common that their paths are continuous. Is this a necessary condition?
- We want to find more exotic examples as most of the previous examples have been extensively studied through existing theories (of Brownian motion and continuous semi-martingales).

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- All of the previous examples are functional transforms of Brownian motion and have made use of the scaling and Markov properties and (in some cases) isotropic distributional invariance.
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Interlude: (killed) Lévy process

- $(\xi_t, t \geq 0)$ is a (killed) Lévy process if it has stationary and independents with RCLL paths (and is sent to a cemetery state after and independent and exponentially distributed time).
- Process is entirely characterised by its one-dimensional transitions, which are coded by the Lévy–Khinchine formula

$$\mathbb{E}[e^{i\theta\xi_t}] = e^{-\Psi(\theta)t}, \qquad \theta \in \mathbb{R},$$

where,

$$\Psi(\theta) = q + ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}^d} (1 - e^{i\theta x} + i\theta x \mathbf{1}_{(|x|<1)})\Pi(dx),$$

where $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and Π is a measure satisfying $\int_{\mathbb{R}^d}(1 \wedge |x|^2)\Pi(dx) < \infty$. Think of Π as the intensity of jumps in the sense of

$$\mathbb{P}(X \text{ has jump at time } t \text{ of size } dx) = \Pi(dx)dt + o(dt).$$

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Interlude: (killed) Lévy process

- In one dimension the path of a Lévy process can be monotone, in which case it is called a *subordinator* and we work with the Laplace exponent

$$\mathbb{E}[\mathrm{e}^{-\lambda\xi_t}] = \mathrm{e}^{-\Phi(\lambda)t}, \qquad t \geq 0$$

where

$$\Phi(\lambda) = q + \delta\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x})\Upsilon(\mathrm{d}x), \qquad \lambda \geq 0.$$

Lévy process: One dimension

- If ξ has a characteristic exponent Ψ then necessarily

$$\Psi(\theta) = \kappa(-\mathrm{i}\theta)\hat{\kappa}(\mathrm{i}\theta), \qquad \theta \in \mathbb{R}.$$

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- The factorisation has a physical interpretation:
  - range of the κ-subordinator agrees with the range of $\sup_{s\leq t} \xi_s$, $t \geq 0$
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α-stable process

Definition

A Lévy process X is called (strictly) α-stable if it is also a self-similar Markov process.

- Necessarily $\alpha \in (0,2]$. [$\alpha = 2 \to$ BM, exclude this.]
- The characteristic exponent $\Psi(\theta) := -t^{-1} \log \mathbb{E}(e^{i\theta X_t})$ satisfies

$$\Psi(\theta) = |\theta|^{\alpha}(e^{\pi i\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta>0)} + e^{-\pi i\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta<0)}), \qquad \theta \in \mathbb{R}.$$

where $\rho = P_0(X_t \geq 0)$ will frequently appear as will $\hat{\rho} = 1 - \rho$

- Assume jumps in both directions ($0 < \alpha\rho, \alpha\hat{\rho} < 1$), so that the Lévy **density** takes the form

$$\frac{\Gamma(1+\alpha)}{\pi} \frac{1}{|x|^{1+\alpha}} \left(\sin(\pi\alpha\rho)\mathbf{1}_{\{x>0\}} + \sin(\pi\alpha\hat{\rho})\mathbf{1}_{\{x<0\}}\right)$$

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Your new friends

Suppose $X = (X_t : t \geq 0)$ is within the assumed class of α-stable processes in one-dimension and let $\underline{X}_t = \inf_{s \leq t} X_s$. Your new friends are:

- $Z = X$
- $Z = X - (-x \wedge \underline{X})$, $x > 0$.
- $Z = X\mathbf{1}_{(\underline{X} > 0)}$
- $Z = |X|$ providing $\rho = 1/2$
- What about $Z =$"X conditioned to stay positive"?

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Conditioned α-stable processes

- Recall that each Lévy processes, $\xi = \{\xi_t : t \geq 0\}$, enjoys the Wiener-Hopf factorisation i.e. up to a multiplicative constant, $\Psi_\xi(\theta) := t^{-1} \log \mathbb{E}[\mathrm{e}^{\mathrm{i}\theta\xi_t}]$ respects the factorisation

$$\Psi_\xi(\theta) = \kappa(-\mathrm{i}\theta)\hat{\kappa}(\mathrm{i}\theta), \qquad \theta \in \mathbb{R},$$

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$$\kappa(\lambda) = q + \mathrm{a}\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x})\nu(\mathrm{d}x), \qquad \lambda \geq 0$$

where ν is a measure satisfying $\int_{(0,\infty)} 1 \wedge x\nu(\mathrm{d}x) < \infty$.

- The probabilistic significance of these subordinators, is that their range corresponds precisely to the range of the running maximum of ξ and of $-\xi$ respectively.
- In the case of α-stable processes, up to a multiplicative constant,

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Conditioned α-stable processes

- Associated to the descending ladder subordinator $\hat{\kappa}$ is its potential measure $\hat{U}$, which satisfies

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda x} \hat{U}(\mathrm{d}x) = \frac{1}{\hat{\kappa}(\lambda)}, \qquad \lambda \geq 0.$$

- It can be shown that for a Lévy process which satisfies $\limsup_{t\to\infty} \xi_t = \infty$, for $A \in \sigma(\xi_u : u \leq t)$,

$$\mathbb{P}_x^{\uparrow}(A) = \lim_{s\to\infty} \mathbb{P}_x(A | \underline{X}_{t+s} > 0) = \mathbb{E}_x\left[\frac{\hat{U}(X_t)}{\hat{U}(x)} \mathbf{1}_{(\underline{X}_t > 0)} \mathbf{1}_{(A)}\right]$$

- In the α-stable case $\hat{U}(x) \propto x^{\alpha\hat{\rho}}$
[Note in the excluded case that $\alpha = 2$ and $\rho = 1/2$, i.e. Brownian motion, $\hat{U}(x) = x$.]

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[Note in the excluded case that $\alpha = 2$ and $\rho = 1/2$, i.e. Brownian motion, $\hat{U}(x) = x$.]

Conditioned α-stable processes

- Associated to the descending ladder subordinator $\hat{\kappa}$ is its potential measure $\hat{U}$, which satisfies

$$\int_{[0,\infty)} e^{-\lambda x} \hat{U}(dx) = \frac{1}{\hat{\kappa}(\lambda)}, \qquad \lambda \geq 0.$$

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Conditioned α-stable processes

- For $c, x > 0$, $t \geq 0$ and appropriately bounded, measurable and non-negative f, we can write,

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\begin{aligned}
&\mathbb{E}^{\uparrow}_x[f(\{cX_{c^{-\alpha}s} : s \leq t\})] \\
&\quad = \mathbb{E}\left[f(\{cX^{(x)}_{c^{-\alpha}s} : s \leq t\}) \frac{(X^{(x)}_{c^{-\alpha}t})^{\alpha\hat{\rho}}}{x^{\alpha\hat{\rho}}} \mathbf{1}_{(\underline{X}^{(x)}_{c^{-\alpha}t} \geq 0)} \right] \\
&\quad = \mathbb{E}\left[f(\{X^{(cx)}_s : s \leq t\} \frac{(X^{(cx)}_t)^{\alpha\hat{\rho}}}{(cx)^{\alpha\hat{\rho}}} \mathbf{1}_{(\underline{X}^{(cx)}_t \geq 0)} \right] \\
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\end{aligned}
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- This also makes the process $(X, \mathbb{P}^{\uparrow}_x)$, $x > 0$, a self-similar Markov process on $[0, \infty)$.
- Unlike the case of Brownian motion, the conditioned stable process does not have the law of the radial part of a 3-dimensional stable process.

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Notation

- Use $\xi := \{\xi_t : t \geq 0\}$ to denote a Lévy process which is killed and sent to the cemetery state $-\infty$ at an independent and exponentially distributed random time, $\mathbf{e}_q$, with rate in $q \in [0, \infty)$. The characteristic exponent of ξ is thus written

$$-\log E(\mathrm{e}^{\mathrm{i}\theta\xi_1}) = \Psi(\theta) = q + \text{Lévy–Khintchine}$$

- Define the associated integrated exponential Lévy process

$$I_t = \int_0^t \mathrm{e}^{\alpha\xi_s}\mathrm{d}s, \qquad t \geq 0. \tag{1}$$

and its limit, $I_\infty := \lim_{t\uparrow\infty} I_t$.

- Also interested in the inverse process of I:

$$\varphi(t) = \inf\{s > 0 : I_s > t\}, \qquad t \geq 0. \tag{2}$$

As usual, we work with the convention $\inf\emptyset = \infty$.

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Lamperti transform for POSITIVE ssMp

Theorem (Part (i))

Fix $\alpha > 0$. If $Z^{(x)}$, $x > 0$, is a positive self-similar Markov process with index of self-similarity α, then up to absorption at the origin, it can be represented as follows. For $x > 0$,

$$Z_t^{(x)} \mathbf{1}_{(t < \zeta^{(x)})} = x \exp\{\xi_{\varphi(x^{-\alpha} t)}\}, \qquad t \geq 0,$$

where $\zeta^{(x)} = \inf\{t > 0 : Z_t^{(x)} = 0\}$ and either

(1) $\zeta^{(x)} = \infty$ *almost surely for all $x > 0$, in which case ξ is a Lévy process satisfying* $\limsup_{t\uparrow\infty} \xi_t = \infty$,

(2) $\zeta^{(x)} < \infty$ *and* $Z^{(x)}_{\zeta^{(x)}-} = 0$ *almost surely for all $x > 0$, in which case ξ is a Lévy process satisfying* $\lim_{t\uparrow\infty} \xi_t = -\infty$, *or*

(3) $\zeta^{(x)} < \infty$ *and* $Z^{(x)}_{\zeta^{(x)}-} > 0$ *almost surely for all $x > 0$, in which case ξ is a Lévy process killed at an independent and exponentially distributed random time.*

In all cases, we may identify $\zeta^{(x)} = x^\alpha I_\infty$.

Lamperti transform for POSITIVE ssMp

Theorem (Part (ii))

Conversely, suppose that ξ is a given (killed) Lévy process. For each $x > 0$, define

$$Z_t^{(x)} = x\exp\{\xi_{\varphi(x^{-\alpha}t)}\}\mathbf{1}_{(t<x^{\alpha}I_\infty)}, \qquad t \geq 0.$$

Then $Z^{(x)}$ defines a positive self-similar Markov process, up to its absorption time $\zeta^{(x)} = x^{\alpha}I_\infty$, with index α.

Lamperti transform for POSITIVE ssMp

$(Z, \mathrm{P}_x)_{x>0}$ pssMp $\leftrightarrow$ $(\xi, \mathbb{P}_y)_{y\in\mathbb{R}}$ killed Lévy

$Z_t = \exp(\xi_{S(t)})$, $\xi_s = \log(Z_{T(s)})$,

S a random time-change T a random time-change

$$\left.\begin{array}{c} Z \text{ never hits zero} \\ Z \text{ hits zero continuously} \\ Z \text{ hits zero by a jump} \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} \xi \to \infty \text{ or } \xi \text{ oscillates} \\ \xi \to -\infty \\ \xi \text{ is killed} \end{array}\right.$$

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Stable process killed on entry to $(-\infty, 0)$

- The stable process cannot 'creep' downwards across the threshold 0 and so must do so with a jump.
- This puts $Z_t^* := X_t \mathbf{1}_{(\underline{X}_t > 0)}$, $t \geq 0$, in the class of pssMp for which the underlying Lévy process experiences exponential killing.
- Write $\xi^* = \{\xi_t^* : t \geq 0\}$ for the underlying Lévy process and denote its killing rate by q^*.
- Let's try and decode the characteristics of ξ^*.

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- We know that the α-stable process experiences downward jumps at rate

$$\frac{\Gamma(1+\alpha)}{\pi}\sin(\pi\alpha\hat{\rho})\frac{1}{|x|^{1+\alpha}}\mathrm{d}x, \qquad x<0.$$

- Given that we know the value of Z^*_{t-}, on $\{\underline{X}_t > 0\}$, the stable process will pass over the origin at rate

$$\frac{\Gamma(1+\alpha)}{\pi}\sin(\pi\alpha\hat{\rho})\left(\int_{Z^*_{t-}}^{\infty}\frac{1}{|x|^{1+\alpha}}\mathrm{d}x\right) = \frac{\Gamma(1+\alpha)}{\alpha\pi}\sin(\pi\alpha\hat{\rho})(Z^*)^{-\alpha}_{t-}.$$

- On the other hand, the Lamperti transform says that on $\{t<\zeta\}$, as a pssMp, Z is sent to the origin at rate

$$q^*\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = q^*\mathrm{e}^{-\alpha\xi^*_{\varphi(t)}} = q^*(Z^*)^{-\alpha}_t.$$

- Comparing gives us

$$q^* = \Gamma(\alpha)\sin(\pi\alpha\hat{\rho})/\pi = \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}.$$

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- Referring again to the Lamperti transform, we know that, under $\mathbb{P}_1$ (so that $\mathbb{P}_1(\xi^*_0 = 0) = 1$),

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where $\mathbf{e}_{q^*}$ is an exponentially distributed random variable with rate q^*.

- This motivates the computation

$$\mathbb{E}_1[(Z^*_{\zeta-})^{\mathrm{i}\theta}] = \mathbb{E}_1[\mathrm{e}^{\mathrm{i}\theta\xi^*_{\mathbf{e}_{q^*}-}}] = \frac{q^*}{(\Psi^*(z) - q^*) + q^*}, \qquad \theta \in \mathbb{R},$$

where Ψ^* is the characteristic exponent of ξ^*.

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Stable process killed on entry to $(-\infty, 0)$

Setting

$$K = \frac{\sin \alpha\hat{\rho}\pi}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})},$$

Remembering the “overshoot-undershoot” distributional law at first passage (well known in the literature) and deduce that, for all $v \in [0,1]$,

$$\begin{aligned}
&\mathbb{P}_1(X_{\tau_0^- -} \in \mathrm{d}v) \\
&= \hat{\mathbb{P}}_0(1 - X_{\tau_1^+ -} \in \mathrm{d}v) \\
&= K \left(\int_0^\infty \int_0^\infty \mathbf{1}_{(y \le 1 \wedge v)} \frac{(1-y)^{\alpha\hat{\rho}-1}(v-y)^{\alpha\rho-1}}{(v+u)^{1+\alpha}} \mathrm{d}u\mathrm{d}y \right) \mathrm{d}v \\
&= \frac{K}{\alpha} \left(\int_0^1 \mathbf{1}_{(y\le v)} v^{-\alpha}(1-y)^{\alpha\hat{\rho}-1}(v-y)^{\alpha\rho-1}\mathrm{d}y \right) \mathrm{d}v,
\end{aligned}$$

where $\hat{\mathbb{P}}_0$ is the law of $-X$ issued from 0.

Stable process killed on entry to $(-\infty, 0)$

We are led to the conclusion that

$$
\begin{aligned}
\frac{q*}{\Psi^*(\theta)} &= \frac{K}{\alpha}\int_0^1 (1-y)^{\alpha\hat\rho-1}\int_0^\infty \mathbf{1}_{(y\le v)} v^{i\theta-\alpha\hat\rho-1}\left(1-\frac{y}{v}\right)^{\alpha\rho-1} dv dy \\
&= \frac{K}{\alpha}\int_0^1 (1-y)^{\alpha\hat\rho-1} y^{i\theta-\alpha\hat\rho} dy \frac{\Gamma(\alpha\hat\rho-i\theta)\Gamma(\alpha\rho)}{\Gamma(\alpha-i\theta)} \\
&= \frac{\Gamma(\alpha\hat\rho-i\theta)\Gamma(\alpha\rho)\Gamma(1-\alpha\hat\rho+i\theta)\Gamma(\alpha\hat\rho)\Gamma(\alpha+1)}{\alpha\Gamma(\alpha\rho)\Gamma(\alpha\hat\rho)\Gamma(1-\alpha\hat\rho)\Gamma(\alpha\hat\rho)\Gamma(1+i\theta)\Gamma(\alpha-i\theta)},
\end{aligned}
$$

where in the first equality Fubini's Theorem has been used, in the second equality a straightforward substitution $w = y/v$ has been used for the inner integral on the preceding line together with the classical beta integral and, finally, in the third equality, the Beta integral has been used for a second time. Inserting the respective values for the constants q^* and K, we come to rest at the following result:

Stable process killed on entry to $(-\infty, 0)$

Theorem

For the pssMp constructed by killing a stable process on first entry to $(-\infty, 0)$*, the underlying* *killed* *Lévy process,* ξ^**, that appears through the Lamperti transform has characteristic exponent given by*

$$\Psi^*(z) = \frac{\Gamma(\alpha - \mathrm{i}z)}{\Gamma(\alpha\hat{\rho} - \mathrm{i}z)} \frac{\Gamma(1 + \mathrm{i}z)}{\Gamma(1 - \alpha\hat{\rho} + \mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

Stable processes conditioned to stay positive

- Use the Lamperti representation of the α-stable process X to write, for $A \in \sigma(X_u : u \le t)$,

$$\mathbb{P}^{\uparrow}_x(A) = \mathbb{E}_x\left[\frac{X_t^{\alpha\hat\rho}}{x^{\alpha\hat\rho}}\mathbf{1}_{(\underline{X}_t>0)}\mathbf{1}_{(A)}\right] = E\left[\mathrm{e}^{\alpha\hat\rho\xi^*_\tau}\mathbf{1}_{(\tau<\mathbf{e}_{q^*})}\mathbf{1}_{(A)}\right],$$

where $\tau = \varphi(x^{-\alpha}t)$ is a stopping time in the natural filtration of ξ^*.

- Noting that $\Psi^*(-\mathrm{i}\alpha\hat\rho) = 0$, the change of measure constitutes an Esscher transform at the level of ξ^*.

Theorem

The underlying Lévy process, $\xi^\uparrow$, that appears through the Lamperti transform applied to $(X, \mathbb{P}^\uparrow_x)$, $x > 0$, has characteristic exponent given by

$$\Psi^\uparrow(z) = \frac{\Gamma(\alpha\rho - \mathrm{i}z)}{\Gamma(-\mathrm{i}z)}\frac{\Gamma(1+\alpha\hat\rho+\mathrm{i}z)}{\Gamma(1+\mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

- In particular $\Psi^\uparrow(z) = \Psi^*(z - \mathrm{i}\alpha\hat\rho)$, $z \in \mathbb{R}$ so that $\Psi^\uparrow(0) = 0$ (i.e. no killing!)
- One can also check by hand that $\Psi^{\uparrow\prime}(0+) = E[\xi^\uparrow_1] > 0$ so that $\lim_{t\to\infty}\xi^\uparrow_t = \infty$.

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Did you spot the other root?

- In essence, the case of the stable process conditioned to stay positive boils down to an Esscher transform in the underlying (Lamperti-transformed) Lévy process.
- It was important that we identified a root of $\Psi^*(z) = 0$ in order to avoid involving a 'time component' of the Esscher transform.
- However, there is another root of the equation

$$\Psi^*(z) = \frac{\Gamma(\alpha - \mathrm{i}z)}{\Gamma(\alpha\hat{\rho} - \mathrm{i}z)} \frac{\Gamma(1 + \mathrm{i}z)}{\Gamma(1 - \alpha\hat{\rho} + \mathrm{i}z)} = 0,$$

namely $z = -\mathrm{i}(1 - \alpha\hat{\rho})$.

- And this means that

$$\mathrm{e}^{(1-\alpha\hat{\rho})\xi^*}, \qquad t \geq 0,$$

is a unit-mean Martingale, which can also be used to construct an Esscher transform:

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Reverse engineering

- What now happens if we define for $A \in \sigma(X_u : u \leq t)$,

$$\mathbb{P}_x^{\downarrow}(A) = E\left[\mathrm{e}^{(1-\alpha\hat{\rho})\xi^*_\tau}\mathbf{1}_{(\tau<\mathbf{e}_{q^*})}\mathbf{1}_{(A)}\right] = \mathbb{E}_x\left[\frac{X_t^{(1-\alpha\hat{\rho})}}{x^{(1-\alpha\hat{\rho})}}\mathbf{1}_{(\underline{X}_t>0)}\mathbf{1}_{(A)}\right],$$

  where $\tau = \varphi(x^{-\alpha}t)$ is a stopping time in the natural filtration of ξ^*.
- In the same way we checked that $(X, \mathbb{P}_x^{\uparrow})$, $x > 0$, is a pssMp, we can also check that $(X, \mathbb{P}_x^{\downarrow})$, $x > 0$ is a pssMp.
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ξ^*, $\xi^\uparrow$ and $\xi^\downarrow$

- The three examples of pssMp offer quite striking underlying Lévy processes
- Is this exceptional?

Censored stable processes

- Start with X, the stable process.
- Let $A_t = \int_0^t \mathbf{1}_{(X_t>0)}\,\mathrm{d}t$.
- Let γ be the right-inverse of A, and put $\check{Z}_t := X_{\gamma(t)}$.
- Finally, make zero an absorbing state: $Z_t = \check{Z}_t \mathbb{1}_{(t<T_0)}$ where

$$T_0 = \inf\{t > 0 : X_t = 0\}.$$

  Note $T_0 < \infty$ a.s. if and only if $\alpha \in (1,2)$ and otherwise $T_0 = \infty$ a.s.
- This is the censored stable process.

Censored stable processes

Theorem

Suppose that the underlying Lévy process for the censored stable process is denoted by $\overset{\leadsto}{\xi}$. Then $\overset{\leadsto}{\xi}$ is equal in law to $\xi^{**} \oplus \xi^{C}$, with

- ξ^{**} *equal in law to* ξ^{*} *with the killing removed,*
- ξ^{C} *a compound Poisson process with jump rate* $q^{*} = \Gamma(\alpha)\sin(\pi\alpha\hat{\rho})/\pi$.

Moreover, the characteristic exponent of $\overset{\leadsto}{\xi}$ is given by

$$\overset{\leadsto}{\Psi}(z) = \frac{\Gamma(\alpha\rho - iz)}{\Gamma(-iz)}\frac{\Gamma(1-\alpha\rho+iz)}{\Gamma(1-\alpha+iz)}, \qquad z \in \mathbb{R}.$$

The radial part of a stable process

- Suppose that X is a symmetric stable process, i.e $\rho = 1/2$.
- We know that $|X|$ is a pssMp.

Theorem

Suppose that the underlying Lévy process for $|X|$ is written $\xi^{\odot}$, then it characteristic exponent is given by

$$\Psi^{\odot}(z) = 2^{\alpha} \frac{\Gamma(\frac{1}{2}(-\mathrm{i}z + \alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)} \frac{\Gamma(\frac{1}{2}(\mathrm{i}z + 1))}{\Gamma(\frac{1}{2}(\mathrm{i}z + 1 - \alpha))}, \qquad z \in \mathbb{R}.$$

Hypergeometric Lévy processes

Definition (and Theorem)

For $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$ in

$$\left\{ \ \beta \leq 2, \ \gamma, \hat{\gamma} \in (0,1) \ \hat{\beta} \geq -1, \text{ and } 1 - \beta + \hat{\beta} + \gamma \wedge \hat{\gamma} \geq 0 \ \right\}$$

there exists a (killed) Lévy process, henceforth refered to as a hypergeometric Lévy process, having the characteristic function

$$\Psi(z) = \frac{\Gamma(1-\beta+\gamma-\mathrm{i}z)}{\Gamma(1-\beta-\mathrm{i}z)} \frac{\Gamma(\hat{\beta}+\hat{\gamma}+\mathrm{i}z)}{\Gamma(\hat{\beta}+\mathrm{i}z)} \qquad z \in \mathbb{R}.$$

The Lévy measure of Y has a density with respect to Lebesgue measure is given by

$$\pi(x) = \begin{cases} -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\hat{\gamma})\Gamma(-\gamma)} \mathrm{e}^{-(1-\beta+\gamma)x} {}_2F_1\left(1+\gamma, \eta; \eta-\hat{\gamma}; \mathrm{e}^{-x}\right), & \text{if } \ x > 0, \\[2ex] -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\gamma)\Gamma(-\hat{\gamma})} \mathrm{e}^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1\left(1+\hat{\gamma}, \eta; \eta-\gamma; \mathrm{e}^{x}\right), & \text{if } \ x < 0, \end{cases}$$

where $\eta := 1 - \beta + \gamma + \hat{\beta} + \hat{\gamma}$, for $|z| < 1$, ${}_2F_1(a, b; c; z) := \sum_{k \geq 0} \frac{(a)_k (b)_k}{(c)_k k!} z^k$.

- So far we only spoke about $[0, \infty)$.
- What can we say about $\mathbb{R}$-valued self-similar Markov processes.
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Markov additive processes (MAPs)

- E is a finite state space
- $(J(t))_{t\geq 0}$ is a continuous-time, irreducible Markov chain on E
- process (ξ, J) in $\mathbb{R} \times E$ is called a *Markov additive process (MAP)* with probabilities $\mathbb{P}_{x,i}$, $x \in \mathbb{R}$, $i \in E$, if, for any $i \in E$, $s, t \geq 0$: Given $\{J(t) = i\}$,
  - $(\xi(t+s) - \xi(t), J(t+s)) \perp \{(\xi(u), J(u)) : u \leq t\}$,
  - $(\xi(t+s) - \xi(t), J(t+s)) \stackrel{d}{=} (\xi(s), J(s))$ with $(\xi(0), J(0)) = (0, i)$.

Pathwise description of a MAP

The pair (ξ, J) is a Markov additive process if and only if, for each $i, j \in E$,

- there exist a sequence of iid Lévy processes $(\xi_i^n)_{n\geq 0}$
- and a sequence of iid random variables $(U_{ij}^n)_{n\geq 0}$, independent of the chain J,
- such that if $T_0 = 0$ and $(T_n)_{n\geq 1}$ are the jump times of J,

the process ξ has the representation

$$\xi(t) = \mathbb{1}_{(n>0)}(\xi(T_n-) + U^n_{J(T_n-),J(T_n)}) + \xi^n_{J(T_n)}(t - T_n),$$

for $t \in [T_n, T_{n+1})$, $n \geq 0$.

Lamperti-Kiu transform

- Take J to be irreducible on $E = \{1, -1\}$.
- Let

$$Z_t = |x| e^{\xi(\tau(|x|^{-\alpha} t))} J(\tau(|x|^{-\alpha} t)) \qquad 0 \le t < T_0,$$

where

$$\tau(t) = \inf\left\{ s > 0 : \int_0^s \exp(\alpha \xi(u)) \mathrm{d}u > t \right\}$$

and

$$T_0 = |x|^{-\alpha} \int_0^\infty e^{\alpha \xi(u)} \mathrm{d}u.$$

- Then Z_t is a real-valued self-similar Markov process in the sense that the law of $(cZ_{tc^{-\alpha}} : t \ge 0)$ under $\mathbb{P}_x$ is $\mathbb{P}_{cx}$.
- The converse (within a special class of rssMps) is also true.

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Characteristics of a MAP

- Denote the transition rate matrix of the chain J by $\mathbf{Q} = (q_{ij})_{i,j \in E}$.
- For each $i \in E$, the Laplace exponent of the Lévy process ξ_i will be written ψ_i (when it exists).
- For each pair of $i, j \in E$, define the Laplace transform $G_{ij}(z) = \mathbb{E}(e^{zU_{ij}})$ of the jump distribution U_{ij} (when it exists).
- Write $G(z)$ for the $N \times N$ matrix whose (i,j)th element is $G_{ij}(z)$.
- Let

$$\boldsymbol{\Psi}(z) = \mathrm{diag}(\psi_1(z), \ldots, \psi_N(z)) + \mathbf{Q} \circ G(z),$$

  (when it exists), where $\circ$ indicates elementwise multiplication.
- The matrix exponent of the MAP (ξ, J) is given by

$$\mathbb{E}_{0,i}(e^{z\xi(t)}; J(t) = j) = \left(e^{\boldsymbol{\Psi}(z)t}\right)_{i,j}, \qquad i, j \in E,$$

  (when it exists).

An α-stable process is a rssMp

- An α-stable process up to absorption in the origin is a rssMp.
- When $\alpha \in (0,1]$, the process never hits the origin a.s.
- When $\alpha \in (1,2)$, the process is absorbs at the origin a.s.
- The matrix exponent of the underlying MAP is given by:

$$\begin{bmatrix} \frac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho} - z)\Gamma(1-\alpha\hat{\rho}+z)} & -\frac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \\ -\frac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & \frac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\rho - z)\Gamma(1-\alpha\rho+z)} \end{bmatrix},$$

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Esscher transform for MAPs

- If $\mathbf{\Psi}(z)$ is well defined then it has a real simple eigenvalue $\chi(z)$, which is larger than the real part of all its other eigenvalues.
- Furthermore, the corresponding right-eigenvector $\mathbf{v}(z) = (v_1(z), \cdots, v_N(z))$ has strictly positive entries and may be normalised such that $\pi \cdot \mathbf{v}(z) = 1$.

Theorem

Let $\mathcal{G}_t = \sigma\{(\xi(s), J(s)) : s \leq t\}$, $t \geq 0$, *and*

$$M_t := \mathrm{e}^{\gamma\xi(t) - \chi(\gamma)t} \frac{v_{J(t)}(\gamma)}{v_i(\gamma)}, \qquad t \geq 0,$$

for some $\gamma \in \mathbb{R}$ *such that* $\chi(\gamma)$ *is defined. Then,* M_t, $t \geq 0$, *is a unit-mean martingale. Moreover, under the change of measure*

$$\mathrm{d}\mathbf{P}^{\gamma}_{0,i}\Big|_{\mathcal{G}_t} = M_t \, \mathrm{d}\mathbf{P}_{0,i}\Big|_{\mathcal{G}_t}, \qquad t \geq 0,$$

the process (ξ, J) *remains in the class of MAPs with new exponent given by*

$$\mathbf{\Psi}_\gamma(z) = \mathbf{\Delta}_v(\gamma)^{-1}\mathbf{\Psi}(z+\gamma)\mathbf{\Delta}_v(\gamma) - \chi(\gamma)\mathbf{I}.$$

Here, $\mathbf{I}$ *is the identity matrix and* $\mathbf{\Delta}_v(\gamma) = \mathrm{diag}(\boldsymbol{v}(\gamma))$.

Esscher and drift

- Suppose that χ is defined in some open interval D of $\mathbb{R}$, then, it is smooth and convex on D.
- Since $\mathbf{\Psi}(0) = -\mathbf{Q}$, we always have $\chi(0) = 0$ and $\mathbf{v}(0) = (1, \cdots, 1)$. So $0 \in D$ and $\chi'(0)$ is well defined and finite.
- With all of the above

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Esscher and the stable-MAP

- For the MAP that underlies the stable process $D = (-1, \alpha)$, it can be checked that $\det \Psi(\alpha - 1) = 0$ i.e. $\chi(\alpha - 1) = 0$, which makes

$$\mathbf{\Psi}^{\bullet}(z) = \mathbf{\Delta}^{-1}\mathbf{\Psi}(z+\alpha-1)\mathbf{\Delta} = \begin{bmatrix} \dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\rho-z)\Gamma(\alpha\rho+z)} & -\dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} \\ -\dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} & \dfrac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\hat{\rho}-z)\Gamma(\alpha\hat{\rho}+z)} \end{bmatrix},$$

  where $\mathbf{\Delta} = \text{diag}(\sin(\pi\alpha\hat{\rho}), \sin(\pi\alpha\rho))$.
- When $\alpha \in (0, 1)$, $\chi'(0) > 0$ (because the stable process never touches the origin a.s.) and $\mathbf{\Psi}^{\bullet}(z)$-MAP drifts to $-\infty$
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Riesz-Bogdan-Zak transform

Theorem (Riesz–Bogdan–Zak transform)

Suppose that X is an α-stable process as outlined in the introduction. Define

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} \mathrm{d}u > t\}, \qquad t \geq 0.$$

Then, for all $x \in \mathbb{R}\backslash\{0\}$, $(-1/X_{\eta(t)})_{t\geq 0}$ under $\mathbb{P}_x$ is equal in law to $(X, \mathbb{P}^\bullet_{-1/x})$, where

$$\left.\frac{\mathrm{d}\mathbb{P}^\bullet_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \left(\frac{\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\mathrm{sgn}(X_t)}{\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\mathrm{sgn}(x)}\right) \left|\frac{X_t}{x}\right|^{\alpha-1} \mathbf{1}_{(t<\tau^{\{0\}})}$$

and $\mathcal{F}_t := \sigma(X_s : s \leq t)$, $t \geq 0$. Moreover, the process $(X, \mathbb{P}^\bullet_x)$, $x \in \mathbb{R}\backslash\{0\}$ is a self-similar Markov process with underlying MAP via the Lamperti-Kiu transform given by $\mathbf{\Psi}^\bullet(z)$.

What is the $\mathbf{\Psi}^\bullet$-MAP?

Thinking of the affect on the long term behaviour of the underlying MAP of the Esscher transform

- When $\alpha \in (0,1)$, $(X, \mathbb{P}^\bullet_x)$, $x \neq 0$ has the law of the the stable process conditioned to absorb continuously at the origin.
- When $\alpha \in (1,2)$, $(X, \mathbb{P}^\bullet_x)$, $x \neq 0$ has the law of the stable process conditioned to avoid the origin.

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