Anomalous subdiffusion and time-fractional differential equations II

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Framework

Recall $S_t = \kappa t + \overline{S}_t$ is a subordinator that is not compounded Poisson; i.e. either ν is infinite or $\kappa > 0$. Define $w(x) = \nu[x, \infty)$.

$$\phi(\lambda) = \kappa\lambda + \int_0^\infty (1 - e^{-\lambda x})\nu(dx) =: \kappa\lambda + \phi_0(\lambda)$$

Facts: (i) $t \mapsto S_t$ is strictly increasing so its inverse subordinator L_t is continuous in t.

(ii) $\mathbb{E}\left[e^{\lambda L_t}\right] < \infty$ for any $\lambda \geq 1$.

Suppose that $\{T_t; t \geq 0\}$ is a strongly continuous semigroup with infinitesimal generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ in some Banach space $(\mathbb{B}, \|\cdot\|)$. Note $\|T_t\| \leq c\, e^{\alpha t}$ for some $c, \alpha > 0$.

E.g. Markov transition semigroups; Schrödinger semigroups.

E.g. $(\mathbb{B}, \|\cdot\|) = L^p(\mathcal{X}; \mu)$ for $p \geq 1$ or $(C_\infty(\mathcal{X}), \|\cdot\|_\infty)$.

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Main result

Theorem (C. CSF '17, CJAPS '24)

For every $f \in \mathcal{D}(\mathcal{L})$, $u(t,x) := \mathbb{E}\left[T_{L_t} f(x)\right]$ *is the unique solution in* $(\mathbb{B}, \|\cdot\|)$ *to*

$$(\kappa\partial_t + \partial_t^w)\, u(t,x) = \mathcal{L}u(t,x) \quad \text{with } u(0,x) = f(x)$$

in the following sense:

1. $u(t) \in \mathcal{D}(\mathcal{L})$, $\|u(t)\| + \|\mathcal{L}u(t)\| \le c\, e^{\alpha t}$, *and* $t \mapsto u(t)$ *is continuous in* $(\mathbb{B}, \|\cdot\|)$;
2. *for every* $t > 0$, $I_t^w(u) := \int_0^t w(t-s)(u(s) - f(x))ds$ *converges absolutely in* $(\mathbb{B}, \|\cdot\|)$ *and*

$$\lim_{\delta\to 0} \frac{1}{\delta}\left(\kappa(u(t+\delta) - u(t)) + I_{t+\delta}^w(u) - I_t^w(u)\right) = \mathcal{L}u(t)$$

in $(\mathbb{B}, \|\cdot\|)$.

Main result

Theorem (C. 17 & 24 (continued))

In addition, $t \mapsto \mathcal{L}u(t)$ are continuous in $(\mathbb{B}, \|\cdot\|)$. When $\kappa > 0$, $t \mapsto u(t)$ is globally Lipschitz continuous in $(\mathbb{B}, \|\cdot\|)$, and both $\partial_t u(t)$ and $\frac{d}{dt} I_t^w(u)$ exists as a continuous function taking values in $(\mathbb{B}, \|\cdot\|)$.

Conversely, if $u(t)$ is a solution in the sense of (i) and (ii) above with $f \in \mathcal{D}(\mathcal{L})$, then $u(t) = \mathbb{E}\left[T_{L_t} f(x)\right]$ in $\mathbb{B}$ for every $t \geq 0$.

Uniqueness

The Laplace transform of w is

$$\int_0^\infty e^{-\lambda x} w(x)dx = \frac{1}{\lambda}\int_0^\infty \left(1 - e^{-\lambda\xi}\right)\mu(d\xi) = \frac{\phi_0(\lambda)}{\lambda}.$$

Suppose that $v(t,x)$ is a solution to the time fractional equation with $v(0,x) = 0$. Hence we have for every $t > 0$,

$$\kappa v(t,x) + \int_0^t w(t-r)v(r,x)dr = \int_0^t \mathcal{L}v(s,x)ds.$$

Uniqueness

Taking Laplace transform on both sides and denoting by $V(\lambda, x)$ the Laplace transform of $t \mapsto v(t,x)$, we have

$$\begin{aligned} V(\lambda, x)\left(\kappa + \int_0^\infty e^{-\lambda x} w(x)dx\right) &= \frac{1}{\lambda}\int_0^\infty e^{-\lambda t}\mathcal{L}v(t,x)dt \\ &= \frac{\mathcal{L}V(\lambda, x)}{\lambda}. \end{aligned}$$

It follows that $(\phi(\lambda) - \mathcal{L})V(\lambda, x) = 0$ for every $\lambda > 0$. Since $\mathcal{L}$ is the generator of $\{T_t, t \geq 0\}$, $\exists \alpha_0 > 0$ so that for every $\alpha > \alpha_0$, the resolvent $G_\alpha = \int_0^\infty e^{-\alpha t} T_t dt$ is well defined and is the inverse to $\alpha - \mathcal{L}$. Thus $V(\lambda, \cdot) = 0$ in $\mathbb{B}$ for every $\lambda > \phi^{-1}(\alpha_0)$. By the uniqueness of Laplace transform, we have $v(t, \cdot) = 0$ in $\mathbb{B}$ for every $t > 0$.

Approach to Existence

We first investigate some key properties of subordinators.

Lemma (C. '17 & '24)

For every $t > 0$ and $s > 0$,

$$\mathbb{P}(\overline{S}_s \geq t) = \int_0^s \mathbb{E}\left[w(t - \overline{S}_r) 1_{\{t > \overline{S}_r\}}\right] dr.$$

Proof: Using change of variable formula for non-decreasing functions and Fubini's theorem.

Some identities

Define $G(0) = 0$ and $G(x) = \int_0^x w(y)dy$.

Corollary (C. '17 & '24)

For every $t, s > 0$,

(i) $\int_0^\infty \mathbb{E}\left[w(t - \overline{S}_r)1_{\{t > \overline{S}_r\}}\right] dr = 1.$

(ii) $\int_0^\infty \mathbb{E}\left[G(t - \overline{S}_r)1_{\{t \geq \overline{S}_r\}}\right] dr = t$ *for every* $t > 0$.

(iii) $\int_0^\infty \mathbb{E}\left[G(t - S_r)1_{\{t \geq S_r\}}\right] dr \leq t$ *for every* $t > 0$.

Proof: (i) follows from the lemma by taking $s \to \infty$.

Proof of Corollary

(ii) follows from (i) and Fubini theorem that

$$\begin{aligned} t &= \int_0^t \left(\int_0^\infty \mathbb{E}\left[w(s-\overline{S}_r) 1_{\{s>S_r\}} \right] dr \right) ds \\ &= \int_0^\infty \mathbb{E}\left[G(t-\overline{S}_r) 1_{\{t>\overline{S}_r\}} \right] dr. \end{aligned}$$

(iii) Since $G(x)$ is an increasing function in x, we have by (ii)

$$\int_0^\infty \mathbb{E}\left[G(t-S_r) 1_{\{t>S_r\}} \right] dr \leq \int_0^\infty \mathbb{E}\left[G(t-\overline{S}_r) 1_{\{t>\overline{S}_r\}} \right] dr \leq t.$$

Existence and probabilistic representation

(i) By the integration by parts formula, one can show that

$$\int_0^t w(t-r)\mathbb{P}(S_s > r)dr = G(t) - \mathbb{E}\left[G(t-S_s)1_{\{t\geq S_s\}}\right].$$

(ii) For $u(t,x) := \mathbb{E}_x\left[f(T_{L_t}f(x)\right]$, note $\mathbb{P}(L_r \leq s) = \mathbb{P}(S_s \geq r)$.
Using above identity and an integration by parts,

$$\begin{aligned}
& \int_0^t w(t-r)(u(r,x)-u(0,x))dr \\
= & \int_0^t w(t-r)\left(\int_0^\infty (T_sf(x)-f(x))d_s\mathbb{P}(S_s \geq r)\right)dr \\
= & \int_0^\infty (T_sf(x)-f(x))d_s\left(\int_0^t w(t-r)\mathbb{P}(S_s > r)dr\right) \\
= & -\int_0^\infty (T_sf(x)-f(x))d_s\mathbb{E}\left[G(t-S_s)1_{\{t\geq S_s\}}\right] \\
= & \int_0^\infty \mathbb{E}\left[G(t-S_s)1_{\{t\geq S_s\}}\right]\mathcal{L}T_sf(x)ds. \qquad (0.1)
\end{aligned}$$

Existence and probabilistic representation

(iii) we have by the Lemma 1 that

$$\begin{aligned}
&\mathbb{P}(S_r \geq s) = \mathbb{P}(\overline{S}_r \geq s - \kappa r) \\
= \quad & \mathbf{1}_{\{\kappa r \geq s\}} + \mathbf{1}_{\{\kappa r < s\}} \int_0^r \mathbb{E}\left[w(s - \kappa r - \overline{S}_y) 1_{\{s - \kappa r > \overline{S}_y\}} \right] dy.
\end{aligned}$$

So for every $t > 0$,

$$\int_0^t \mathbb{P}(S_r \geq s) ds = (\kappa r) \wedge t + 1_{\{\kappa r < t\}} \mathbb{E} \int_0^r G(t - \kappa r - \overline{S}_y) 1_{\{t - \kappa r > \overline{S}_y\}} dy.$$

Thus

$$\begin{aligned}
\int_0^t \mathcal{L}u(s,x) ds &= \int_0^t \left(\int_0^\infty T_r \mathcal{L}f(x) d_r \mathbb{P}(S_r \geq s) \right) ds \\
&= \int_0^\infty T_r \mathcal{L}f(x) d_r \left(\int_0^t \mathbb{P}(S_r \geq s) ds \right).
\end{aligned}$$

$$= \quad \cdots\cdots$$

$$= \int_0^\infty T_r \mathcal{L} f(x) \mathbb{E}\left[G(t - S_r) 1_{\{t \geq S_r\}} \right] dr + \kappa(u(t,x) - u(0,x)).$$

This together with (0.1) gives

$$\kappa(u(t,x) - u(0,x)) + \int_0^t w(t-r)(u(r,x) - u(0,x))dr = \int_0^t \mathcal{L}u(s,x)ds.$$

Consequently, $(\kappa \partial_t + \partial_t^w)\, u(t,x) = \mathcal{L}u(t,x)$ in $\mathbb{B}$ as $t \mapsto \mathcal{L}u(t,\cdot)$ is continuous in $(\mathbb{B}, \|\cdot\|)$.

Fundamental solution

When the strongly continuous semigroup $\{T_t; t \geq 0\}$ has an integral kernel $p_0(t, x, y)$ with respect to some measure $m(dx)$, then there is a kernel $p(t, x, y)$ so that

$$u(t, x) := \mathbb{E}\left[T_{L_t} f(x)\right] = \int_{\mathcal{X}} p(t, x, y) f(y) m(dy);$$

in other words,

$$p(t, x, y) := \mathbb{E}\left[p_0(L_t, x, y)\right] = \int_0^\infty p_0(s, x, y) d_s \mathbb{P}(L_t \leq s)$$

is the fundamental solution to the time fractional equation $(\kappa \partial_t + \partial_t^w) u = \mathcal{L} u$.

In [C.-Kim-Kumagai-Wang, Forum Math. '18], two-sided estimates on $p(t, x, y)$ are obtained when $\kappa = 0$ and $\{T_t; t \geq 0\}$ is the transition semigroup of a diffusion process that satisfies two-sided Gaussian-type estimates or of a stable-like process on metric measure spaces.

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Estimates of fundamental solution

Theorem (C.-Kim-Kumagai-Wang 2018)

(i) *When X is a diffusion having* $\mathrm{HK}(\alpha)$ *with* $\alpha \geq 2$,

$$p(t,x,y) \simeq H_{\leq 1}(t,d(x,y)) \quad \text{if } d(x,y) \leq t^{\beta/\alpha},$$
$$p(t,x,y) \asymp H^{(c)}_{\geq 1}(t,d(x,y)) \quad \text{if } d(x,y) \geq t^{\beta/\alpha}.$$

(ii) *When X is an α-stable-like process with* $0 < \alpha < 2$,

$$p(t,x,y) \simeq H_{\leq 1}(t,d(x,y)) \quad \text{if } d(x,y) \leq t^{\beta/\alpha},$$
$$p(t,x,y) \simeq H^{(j)}_{\geq 1}(t,d(x,y)) \quad \text{if } d(x,y) \geq t^{-\beta/\alpha}.$$

$$H_{\leq 1}(t,d(x,y)) = \begin{cases} t^{-\beta d/\alpha}, & d < \alpha, \\ t^{-\beta} \log\left(\dfrac{2t^{\beta}}{d(x,y)^{\alpha}}\right), & d = \alpha, \\ = t^{\beta}/d(x,y)^{d-\alpha}, & d > \alpha, \end{cases}$$

$H^{(c)}_{\geq 1}(t,d(x,y)) = t^{-\beta d/\alpha} \exp\left(-\left(d(x,y)^{\alpha}/t^{\beta}\right)^{1/(\alpha-\beta)}\right)$, $\quad H^{(j)}_{\geq 1}(t,d(x,y)) = t^{\beta}/d(x,y)^{d+\alpha}$.

Observation

When $x \neq y$, $\lim_{t\to 0} p(t,x,y) = 0$ but $\lim_{t\to 0} p(t,x,x) = \infty$ if $d < \alpha$ and

$$p(t,x,x) = \infty \quad \text{for every } t > 0 \ \text{ if } d \geq \alpha.$$

$p(t,x,y)$ is sub-exponential decay in $d(x,y)$ at infinity in the local case and polynomial decay in non-local case.

Poisson equation

We now turn to equations with a source term. Under suitable conditions, the solution to

$$\frac{\partial u}{\partial t} = \mathcal{L}u + f(t,x) \quad \text{with } u(0,x) = \phi(x)$$

is given by

$$\begin{aligned} u(t,x) &= T_t\phi(x) + \int_0^t T_{t-s}f(s,\cdot)(x)ds \\ &= \mathbb{E}_x\phi(X_t) + \mathbb{E}_x \int_0^t f(t-s, X_s)ds. \end{aligned}$$

Why? Formally,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}T_t\phi(x) + T_0 f(t,\cdot)(x) + \int_0^t \frac{\partial}{\partial t}T_{t-s}f(s,\cdot)(x)ds \\ &= \mathcal{L}T_t\phi(x) + f(t,x) + \int_0^t \mathcal{L}T_{t-s}f(s,\cdot)(x)ds \\ &= \mathcal{L}u(t,\cdot)(x) + f(t,x). \end{aligned}$$

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Time fractional Poisson equation

The goal of the remaining part of this talk is to study

$$\partial_t^w v = \mathcal{L} v + f(t,x),$$

where

$$\partial_t^w g(t) := \frac{d}{dt}\int_0^t w(t-s)(g(s)-g(0))ds.$$

Here $w \in L^1_{loc}([0,\infty))$ is an unbounded decreasing function with $w(0)=\infty$.

Fractional time Poisson equation

Let $0 < \beta < 1$. How to solve

$$\partial_t^\beta u(t,x) = \Delta u(t,x) + f(t,x)$$

with $u(0,x) = 0$?

We know from above $p(t,x,y) = \mathbb{E} p_0(L_t, x, y)$ is the fundamental solution of $\partial_t^\beta u(t,x) = \Delta u(t,x)$, where $p_0(t,x,y) = (4\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{4t}\right)$. Define

$$q(t,x,y) = \partial_t^{1-\beta} p(\cdot, x, y)(t).$$

It is known in literature (Eidelman, Ivasyshen, Kouchubei, Umarov, Saydamatov, ...) that

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} q(t-s,x,y) f(s,y)\, dy ds$$

solves the Poisson equation. (Duhamel's formula)

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Questions

• Solution in which sense?

• Positivity: If $f(t, x, y) \geq 0$, is the solution $u(t, x) \geq 0$?

• What happens for general spatial generator $\mathcal{L}$ and for general time fractional derivatives ∂_t^w?

Caution: $p(t, x, y)(t)$ is singular at $t = 0$ and at $x = y$.

Set up

Assume that $\{S_t, \mathbb{P}; t \geq 0\}$ is a driftless subordinator with infinite Lévy measure ν and having bounded density $\overline{p}(r, \cdot)$ for each $r > 0$. A sufficient condition for the latter is

$$\lim_{s\to\infty} \frac{\phi(s)}{\ln(1+s)} = \lim_{s\to\infty} \frac{1}{\ln(1+s)} \int_0^\infty (1 - e^{-sx})\, \nu(dx) = \infty.$$

(Hartman and Wintner's condition.)

Suppose that $\{P_t^0;\ t \geq 0\}$ is a strongly continuous semigroup in some Banach space $(\mathbb{B}, \|\cdot\|)$ and $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is its infinitesimal generator.

- $\{P_t^0;\ t \geq 0\}$ is not required to be *uniformly bounded.*

Poisson equation

Theorem (C.-Kim-Kumagai-Wang, JFA '20; C. '24)

Let $g \in \mathcal{D}(\mathcal{L})$ *and* $f(t,x)$ *on* $(0, T_0] \times \mathcal{X}$ *so that for a.e.* $t \in (0, T_0]$, $f(t,\cdot) \in \mathcal{D}(\mathcal{L})$ *and* $\mathrm{esssup}_{t\in[0,T_0]}\|f(t,\cdot)\| + \int_0^{T_0} \|\mathcal{L}f(t,\cdot)\|dt < \infty$.
The function

$$\begin{aligned} u(t,x) &= \mathbb{E}\left[P^0_{L_t} g(x)\right] + \mathbb{E}\left[\int_0^\infty 1_{\{S_r<t\}} P^0_r f(t-S_r,\cdot)(x)dr\right] \\ &= \mathbb{E}\left[P^0_{L_t} g(x)\right] + \int_{s=0}^t \int_{r=0}^\infty P^0_r f(t-s,\cdot)(x)\overline{p}(r,s)\,dr\,ds \end{aligned}$$

is the unique (strong) solution of $\partial^w_t u = \mathcal{L}u + f(t,x)$ *on* $(0, T_0] \times \mathcal{X}$ *with* $u(0,x) = g(x)$.

Classical case: $u(t,x) = P^0_t \phi(x) + \int_0^\infty 1_{\{s<t\}} P^0_s f(t-s,\cdot)(x)ds$ solves $\frac{\partial u}{\partial t} = \mathcal{L}u + f(t,x)$ with $u(0,x) = \phi(x)$.

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Classical case: $u(t,x) = P^0_t \phi(x) + \int_0^\infty 1_{\{s<t\}} P^0_s f(t-s,\cdot)(x)ds$ solves $\frac{\partial u}{\partial t} = \mathcal{L}u + f(t,x)$ with $u(0,x) = \phi(x)$.

Another fundamental solution

Suppose that $(\mathbb{B}, \|\cdot\|) = L^p(\mathcal{X};\nu)$ or $C_\infty(\mathcal{X})$, and the semigroup $\{P_t^0; t \geq 0\}$ has an integrable kernel $p_0(t,x,y)$ with respect to some measure $\mu(dx)$ on $\mathcal{X}$. Define

$$q(t,x,y) = \int_0^\infty p_0(r,x,y)\overline{p}(r,t)dr.$$

Then the unique solution in above theorem can be expressed as

$$u(t,x) = \int_{\mathcal{X}} p(t,x,y)g(y)\mu(dy) + \int_0^t \int_{\mathcal{X}} q(s,x,y)f(t-s,y)\mu(dy)ds.$$

(Recall $p(t,x,y) = \mathbb{E}\left[p_0(L_t,x,y)\right]$.)

Remarks

• Positivity of $q(t, x, y)$.

• Two-sided estimates of $q(t, x, y)$.

• Stability of $p(t, x, y)$ and $q(t, x, y)$.

• An analogous probabilistic representation for solutions of Poisson equation has been obtained recently by M. E. Hernández-Hernández, V. N. Kolokoltsov and L. Toniazzi (2017) and L. Toniazzi (2018) using a different approach and in restrictive settings (Feller generator $\mathcal{L}$ in space $\mathbb{R}^d$, using Mittag-Leffer functions).

Potential measure

S: driftless subordinator having infinite Lévy measure ν and Laplace exponent ϕ. Its *potential measure U*:

$$U(A) = \mathbb{E} \int_0^\infty \mathbf{1}_A(S_r)dr = \int_0^\infty \mathbb{P}(S_r \in A)dr.$$

Facts:

(i) U is diffusive: $U(\{x\}) = 0$ for every $x \geq 0$;

(ii) $\frac{c_1}{\phi(1/t)} \leq U([0,t]) \leq \frac{c_2}{\phi(1/t)}$.

Lemma (C. CJAPS '24)

For every $t > 0$, $w * U(t) := \int_0^t w(t-s)U(ds) = 1$.

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(i) U is diffusive: $U(\{x\}) = 0$ for every $x \geq 0$;

(ii) $\dfrac{c_1}{\phi(1/t)} \leq U([0,t]) \leq \dfrac{c_2}{\phi(1/t)}$.

Lemma (C. CJAPS '24)

For every $t > 0$, $w * U(t) := \int_0^t w(t-s)U(ds) = 1$.

A connection

Theorem (C.-Kim-Kumagai-Wang JFA '20, C. CJAPS '24)

Suppose S_t has density function $\overline{p}(r,t)$. Then for every $t>0$ and $x,y\in\mathcal{X}$,

$$\int_0^t q(s,x,y)\,ds = \int_0^t p(t-s,x,y)U(ds).$$

For m-a.e. $x\in\mathcal{X}$ and m-a.e. $y\in\mathcal{X}\setminus\{x\}$, the above integrals are finite for all $t>0$. For those $x\neq y$, for all $t>0$,

$$\partial_t^{*,w}p(\cdot,x,y)(t) := \frac{d}{dt}\int_0^t p(t-s,x,y)U(ds)$$

exists for a.e. $t>0$ and

$$q(t,x,y) = \partial_t^{*,w}p(\cdot,x,y)(t).$$

- When $\phi(r)=r^\beta$, $\partial_t^{*,w}=\partial_t^{1-\beta}$.

Where these formula come from?

Observations: Suppose g is locally Lipschitz on $[0, \infty)$.

(i) $\partial_t^w g(t)$ exists for a.e. $t > 0$ and

$$\partial_t^w g(t) = \int_0^t w(t-s)g'(s)ds.$$

(ii) Extending $g(s) = g(0)$ for $s < 0$, then

$$\partial_t^w g(t) = -\int_0^\infty (g(t-z) - g(t))\nu(dz) = -\mathcal{A}^* g(t).$$

Here $\mathcal{A}^*$ is the infinitesimal generator of the Lévy process $-S_t$.

Space-time process

Key observation: $-\partial_t^w + \mathcal{L}$ is the infinitesimal generator of $(-S_t, X_t)$.

Suppose that $u(t,x)$ is a solution to $\partial_t^w u = \mathcal{L}u + f(t,x)$ on $(0, T_0] \times \mathcal{X}$ with $u(0,x) = g(x)$. For each fixed $T \in (0, T_0]$, consider $u(T - S_t, X_t)$. Then

$$\begin{aligned} M_t &= u(T - S_t, X_t) - \int_0^t (-\partial_t^w + \mathcal{L})u(T - S_t, X_t)dt \\ &= u(T - S_t, X_t) + \int_0^t f(T - S_t, X_t)dt \end{aligned}$$

is a martingale. So $\mathbb{E}_x M_0 = \mathbb{E}_x M_{L_T}$. That is,

$$\begin{aligned} u(T,x) &= \mathbb{E}_x g(X_{L_T}) + \mathbb{E}_x \int_0^{L_T} f(T - S_t, X_t)dt \\ &= \mathbb{E} P_{L_T} g(x) + \mathbb{E} \int_0^\infty 1_{\{S_t < T\}} P_t f(T - S_t, \cdot)(x)dt. \end{aligned}$$

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Approach

However, there is a problem!

We do not know a prior if $u(T-t,x)$ is in the domain of the generator $-\partial_t^w + \mathcal{L}$.

To rigorously prove these formulas, we use a different approach by studying the properties of subordinator and inverse subordinator.

Estimate

Formula for the 2nd fundamental solution

$$q(t,x,y)=\int_0^\infty p_0(r,x,y)\overline{p}(r,t)dr.$$

allows us to obtain estimates and stability results on the solutions to the Poisson equation.

Particular case: $S_t=\beta$-subordinator, or Caputo derivative ∂_t^β.

Define

$$\widetilde{H}_{\leq 1}(t,d(x,y))=\begin{cases} t^{\beta-1-\beta d/\alpha}, & d<2\alpha, \\ t^{-1-\beta}\log\left(\frac{2t^\beta}{d(x,y)^\alpha}\right), & d=2\alpha, \\ = t^{-1-\beta}/d(x,y)^{d-2\alpha}, & d>2\alpha, \end{cases}$$

$$\widetilde{H}^{(c)}_{\geq 1}(t,d(x,y))=t^{\beta-1-\beta d/\alpha}\exp\left(-(d(x,y)^\alpha/t^\beta)^{1/(\alpha-\beta)}\right),$$

$$\widetilde{H}^{(j)}_{\geq 1}(t,d(x,y))=t^{2\beta-1}/d(x,y)^{d+\alpha}.$$

Estimates of fundamental solution q

Theorem (C.-Kim-Kumagai-Wang, JFA '20)

(i) *When X is a diffusion having $\mathrm{HK}(\alpha)$ with $\alpha \geq 2$,*

$$q(t,x,y) \simeq \widetilde{H}_{\leq 1}(t, d(x,y)) \quad \text{if } d(x,y) \leq t^{\beta/\alpha},$$
$$q(t,x,y) \asymp \widetilde{H}^{(c)}_{\geq 1}(t, d(x,y)) \quad \text{if } d(x,y) \geq t^{\beta/\alpha}.$$

(ii) *When X is an α-stable-like process with $0 < \alpha < 2$,*

$$q(t,x,y) \simeq \widetilde{H}_{\leq 1}(t, d(x,y)) \quad \text{if } d(x,y) \leq t^{\beta/\alpha},$$
$$q(t,x,y) \simeq \widetilde{H}^{(j)}_{\geq 1}(t, d(x,y)) \quad \text{if } d(x,y) \geq t^{\beta/\alpha}.$$

(i) When $\{S_t; t \geq 0\}$ is a β-subordinator with $0 < \beta < 1$ with Laplace exponent $\phi(\lambda) = \lambda^\beta$, Then S_t has no drift (i.e. $\kappa = 0$) and its Lévy measure is $\mu(dx) = \frac{\beta}{\Gamma(1-\beta)} x^{-(1+\beta)} dx$. Hence

$$w(x) := \mu(x, \infty) = \int_x^\infty \frac{\beta}{\Gamma(1-\beta)} y^{-(1+\beta)} dy = \frac{x^{-\beta}}{\Gamma(1-\beta)}.$$

Thus the time fractional derivative $\partial_t^w f$ is exactly the Caputo derivative of order β. In this case, our Theorem recovers the main result of Baeumer-Meerschaert (2001) and Meerschaert-Scheffler (2004).

Truncated stable-subordinator

(ii) A truncated β-stable subordinator $\{S_t; t \geq 0\}$ is driftless and has Lévy measure

$$\mu_\delta(dx) = \frac{\beta}{\Gamma(1-\beta)} x^{-(1+\beta)} 1_{(0,\delta]}(x) dx$$

for some $\delta > 0$. In this case,

$$\begin{aligned} w_\delta(x) &:= \mu_\delta(x,\infty) = 1_{\{0<x\leq\delta\}} \int_x^\delta \frac{\beta}{\Gamma(1-\beta)} y^{-(1+\beta)} dy \\ &= \frac{1}{\Gamma(1-\beta)} \left(x^{-\beta} - \delta^{-\beta} \right) 1_{(0,\delta]}(x). \end{aligned}$$

The corresponding the fractional derivative is

$$\partial_t^{w_\delta} f(t) := \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_{(t-\delta)^+}^t \left((t-s)^{-\beta} - \delta^{-\beta} \right) (f(s) - f(0)) ds.$$

Clearly, as $\lim_{\delta\to\infty} w_\delta(x) = w(x) := \frac{1}{\Gamma(1-\beta)} x^{-\beta}$. Consequently, $\partial_t^{w_\delta} f(t) \to \partial_t^{w} f(t)$, the Caputo derivative of f of order β, in the distributional sense as $\delta \to 0$. Using the probabilistic representation in the main Theorem, one can deduce that as $\delta \to \infty$, the solution to the equation $\partial_t^{w_\delta} u = \mathcal{L}u$ with $u(0,x) = f(x)$ converges to the solution of $\partial_t^{\beta} u = \mathcal{L}u$ with $u(0,x) = f(x)$.

Stability

(iii) If we define

$$\eta_\delta(r) = \frac{\Gamma(2-\beta)\,\delta^{\beta-1}}{\beta} w_\delta(r) = (1-\beta)\delta^{\beta-1}\left(x^{-\beta} - \delta^{-\beta}\right) 1_{(0,\delta]}(x),$$

then $\eta_\delta(r)$ converges weakly to the Dirac measure concentrated at 0 as $\delta \to 0$. So $\partial_t^{\eta_\delta} f(t)$ converges to $f'(t)$ for every differentiable f. It can be shown that the subordinator corresponding to η_δ, that is, subordinator with Lévy measure

$$\nu_\delta(dx) := \frac{(1-\beta)\,\delta^{\beta-1}}{\beta} x^{-(1+\beta)} 1_{(0,\delta]}(x) dx,$$

converges as $\delta \to 0$ to deterministic motion t moving at constant speed 1. Using the main Theorem, one can show that the solution to the equation $\partial_t^{\eta_\delta} u(t,x) = \mathcal{L}u(t,x)$ with $u(0,x) = f(x)$ converges to the solution of the heat equation $\partial_t u = \mathcal{L}u$ with $u(0,x) = f(x)$.

Thank you!