Anomalous subdiffusion and time-fractional differential equations III

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Fundamental solution

When the strongly continuous semigroup $\{T_t; t \geq 0\}$ has an integral kernel $p_0(t, x, y)$ with respect to some measure $m(dx)$, then there is a kernel $p(t, x, y)$ so that

$$u(t, x) := \mathbb{E}\left[T_{L_t} f(x)\right] = \int_{\mathcal{X}} p(t, x, y) f(y) m(dy);$$

in other words,

$$p(t, x, y) := \mathbb{E}\left[p_0(L_t, x, y)\right] = \int_0^\infty p_0(s, x, y) d_s \mathbb{P}(L_t \leq s)$$

is the fundamental solution to the time fractional equation $\left(\kappa \partial_t + \partial_t^w\right) u = \mathcal{L} u$.

Estimates of fundamental solution

In [C.-Kim-Kumagai-Wang, Forum Math. '18], two-sided estimates on $p(t,x,y)$ are obtained when $\kappa = 0$ and $\{T_t; t \geq 0\}$ is the transition semigroup of a diffusion process that satisfies two-sided Gaussian-type estimates or of a stable-like process on metric measure spaces.

When $\kappa = 0$, S is the β-stable subordinator with $0 < \beta < 1$, and X is Brownian motion, estimates of $p(t,x)$ were obtained (e.g. Eidelman-Kochubei, 2004) via

$E_\beta(z) = \sum_{k=1}^\infty \frac{z^k}{\Gamma(\beta k+1)}$: Mittag-Leffler function

$p(t,x) = \mathcal{F}^{-1}(E_\beta(|\xi|^2 t^\beta))$ and using Fourier analysis.

Diffusions having $\mathrm{HK}(d_w)$ with $d_2 \geq 2$

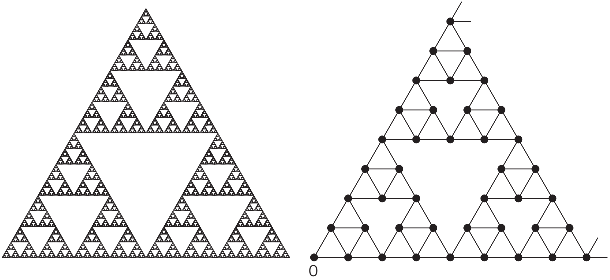
$\mathrm{HK}(d_w)$: d_f is the Hausdorff dimension of $\mathcal{X}$:

$$p_0(t,x,y) \asymp c_3 t^{-d_f/d_w} \exp\Big(- c_4 \Big(\frac{d(x,y)^{d_w}}{t} \Big)^{\frac{1}{d_w - 1}} \Big)$$

Examples:

1. $\mathcal{X} = \mathbb{R}^d$ and $\mathcal{L} = \mathrm{div}(A(x)\nabla)$, where $\lambda^{-1} I_{d\times d} \leq A(x) \leq \lambda I_{d\times d}$. (Aronson's estimate)
2. BM on manifolds with non-negative Ricci curvature satisfying VD and PI(2).
3. (sub-diffusive) BM on some fractals, e.g., on unbounded Sierpinski gasket (Barlow-Perkins '88, where $d_f = \log 3/\log 2$ and $d_w = \log 5/\log 3 > 0$), and unbounded Sierpinski carpet (Barlow-Bass 1992/1999, $d_f = \log 8/\log 3$, where $d_w > 2$ (between 2.008 and 2.012)).

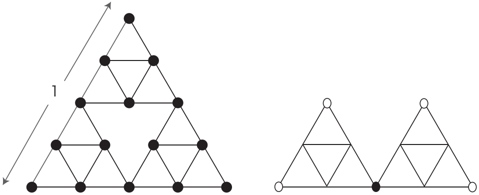
Sierpinski gasket



G: Sierpinski gasket graph, RHS extended to infinity

$\{S_n : n = 0, 1, 2, \cdots\}$: simple RW on G
In each step, S_n moves to one of the nearest neighbors with equal probability.

Brownian motion on Sierpinski gasket



What is the average time for $2^{-1}S_n$ to reach $\circ$ from $\bullet$?
Answer: 5

For scaled RW $2^{-m}S_{[5^m t]}$, the average time to reach $\{-1, 1\}$ from 0 is 1.

Goldstein, Kusuoka '87: $\{2^{-m}S_{[5^m t}\} \Longrightarrow \{B_t\}$ (BM on G.

The corresponding Laplacian on G (Kigami '89):

$$\frac{5^m}{v(x)} \sum_{y \in 2^{-m}G: y \sim y} (f(y) - f(x)) \to \mathcal{L}f(x)$$

Stable-like processes

Let X be a symmetric jump processes on $\mathcal{X}$ that has jointly continuous transition density function $p_0(t, x, y)$:

$$p_0(t, x, y) \asymp t^{-d_f/\alpha} \wedge \frac{t}{d(x, y)^{d_f+\alpha}}.$$

Examples:

1. $\mathcal{X} = \mathbb{R}^d$ and $\mathcal{L}f(x) = \mathrm{p.v.} \int_{\mathbb{R}^d} (f(y) - f(x)) \frac{1}{|y-x|^{d+\alpha}} dy$ for $0 < \alpha < 2$. (C.-Kumagai 2003)
2. Same holds on Ahlfors d-regular set in $\mathbb{R}^n$.
3. β-stable subordination of diffusions having $\mathrm{HK}(d_w)$ estimates. In this case, $\alpha = \beta d_w \in (0, d_w)$. So α can be larger than 2 on fractals such as Sierpinski gasket or carpet.
4. Analytic characterization of stable-like HKE on metric measure space is obtained in C.-Kumagai-Wang 2016/2021. (MAMS)

Estimates of fundamental solution

Theorem (C.-Kim-Kumagai-Wang 2018)

(Special case when S is a β-stable subordinator)

(i) *When X is a diffusion having $\mathrm{HK}(\alpha)$ with $\alpha \geq 2$,*

$$p(t,x,y) \simeq H_{\leq 1}(t, d(x,y)) \quad \text{if } d(x,y) \leq t^{\beta/\alpha},$$
$$p(t,x,y) \asymp H_{\geq 1}^{(c)}(t, d(x,y)) \quad \text{if } d(x,y) \geq t^{\beta/\alpha}.$$

(ii) *When X is an α-stable-like process with $0 < \alpha < 2$,*

$$p(t,x,y) \simeq H_{\leq 1}(t, d(x,y)) \quad \text{if } d(x,y) \leq t^{\beta/\alpha},$$
$$p(t,x,y) \simeq H_{\geq 1}^{(j)}(t, d(x,y)) \quad \text{if } d(x,y) \geq t^{-\beta/\alpha}.$$

$$H_{\leq 1}(t, d(x,y)) = \begin{cases} t^{-\beta d/\alpha}, & d < \alpha, \\ t^{-\beta} \log\left(\dfrac{2t^{\beta}}{d(x,y)^{\alpha}}\right), & d = \alpha, \\ = t^{-\beta}/d(x,y)^{d-\alpha}, & d > \alpha, \end{cases}$$

$H_{\geq 1}^{(c)}(t, d(x,y)) = t^{-\beta d/\alpha} \exp\left(-\left(d(x,y)^{\alpha}/t^{\beta}\right)^{1/(\alpha-\beta)}\right)$, $\quad H_{\geq 1}^{(j)}(t, d(x,y)) = t^{\beta}/d(x,y)^{d+\alpha}$.

Observation

When $x \neq y$, $\lim_{t\to 0} p(t,x,y) = 0$ but $\lim_{t\to 0} p(t,x,x) = \infty$ if $d < \alpha$ and

$$p(t,x,x) = \infty \quad \text{for every } t > 0 \ \text{ if } d \geq \alpha.$$

$p(t,x,y)$ is sub-exponential decay in $d(x,y)$ at infinity in the local case and polynomial decay in non-local case.

Key proposition

We in fact have estimates for $p(t,x,y)$ with more general subordinators in terms of their Laplace exponent ϕ.

Proposition (C.-Kim-Kumagai-Wang 2018)

$$\mathbb{P}(L_t \le r) = \mathbb{P}(S_r \ge t) \asymp r\phi(t^{-1}) \quad \textit{if } r\phi(t^{-1}) \ll 1,$$
$$\mathbb{P}(L_t \le r) = \mathbb{P}(S_r \le t) \asymp \exp(-t(\phi')^{-1}(t/r)) \quad \textit{if } r\phi(t^{-1}) \gg 1.$$

Remark: Roughly speaking, $L_t \approx 1/\phi(t^{-1})$ with large probability so the estimates on $p(t,x,y) = \mathbb{E}[p_0(L_t,x,y)]$ is obtained from $p_0(t,x,y)$ by taking $t \to 1/\phi(t^{-1})$.

Poisson equation

We now turn to equations with a source term. Under suitable conditions, the solution to

$$\frac{\partial u}{\partial t} = \mathcal{L}u + f(t,x) \quad \text{with } u(0,x) = \phi(x)$$

is given by

$$\begin{aligned} u(t,x) &= T_t\phi(x) + \int_0^t T_{t-s}f(s,\cdot)(x)ds \\ &= \mathbb{E}_x\phi(X_t) + \mathbb{E}_x \int_0^t f(t-s, X_s)ds. \end{aligned}$$

Why? Formally,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}T_t\phi(x) + T_0 f(t,\cdot)(x) + \int_0^t \frac{\partial}{\partial t}T_{t-s}f(s,\cdot)(x)ds \\ &= \mathcal{L}T_t\phi(x) + f(t,x) + \int_0^t \mathcal{L}T_{t-s}f(s,\cdot)(x)ds \\ &= \mathcal{L}u(t,\cdot)(x) + f(t,x). \end{aligned}$$

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Time fractional Poisson equation

The goal of the remaining part of this talk is to study

$$\partial_t^w v = \mathcal{L} v + f(t,x),$$

where

$$\partial_t^w g(t) := \frac{d}{dt} \int_0^t w(t-s)(g(s)-g(0))ds.$$

Here $w \in L^1_{loc}([0,\infty))$ is an unbounded decreasing function with $w(0)=\infty$.

Fractional time Poisson equation

Let $0 < \beta < 1$. How to solve

$$\partial_t^\beta u(t,x) = \Delta u(t,x) + f(t,x)$$

with $u(0,x) = 0$?

We know from above $p(t,x,y) = \mathbb{E} p_0(L_t,x,y)$ is the fundamental solution of $\partial_t^\beta u(t,x) = \Delta u(t,x)$, where $p_0(t,x,y) = (4\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{4t}\right)$. Define

$$q(t,x,y) = \partial_t^{1-\beta} p(\cdot,x,y)(t).$$

It is known in literature (Eidelman, Ivasyshen, Kouchubei, Umarov, Saydamatov, ...) that

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} q(t-s,x,y) f(s,y)\, dyds$$

solves the Poisson equation. (Duhamel's formula)

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Questions

• Solution in which sense?

• Positivity: If $f(t,x,y) \geq 0$, is the solution $u(t,x) \geq 0$?

• What happens for general spatial generator $\mathcal{L}$ and for general time fractional derivatives ∂_t^w?

Caution: $p(t,x,y)(t)$ is singular at $t = 0$ and at $x = y$.

Set up

Assume that $\{S_t, \mathbb{P}; t \geq 0\}$ is a driftless subordinator with infinite Lévy measure ν and having bounded density $\overline{p}(r, \cdot)$ for each $r > 0$. A sufficient condition for the latter is

$$\lim_{s\to\infty} \frac{\phi(s)}{\ln(1+s)} = \lim_{s\to\infty} \frac{1}{\ln(1+s)} \int_0^\infty (1 - e^{-sx})\,\nu(dx) = \infty.$$

(Hartman and Wintner's condition.)

Suppose that $\{P_t^0;\ t \geq 0\}$ is a strongly continuous semigroup in some Banach space $(\mathbb{B}, \|\cdot\|)$ and $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is its infinitesimal generator.

- $\{P_t^0;\ t \geq 0\}$ is not required to be *uniformly bounded.*

Poisson equation

Theorem (C.-Kim-Kumagai-Wang, JFA '20; C. '24)

Let $g \in \mathcal{D}(\mathcal{L})$ and $f(t,x)$ on $(0, T_0] \times \mathcal{X}$ so that for a.e. $t \in (0, T_0]$, $f(t,\cdot) \in \mathcal{D}(\mathcal{L})$ and $\mathrm{esssup}_{t \in [0,T_0]} \|f(t,\cdot)\| + \int_0^{T_0} \|\mathcal{L} f(t,\cdot)\| dt < \infty$. The function

$$\begin{aligned} u(t,x) &= \mathbb{E}\left[P^0_{L_t} g(x)\right] + \mathbb{E}\left[\int_0^\infty 1_{\{S_r < t\}} P^0_r f(t - S_r, \cdot)(x) dr\right] \\ &= \mathbb{E}\left[P^0_{L_t} g(x)\right] + \int_{s=0}^t \int_{r=0}^\infty P^0_r f(t-s,\cdot)(x) \overline{p}(r,s)\, dr\, ds \end{aligned}$$

is the unique (strong) solution of $\partial^w_t u = \mathcal{L} u + f(t,x)$ on $(0, T_0] \times \mathcal{X}$ with $u(0,x) = g(x)$. in the following sense.

Classical case: $u(t,x) = P^0_t \phi(x) + \int_0^\infty 1_{\{s<t\}} P^0_s f(t-s,\cdot)(x) ds$ solves $\frac{\partial u}{\partial t} = \mathcal{L} u + f(t,x)$ with $u(0,x) = \phi(x)$.

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Classical case: $u(t,x) = P^0_t \phi(x) + \int_0^\infty 1_{\{s<t\}} P^0_s f(t-s,\cdot)(x)ds$ solves $\frac{\partial u}{\partial t} = \mathcal{L}u + f(t,x)$ with $u(0,x) = \phi(x)$.

Poisson equation

Theorem (C.-Kim-Kumagai-Wang, 2018+)

1. *$u(t,\cdot)$ is well defined as an element in $\mathbb{B}$ for each $t \in (0, T_0]$ such that $\sup_{t\in(0,T_0]} \|u(t,\cdot)\| < \infty$, $t \mapsto u(t,x)$ is continuous in $(\mathbb{B}, \|\cdot\|)$ and $\lim_{t\to 0} \|u(t,\cdot) - g\| = 0$.*
2. *For a.e. $t \in (0, T_0]$, $u(t,\cdot) \in \mathcal{D}(\mathcal{L})$ and $\mathcal{L}u(t,\cdot)$ exists in the Banach space $\mathbb{B}$ with $\int_0^{T_0} \|\mathcal{L}u(t,\cdot)\| dt < \infty$.*
3. *For every $T \in (0, T_0]$,*

$$\int_0^T w(T-t)(u(t,\cdot)-g)\, dt = \int_0^T (f(t,\cdot) + \mathcal{L}u(t,\cdot))\, dt \quad \text{in } \mathbb{B}.$$

We also have corresponding result for weak solutions.

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Another fundamental solution

Suppose that $(\mathbb{B}, \|\cdot\|) = L^p(\mathcal{X};\nu)$ or $C_\infty(\mathcal{X})$, and the semigroup $\{P_t^0; t \geq 0\}$ has an integrable kernel $p_0(t,x,y)$ with respect to some measure $\mu(dx)$ on $\mathcal{X}$. Define

$$q(t,x,y) = \int_0^\infty p_0(r,x,y)\overline{p}(r,t)dr.$$

Then the unique solution in above theorem can be expressed as

$$u(t,x) = \int_{\mathcal{X}} p(t,x,y)g(y)\mu(dy) + \int_0^t \int_{\mathcal{X}} q(s,x,y)f(t-s,y)\mu(dy)ds.$$

(Recall $p(t,x,y) = \mathbb{E}\left[p_0(L_t,x,y)\right]$.)

Remarks

• Positivity of $q(t, x, y)$.

• Two-sided estimates of $q(t, x, y)$.

• Stability of $p(t, x, y)$ and $q(t, x, y)$.

• An analogous probabilistic representation for solutions of Poisson equation has been obtained recently by M. E. Hernández-Hernández, V. N. Kolokoltsov and L. Toniazzi (2017) and L. Toniazzi (2018) using a different approach and in restrictive settings (Feller generator $\mathcal{L}$ in space $\mathbb{R}^d$, using Mittag-Leffer functions).

Potential measure

S: driftless subordinator having infinite Lévy measure ν and Laplace exponent ϕ. Its *potential measure U*:

$$U(A) = \mathbb{E}\int_0^\infty \mathbf{1}_A(S_r)dr = \int_0^\infty \mathbb{P}(S_r \in A)dr.$$

Facts:

(i) U is diffusive: $U(\{x\}) = 0$ for every $x \geq 0$;

(ii) $\dfrac{c_1}{\phi(1/t)} \leq U([0,t]) \leq \dfrac{c_2}{\phi(1/t)}$.

Lemma (C. CJAPS '24)

For every $t > 0$, $w * U(t) := \int_0^t w(t-s)U(ds) = 1$.

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A connection

Theorem (C.-Kim-Kumagai-Wang JFA '20, C. CJAPS '24)

Suppose S_t has density function $\overline{p}(r,t)$. Then for every $t>0$ and $x, y \in \mathcal{X}$,

$$\int_0^t q(s,x,y)\,ds = \int_0^t p(t-s,x,y)U(ds).$$

For m-a.e. $x \in \mathcal{X}$ and m-a.e. $y \in \mathcal{X} \setminus \{x\}$, the above integrals are finite for all $t>0$. For those $x \neq y$, for all $t>0$,

$$\partial_t^{*,w} p(\cdot,x,y)(t) := \frac{d}{dt}\int_0^t p(t-s,x,y)U(ds)$$

exists for a.e. $t>0$ and

$$q(t,x,y) = \partial_t^{*,w} p(\cdot,x,y)(t).$$

- When $\phi(r) = r^\beta$, $\partial_t^{*,w} = \partial_t^{1-\beta}$.

Where these formula come from?

Observations: Suppose g is locally Lipschitz on $[0, \infty)$.

(i) $\partial_t^w g(t)$ exists for a.e. $t > 0$ and

$$\partial_t^w g(t) = \int_0^t w(t-s) g'(s) ds.$$

(ii) Extending $g(s) = g(0)$ for $s < 0$, then

$$\partial_t^w g(t) = -\int_0^\infty (g(t-z) - g(t)) \nu(dz) = -\mathcal{A}^* g(t).$$

Here $\mathcal{A}^*$ is the infinitesimal generator of the Lévy process $-S_t$.

Space-time process

Key observation: $-\partial_t^w + \mathcal{L}$ is the infinitesimal generator of $(-S_t, X_t)$.

Suppose that $u(t,x)$ is a solution to $\partial_t^w u = \mathcal{L}u + f(t,x)$ on $(0, T_0] \times \mathcal{X}$ with $u(0,x) = g(x)$. For each fixed $T \in (0, T_0]$, consider $u(T - S_t, X_t)$. Then

$$
\begin{aligned}
M_t &= u(T - S_t, X_t) - \int_0^t (-\partial_t^w + \mathcal{L}) u(T - S_t, X_t) dt \\
&= u(T - S_t, X_t) + \int_0^t f(T - S_t, X_t) dt
\end{aligned}
$$

is a martingale. So $\mathbb{E}_x M_0 = \mathbb{E}_x M_{L_T}$. That is,

$$
\begin{aligned}
u(T, x) &= \mathbb{E}_x g(X_{L_T}) + \mathbb{E}_x \int_0^{L_T} f(T - S_t, X_t) dt \\
&= \mathbb{E} P_{L_T} g(x) + \mathbb{E} \int_0^\infty 1_{\{S_t < T\}} P_t f(T - S_t, \cdot)(x) dt.
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$$\begin{aligned} M_t &= u(T - S_t, X_t) - \int_0^t (-\partial_t^w + \mathcal{L}) u(T - S_t, X_t) dt \\ &= u(T - S_t, X_t) + \int_0^t f(T - S_t, X_t) dt \end{aligned}$$

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Approach

However, there is a problem!

We do not know a prior if $u(T-t,x)$ is in the domain of the generator $-\partial_t^w + \mathcal{L}$.

To rigorously prove these formulas, we use a different approach by studying the properties of subordinator and inverse subordinator.

Thank you!