Anomalous subdiffusion and time-fractional differential equations IV

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Poisson equation

Recall

Theorem (C.-Kim-Kumagai-Wang, JFA '20; C. '24)

Let $g \in \mathcal{D}(\mathcal{L})$ *and* $f(t,x)$ *on* $(0, T_0] \times \mathcal{X}$ *so that for a.e.* $t \in (0, T_0]$, $f(t, \cdot) \in \mathcal{D}(\mathcal{L})$ *and* $\mathrm{esssup}_{t\in[0,T_0]}\|f(t,\cdot)\| + \int_0^{T_0} \|\mathcal{L}f(t,\cdot)\| dt < \infty$.
The function

$$u(t,x) = \mathbb{E}\left[P^0_{L_t} g(x)\right] + \mathbb{E}\left[\int_0^\infty 1_{\{S_r<t\}} P^0_r f(t - S_r, \cdot)(x) dr\right]$$

$$= \int_{\mathcal{X}} p(t,x,y) g(y)\mu(dy) + \int_0^t \int_{\mathcal{X}} q(s,x,y) f(t-s,y)\mu(dy)ds$$

is the unique (strong) solution of $\partial^w_t u = \mathcal{L}u + f(t,x)$ *on* $(0, T_0] \times \mathcal{X}$ *with* $u(0,x) = g(x)$.

Here $p(t,x,y) = \mathbb{E}\left[p_0(L_t, x, y)\right]$ and

$$q(t,x,y) = \int_0^\infty p_0(r,x,y)\overline{p}(r,t)dr.$$

Estimates

Formula for the 2nd fundamental solution

$$q(t,x,y) = \int_0^\infty p_0(r,x,y)\overline{p}(r,t)dr.$$

allows us to obtain estimates and stability results on the solutions to the Poisson equation.

Particular case: $S_t = \beta$-stable subordinator, or Caputo derivative ∂_t^β.

Define

$$\widetilde{H}_{\le 1}(t,d(x,y)) = \begin{cases} t^{\beta-1-\beta d/\alpha}, & d < 2\alpha, \\ t^{-1-\beta}\log\left(\dfrac{2t^\beta}{d(x,y)^\alpha}\right), & d = 2\alpha, \\ = t^{-1-\beta}/d(x,y)^{d-2\alpha}, & d > 2\alpha, \end{cases}$$

$$\widetilde{H}^{(c)}_{\ge 1}(t,d(x,y)) = t^{\beta-1-\beta d/\alpha}\exp\Big(-(d(x,y)^\alpha/t^\beta)^{1/(\alpha-\beta)}\Big),$$

$$\widetilde{H}^{(j)}_{\ge 1}(t,d(x,y)) = t^{2\beta-1}/d(x,y)^{d+\alpha}.$$

Estimates of fundamental solution q

Theorem (C.-Kim-Kumagai-Wang, JFA '20)

(i) *When X is a diffusion having* $\mathrm{HK}(\alpha)$ *with* $\alpha \geq 2$,

$$\begin{aligned} q(t,x,y) &\simeq \widetilde{H}_{\leq 1}(t, d(x,y)) \quad &&\text{if } d(x,y) \leq t^{\beta/\alpha}, \\ q(t,x,y) &\asymp \widetilde{H}^{(c)}_{\geq 1}(t, d(x,y)) \quad &&\text{if } d(x,y) \geq t^{\beta/\alpha}. \end{aligned}$$

(ii) *When X is an α-stable-like process with* $0 < \alpha < 2$,

$$\begin{aligned} q(t,x,y) &\simeq \widetilde{H}_{\leq 1}(t, d(x,y)) \quad &&\text{if } d(x,y) \leq t^{\beta/\alpha}, \\ q(t,x,y) &\simeq \widetilde{H}^{(j)}_{\geq 1}(t, d(x,y)) \quad &&\text{if } d(x,y) \geq t^{\beta/\alpha}. \end{aligned}$$

Observation

When $x \neq y$, $\lim_{t\to 0} q(t,x,y) = 0$ but $\lim_{t\to 0} q(t,x,x) = \infty$ if $d < 2\alpha$ and

$$q(t,x,x) = \infty \quad \text{for every } t > 0 \ \text{ if } d \geq 2\alpha.$$

$q(t,x,y)$ is sub-exponential decay in $d(x,y)$ at infinity in the local case and polynomial decay in non-local case.

Condition on the subordinators

A key to the above estimates on $q(t,x,y)$ is the estimates on $\overline{p}(r,t)$. Suppose

1. The Laplace exponent ϕ of S satisfies

   $$c_1\kappa^{\beta_1} \le \frac{\phi(\kappa\lambda)}{\phi(\lambda)} \le c_2\kappa^{\beta_2} \quad \text{for all } \kappa, \lambda \ge 1,$$

   where $0 < \beta_1 \le \beta_2 < 1$.
2. The Lévy measure ν of S has the property that $\nu(dz) = \nu(z)dz$ with $z\nu(z)$ non-increasing on $(0,\infty)$.

Clearly, any β-stable subordinator satisfies the above condition.

Estimates for the subordinators

Theorem (C.-Kim-Kumagai-Wang, JFA '20)

Under the above assumption for the subordinator, for every $L > 0$, $\exists c_i = c_i(K) \geq 1$ *so that*

1. *When* $r\phi(1/t) \leq L$,

$$c_1^{-1}\frac{r\phi(1/t)}{t} \leq \overline{p}(r,t) \leq c_1\frac{r\phi(1/t)}{t}.$$

2. *When* $r\phi(1/t) \geq L$,

$$\frac{\exp(-c_3 t(\phi')^{-1}(t/r))}{c_2 t} \leq \overline{p}(r,t) \leq c_2\frac{\exp(-c_3 t(\phi')^{-1}(t/r))}{t},$$

where $(\phi')^{-1}(\lambda) := \inf\{s > 0 : \phi'(s) \leq \lambda\}$.

β-stable subordinator

When $\phi(\lambda)=\lambda^\beta$, $r\phi(1/t)=r/t^\beta$, and $(\phi')^{-1}(t)=(t/\beta)^{-1/(1-\beta)}$ and so $t(\phi')^{-1}(t/r)=(\beta r/t^\beta)^{1/(1-\beta)}$. Thus we have from the above theorem that

Corollary

1. *When* $r\le Lt^\beta$,

$$c_1^{-1}\frac{r}{t^{1+\beta}}\le \overline{p}(r,t)\le c_1\frac{r}{t^{1+\beta}}.$$

2. *When* $r\ge Lt^\beta$,

$$\frac{\exp(-c_3(r/t^\beta)^{1/(1-\beta)})}{c_2 t}\le \overline{p}(r,t)\le c_2\frac{\exp(-c_3(r/t^\beta)^{1/(1-\beta)})}{t}.$$

Examples

(i) When $\{S_t; t \geq 0\}$ is a β-stable subordinator with $0 < \beta < 1$ with Laplace exponent $\phi(\lambda) = \lambda^\beta$, Then S_t has no drift (i.e. $\kappa = 0$) and its Lévy measure is $\mu(dx) = \frac{\beta}{\Gamma(1-\beta)} x^{-(1+\beta)} dx$. Hence

$$w(x) := \mu(x, \infty) = \int_x^\infty \frac{\beta}{\Gamma(1-\beta)} y^{-(1+\beta)} dy = \frac{x^{-\beta}}{\Gamma(1-\beta)}.$$

Thus the time fractional derivative $\partial_t^w f$ is exactly the Caputo derivative of order β. In this case, our Theorem recovers the main result of Baeumer-Meerschaert (2001) and Meerschaert-Scheffler (2004).

Truncated stable-subordinator

(ii) A truncated β-stable subordinator $\{S_t; t \geq 0\}$ is driftless and has Lévy measure

$$\mu_\delta(dx) = \frac{\beta}{\Gamma(1-\beta)} x^{-(1+\beta)} 1_{(0,\delta]}(x)dx$$

for some $\delta > 0$. In this case,

$$\begin{aligned} w_\delta(x) &:= \mu_\delta(x,\infty) = 1_{\{0<x\leq\delta\}} \int_x^\delta \frac{\beta}{\Gamma(1-\beta)} y^{-(1+\beta)} dy \\ &= \frac{1}{\Gamma(1-\beta)} \left(x^{-\beta} - \delta^{-\beta} \right) 1_{(0,\delta]}(x). \end{aligned}$$

The corresponding the fractional derivative is

$$\partial_t^{w_\delta} f(t) := \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_{(t-\delta)^+}^t \left((t-s)^{-\beta} - \delta^{-\beta} \right) (f(s) - f(0)) ds.$$

Stability

Clearly, as $\lim_{\delta\to\infty} w_\delta(x) = w(x) := \frac{1}{\Gamma(1-\beta)} x^{-\beta}$. Consequently, $\partial_t^{w_\delta} f(t) \to \partial_t^{w} f(t)$, the Caputo derivative of f of order β, in the distributional sense as $\delta \to 0$. Using the probabilistic representation in the main Theorem, one can deduce that as $\delta \to \infty$, the solution to the equation $\partial_t^{w_\delta} u = \mathcal{L} u$ with $u(0,x) = f(x)$ converges to the solution of $\partial_t^{\beta} u = \mathcal{L} u$ with $u(0,x) = f(x)$.

Stability

(iii) If we define

$$\eta_\delta(r) = \frac{\Gamma(2-\beta)\,\delta^{\beta-1}}{\beta} w_\delta(r) = (1-\beta)\delta^{\beta-1}\left(x^{-\beta} - \delta^{-\beta}\right) 1_{(0,\delta]}(x),$$

then $\eta_\delta(r)$ converges weakly to the Dirac measure concentrated at 0 as $\delta \to 0$. So $\partial_t^{\eta_\delta} f(t)$ converges to $f'(t)$ for every differentiable f. It can be shown that the subordinator corresponding to η_δ, that is, subordinator with Lévy measure

$$\nu_\delta(dx) := \frac{(1-\beta)\,\delta^{\beta-1}}{\beta}\, x^{-(1+\beta)} 1_{(0,\delta]}(x) dx,$$

converges as $\delta \to 0$ to deterministic motion t moving at constant speed 1. Using the main Theorem, one can show that the solution to the equation $\partial_t^{\eta_\delta} u(t,x) = \mathcal{L}u(t,x)$ with $u(0,x) = f(x)$ converges to the solution of the heat equation $\partial_t u = \mathcal{L}u$ with $u(0,x) = f(x)$.

Time-fractional equation for Schrödinger operator

Suppose X is a Markov process on $\mathcal{X}$ with generator $\mathcal{L}$ and κ is a bounded function on $\mathcal{X}$. Then the Feynman-Kac semigroup

$$T_t f(x) := \mathbb{E}_x \left[e^{-\int_0^t \kappa(X_s)ds} f(X_t) \right]$$

has infinitesimal generator $\mathcal{L} - \kappa$. Hence

$$u(t,x) = \mathbb{E}[T_{L_t} f(x)] = \mathbb{E}_x \left[e^{-\int_0^{L_t} \kappa(X_s)ds} f(X_{L_t}) \right]$$

is the unique solution to

$$\partial_t^w u = (\mathcal{L} - \kappa) u \quad \text{with } u(0,x) = f(x).$$

Feynman-Kac transform for sub-diffusions

Theorem (C.-Deng-Peng, SIAM J. Math. Anal. '21)

Suppose $\kappa(x)$ is a bounded function on $\mathcal{X}$. Let $f \in \mathbb{B}$ and define

$$u(t,x) := \mathbb{E}_x \left[e^{-\int_0^t \kappa(X_{L_s})ds} f(X_{L_t}) \right].$$

Then $u(t,x)$ is the unique mild solution in $(\mathbb{B}, \|\cdot\|)$ to *Kolmogorov backward equation*

$$\partial_t^{w,\kappa(x)} u(t,x) = \mathcal{L}u(t,x) - \kappa(x) I_t^{w,\kappa(x)} u(t,x),$$

with $u(0,x) = f(x)$. Here

$$\begin{aligned} I_t^{w,\kappa}\varphi &:= \int_0^t \varphi(t-s) e^{-\kappa s} w(s)ds, \\ \partial_t^{w,\kappa}\varphi &:= \frac{d}{dt} I_t^{w,\kappa}(\varphi - \varphi(0)). \end{aligned}$$

Mild solution

We say $u(t,x)$ is a mild solution to

$$\partial_t^{w,\kappa(x)} u(t,x) = \mathcal{L}u(t,x) - \kappa(x) I_t^{w,\kappa(x)} u(t,x),$$

with $u(0,x) = f(x)$ if

(a) $t \mapsto u(t,\cdot)$ is continuous on $[0,\infty)$ taking values in $\mathbb{B}$ with $u(0,x) = f(x)$ and t $\exists C_1, C_2 > 0$ so that

$$\|u(t,\cdot)\| \leq C_1 e^{C_2 t} \quad \text{for every } t \geq 0;$$

(b) for every $t > 0$, $\int_0^t u(s,x)ds \in \mathcal{D}(\mathcal{L})$ and

$$\int_0^t e^{-\kappa(x)(t-s)} w(t-s)(u(s,x) - u(0,x))ds$$

$$= \mathcal{L}\int_0^t u(s,x)ds - \kappa(x)\int_0^t \int_0^s e^{-\kappa(x)(s-r)} w(s-r)u(r,x)drds.$$

Dual Feynman-Kac transform

Suppose that X is in weak duality with $\widehat{X}$ w.r.t. m on $\mathcal{X}$.

Theorem (Zhang-C. SPA '22)

Suppose that $\kappa(x)$ is a bounded function on $\mathcal{X}$. For $f, g \in L^2(\mathcal{X}; m)$,

$$\mathbb{E}_{gm}\left[e^{-\int_0^t \kappa(X_{L_s})ds} f(X_{L_t})\right] = \int_{\mathcal{X}} f(y) v_g(t,y) m(dy),$$

where

$$v_g(t,y) = \widehat{\mathbb{E}}_y\left[e^{-\int_0^t \kappa(\widehat{X}_{L_t - L_r})dr} g(\widehat{X}_{L_t})\right]$$

Thus under the measure $\mathbb{P}_{gm}$, the "distribution" of $Y_t = X_{L_t}$ under the Feynman-Kac transform $e^{-\int_0^t \kappa(X_{L_s})ds}$ is $v_g(t,y)m(dy)$.

Fokker-Planck equation

Theorem (Zhang-C. SPA '22)

Suppose $\kappa(x)$ *is bounded on* $\mathcal{X}$*. Define for* $g \in L^2(\mathcal{X}; m)$

$$v_g(t,x):=\widehat{\mathbb{E}}_x\left[e^{-\int_0^t \kappa(\widehat{X}_{L_t-L_s})ds}\, g(\widehat{X}_{L_t})\right].$$

Then $v(t,x) := v_g(t,x)$ *is the unique mild solution in* $L^2(\mathcal{X}; m)$ *to the non-local Fokker-Planck equation*

$$\frac{\partial}{\partial t}v(t,y) = \mathcal{L}^*\left(\frac{\partial}{\partial t}+\kappa(y)\right) I_t^{*,w,\kappa(y)}v(\cdot,y) - \kappa(y)v(t,y)$$

with $v(0,x) = g(x)$*. Here*

$$I_t^{*,w,\kappa}\varphi:=\int_0^t \varphi(t-s)e^{-\kappa s}U(ds).$$

Define $\partial_t^{*,w,\kappa}\varphi := \frac{d}{dt} I_t^{*,w,\kappa}\left(\varphi-\varphi(0)\right)$.

Kolmogorov backward equation (revisited)

Theorem (Zhang-C. SPA '22)

Suppose $\kappa(x)$ *is a bounded function on* $\mathcal{X}$. *Let* $f \in L^2(\mathcal{X}; m)$ *and define*

$$u(t,x) := \mathbb{E}_x \left[e^{-\int_0^t \kappa(X_{L_s}) ds} f(X_{L_t}) \right].$$

Then $u(t,x)$ *is the unique mild solution in* $L^2(\mathcal{X}; m)$ *to* *Kolmogorov backward equation*

$$\frac{\partial}{\partial t} u(t,x) = \left(\frac{\partial}{\partial t} + \kappa(x) \right) I_t^{*,w,\kappa(x)} \mathcal{L} u(\cdot, x) - \kappa(x) u(t,x)$$

with $u(0,x) = f(x)$.

Non-local Fokker-Planck equation or Kolmogorov forward equation:

$$\frac{\partial}{\partial t} v(t,y) = \mathcal{L}^* \left(\frac{\partial}{\partial t} + \kappa(y) \right) I_t^{*,w,\kappa(y)} v(\cdot, y) - \kappa(y) v(t,y)$$

Another version of the time-fractional equation

Taking $\kappa = 0$ in the above Theorem, we in particular have

Corollary

Suppose $\kappa(x)$ is a bounded function on $\mathcal{X}$. Let $f \in L^2(\mathcal{X}; m)$ and define $u(t,x) := \mathbb{E}_x \left[f(X_{L_t}) \right]$. Then $u(t,x)$ is the unique mild solution in $L^2(\mathcal{X}; m)$ to Kolmogorov backward equation

$$\frac{\partial}{\partial t} u(t,x) = \frac{\partial}{\partial t} I_t^{*,w} \mathcal{L} u(\cdot, x)$$

with $u(0,x) = f(x)$.

- $\partial_t^w u(\cdot, x) = \mathcal{L} u(t,x)$ with $u(0,x) = f(x)$.

An identity

$$I_t^{*,w}(I_\cdot^w\varphi) = I_t^w(I_\cdot^{*,w}\varphi) = \int_0^t \varphi(s)ds.$$

For φ locally Lipschitz on $[0,\infty)$,

$$\frac{d}{dt}I_t^{*,w}(I_\cdot^w\varphi') = \varphi'.$$

So we have the identity for locally Lipschitz φ on $[0,\infty)$,

$$\frac{d}{dt}\varphi = \frac{d}{dt}I_t^{*,w}(\partial_\cdot^w\varphi) = \partial_t^{*,w}(\partial_\cdot^w\varphi).$$

Note in this case, $\partial_t^w\varphi = I_t^w(\varphi')$, which is 0 when $t=0$.

Black-Scholes model for bear markets

During bear market, financial activities are less frequent. We propose to model stock price by sub-diffusions:

$$d\mathcal{S}_t = \mathcal{S}_t \left(r_t dt + \overline{\mu}_t dL_{(t-a)^+} + \sigma_t dB_{L_{(t-a)^+}} \right),$$

where $a \geq 0$ is the initial wake up time for the market, $r_t \geq 0$ is the interest rate.

How to price European call option $(\mathcal{S}_T - K)^+$ with maturity T and strike price K?

S. Zhang and C. Fractional Black-Scholes model and Girsanov transform for sub-diffusions. Preprint 2025.

Stochastic optimal control

Joint with Shuaiqi Zhang

1. Stochastic maximum principle for sub-diffusions and its applications. *SIAM J. Control Optim.* **62** (2024), 953-981.
2. Fully coupled forward-backward stochastic differential equations driven by sub-diffusions. *J. Differential Equations* **405** (2024), 337-358
3. Stochastic maximum principle for fully coupled forward-backward stochastic differential equations driven by sub-diffusion. *SIAM J. Control Optim.* **62** (2024), 2433-2455.

Thank you!