Encounters with Imprecise Probabilities

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What is imprecise probability (IP)?

Is it reasonable to say "the probability of rain tomorrow is 0.364927401482906398710 4"?

What does it mean when a weather forecaster says "the probability of rain tomorrow is 0.4"?

- One guess is that the probability of rain is in the interval $(0.35, 0.45)$.
- Another is that it has something to do with a dozen or so ensemble runs.

The Reality: Almost all real world probabilities are imprecise; it is often better to think in terms of intervals of probability.

Also losses are usually imprecise, models are imprecise, everything is imprecise.

The *Society for Imprecise Probability (SIPTA)* consists of many scientists from a variety of disciplines that try to address IP.

Outline

- My view on dealing with IP *in statistics,* molded by interactions with Herman Rubin and Jack Good.
- Four applications of this IP reasoning:
  - the problem of interval probabilities;
  - to correct the p-value problem in science;
  - to provide a sound analysis for any normal hierarchical model;
  - to tackle *uncertainty quantification (UQ)*, the intersection of mathematical modeling of processes and data.

Herman Rubin and Jack Good

Herman Rubin: *A weak system of axioms for 'rational' behavior and the non-separability of utility from prior*, shows that weak imprecise choice axioms require compatibility with some (not unique) Bayesian analysis.

An implementation: Model the (say) imprecise probabilities by the class $\mathcal{P}$ of compatible probability distributions (today called *credal sets*), and make interesting statements for the class (if possible); this should be the gold standard for IP.

Example 1 (Lenny's problem): Suppose climate science gives a predictive probability distribution $p(y)$ for temperature y in 2040. But we assess that there is a 20% chance of the 'big surprise' i.e., we are completely wrong. This can be represented by the class of probability distributions

$$\mathcal{P} = \{0.2q(y) + 0.8p(y);\ q(y) \text{ being any distribution}\}\,.$$

If, say, $p(y)$ is $N(30, 2^2)$, one could then make the valid statement

$$Pr(Y < 34) = 0.2Pr(Y < 34 \mid q) + 0.8P(Y < 34 \mid p) \geq (0.8)(97.5) = 0.78\,.$$

Jack Good (who, while theoretical, tended to focus on practical ideas): When handling imprecise probabilities, "use probabilities of a higher type."

Example 2 (Genome-Wise Association Studies - GWAS):

- A typical GWAS study looks at, say, 20 (related) diseases and 1,000,000 genes (or SNPs), and attempts to determine which genes are associated with which diseases. (Note: 20,000,000 tests are being done here.)
- GWAS studies from 1997-2007 (about 50,000 published papers) had an extremely high rate of replication failure, because most were not adjusting enough for multiple testing, thinking that using 'strict' p-values such 10^{-3} or 10^{-4} would be enough.
- A very influential paper in Nature (2007), by the Wellcome Trust Case Control Consortium, argued for a cutoff of $p < 5 \times 10^{-7}$ for claiming discovery of an association (later shifted to 5×10^{-8} and ??? today).
- *Key step:* They did a subjective Bayesian assessment that the prior odds of a true association to false association are 1/100,000, stating this could be off by a factor of 10 either way (which they subsequently ignored).

Jack Good's solution would have been

- p, the probability of a disease/gene association should be considered a 'logical' unknown probability, to be handled at a 'higher level.'
- At the higher level, assign a prior distribution; for instance a $Gamma(1, 10^5)$ prior is compatible with the prior information of the medical geneticists.
- This would be irrelevant if there were no data (and p entered the subsequent analysis linearly) but there is lots of data.
- This fits into the class of priors framework by defining

$$\mathcal{P} = \{\text{point mass distributions at } p, \quad 0 \leq p \leq 1\}\,.$$

While dealing with $\mathcal{P}$ by placing a single prior distribution on the class logically still corresponds to just a single overall distribution, Good argued that answers are much less sensitive to such higher level distributions.

These examples outline the way I have always approached IP in practice.

- Model the imprecision through a class $\mathcal{P}$ of probability distributions (today called a credal set), and proceed by either
  - Making interesting probability statements that are valid for any distribution in $\mathcal{P}$ (called *robust Bayesian analysis* in the old days);
  - Placing a probability distribution over $\mathcal{P}$ and proceeding via Bayes (*hierarchical Bayesian analysis*).
- Robust Bayesian analysis is sometimes very effective; hierarchical Bayesian analysis is usually very effective.

An Aside: There are other versions of robust Bayesian analysis:

- Choose a 'robust' prior distribution in $\mathcal{P}$ to use.
- Choose the most 'objective' prior distribution in $\mathcal{P}$, the extreme of which, when $\mathcal{P}$ = all distributions, is *objective Bayesian analysis.*
- Choose the empirical Bayes prior distribution in $\mathcal{P}$ (almost always worse than hierarchical Bayes).

I. Dealing with interval probabilities

The problem: We know $p_i \in (a_i, b_i)$, $i = 1, \ldots, m$. Of interest: $P = \prod_{i=1}^{m} p_i$.

Example: A device has m components, each of which will function with independent probability p_i. The device functions only if all components function, which has probability P.

Standard IP answer: State that $P \in (\prod_i a_i, \prod_i b_i)$.

Example: $P \in (0.4, 0.98)$, a probably useless answer.

Bad alternative: use *midpoints*, so $P = \prod_i [(a_i + b_i)/2]$.

Example: $P = 0.92$, ignoring the IP issue.

Laplace alternative (inverse probability): $p_i \sim \text{Uniform}(a_i, b_i)$, find the equal-tailed 95% Bayesian confidence interval for P.

Example: $P \in (0.91, 0.94)$, sensible, but not obviously a fully IP solution.

Higher level IP: Model actual beliefs about $p_i \in (a_i, b_i)$, but stay within the IP framework.

- Values near the midpoints are often more likely than the endpoints.
- Beliefs are typically symmetric and unimodal in the intervals.
- Reflect IP concerns by forming the credal sets $\mathcal{P}_i$ of all distributions with the above two properties for p_i and let $\mathcal{P} = \mathcal{P}_1 \bigotimes \mathcal{P}_2 \bigotimes \ldots \bigotimes \mathcal{P}_m$.
  - Often the p_i are dependent, which would have to be incorporated into the overall class $\mathcal{P}$ of possible priors, challenging to do in an IP way.
- Find the extremal 95% confidence interval – here, the union of all 95% equal-tailed Bayesian confidence intervals from priors in $\mathcal{P}$.

Example: If the p_i are independent, $P \in (0.91, 0.94)$. (The extremal 95% CI over this 'optimal' credal set $\mathcal{P}$ happens to be the same as that from the uniform priors.)

II. The p-value issue

- Significance testing using p-values, declaring a 'discovery' if $p \leq 0.05$, is by far the dominant method of testing in science.
- Its standard uncritical use is viewed by many as being a major source of the problems of reproducibility of science.
- Everyone is talking about it:
  - articles in all the major science journals;
  - changes in editorial policy (the journal *Basic and Applied Social Psychology* banned p-values);
  - the recent ASA position statement about p-values and discussion; the recent *The American Statistician* issue on the subject.
  - the article that appeared in Fall 2017 *Nature Human Behavior*, with over 70 leading scientists in a variety of fields, recommending changing 'statistical significance' from $p \leq 0.05$ to $p \leq 0.005$.
- The better solution is to use Bayesian odds of hypotheses, which are much more readily interpretable.

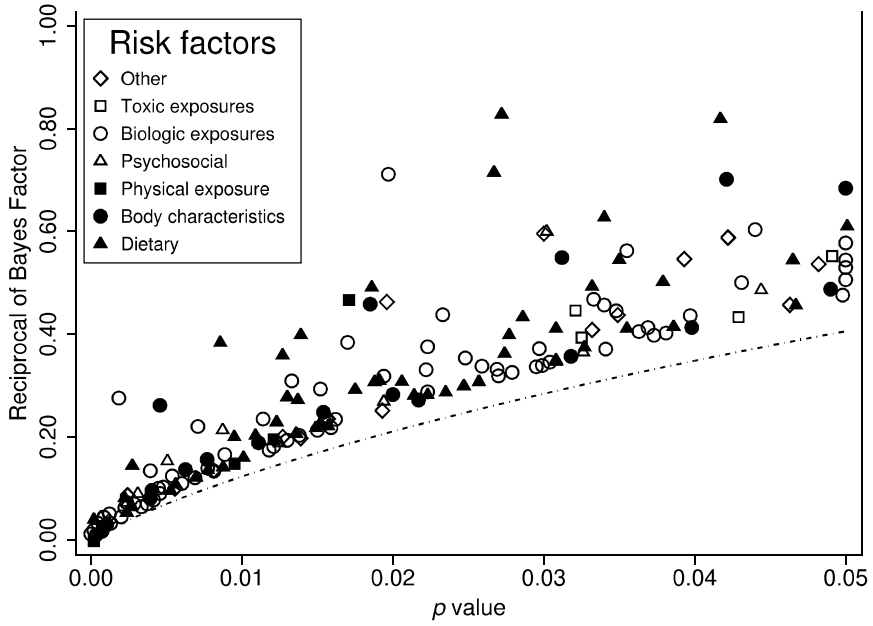
Comparison of p-values with Bayesian odds of hypotheses

Suppose data x arises from the density $f(x \mid \theta)$, and we are interested in testing $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$ with $\pi(\theta)$ being a specified prior distribution of θ under H_1. The *Bayesian odds (Bayes factor)* of H_0 to H_1 is

$$B_{01} = \frac{f(x \mid 0)}{\int f(x \mid \theta)\pi(\theta)\, d\theta}\,.$$

The following investigations compared the p-values from published studies with B_{01}.

- They looked at a large collections of published studies where $0 < p < 0.05$;
- computed B_{01} for each study;
- graphed B_{01} versus the corresponding p-values.
- The first two graphs are for 272 'significant' epidemiological studies with two different choices of the prior; the third for 50 'significant' meta-analyses (these three from J.P. Ioannides, Am J Epidemiology, 2008); and the last is for 314 ecological studies (reported in Elgersma and Green, 2011).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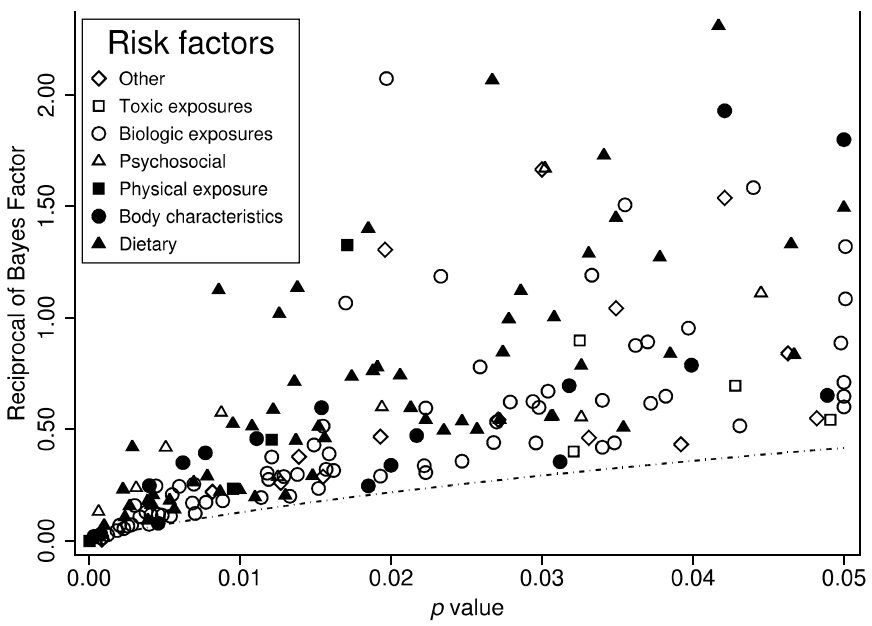
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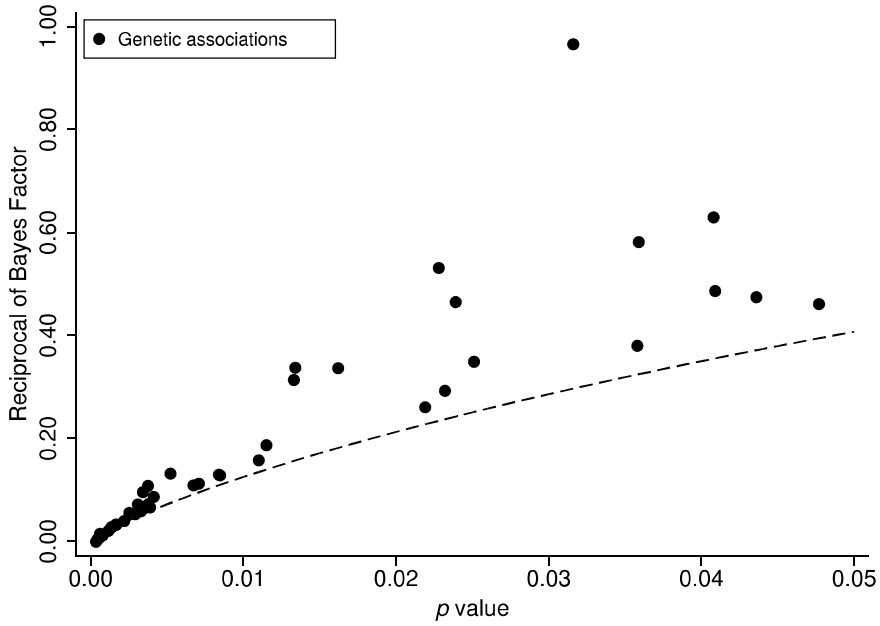
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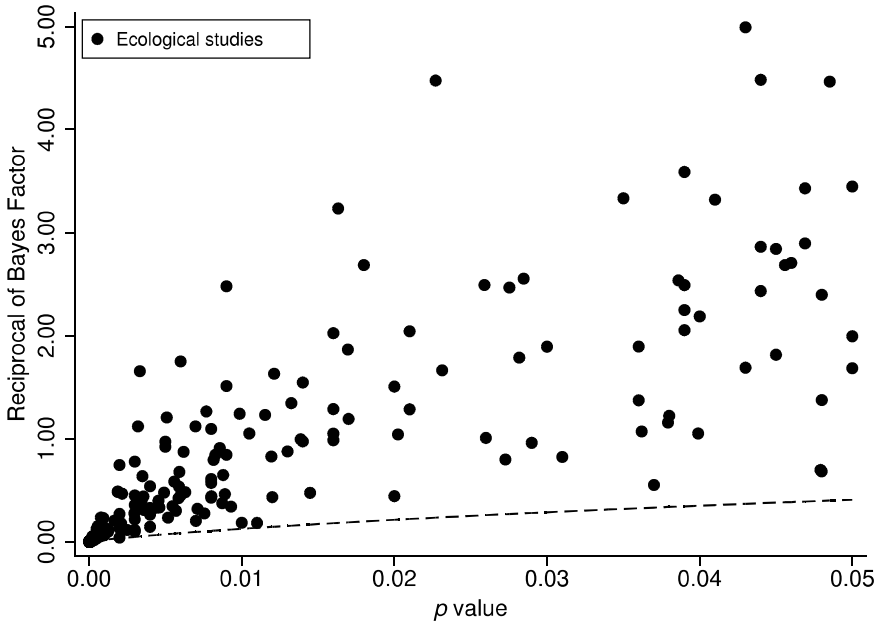
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A quick IP fix of the p-value issue

Determination of the Bayesian odds of H_0 to H_1 can be a challenging problem, because of the typical need for proper priors.

But *robust Bayesian* theory can be used (Sellke, Bayarri and Berger, 2001) to give a bound on the odds of H_0 to H_1, for each given p-value:

Theorem 1 *A proper p-value satisfies* $H_0 : p(X) \sim Uniform(0,1)$, *so test this versus* $H_1 : p \sim g(p)$, *where* g *is in the class of distributions* $\mathcal{P} = \{\text{densities for } p \text{ such that } Y = -\log(p) \text{ has a non-increasing failure rate}\}$, *a natural non-parametric condition on* g. *Then*

$$B_{01} \geq \inf_{g \in \mathcal{P}} 1/g(p) = -e\, p\, \log(p) \quad \text{for } p < e^{-1}.$$

(Vovk (1993) proved this for $\mathcal{P} = \{\text{Beta}(\xi, 1),\ 0 < \xi < 1\}$.)

p	.2	.1	.05	.01	.005	.001	.0001	.00001
$-ep\log(p)$	.879	.629	.409	.123	.072	.0189	.0025	.00031

Note: This bound is the graphed dotted line in the previous figures.

A lower bound over $\mathcal{P} = \{\text{all prior distributions}\}$.

Theorem 2 *Consider testing* $H_0 : \theta = \theta_0$ *vs* $H_1 : \theta \neq \theta_0$ *based on test statistic* $T(x)$, *with* $p(x) = P(T(X) > T(x) \mid \theta_0) \equiv 1 - F(T(x) \mid \theta_0)$ *and* f *being the density corresponding to* F. *For* any *prior* $\pi(\theta)$,

$$B_{01} \geq \frac{f(F^{-1}(1-p) \mid \theta_0)}{\sup_\theta f(F^{-1}(1-p) \mid \theta)} .$$

p	0.1	0.05	0.01	0.005	0.001	0.0001	0.00001	5×10^{-7}
$-ep\log(p)$	.629	.409	.123	.072	.0189	.0025	.00031	5×10^{-6}
$\sup_\theta$, Normal	.345	.227	.068	.039	.0088	.0010	.00014	6×10^{-6}

III. Optimal hyperpriors for normal hierarchical models

(with Chengyuan Song and Dongchu Sun)

For $i = 1, 2, \ldots, m$,

- $\boldsymbol{X}_i = \boldsymbol{\theta}_i + \epsilon_i, \qquad \epsilon_i \sim N_k(\cdot \mid \mathbf{0}, \boldsymbol{\Sigma}_i)$,
  the $\boldsymbol{X}_i$ and $\boldsymbol{\theta}_i$ being $k \times 1$ vectors, $k \geq 2$, with the $\boldsymbol{\Sigma}_i$ known.

  Example: At hospital i,

  - $\boldsymbol{X}_i = (X_{i1}, \ldots, X_{ik})$ is the sample averages of the costs of k different medical treatments;
  - $\boldsymbol{\theta}_i$ is the corresponding unknown vector of true mean costs of the treatments at the hospital;
  - $\boldsymbol{\Sigma}_i$ is the associated (estimated) covariance matrix.

  Note: If $\boldsymbol{X}_i = \boldsymbol{B}_i\boldsymbol{\theta}_i + \epsilon_i$ for given design matrix $\boldsymbol{B}_i$, transform to $\boldsymbol{X}_i^* = (\boldsymbol{B}_i^t\boldsymbol{\Sigma}_i^{-1}\boldsymbol{B}_i)^{-1}\boldsymbol{B}_i^t\boldsymbol{\Sigma}_i^{-1}\boldsymbol{X}_i$, which will be distributed as above.

- $\boldsymbol{\theta}_i = \boldsymbol{z}_i\boldsymbol{\beta} + \epsilon_i^*, \quad \epsilon_i^* \sim N_k(\cdot \mid 0, \boldsymbol{V})$,
  with the $\boldsymbol{z}_i$ being specified $k \times l$ covariate matrices.
  - $\boldsymbol{\beta}$ is an $l \times 1$ unknown 'hyper-mean' vector, $l \geq 2$;
  - $\boldsymbol{V}$ is an unknown $k \times k$ 'hyper-covariance matrix'.

  Example continued: Because all hospitals are related, the $\boldsymbol{\theta}_i$ are assigned a hierarchical prior referring to the 'population' of hospitals.

  The $\boldsymbol{z}_{ij}$ are known covariates, giving hospital i's characteristics for treatment j, such as the number of patients receiving the treatment, the average severity of the condition of the patients, the average income of the patients, etc.

We have specified a credal class of priors

$$\mathcal{P} = \{\pi(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m) \mid \boldsymbol{\beta}, \boldsymbol{V}), \quad \boldsymbol{\beta} \in \mathcal{R}^k, \ \boldsymbol{V}_{k\times k} \text{ positive definite}\}.$$

No way to do optimal IP here; traditional efforts include

- Empirical Bayes: estimate $\boldsymbol{\beta}$ and $\boldsymbol{V}$ (often seriously problematical).
- (Laplace) hierarchical Bayes: assign the higher level prior $\pi(\boldsymbol{\beta}, \boldsymbol{V}) = 1$.
  - But the constant prior is bad here.

Goal: Find good hyperpriors $\pi(\boldsymbol{\beta}, \boldsymbol{V}) = \pi(\boldsymbol{\beta})\pi(\boldsymbol{V})$ (independence assumed).

Recommended prior: After standardizing the covariates, use

$$\begin{aligned} \pi(\boldsymbol{\beta}) &\propto \frac{1}{(1+||\boldsymbol{\beta}||^2)^{(p-1)/2}}, \ \boldsymbol{\beta} \in \mathbb{R}^p, \\ \pi(\boldsymbol{V}) &\propto \frac{1}{|\boldsymbol{V}|^{1-1/(2k)} \prod_{1 \le i < j \le k} (v_i - v_j)}, \ \boldsymbol{V} > 0\,, \end{aligned}$$

where $v_1 > v_2 > \ldots > v_k$ are the eigenvalues of $\boldsymbol{V}$.

- These are related to *reference priors* which are highly recommended in the objective Bayesian literature.
- The priors can be efficiently implemented with MCMC algorithms.
- Using the priors will result in *admissible* frequentist shrinkage estimators of $(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m)$, under quadratic loss, and they are the vaguest priors which do so.
- These priors can be used for any means and covariance matrices that occur in any normal hierarchical model, no matter how many levels.

The choice of the hyperprior is important. For instance, in hierarchical normal models the current standard choice of the hyperprior is $\pi(\boldsymbol{\beta}, \boldsymbol{V}) = 1$.

- The constant prior requires more than $2k$ vector observations for posterior propriety, while the recommended prior requires only 2 vector observations (transforming to $\boldsymbol{V} = \boldsymbol{O}'\boldsymbol{D}\boldsymbol{O}$, where $\boldsymbol{D}$ is diagonal and $\boldsymbol{O}$ is orthogonal, yields $\pi(\boldsymbol{V})d\boldsymbol{V} = |\boldsymbol{D}|^{1-1/(2k)}\mathbf{1}_{\{d_1>d_2>\ldots>d_k\}}d\boldsymbol{D}\ d\boldsymbol{O}$).
- The constant prior for $\boldsymbol{V}$ has the bizarre property of strongly forcing the eigenvalues of $\boldsymbol{V}$ apart ($1d\boldsymbol{V} = \prod_{i<j}(d_i - d_j)\ \mathbf{1}_{\{d_1>d_2>\ldots>d_k\}}d\boldsymbol{D}\ d\boldsymbol{O}$).
- The constant prior yields estimates of $(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m)$ that are much worse:

Table 3. The mean square error of estimates of $(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m)$ for the Constant prior (C) and the Recommended prior (R) in the following eight scenarios:

$k_1 = 4, m_1 = 10$, $k_2 = 5, m_2 = 15$; $\boldsymbol{\beta}_1 = \mathbf{1}_k$, $\boldsymbol{\beta}_2 = 50\mathbf{1}_k$; $\boldsymbol{V}_1 = \boldsymbol{I}_k$, $\boldsymbol{V}_2 = diag\{8k-7, \ldots, 9, 1\}$

Prior	$k_1\beta_1V_1$	$k_1\beta_2V_1$	$k_1\beta_1V_2$	$k_1\beta_2V_2$	$k_2\beta_1V_1$	$k_2\beta_2V_1$	$k_2\beta_1V_2$	$k_2\beta_2V_2$
C	68.481	71.552	76.541	84.039	111.507	128.434	134.340	145.854
R	42.735	44.746	63.311	76.338	77.129	107.277	123.973	134.529

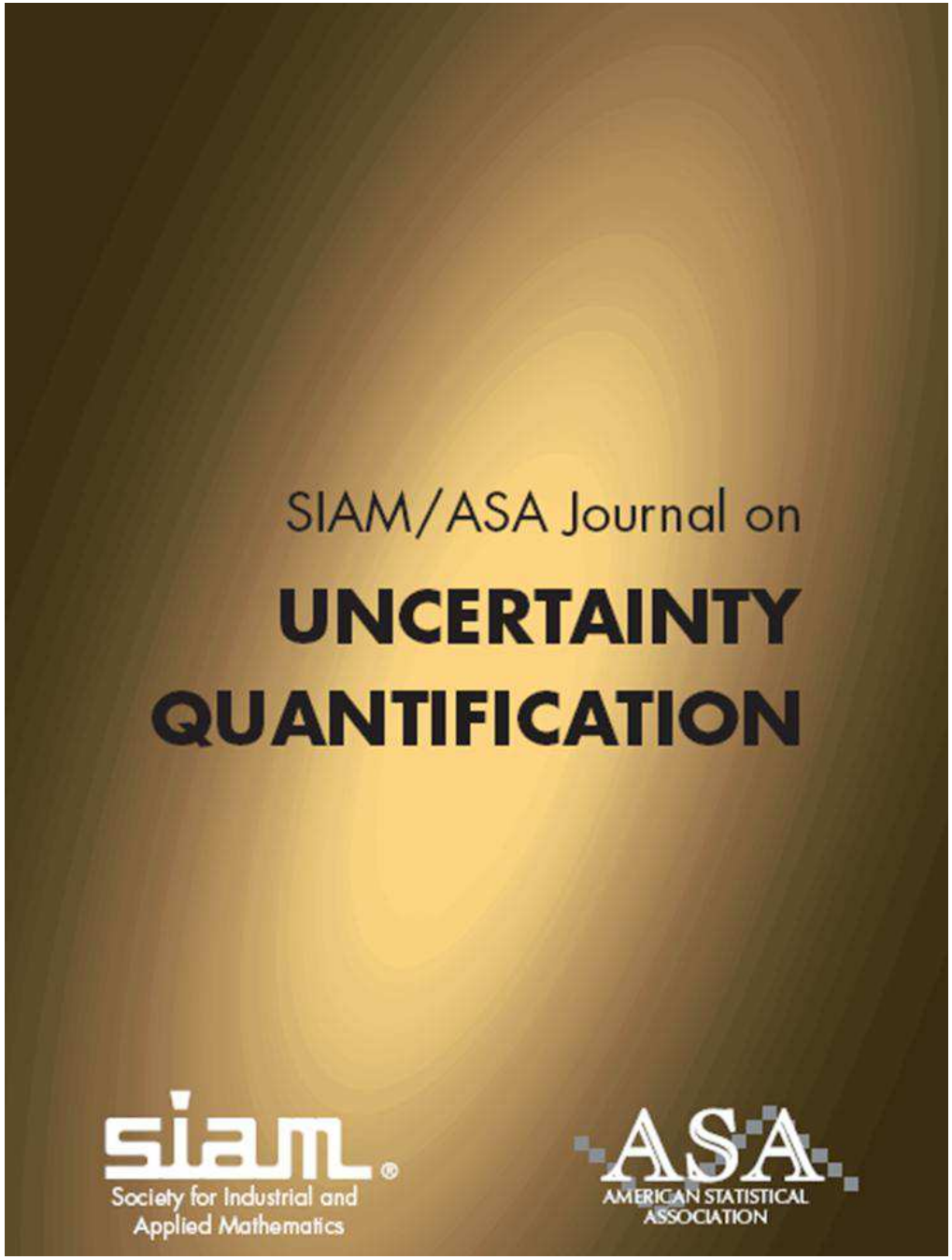
IV. Uncertainty Quantification (UQ): Dealing with uncertainties involved in math modeling of processes

- *Real process:* $y^R(\mathbf{x})$, with input $\mathbf{x}$.
  - Example: $y^R(\mathbf{x})$ is future temperature under carbon forcing $\mathbf{x}$.
- *Computer model output:* $y^M(\mathbf{x}, \mathbf{u})$, with unknown parameters $\mathbf{u}$.
  - Example: y^M is prediction of future temperature from a climate model; $\mathbf{u}$ is the hundreds of unknown parameters in the model.
- *Observations of the real process:* $y^O(\mathbf{x})$
  - Example: Alas, we only have one (partial) observation of climate.
- *Classical formulation:* $y^O(\mathbf{x}) = y^M(\mathbf{x}, \mathbf{u}) + \epsilon$, with ϵ being random error.

Twenty years ago Tony O'Hagan said, "no, the math model is virtually always *imprecise*, so the correct formulation is

$$y^O(\mathbf{x}) = y^M(\mathbf{x}, \mathbf{u}) + \mathbf{b}(\mathbf{x}, \mathbf{u}) + \epsilon\,,$$

where $b(\mathbf{x}, \mathbf{u}) = y^R(\mathbf{x}) - y^M(\mathbf{x}, \mathbf{u})$ is *model discrepancy (bias).*"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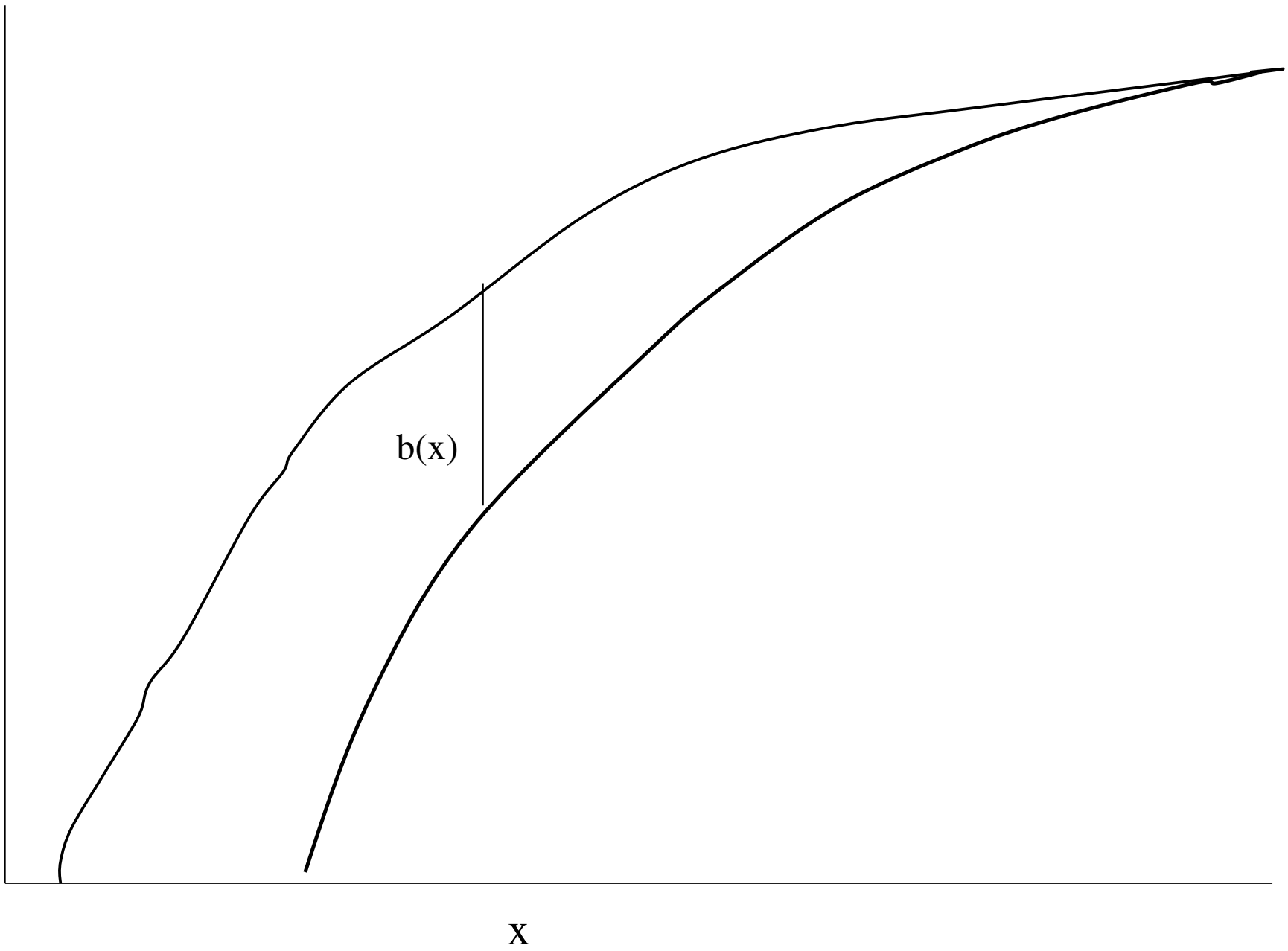
b(x)
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The Jeffreys-Lindley Paradox

Suppose we have n i.i.d. observations from density $f(x \mid \theta)$ and wish to test $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$.

Suppose a Bayesian has fixed (nonzero) prior probabilities of the hypotheses and a fixed bounded prior density on θ under H_1.

Let $p = p(x_1, \ldots, x_n)$ be the p-value for the test, corresponding (say) to the likelihood ratio test.

The 'Paradox': Suppose n is increasing and $(x_1, \ldots, x_n)$ are such that $p(x_1, \ldots, x_n)$ is fixed at $p = 10^{-10}$; then $Pr(H_0 \mid x_1, \ldots, x_n) \to 1$ as $n \to \infty$.

Model bias: Rarely is $f(x \mid \theta)$ exactly true. Suppose, instead, that $x_i - b$ has density $f(\cdot \mid \theta)$, where b reflects a model bias (alternatively, suppose that H_0 is really $H_0 : |\theta| < b$.) This bias won't be known, but its prior distribution will typically lie in the credal class

$$\mathcal{P} = \{\text{all prior distributions that do not have a postive probability mass at } 0\}.$$

Result: For any $\pi \in \mathcal{P}$ and fixed p, $Pr(H_0 \mid x_1, \ldots, x_n, \pi) \to c_\pi < 1$ as $n \to \infty$.

Example: UQ for a computer model of road-load dynamics

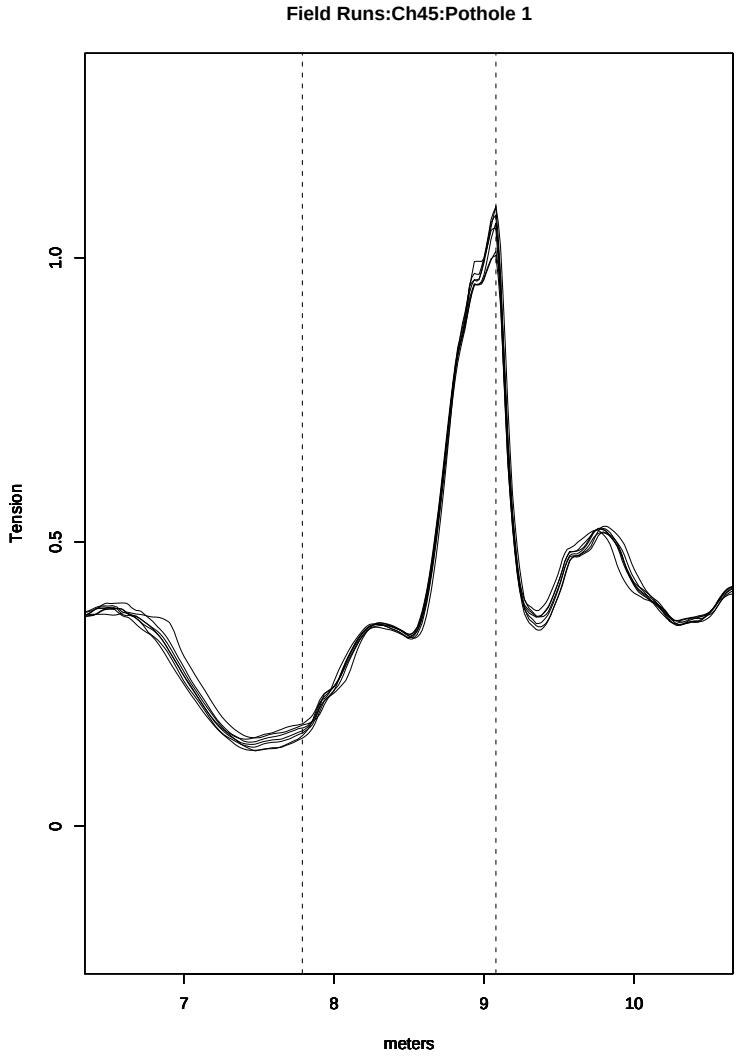
(with M.J. Bayarri, J. Cafeo, G. Garcia-Donato, F. Liu, J. Palomo, R.J. Parthasarathy, R. Paulo, J. Sacks, and D. Walsh: *AOS, 2007*)

Consider a vehicle being driven over a road with two major potholes.

- $\boldsymbol{x} = (x_1, \ldots, x_7)$ is the vector of key vehicle characteristics, unknown because of manufacturing variability.
- $y^R(\boldsymbol{x}; t)$ is the time-history curve of resulting forces.

A finite element PDE computer model of the vehicle being driven over the road

- depends on $\boldsymbol{x} = (x_1, \ldots, x_7)$ and unknown calibration parameters $\boldsymbol{u} = (u_1, u_2)$;
- yields time-history force curve $y^M(\boldsymbol{x}, \boldsymbol{u}; t)$.



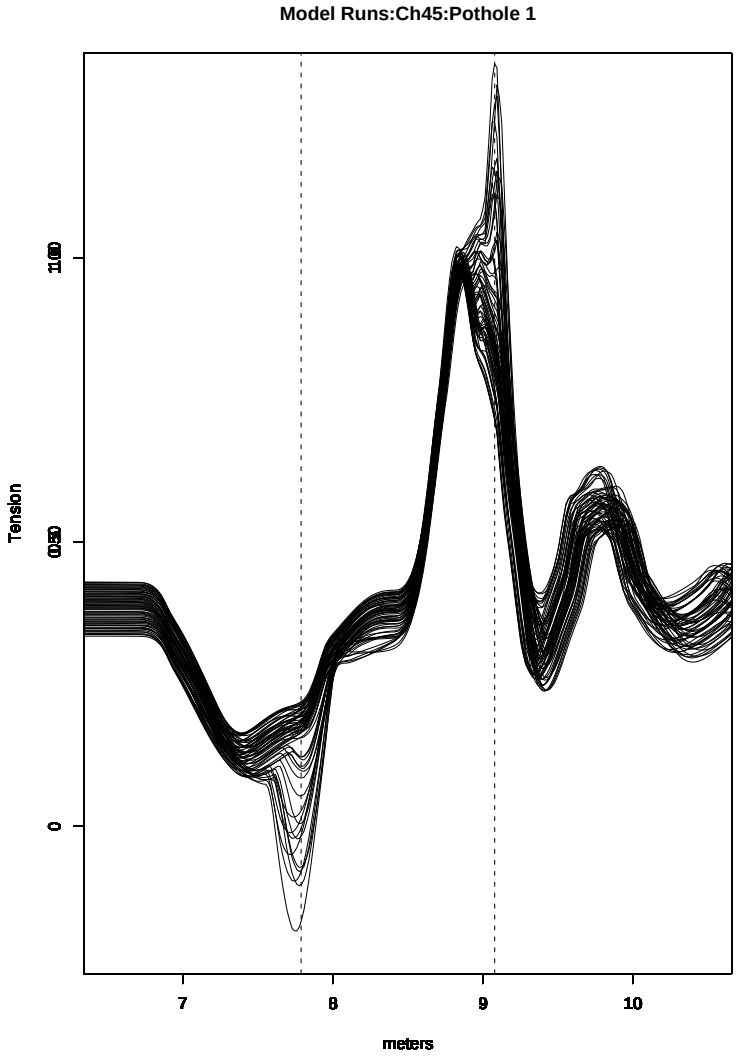


Figure 1: Force curves from 7 field runs (left) and 65 computer model runs (right) for one of the potholes.

Analysis proceeded by

- registration (aligning) of field and model force curves;
- employing a wavelet representation of all force curves, so that

$$y^M(\boldsymbol{x},\boldsymbol{u};t) \approx \sum_{i=1}^{289} w_i^M(\boldsymbol{x},\boldsymbol{u})\psi_i(t), \quad y_r^F(\boldsymbol{x}^*;t) \approx \sum_{i=1}^{289} w_{ir}^F(\boldsymbol{x}^*)\psi_i(t)\,,$$

  where the $w_i^M(\boldsymbol{x},\boldsymbol{u})$ and $w_{ir}^F(\boldsymbol{x}^*)$ are the coefficients computed through the wavelet decomposition and the $\psi_i(t)$ are known basis functions;
- introducing model bias having a zero mean Gaussian process prior;
- for computational reasons and to allow inference at new inputs, replacing each $w_i^M(\boldsymbol{x},\boldsymbol{u})$ by an emulator (a Gaussian process approximation to that part of the computer model);
- assigning priors (a mix of subjective and objective) to all unknowns;
- employing modularized MCMC to determine the posterior distribution of all unknowns and to make future predictions.

Parameter	**Type**	**Uncertainty Range**	**Prior**
$Damping_1$	Calibration	[0.125, 0.875]	Uniform
$Damping_2$	Calibration	[0.125, 0.875]	Uniform
x_1	Nominal+Variation	[0.1667, 0.8333]	Normal
x_2	Nominal+Variation	[0.1667, 0.8333]	Normal
x_3	Nominal+Variation	[0.2083, 0.7917]	Normal
x_4	Nominal+Variation	[0.1923, 0.8077]	Normal
x_5	Nominal+Variation	[0.3529, 0.6471]	Normal
x_6	Nominal+Variation	[0.1471, 0.8529]	Normal
x_7	Nominal+Variation	[0.1923, 0.8077]	Normal

Table 1: Uncertainty ranges for calibration parameters and parameters subject to manufacturing variation, and utilized priors

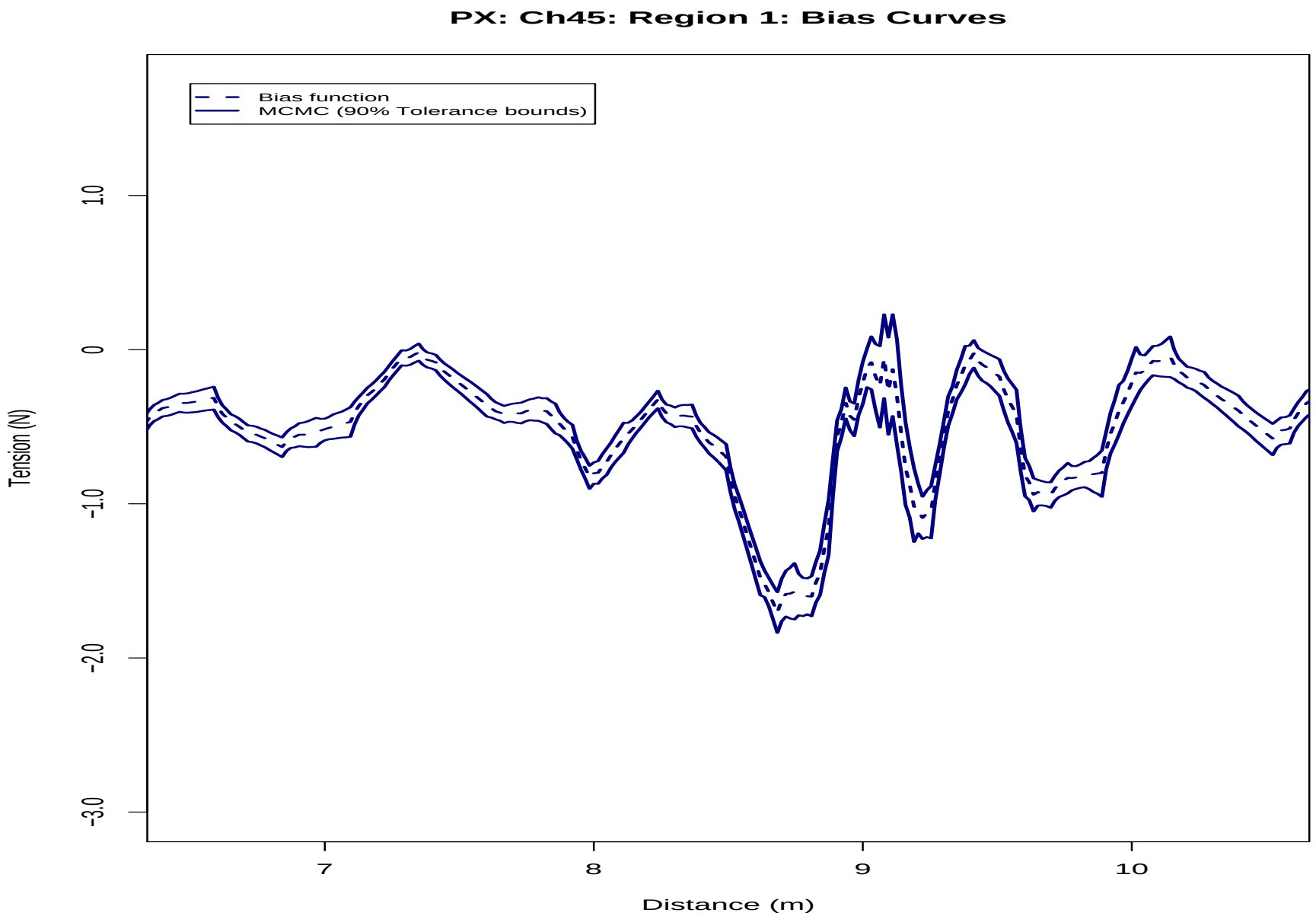


Figure 2: Posterior bias curve estimate and 90% tolerance bands.

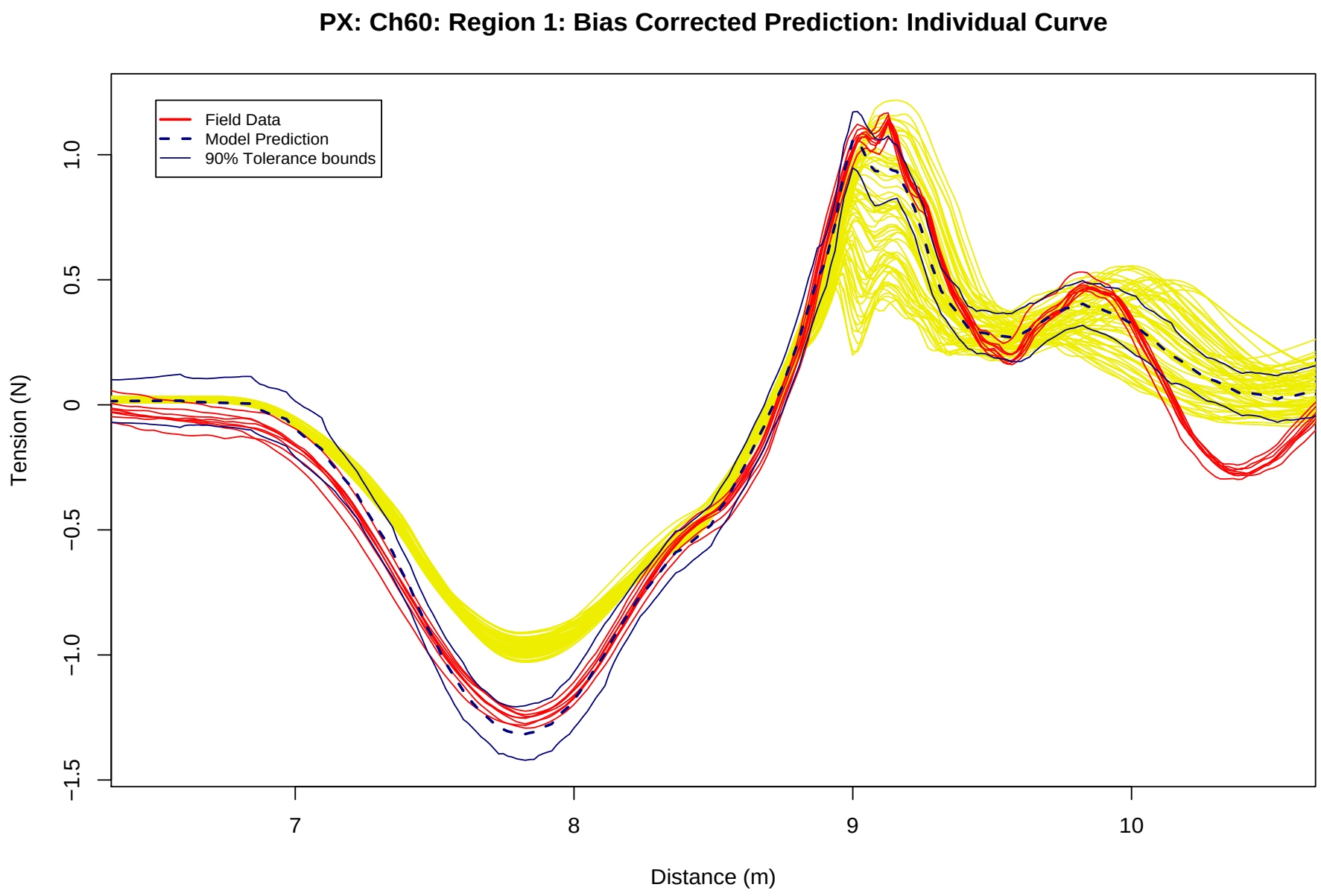


Figure 3: Extrapolation of bias to predict the force curve for Vehicle B.

IP features of UQ

- It is often crucial to acknowledge that the computer model is imprecise, and incorporate a discrepancy (bias) term.
- Unknown model parameters and model inputs are often constrained to lie in known intervals; distributions (typically uniform or normal) need to be assigned to these intervals, more general IP analysis not being feasible.
- Fully Bayesian analysis of all unknowns is typically required, because of sparsity of data on reality.
  - This typically requires development of an emulator, a statistical approximation to the computer model.

Thanks!