Selected Topics in Stochastic Partial Differential Equations

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Abstract

Through a few selected examples (and counterexamples) we will explore how stochastic PDEs arise naturally in the study of spatial stochastic processes. We will introduce models from population genetics, the evolution of phase fields, quantum field theory and growth models, with an emphasis on the connection between small-scale dynamics and macroscopic behavior. We will obtain on overview of the methods available to solve stochastic PDEs, study their qualitative behavior.

Contents

1 Branching Brownian Motions

In the study of the evolution of populations, what is the driving force between the diversity of species that we observe in nature? Spatial dispersion (this is just one aspect, on which we choose to focus) plays a key rôle: as members of the same species separate over space, they develop mutations independently of one another, leading to the evolution of different species.

A crucial question is whether there are fundamental (universal) laws that apply to the evolution of populations over space and time (see for example the Wright–Malécot formula for the probability of common descent). Small populations are driven by "local" laws: the particular geography in which they live for example. Or the structure of the randomness in the reproduction mechanism, which can be so strong as to lead to extinction of a species (so-called *genetic* drift). The quest for universality is to identify how these forces act on *large* populations. While one can come up with a miriad of different "local" laws, most of them share more or less the same effect on large populations, perhaps up to a couple of parameters.

Mathematically, this leads to generalisations of the law of large numbers and the central limit theorem to processes that evolve both in time and in space. Population genetics is but one setting in which such studies are meaningful: other applications of similar ideas are fundamental in the study of phase fields, quantum field theory, growth processes, etc...

We start by looking at these questions from the perspective of population genetics, so let us introduce a few classical models. For this chapter, we refer to the very nice exposition [Ethoo] for further reading.

1.1 The Moran model

The Moran model considers the evolution of $n \in \mathbb{N}$ particles that form a population. Each particle can be either of type A or of type B. Evolutionary reproductive events take place at random exponential times. When such events take place, one member of the population, chosen uniformly at random, dies and another reproduces (the type of the new particle is inherited from its parent). Let X_t^n be the number of particles of type A. Then X^n is a Markov process on $\{0, \ldots, n\}$ with generator

$$\mathscr{L}(f)(x) = p(x)(1 - p(x)) \sum_{y=\pm 1} (f(x+y) - f(x)) , \qquad p(x) = \frac{x}{n} .$$

In general, continuous-time jump Markov processes are characterised by their generators, which in the simplest setting take the form:

$$\mathscr{L}(f)(x) = \int_S (f(y) - f(x)) q(x, \mathrm{d}y) , \tag{1.1}$$

where $q(x, \mathrm{d}y)$ is a measure that indicates the rate of transition from x to y and S is the state space. If $q(x, \mathrm{d}y)$ is a finite measure (as in our case) one can think of

$$p(x, \mathrm{d}y) = \frac{1}{\int q(x, \mathrm{d}y)} q(x, \mathrm{d}y) ,$$

as the transition probabilities from x to y of a discrete time Markov chain. And $\int q(x, \mathrm{d}y)$ indicates the total speed (or rate) at which a transition (chosen according to p) happens. Crucially, $\mathscr{L}$ helps us in computing expectations:

$$\partial_t \mathbb{E}[f(X_t)] = \mathbb{E}[\mathscr{L} f(X_t)] ,$$

and we will make use of the following result (we will be a bit reckless and omit writing domains of generators and other assumptions: for now, take the state space S to be finite).

Theorem 1.1 *If $(X_t)_{t \geqslant 0}$ is a jump Markov process with generator $\mathscr{L}$ of the form* (1.1)*, with q a finite measure, then*

$$M_t^f = f(X_t) - \int_0^t \mathscr{L}(f)(X_s) \, \mathrm{d}s ,$$

is a martingale with predictable quadratic variation

$$\langle M^f \rangle_t = \int_0^t \mathscr{L}(f^2)(X_s) - 2f\mathscr{L}(f)(X_s) \, \mathrm{d}s \ .$$

Proof. For a proof of the first statement see [EK86, Proposition 4.1.7]. For the second statement, we can compute

$$(M_t^f)^2 = f(X_t)^2 - 2f(X_t) \int_0^t \mathscr{L}(f)(X_s) \, \mathrm{d}s + \left(\int_0^t \mathscr{L}(f)(X_s) \, \mathrm{d}s \right)^2 . \quad (1.2)$$

We need only concentrate on the last two terms. Here we have

$$\begin{aligned} 2f(X_t) \int_0^t \mathscr{L}(f)(X_s) \, \mathrm{d}s =& 2 \int_0^t f(X_s)\mathscr{L}(f)(X_s) \, \mathrm{d}s + 2 \int_0^t M_{s,t}^f \mathscr{L}(f)(X_s) \, \mathrm{d}s \\ &+ 2 \int_0^t \int_s^t \mathscr{L}(f)(X_u) \, \mathrm{d}u \mathscr{L}(f)(X_r) \, \mathrm{d}r \end{aligned}$$

The second term here cancels via the tower property under expectation (we have defined $M_{s,t}^f = M_t^f - M_s^f$) and the last term cancels with the last term in (1.2). □

It turns out that the martingale problem is a great tool to study scaling limits of Markov processes. For example, from our motivation it is natural to consider the behaviour of X_t^n for $n \gg 1$. Of course, for times of order 1 nothing happens as a huge population will not suddenly change its size. Instead, we must look at the behaviour at large times. The correct scaling is parabolic: we must find $(\alpha_t)_{t \geqslant 0}$ such that

$$\left(\frac{1}{n} X_{n^2 t}^n \right)_{t \geqslant 0} \Rightarrow (\alpha_t)_{t \geqslant 0} \ .$$

We must study therefore the limit of the generator

$$\mathscr{L}^n(f)(\alpha) = n^2 \alpha(1-\alpha) \sum_{y = \pm 1} (f(\alpha + 1/n) - f(\alpha)) \ .$$

Note that this is a discrete approximation of the Laplacian $\Delta f = \sum_{i=1}^d \partial_{x_i}^2 f$.

Exercise 1 *Prove that for $f \in C^2(\mathbb{R}^d;\mathbb{R})$ (meaning that the second derivative is locally continuous)*

$$\lim_{n \to \infty} n^2 \sum_{y \sim_n x} f(x+y) - f(x) = \Delta f(x) \, ,$$

where $y \sim_n x$ if y is of the form $y = x \pm n^{-1} e_i$ for some basis vector $e_i \in \mathbb{R}^d$.

Therefore, if there is any right in the world, the limit $(\alpha_t)_{t \geqslant 0}$ should be the diffusion

$$\mathrm{d}\alpha_t = \sqrt{2\alpha_t(1-\alpha_t)} \, \mathrm{d}W_t \ ,$$

for $(W_t)_{t \geqslant 0}$ a Brownian motion. Indeed, X_t^n is a martingale itself (check this by taking $f(x) = x$). Presuming the limit is continuous, the martingale is characterised by its quadratic variation, which is given via Dynkin's formula.

Exercise 2 *Check that for $f(\alpha) = \alpha$*

$$\mathscr{L}^n(f^2)(\alpha) - 2f\mathscr{L}^n(f)(\alpha) \to 2\alpha(1-\alpha) \, , \qquad n \to \infty \ .$$

1.2 Large scale dynamics with critical branching

The aim of this mini-course is to study the limiting processes that arise in analogy to the example just seen, but in presence of an additional spatial component. This means that for instance particles are not only allowed to reproduce and die out, but can just as well move in space. For instance consider $\mathbf{Z}^d$ the unit lattice and on it a system of particles defined as follows:

- All particles behaves independently of one another.
- Each particle reproduces at rate one (giving birth to a new particle at the same location of the parent).
- Each particle dies out at rate one.
- Each particle performs a random walk with unit rate on the lattice.

The system of branching Brownian motions just described is equivalently defined through the generator:

$$\mathscr{L}(f)(\eta) = \sum_{x \in \mathbf{Z}^d} \eta(x) \left\{ \sum_{y \sim x} f(\eta^{x \to y}) - f(\eta) \right\} \\ + \eta(x)\big\{ f(\eta^{x+}) - f(\eta) \big\} + \eta(x)\big\{ f(\eta^{x-}) - f(\eta) \big\} \; .$$

Here $\eta \colon \mathbf{Z}^d \to \mathbb{N}$ represents the umber of particles alive at x. And in addition

1. $\eta^{x \to y}$ represents the movement of one particle from x to y:

$$\eta^{x \to y}(z) = \begin{cases} \eta(z) & \text{if } z \notin \{x, y\} \; , \\ \eta(x) - 1 & \text{if } z = x \; , \\ \eta(y) + 1 & \text{if } z = y \; . \end{cases}$$

2. $\eta^{x\pm}(z) = \eta(z) \pm 1_{\{x\}}(z)$ stands for the birth or death of a particle at x.

Our Markov process $(\eta_t)_{t \geqslant 0}$ is therefore a sequence of functions $\eta_t \colon \mathbf{Z}^d \to \mathbb{N}$ such that $\eta_t(x)$ counts the number of particles alive at time $t \geqslant 0$ in location x. How can we describe the large-scale effective dynamic that arises from this model?

First, we must consider the evolution of a large cloud of particles. Therefore we fix an initial condition with n^α particles (the parameter $\alpha \in \mathbb{N}$ will be tuned later on). For simplicity, we position all particles at $x = 0$ in the initial state:

$$\eta_0^n(x) = n^\alpha 1_{\{0\}}(x) \; .$$

Then we associate to its evolution η_t^n the empirical density of the particles

$$\mu_t^n = \frac{1}{n^\alpha} \sum_{x \in n^{-1}\mathbf{Z}^d} \eta^n_{n^2 t}(nx) \delta_x \; .$$

Note that here we have zoomed out in space and time: particles seen from far away live on a smaller lattice $n^{-1}\mathbf{Z}^d$, and we have scaled time parabolically by a factor n^2. Then $\mu_t^n \in \mathscr{M}(\mathbb{R}^d)$, the space of finite positive measures on $\mathbb{R}^d$. For later convenience we note that this is a topological space

$$(\mathscr{M}(\mathbb{R}^d), \tau_{\mathrm{v}}) \; ,$$

when endowed with the topology τ_{v} of *vague* convergence, meaning that $\mu^n \to \mu$ vaguely if

$$\langle \mu^n, \varphi \rangle \to \langle \mu, \varphi \rangle \;, \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}) \;,$$

we denote with $C_c^\infty(\mathbb{R}^d; \mathbb{R})$ is the space of such functions with compact support.

Our objective is to show that the empirical measure of the critical spatial branching process $(\mu_t^n)_{t \geqslant 0}$ has a limit in distribution as a process with values in $\mathcal{M}(\mathbb{R}^d)$. To see this, we can test the empirical measure against continuous and bounded functions. We find

$$\langle \mu_t^n, \varphi \rangle = \frac{1}{n^\alpha} \sum_x \eta_{n^2 t}^n(nx) \varphi(x) \;.$$

To see how $\langle \mu_t^n, \varphi \rangle$ evolves in time we can use the generator:

$$\mathrm{d}\langle \mu_{n^2 t}^n, \varphi \rangle = \langle \mu_{n^2 t}^n, \Delta^n \varphi \rangle \,\mathrm{d}t + \mathrm{d}M_t^{n,\varphi} \;, \tag{1.3}$$

for some cádlág martingale $(M_t^{n,\varphi})_{t \geqslant 0}$. For example, if we choose $\varphi \equiv 1$, we obtain an a-priori bound on the total mass of the process.

Lemma 1.2 *Let μ^n be defined as above and fix $\alpha \geqslant 2$. Then for any $t \geqslant 0$*

$$\sup_{n \in \mathbb{N}} \mathbb{E} \Big| \sup_{0 \leqslant s \leqslant t} \langle \mu_t^n, 1 \rangle \Big|^2 < \infty \;.$$

Proof. From (1.3), Theorem 1.1, and a short calculation, we know that $N_t^n = \langle \mu_t^n, 1 \rangle$ is a martingale, with predictable quadratic variation

$$\mathrm{d}\langle N^n \rangle_t = 2 n^{2-\alpha} \langle \mu_t^n, 1 \rangle \,\mathrm{d}t \;.$$

If follows that

$$\mathbb{E} \sup_{0 \leqslant s \leqslant t} \langle \mu_s^n, 1 \rangle^2 \lesssim \langle \mu_0^n, 1 \rangle^2 + \mathbb{E} \langle N_t^n \rangle \simeq 1 + n^{2-\alpha} t \;.$$

□

As a consequence of this a-priori estimate we obtain tightness of the sequence μ^n.

Corollary 1.3 *For $\alpha \geqslant 2$, the sequence $(\mu_t^n)_{t \geqslant 0}$ is tight in the space $\mathbb{D}([0,\infty); \mathcal{M}(\mathbb{R}^d))$. In addition any limit point lives in $C([0,\infty); \mathcal{M}(\mathbb{R}^d))$.*

The proof of this corollary is slightly technical, because it involves understanding tight sets in the Skorokhod space: we refer the reader to [Ethoo, Proposition 1.19]. The fundamental question we must ask now is whether there exists a *unique* limit point μ of μ^n, and how to characterise it. Taking naively the limit $n \to \infty$ in (1.3) we would hope that any limit μ satisfies identities of the following form for any $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$:

$$\mathrm{d}\langle \mu_t, \varphi \rangle = \langle \mu_t, \Delta \varphi \rangle \,\mathrm{d}t + \mathrm{d}M_t^\varphi \;, \tag{1.4}$$

In addition, the limiting martingale should have quadratic variation given by the limit of the discrete quadratic variations, by Theorem 1.1:

$$\mathrm{d}\langle M^{n,\varphi} \rangle_t = \langle \mu_t^n, x \mapsto \sum_{y \sim x} |\varphi(y) - \varphi(x)|^2 \rangle n^2 n^{-\alpha} \,\mathrm{d}t$$

$$+ \langle \mu_t^n, \varphi^2 \rangle n^2 n^{-\alpha} \,\mathrm{d}t \,.$$

Then choose $\alpha = 2$. The first term does not contribute, and the limiting variance is

$$\mathrm{d}\langle M^\varphi \rangle_t = \langle \mu_t, \varphi^2 \rangle \,\mathrm{d}t \,.$$

Instead for $\alpha > 2$ the limiting variance is zero (we therefore expect the limit μ to be deterministic in this case, corresponding to a law of large numbers). We have concluded the following

- A law of large numbers if the number of particles is very large ($\alpha > 2$).
- A limiting stochastic process if the number of particles is chosen correctly ($\alpha = 2$).

For the limit we have furthermore identified a martingale problem.

Definition 1.4 *We say that a Markov process $(\mu_t)_{t \geqslant 0} \in C([0,\infty); \mathcal{M}(\mathbb{R}^d))$ solves the Dawson–Watanabe martingale problem, if for any $\varphi \in C_c^\infty(\mathbb{R}^{d+1}; \mathbb{R})$ we have for some continuous square-integrable martingale $(M_t^\varphi)_{t \geqslant 0}$*

$$\mathrm{d}\langle \mu_t, \varphi \rangle = \langle \mu_t, \Delta\varphi \rangle \,\mathrm{d}t + \mathrm{d}M_t^\varphi \,, \qquad \mathrm{d}\langle M^\varphi \rangle_t = \langle \mu_t, \varphi^2 \rangle \,\mathrm{d}t \,.$$

Can we be more precise in characterising the limit? Is there a unique solution to the Dawson–Watanabe martingale problem?

1.2.1 The law of large numbers

Before we proceed to answer this question, let us obtain a complete picture in the case $\alpha > 2$, when the limit is deterministic, as this is in any case useful later on. If we choose $\alpha > 2$, then the limit will be deterministic and satisfy

$$\mathrm{d}\langle \mu_t, \varphi \rangle = \langle \mu_t, \Delta\varphi \rangle \,\mathrm{d}t \,, \qquad \forall t \geqslant 0 \,, \quad \varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}) \,.$$

Since Δ is self-adjoint, this is just the *weak formulation* of the heat equation

$$\partial_t \mu_t(x) = \Delta \mu_t(x) \,, \qquad \forall t \geqslant 0 \,, \quad x \in \mathbb{R}^d \,, \qquad \mu_0(x) = \delta_0(x) \,. \tag{1.5}$$

Remark 1.5 *For the particular choice of initial condition $\mu_0 = \delta_0$, the solution to* (1.5) *is explicit and given by the heat kernel*

$$p(t,x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{2t} \right) .$$

For arbitrary initial condition one deduces that

$$\mu_t(x) = P_t \mu_0(x) \,, \qquad P_t f(x) = \int_{\mathbb{R}^d} p_t(x-y) f(y) \,\mathrm{d}y \,.$$

We call $(P_t)_{t \geqslant 0}$ the heat semigroup.

1.3 Duality

To prove uniqueness of solutions to the martingale problem there is a trick that is very common in the study of population genetics, as it relates to observing the population evolve "backwards in time", and prescribe the evolution of a genealogical tree rather than the evolution of the particles "forward in time". In the case of the Dawson–Watanabe martingale problem the key observation is that by Itô's formula the following holds for $f \in C^2(\mathbb{R};\mathbb{R})$ (now we are working with martingales in continuous time):

$$\mathrm{d}f(\langle \mu_t, \varphi\rangle) = \left\{ f'(\langle \mu_t, \varphi_t\rangle)\langle \mu_t, \Delta\varphi\rangle + \frac{1}{2} f''(\langle \mu_t, \varphi\rangle)\langle \mu_t, \varphi^2\rangle \right\} \mathrm{d}t + \mathrm{d}\overline{M}^f_t \ ,$$

for some continuous martingale $\overline{M}^f$. Because of the linearity in the variance, there is a simpler structure appearing if one chooses $\varphi \geqslant 0$ and $f(x) = \exp(-\lambda x)$. In this case

$$\mathrm{d}f(\langle \mu_t, \varphi\rangle) = \left\{ \langle \mu_t, -\lambda\Delta\varphi + \frac{\lambda^2}{2}\varphi^2\rangle \right\} f(\langle \mu_t, \varphi_t\rangle)\,\mathrm{d}t + \mathrm{d}\overline{M}^f_t \ .$$

This suggests that if φ were the solution to

$$-\Delta\varphi + \frac{\lambda}{2}\varphi^2 = 0 \ , \tag{1.6}$$

then we would be able to characterise the law of $\langle \mu_t, \varphi\rangle$. Of course, only $\varphi \equiv 0$ solves (1.6). Yet, this idea is fruitful: if instead of choosing φ time-independent one considers the solution φ to

$$\partial_t \varphi = \Delta\varphi - \frac{\lambda}{2}\varphi^2 \ , \qquad \varphi(0,\cdot) = \varphi_0(\cdot) \in C^\infty_c(\mathbb{R}^d;\mathbb{R}) \ . \tag{1.7}$$

Then fix $T \in (0,\infty)$ and consider, again for $f(x) = \exp(-\lambda x)$ the process

$$[0,T] \ni t \mapsto f(\langle \mu_t, \varphi_{T-t}\rangle) = X_t \ .$$

We can again apply Itô's formula to obtain

$$\mathrm{d}X_t = f(\langle \mu_t, \varphi_{T-t}\rangle)\left\{ \langle \mu_t, \lambda\partial_t\varphi_{T-t} - \lambda\Delta\varphi_{T-t} + \frac{\lambda^2}{2}\varphi^2_{T-t}\rangle \right\} \mathrm{d}t + \mathrm{d}M^{\lambda,T}_t \ .$$

We see that the drift vanishes because of our choice of φ. Therefore $(X_t)_{t\in[0,T]}$ is a martingale and in particular

$$\mathbb{E}[\exp(-\lambda\langle \mu_T, \varphi_0\rangle)] = \mathbb{E}[X_T] = \mathbb{E}[X_0] = \exp(-\lambda\langle \mu_0, \varphi_T\rangle) \ .$$

In this way, we have characterised uniquely the law of $(\mu_t)_{t\geqslant 0}$. We have proven the following theorem.

Theorem 1.6 *For any $\mu_0 \in \mathcal{M}(\mathbb{R}^d)$ there exists a unique-in-law solution $(\mu_t)_{t\geqslant 0}$ in $C([0,\infty);\mathcal{M}(\mathbb{R}^d))$ to the Dawson–Watanabe martingale problem of Definition 1.4.*

Proof. Existence follows from approximations via particle systems, while uniqueness follows from the previous calculation. Here we use that the law of a measure-valued random variable is characterised by the law of the one-dimensional projections $\{\langle \mu_t, \varphi\rangle \ : \ \varphi \in C^\infty_c(\mathbb{R}^d;\mathbb{R})\}$ (see [Daw93, Lemma 3.2.4]). □

1.4 Weak formulations and white noise

Theorem 1.6 is incredibly strong, but can we obtain a different understanding of the limiting object? A stochastic PDE should be an equation of the kind

$$\partial_t \mu = \Delta \mu + f(\mu) + \sigma(\mu)\xi \ , \tag{1.8}$$

where ξ is some noise - in analogy to stochastic ODEs, which are of the form

$$\partial_t X = f(X) + \sigma(X)\xi \ .$$

Is there a way by which we can cast our equation into the setting of (1.8)? We might expect that (1.4) can also be rewritten as the weak form of some equation of the kind

$$\mathrm{d}\mu_t(x) = \Delta \mu_t(x)\,\mathrm{d}t + \,\mathrm{d}M_t(x) \ ,$$

where M is a martingale, which must somehow depend on space as well. This turns out to be intuitively correct, but is rather delicate. Naively choosing $\varphi(\cdot) = \delta_x(\cdot)$ as a test function leads to $\langle M^{\delta_x}\rangle_t = \infty$, so there is no clear meaning to $M_t(x)$ evaluated at a single point. The correct way to rewrite the martingale term in (1.4) is through space-time white noise.

Definition 1.7 *Space-time white noise on $\mathbb{R}^d$ is equivalently defined as one of the following:*

1. *A sequence of Gaussian random variables ξ_φ indexed by functions $\varphi \in L^2(\mathbb{R}^{d+1})$ and with covariance*

$$\mathbb{E}[\xi_\varphi \xi_\psi] = \langle \varphi, \psi \rangle \ .$$

2. *A random Gaussian distribution (that is with values in $\mathcal{S}'(\mathbb{R}^{d+1})$) such that*

$$\mathbb{E}\langle \xi, \varphi\rangle\langle \xi, \psi\rangle = \langle \varphi, \psi \rangle \ , \qquad \forall \varphi, \psi \in C_c^\infty(\mathbb{R}^{d+1}) \ .$$

Informally, ξ is a random Gaussian field with covariance

$$\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(x-y)\delta(t-s) \ .$$

This is not rigorous, as above for $M(x)$, because ξ can not be evaluated at a point $x \in \mathbb{R}^{d+1}$.

Exercise 3 *Prove that the two points of the definition above are equivalent, namely that given ξ according to the first point there is a $\widetilde{\xi} \in \mathcal{S}'(\mathbb{R}^{d+1})$ such that $\langle \widetilde{\xi}, \varphi\rangle = \langle \xi, \varphi \rangle$ almost surely for all φ, and $\widetilde{\xi}$ satisfies the second property.* Hint: *Choose an orthonormal basis of $L^2(\mathbb{R}^{d+1})$.*

From the definition, it follows that it is possible to give meaning to the integral of space-time white noise against deterministic functions in $L^2(\mathbb{R}^{d+1})$. Much in the same way as for the Itô isometry, we can extend this integration to adapted space-time processes.

Lemma 1.8 *Let $\varphi \colon \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be an* adapted *spatial process such that*

$$\mathbb{E}\int_0^T \int_{\mathbb{R}^d} |\varphi(s,x)|^2 \,\mathrm{d}x\,\mathrm{d}s < \infty \ .$$

Then the stochastic integral

$$M_t = \int_0^t \int_{\mathbb{R}^d} \varphi(s,x)\xi(\,\mathrm{d}x,\,\mathrm{d}s)\,,$$

is well-defined and a continuous martingale on $[0,T]$ *with quadratic variation*

$$\langle M\rangle_t = \int_0^t \|\varphi_s\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}s\;.$$

Proof. By approximation it suffices to prove the result for $\varphi\colon \mathbb{R}^{d+1}\to\mathbb{R}$ an adapted, simple function. That is, of the form (form some $n\in\mathbb{N}$ and $\{t_i\}_{i\in\mathbb{N}}\subseteq\mathbb{R}$)

$$\varphi(t,x) = \sum_{i=0}^n 1_{[t_i,t_{i+1})}(t)\varphi_i(x)\,,$$

where φ_i is $\mathcal{F}_{t_i}$-measurable (the filtration $\mathcal{F}_t$ is formally the one generated by $\xi(s,x)$ such that $s\leqslant t, x\in\mathbb{R}^d$) and $\mathbb{E}\|\varphi_i\|^2_{L^2(\mathbb{R}^d)}<\infty$. Then

$$\begin{aligned} M_t^\varphi = \langle \xi, \varphi 1_{[0,t)}\rangle &= \int_{[0,t)\times\mathbb{R}^d} \varphi(s,x)\xi(s,x)\,\mathrm{d}s\,\mathrm{d}x \\ &= \sum_{i=0}^n \int_{t_i}^{t_{i+1}\wedge t}\int_{\mathbb{R}^d} \varphi_i(x)\xi(s,x)\,\mathrm{d}s\,\mathrm{d}x \end{aligned}$$

is a square-integrable martingale with quadratic variation

$$\langle M\rangle_t = \int_0^t \|\varphi_i\|^2_{L^2(\mathbb{R}^d)} 1_{[t_i,t_{i+1})}(s)\,\mathrm{d}s\;.$$

□

From Lemma 1.8 if follows that our candidate equation for the Dawson–Watanabe martingale problem should be the following:

$$\partial_t\mu = \Delta\mu + \sqrt{\mu}\xi\,, \tag{1.9}$$

where ξ is space-time white noise. The main issue with (1.9) is that in order for it to even make sense, one must make sense of $\sqrt{\mu}$, but μ is a-priori only a measure. Indeed, this issue can only be overcome in dimension $d=1$: the reason is the irregularity of the noise (see later sections).

Theorem 1.9 *If* $d=1$*, then the unique solution to the Dawson–Watanabe process satisfies for all* $\alpha\in(0,1/2)$ *that* $\mu_t\in C^\alpha_{\mathrm{loc}}(\mathbb{R}^d;[0,\infty))$ *for all* $t>0$*. In addition (up to extending the probability space) there exists a space-time white noise* ξ *such that* μ *solves* (1.9)*.*

1.5 Some properties

Super-Brownian Motion (SBM) has a number of interesting properties, which shed light on its qualitative behaviour. The most interesting consequence of these properties is that SBM behaves fundamentally different from the heat equation $(\partial_t-\Delta)\mu=0$. For instance solutions to the latter equation stay strictly positive for all times. Moreover, the equation is critical, in that it satisfies a scaling invariance (as it should, since it appears as the universal large scale limit of systems of branching particles).

1.5.1 Extinction

First, we observe that the noise is so strong that the population almost surely dies out in finite time. This is known as the effect of *genetic drift*, which leads to one type overcoming the other simply because of the inherent randomness of the evolutionary process.

Exercise 4 *Prove that* $\langle \mu_t, 1 \rangle$ *is a weak solution to the Feller diffusion*

$$\mathrm{d}\langle \mu_t, 1 \rangle = \sqrt{\langle \mu_t, 1 \rangle}\, \mathrm{d}W_t \;,$$

where W_t *is a Brownian motion. Then if*

$$\tau = \inf\{t \geqslant 0 \;:\; \langle \mu_t, 1 \rangle = 0\} \;,$$

deduce that $\mathbb{P}(\tau < \infty) = 1$. Hint: *Apply Itô's formula to* $\sqrt{Z_t}$*, with* $Z_t = \langle \mu_t, 1 \rangle$.

What can we say about the probability of extinction? We have that

$$1_{\{\mu_t = 0\}} = \lim_{\lambda \to \infty} \exp\left(-\langle \mu_t, \lambda \rangle\right) \;.$$

Therefore

$$\mathbb{P}(\mu_t = 0) = \lim_{\lambda \to \infty} \exp\left(-\langle \mu_0, \varphi_t^\lambda \rangle\right) \;,$$

where φ_t^λ is the solution to the dual equation (1.7) with initial condition λ. Now observe that the ODE

$$\partial_t \varphi = -\varphi^2 \;,$$

has a unique solution such that $\varphi_0 = \infty$, and it is given by $\varphi_t = t^{-1}$. Hence $\mathbb{P}(\mu_t = 0) = \exp\left(-\langle \mu_0, 1 \rangle t^{-1}\right)$.

1.5.2 Scale invariance

In dimension $d = 1$ the SPDE for super-Brownian motion satisfies formally some scale invariance. Here we will make use of the following fact.

Exercise 5 *For any* $\varepsilon > 0$ *define* $\xi_\varepsilon(t, x) = \varepsilon^{\frac{3}{2}} \xi(\varepsilon^2 t, \varepsilon x)$*. Then, if* $d = 1$*, we have* $\xi_\varepsilon \overset{d}{=} \xi$. Remark: *of course, the previous formulation is purely formal (*ξ *is a distribution) and must be made sense of through a change of variables, in a distributional sense.*

Because of the scale-invariance of white noise we find that if $\mu^{(\varepsilon)}(t, x) = \varepsilon \mu(\varepsilon^2 t, \varepsilon x)$, then (as usual in $d = 2$)

$$(\partial_t - \Delta) \mu^{(\varepsilon)} = \sqrt{\mu^{(\varepsilon)}} \xi_\varepsilon \;,$$

which means that the equation is "critical" in the sense that the heat operator $(\partial_t - \Delta)$ is not of leading order on small scales.

2 An intermezzo with two counterexamples

Let us conclude with two examples of stochastic PDEs in which things can go wrong.

2.1 Dean–Kawasaki fluctuations

Consider the martingale problem associated to the SPDE

$$\partial_t \mu = \Delta \mu + \gamma \operatorname{div}(\sqrt{\mu}\xi) ,$$

where ξ is a vector-valued space-time white noise $\xi = (\xi_1, \xi_2)$ and $\operatorname{div}(\varphi_1, \varphi_2) = \partial_{x_1}\varphi_1 + \partial_{x_2}\varphi_2$. By martingale problem we mean measure-valued solutions $(\mu_t)_{t\geqslant 0}$ such that for all $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$

$$\mathrm{d}\langle \mu_t, \varphi\rangle = \langle \mu_t, \Delta\varphi\rangle \,\mathrm{d}t + \,\mathrm{d}M_t^\varphi , \qquad \mathrm{d}\langle M^\varphi\rangle_t = \gamma^2 \langle \mu_t, |\nabla\varphi|^2\rangle , \tag{2.1}$$

where M_t^φ is a continuous martingale.

Theorem 2.1 *There exists a solution to the martingale problem* (2.1) *if and only if* $2\gamma^{-2} = n \in \mathbb{N}$. *In this case*

$$\mu_t = \frac{1}{n}\sum_{i=1}^n \delta_{W_t^i} ,$$

where $\{W_t^i\}_{i\in\mathbb{N}}$ *is a collection of i.i.d. Brownian Motions with diffusivity* $\sqrt{2}$.

Proof. We only prove that a system of i.i.d. Brownian Motions does solve the equation, and is the unique solution if $n = \gamma^{-2} \in \mathbb{N}$. The non-existence result can be found in [KLvR19, Theorem 2.2].

First, we show that $\mu_t = \frac{1}{n}\sum_{i=1}^n \delta_{W_t^i}$ solves the martingale problem. We can compute

$$\mathrm{d}\langle \mu_t, \varphi\rangle = \frac{1}{n}\sum_{i=1}^n \nabla\varphi(W_t^i)\,\mathrm{d}W_t^i + \Delta\varphi(W_t^i)\,\mathrm{d}t .$$

Now, the first term is a martingale, and it has quadratic variation

$$\mathrm{d}\langle M^\varphi\rangle_t = \frac{2}{n^2}\sum_{i=1}^n |\nabla\varphi(W_t^i)|^2 = \frac{2}{n}\langle \mu_t, |\nabla\varphi|^2\rangle .$$

For uniqueness, we follow the same strategy as for super-Brownian motion, by finding a dual. If we consider $f(x) = \exp(-\lambda x)$ for $\lambda > 0$ and φ smooth in time and space on $[0, T] \times \mathbb{R}^d$ and positive, then we obtain for all $t \in [0, T]$

$$\begin{aligned} \mathrm{d}f(\langle \mu_t, \varphi_{T-t}\rangle) =& f(\langle \mu_t, \varphi_{T-t}\rangle)\big\{-\lambda\langle \mu_t, -\partial_t\varphi_{T-t} + \Delta\varphi_{T-t} - \lambda\gamma^2|\nabla\varphi_{T-t}|^2\rangle\big\}\,\mathrm{d}t \\ &+ \mathrm{d}M_t . \end{aligned}$$

Therefore, the law of the solution will be unique if we can solve the dual equation

$$\partial_t\varphi = \Delta\varphi - \lambda\gamma^2|\nabla\varphi|^2 .$$

The solution to this equation is given by the Cole-Hopf transformation of the solution to a linear equation.

$$\partial_t u = \Delta u , \qquad u_0 \geqslant 0 ,$$

Then $\psi = \log u$ solves

$$\partial_t\psi = \Delta\psi - |\nabla\psi|^2 , \qquad \psi_0 = \log(u_0) .$$

Then $\varphi = (\lambda\gamma^2)^{-1}\psi$, with matching initial conditions. □

2.2 Itô vs Stratonovich

Next we consider a toy model in stochastic fluid dynamics, namely viscous "passive scalar advection" driven by a white-in-time noise

$$\partial_t \varrho = \Delta \varrho + \gamma(\xi \cdot \nabla)\varrho \,. \tag{2.2}$$

The physically most relevant case appears when the noise ξ is incompressible. For the moment let us choose $\xi = (\mathrm{d}\beta^1, \mathrm{d}\beta^2)$ for a couple of real-valued independent Brownian motions (independent of space, so that naturally the incompressibility condition $\mathrm{div}(\xi) = 0$ is satisfied). Of course, this being a multiplicative equation there is a choice between Itô and Stratonovich noise. Following the (wrong) intuition from stochastic ODEs, we may try to treat the Itô case as it might be somewhat "simpler".

Theorem 2.2 *Equation* (2.2) *admits a solution for all* $\varrho_0 \in C^\infty_c$ *if* $\gamma^2 \leqslant 2$.

Remark 2.3 *The result is tight, in the sense that (as will appear from the proof), for* $\gamma^2 > 2$ *the problem is linked to solving the heat equation backwards in time.*

Proof of Theorem 2.2. The reason for the restriction to $\gamma \leqslant 2$ comes from the Itô–Stratonovich corrector. Recall that the formula for the corrector is as follows for a semimartingale $(X_s)_{s\geqslant 0}$:

$$\int_0^t X_s \circ \mathrm{d}W_s = \int_0^t X_s \,\mathrm{d}W_s + \frac{1}{2}\langle X, W\rangle_t \,. \tag{2.3}$$

We want to apply this formula to $\partial_{x_1}\varrho(t,x)\,\mathrm{d}\beta^1_t$. We find

$$\partial_t \partial_{x_1}\varrho = \Delta\partial_{x_1}\varrho + \gamma(\partial^2_{x_1}\varrho)\,\mathrm{d}\beta^1 + \gamma\partial_{x_1}\partial_{x_2}\varrho \,.$$

And hence by (2.3) we obtain

$$\gamma(\xi\cdot\nabla)\varrho = \gamma(\xi\cdot\nabla)\circ\varrho - \frac{\gamma^2}{2}\Delta\varrho \,.$$

We can therefore reformulate (2.2) as

$$\partial_t\varrho = (1-\gamma^2/2)\Delta\varrho + \gamma(\xi\cdot\nabla)\circ\varrho \,.$$

The solution theory to this equation follows the one for deterministic PDEs from fluid dynamics, via energy estimates. Indeed we immediately find that

$$\frac{1}{2}\partial_t\|\varrho_t\|^2_{L^2} - (1-\gamma^2/2)\|\nabla\varrho_t\|^2_{L^2} \leqslant 0 \,.$$

□

3 The Φ^4_d equation and related problems

In this second part of the course, we study (and motivate the study of) solutions to the Φ^4_d equation

$$\partial_t u = \Delta u + mu - u^3 + \xi \,, \quad u(0,x) = u_0(x) \,. \tag{3.1}$$

There $(t,x) \in [0,\infty)\times\mathbb{R}^d$ and $d \in \mathbb{N}$ is for now arbitrary. The noise ξ will be space-time white noise. In absence of noise, and when $m > 0$, the evolution of this equation is roughly described by the evolution of phase fields on which $u \simeq \pm 1$. In particular, the interface that separates two phases is expected to evolve (at large scales) via mean curvature flow. It is therefore not entirely surprising that the Φ^4 equation (with $m > 0$) is linked to a classical microscopic model for the description of two phases: the Ising model. Indeed, the link appears in presence of "intermediate range" correlations, in what is called the Ising model with Kac interaction.

3.1 The Ising model with Kac interaction

In this section we start the study of the Glauber dynamics of the Ising model with Kac interaction. This will be a jump Markov process on the state space of all possible spin configurations. A spin configuration is a map $\sigma\colon \mathbf{Z}^d \to \{-1,1\}$, or alternatively a collection $(\sigma_x)_{x\in\mathbf{Z}^d} \in \{-1,1\}^{\mathbf{Z}^d}$. Now we prescribe an evolution on these spin configurations - these are usually called the Glauber dynamics associated to the Ising model. Namely, we define the following generator, for a parameter $\gamma \in (0,1)$:

$$\mathscr{L}_\gamma(f)(\sigma) = \sum_{x\in\mathbf{Z}^d} c_\gamma(x,\sigma)\{f(\tau^x\sigma) - f(\sigma)\} .$$

Here the operator $\tau^x\sigma$ flips the value of the spin at the location x:

$$(\tau^x\sigma)_y = \begin{cases} \sigma_y & \text{if } y \neq x , \\ -\sigma_x & \text{if } y = x . \end{cases}$$

Furthermore, the rate associated to the flip is given by

$$c_\gamma(x,\sigma) = e^{-\sigma_x h_\gamma(x,\sigma)}\big[e^{-h_\gamma(x,\sigma)} + e^{h_\gamma(x,\sigma)}\big]^{-1} , \tag{3.2}$$

where the interaction term h_γ depends on the configuration of spins in a ball of distance of order γ^{-1} about x:

$$h_\gamma(x,\sigma) = \sum_y J_\gamma(x,y)\sigma(y) .$$

Here

$$J_\gamma(x,y) = \zeta_\gamma^{-1}\gamma^d J(\gamma|x-y|) , \qquad \forall x \neq y , \qquad \text{and} \qquad J_\gamma(x,x) = 0$$

where J is a smooth radial function with compact support and ζ_γ is the normalisation

$$\zeta_\gamma = \sum_{x\neq 0} \gamma^d J(\gamma|x|) ,$$

and we observe that $\zeta_\gamma \to 1$ as $\gamma \to 0$. Understanding the large-scale dynamics of this model is quite challenging, so we start with a simplification of the model, which can be considered as a form of "linearisation".

3.2 The voter model as high temperature limit of Ising

In particular, we will consider an expansion of (3.2) in the case $h_\gamma(x,\sigma) \ll 1$. Note that when $h_\gamma = 0$, each spin behaves independently of one another, so this is one way to dampen the interactions between the particles in the model. At first order in $h_\gamma(x,\sigma)$, the rate $c_\gamma(x,\sigma)$ is given by

$$c^0_\gamma(x,\sigma) = \frac{1}{2}(1 - \sigma_x h_\gamma(x,\sigma)) = c_\gamma(x,\sigma) + o(h_\gamma^2) .$$

We then start with studying the large-scale dynamics of the generator $\mathscr{L}^0_\gamma$ associated to c^0_γ, and given by

$$\mathscr{L}^0_\gamma(f)(\sigma) = \sum_{x\in\mathbf{Z}^d} c^0_\gamma(x,\sigma)\{f(\tau^x\sigma) - f(\sigma)\} .$$

Now if we look more closely, the transition rates are given by

$$c^0_\gamma(x,\sigma) = \frac{1}{2}\sum_y J_\gamma(x,y)(1-\sigma_x\sigma_y) = \sum_y J_\gamma(x,y)1_{\{\sigma_x\neq\sigma_y\}} ,$$

were we used that J_γ is normalised to a probability measure. This is quite naturally called the *voter* model, in which a particle changes opinion at a rate proportional to the number of neighbouring particles with different opinions. As usual, we study the large scale behaviour of this system through the associated martingale problem. For every smooth function φ with compact support, let us define $\langle\sigma_t,\varphi\rangle = \sum_x \sigma_t(x)\varphi(x)$. Then

$$\mathrm{d}\langle\sigma_t,\varphi\rangle = -\left\{\sum_x c^0_\gamma(x,\sigma)2\sigma_t(x)\varphi(x)\right\}\mathrm{d}t + \mathrm{d}M^\varphi_t ,$$

for a cádlág martingale M^φ_t. We can rewrite the drift term as

$$-c^0_\gamma(x,\sigma)2\sigma_t(x)\varphi(x) = \sum_y J_\gamma(x,y)(\sigma_t(y)-\sigma_t(x))\varphi(x) ,$$

which summing over x becomes (since $J_\gamma(x,y) = J_\gamma(y,x)$):

$$\sum_x \sigma_t(x)\sum_y J_\gamma(x,y)(\varphi(y)-\varphi(x)) .$$

Now we can also compute the quadratic variation of the martingale

$$\begin{aligned}\mathrm{d}\langle M^\varphi\rangle_t &= \sum_x c^0_\gamma(x,\sigma)4\varphi^2(x) = 2\sum_{x,y} J_\gamma(x,y)(1-\sigma_x\sigma_y)\varphi^2(x)\\ &= 2\sum_x \varphi^2(x) - 2\sum_x\varphi^2(x)\sum_{y\neq x} J_\gamma(x,y)\sigma_x\sigma_y .\end{aligned}$$

Here the second term will eventually be irrelevant in our analysis, because neighbouring spins tend to decorrelate. To see this, we must introduce the appropriate scaling.

3.3 Convergence to the stochastic heat equation

To capture the large-scale behaviour of the voter model, we must introduce the appropriate space-time scaling. Let us define the measure-valued process, for a parameter $\alpha > 0$ that we will choose later on:

$$X^\gamma_t = \gamma^\beta\sum_x \sigma_{\gamma^{-2\alpha}t}(x)\delta_{\gamma^{1+\alpha}x} .$$

Then, by the previous calculations we would obtain

$$\mathrm{d}\langle X^\gamma_t,\varphi\rangle = \gamma^{-2\alpha}\sum_x\gamma^\beta\sigma_{\gamma^{-2\alpha}t}(x)\sum_y J_\gamma(x,y)(\varphi(\gamma^{1+\alpha}y)-\varphi(\gamma^{1+\alpha}x)) + \mathrm{d}N^\varphi_t .$$

The drift term is well approximated by

$$\nu^{(\gamma)} X_t^\gamma(\nu \Delta\varphi) \;, \quad \nu^{(\gamma)} = \sum_y J_\gamma(x,y)\gamma^2|x-y|^2 \to \nu = \int_{\mathbb{R}^d} J(|x|)|x|^2 \,\mathrm{d}x \;.$$

Instead, for the quadratic variation we obtain following the previous heuristic:

$$\begin{aligned} \mathrm{d}\langle N^\varphi\rangle_t &= \gamma^{-2\alpha}\gamma^{2\beta} 2\sum_{x,y} J_\gamma(x,y)(1-\sigma_x\sigma_y)\varphi^2(\gamma^{1+\alpha}x) \;, \\ &= \gamma^{-2\alpha}\gamma^{2\beta} 2\sum_{x} \varphi^2(\gamma^{1+\alpha}x) + o(1) \;. \end{aligned}$$

Therefore, we obtain a nontrivial limit under the assumption that

$$-2\alpha + 2\beta = (1+\alpha)d \;,$$

which leads to $\beta = \frac{d}{2} + \frac{d+2}{2}\alpha$. In fact, this heuristic can be made rigorous.

Theorem 3.1 *If $d=1$, then the process X_t^γ converges weakly in $\mathbb{D}([0,\infty); \mathcal{M}(\mathbb{R}^d))$ to the* unique *martingale solution X to*

$$\mathrm{d}\langle X, \varphi\rangle = \langle X, \Delta\varphi\rangle \,\mathrm{d}t + \mathrm{d}N_t^\varphi \;, \qquad \mathrm{d}\langle N^\varphi\rangle_t = 2\langle \varphi,\varphi\rangle \;. \tag{3.3}$$

The complete proof of this result can be found in [BBPS03, Theorem 3.1], together with the proof of the convergence of the full dynamical Ising–Kac model.

Remark 3.2 *The fact that the solution to* (3.3) *is unique follows from the fact that it is Gaussian. Indeed, we can rewrite X (in the case $\nu = 1$ for simplicity) as the solution to*

$$\partial_t X = \Delta X + \xi \;, \tag{3.4}$$

where ξ is space-time white noise. In particular, if $(P_t)_{t\geqslant 0}$ is the heat semigroup, then

$$X_t = P_t X_0 + \int_0^t P_{t-s}\xi \,\mathrm{d}s \;,$$

and the second term is a centered Gaussian for which one can compute explicitly the correlation function (say $X_0 = 0$):

$$\begin{aligned} \mathbb{E}[X_t(x)X_s(y)] &= \int_0^{t\wedge s}\int_{\mathbb{R}^d} p_{t-r}(x,z)p_{s-r}(y,z)\,\mathrm{d}z\,\mathrm{d}r \\ &= \int_0^{t\wedge s} p_{t+s-2r}(x,y)\,\mathrm{d}r \;. \end{aligned}$$

3.4 From linear problems to interactions: stochastic estimates

In this section we introduce a martingale approach to study nonlinear functions of the solution X to (3.4). To simplify the issue, we consider the process on the torus, and at invariance. We write

$$\mathbf{T}^d = \mathbb{R}^d \setminus \mathbf{Z}^d$$

for the d-dimensional Torus, which is obtained from $\mathbb{R}^d$ by quotienting via translations in integer directions. In other words, we are considering PDEs on $[0,1]^d$ with periodic boundary conditions. The torus is particularly convenient, as we can consider the system in Fourier coordinates.

3.4.1 Intermezzo on the Fourier transform

Define the Fourier transform:

$$\hat{\varphi}(k) = \mathscr{F}\varphi(k) = \int_{\mathbf{T}^d} e^{2\pi\iota k\cdot x}\varphi(x)\,\mathrm{d}x = \langle \varphi, e_k\rangle \;,$$

where $\iota = \sqrt{-1}$. Then the Fourier transform satisfies the following properties.

1. The output is a function on the lattice $\mathbf{Z}^d$, namely $\mathscr{F}[\varphi]\colon \mathbf{Z}^d \to \mathbb{R}$.
2. The Fourier transform is invertible (in the space of Schwartz distributions)

$$\mathscr{F}^{-1}[\psi](x) = \sum_{k\in\mathbf{Z}^d} \psi(k)e_k(x) \;.$$

3. The Fourier transform is an Isometry between $L^2(\mathbf{T}^d)$ and $L^2(\mathbf{Z}^d)$:

$$\|\varphi\|_{L^2(\mathbf{T}^d)} = \left(\int_{\mathbf{T}^d} |\varphi(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} = \|\mathscr{F}[\varphi]\|_{L^2(\mathbf{Z}^d)} = \left(\sum_{k\in\mathbf{Z}^d} |\hat{\varphi}(k)|^2\right)^{\frac{1}{2}} \;.$$

We can use the Fourier transform to solve the Heat equation through the following lemma.

Lemma 3.3 *Let φ be the solution to*

$$\partial_t\varphi = \frac{1}{2}\Delta\varphi + f\;, \qquad \varphi(0,\cdot) = \varphi_0(\cdot)\;.$$

Then

$$\hat{\varphi}(t,k) = e^{-2\pi^2 t|k|^2}\hat{\varphi}_0(k) + \int_0^t e^{-2\pi^2(t-s)|k|^2}\hat{f}(s,k)\,\mathrm{d}s\;.$$

3.4.2 A martingale approach to nonlinearities of distributions

Now, let us pass again to analyze non-linear functionals of X_t. As we mentioned, to simplify matters we consider the system at invariance, and with a mean-zero noise. Namely if $\Pi_\times$ is the projection

$$\Pi_\times\varphi(x) = \varphi(x) - \int_{\mathbf{T}^d}\varphi(z)\,\mathrm{d}z\;,$$

then let us consider

$$X_t = \int_{-\infty}^t P_{t-s}\Pi_\times\xi\,\mathrm{d}s\;.$$

Then the law of X is invariant in time. Our objective is to eventually define the non-linear functional

$$X_t^3(x)\;.$$

The issue is that for given $t > 0$, X_t does not even lie in $L^2(\mathbf{T}^d)$, if $d \geqslant 2$. Indeed, we can compute

$$\mathbb{E}\|X_t\|_{L^2}^2 = \int_{\mathbf{T}^d} \mathbb{E}|X_t(x)|^2 \,\mathrm{d}x = \infty \;,$$

because formally

$$\mathbb{E}[X_t(x)X_t(y)] = \int_0^\infty p_{2t}(x-y)\,\mathrm{d}t = \mathscr{G}(x-y) \;,$$

where p_{2t} is the heat kernel of the periodic mean-zero Laplacian and $\mathscr{G}$ the associated Green's function. Since $\mathscr{G}(x-y) \simeq \log|x-y|^{-1}$ in $d = 2$ (and the explosion is even worse in $d \geqslant 3$), we see that the variance at a given point explodes.

Remark 3.4 *Instead, in dimension $d = 1$, these problems do not appear, and almost surely, $X_t \in C^{\frac{1}{2}-\varepsilon}(\mathbf{T}^d)$ for any $t, \varepsilon > 0$. Also, note that in dimension $d = 1$ the invariant measure associated to X_t is a Brownian motion conditioned to have zero mean.*

Since $(X_t)_{t\geqslant 0}$ is invariant in time, we now simply write X for X_0, and consider a time-independent problem. In any dimension, one can still make sense of X as a distribution, since

$$\mathbb{E}[\langle X, \varphi\rangle\langle X, \psi\rangle] = \int_{(\mathbf{T}^d)^2} \mathscr{G}(x-y)\varphi(x)\psi(y)\,\mathrm{d}x\,\mathrm{d}y < \infty \;, \tag{3.5}$$

as long as $\varphi, \psi \in L^\infty$. But this does not help us in defining nonlinear functions of X. Here it is fundamental to use probability theory. Rather than a classical approach via decomposition into homogeneous Itô chaoses, we follow here a new approach via martingales, that has been recently outlined in [BCG23]. To this aim consider the space $\mathscr{M}$ of all continuous martingales:

$$\mathscr{M} = \{(M_u)_{u\geqslant 0} \;:\; M_u \text{ is a real-valued martingale }\} \;.$$

There is a natural notion of product on this space, given by imposing that after a product we still have a martingale:

$$(: MN :)_u = M_u N_u - \langle M, N\rangle_u \;.$$

Here we use colons $: MN :$ to distinguish the product in $\mathscr{M}$ from the product in $\mathbb{R}$. The fundamental idea is to consider M_u an *approximation* of a possible terminal value M_∞ (if this exists - which is not guaranteed, since we are not assuming that the martingales are, for example, uniformly integrable). For example, we may approximate our X as follows

$$X_u = \int_{-\infty}^{-\frac{1}{u}} P_{-s}\Pi_\times \xi \,\mathrm{d}s \overset{d}{=} P_{\frac{1}{u}} \int_{-\infty}^{0} P_{-s}\Pi_\times \xi\,\mathrm{d}s \;.$$

The last calculation shows that for any Given $u > 0$, the distribution X_u is actually a smooth function. Then we can define the product $X(x)X(y) \in \mathscr{M}$ as the martingale

$$(: X(x)X(y) :)_u = X_u(x)X_u(y) - \,\mathrm{d}\langle X(x), X(y)\rangle_u \;.$$

The quadratic variation process is given by a correlation function

$$\langle X(x), X(y)\rangle_u = c_u(x,y) \;,$$

which is deterministic and satisfies $\lim_{u\to\infty} c_u(x,y) = \mathscr{G}(x,y)$. To see that this is the case, we can represent $c_u(x,y) = c_u(x-y)$ in Fourier coordinates

$$\mathscr{F}[c_u](k) = \int_{-\infty}^{-1/u} e^{s2|k|^2}\,\mathrm{d}s \mathbf{1}_{\{|k|\neq 0\}} = \frac{1}{2|k|^2} e^{-|k|^2/u}\mathbf{1}_{\{k\neq 0\}} \to \frac{1}{2|k|^2}\mathbf{1}_{\{k\neq 0\}} \;,$$

where the last limit appears as $u \to \infty$.

Remark 3.5 *Although neither $X(x)$ nor $X(y)$ make sense as random variables, we are still formally able to compute their covariation, which is given by*

$$\begin{aligned}\mathrm{Cov}(X(x), X(y)) &\stackrel{\text{def}}{=} \lim_{u\to\infty} \mathrm{Cov}(X_u(x), X_u(y)) \\ &= \lim_{u\to\infty} c_u(x,y) = \mathscr{G}(x-y)\;, \qquad \forall x\neq y \;.\end{aligned}$$

So far we have given some meaning to $X^2(x)$, not as a random variable, but as a non-closed martingale. The next step is to observe that the $X^2(x)$ we have constructed is actually a spatial distribution.

Remark 3.6 *In the language of Wiener-Itô chaoses, we have approximated X through a smooth family X_u for $u \geqslant 0$. Then, since X is Gaussian, the square X_u^2 has components in a second homogeneous and first inhomogeneous Wiener–Itô chaos. The first chaos component is given by $c_u(x,x)$, which is diverging. Our procedure is a way to remove the diverging zeroth chaos, or expected value (which is referred to as s* renormalisation*), leaving us with a well-defined random field.*

Lemma 3.7 *It holds that*

$$\mathrm{Cov}(: X^2 : (x), : X^2 : (y)) = 2\mathscr{G}^2(x,y)$$

Proof. We find that for some martingale $(M_u)_{u\geqslant 0}$

$$\mathrm{d}(: X^2 : (x))_u (: X^2 : (y))_u = \mathrm{d}\langle : X^2 : (x), : X^2 : (y)\rangle_u + \mathrm{d}M_u \;.$$

As we are only interested in the quadratic covariation term, we find

$$\mathrm{d}\langle : X^2 : (x), : X^2 : (y)\rangle_u = 4X_u(x)X_u(y)\partial_u c_u(x,y)\,\mathrm{d}u \;.$$

Therefore

$$\mathbb{E}\,\mathrm{d}\langle : X^2 : (x), : X^2 : (y)\rangle_u = 4c_u(x,y)\partial_u c_u(x,y)\,\mathrm{d}u$$

Integrating in time, we obtain the desired result. □

Corollary 3.8 *The Wick square $: X^2 :$ is, up to considering a modification, a random distribution, in dimension $d < 4$.*

Proof. This follows since $\mathscr{G}^2(x,y) \simeq |x-y|^{2(d-2)}$ is integrable if and only if $2d-4 < d$. Therefore we obtain

$$\mathbb{E}\big[\langle : X^2 :, \varphi\rangle\langle : X^2 :, \psi\rangle\big] = 2\int_{(\mathbf{T}^d)^2} \mathscr{G}(x,y)\varphi(x)\psi(y)\,\mathrm{d}x\,\mathrm{d}y < \infty \;,$$

for all $\varphi, \psi \in L^\infty$. From here one can construct a random distribution by defining:

$$\hat{Z}(k) \stackrel{\text{def}}{=} \langle : X^2 :, e_k \rangle \;, \qquad Z = \mathscr{F}^{-1} \hat{Z} \;,$$

and by observing that $\langle Z, \varphi \rangle$ is equal in distribution to $\langle : X^2 :, \varphi \rangle$, for any $\varphi \in L^\infty$. □

Similarly we can proceed to define the cube $: X^3 : (x)$. Here the compensation that is required to build a martingale starting from $X(x)$ is given by

$$\mathrm{d}X_u^3(x) = 3X_u(x)\,\mathrm{d}\langle X(x), X(x) \rangle_u + \mathrm{d}M_u \;, \qquad \mathrm{d}M_u(x) = 3X_u^2(x)\,\mathrm{d}X_u(x) \quad (3.6)$$

Therefore it would seem natural to consider in this case the martingale

$$(: X^3 : (x))_u = X_u^3(x) - 3 \int_0^u X_r(x) \partial_r c_r(x,x)\,\mathrm{d}r \;.$$

Unfortunately, this is not enough, you can check that this choice gives rise to an exploding covariance matrix for $: X^3 :$ (covariances are taken formally as the limit in Remark 3.5). Instead, the correct choice of higher degree Wick powers is

$$\mathrm{d}(: X^n(x) :)_u = n(: X^{n-1}(x) :)_u \,\mathrm{d}X_u \qquad (3.7)$$

Exercise 6 *For $n = 3$, one way to obtain the expression* (3.7) *is to start from* (3.6) *and observe that for some martingale N_u*

$$\int_0^u X_r(x) \partial_r c_r(x,x)\,\mathrm{d}r = X_u \int_0^u \partial_r c_r(x,x)\,\mathrm{d}r - N_u \;, \qquad \mathrm{d}N_u = c_u\,\mathrm{d}X_u \;.$$

Hence one can rewrite

$$X_u^3(x) = 3X_u c_u - 3N_u + M_u \;,$$

and $M_u - 3N_u$ is the martingale appearing in (3.7)*. In particular*

$$(: X^3(x) :)_u = X_u(x)(: X^2(x) :)_u = X_u(x)(X_u^2(x) - c_u(x,x)) \;.$$

Now, the expression (3.7) leads to an inductive way of computing the correlation functions of the Wick powers.

Lemma 3.9 *It holds that*

$$\mathrm{Cov}(: X^n(x) :, : X^n(y) :) = n! \mathscr{G}^n(x,y) \;.$$

Proof. This follows if we can show by induction that

$$\mathrm{Cov}(: X^n(x) :_u, : X^n(y) :_u) = n! c_u^n(x,y) \;.$$

If the above identity holds for $n-1$, then

$$\mathrm{Cov}(: X^n(x) :, : X^n(y) :) = \int_0^u n^2 (n-1)! c_r^{n-1}(x,y) \partial_r c_r(x,y)\,\mathrm{d}r = n! c_u^n(x,y) \;.$$

□

Corollary 3.10 *From Lemma 3.9 we conclude that $: X^3(x) :$ is a random distribution in $d \leqslant 2$. Instead, in $d = 3$, the covariance*

$$\mathrm{Cov}(: X^3(x) :, : X^3(y) :) = 6\mathscr{G}^3(x,y) \simeq |x-y|^{-3(d-2)}$$

is not integrable.

3.5 Towards singular stochastic PDEs

We are now ready to give a meaning to (3.1) in dimension $d = 2$ ($d = 1$ is much simpler, and $d = 3$ significantly more complex). For simplicity we stick to the case $m = 0$, since the case $m \neq 0$ follows in exactly the same way. Following the idea of "linearisation" that appeared already in the scaling of the Ising model, we start with the ansatz that the solution φ to (3.1) is a perturbation of the Gaussian process $(X_t)_{t\geqslant 0}$ that solves (3.4). Namely, let us (formally, since the existence of φ is not guaranteed) defined ψ by

$$\varphi = X + \psi \ .$$

Then we expect ψ to solve the equation

$$\partial_t \psi = \Delta\psi - \psi^3 - 3X^2\psi - 3X\psi^2 - X^3 \ , \qquad \psi(0, \cdot) = \varphi_0(\cdot) - X_0(\cdot) \ .$$

As we have learned, the products X^3 and X^2 are not well defined, and we replace them with their (well-defined) Wick products instead:

$$\partial_t \psi = \Delta\psi - \psi^3 - 3 :X^2: \psi - 3X\psi^2 - :X^3: \ , \tag{3.8}$$

which is sometimes formally written as the effect of an infinite renormalisation

$$\partial_t \psi = \Delta\psi - \psi^3 - 3(X^2 - \infty)\psi - 3X\psi^2 - X(X^2 - \infty) \ .$$

Now we are able to use classical PDE arguments to solve (3.8). To this aim, we need a couple of results, which we will not prove. The first one is a quantitative estimate of the (ir-)regularity of the driving noise terms. The second one is a quantitative estimate of the regularising effect of the heat semigroup: these are known as Schauder estimates.

To state these results we need a notion of *negative* regularity (for distributions that are not functions). Here we use Besov spaces $\mathscr{C}^{-\alpha}$ of negative regularity ($\alpha > 0$), which in first approximation can be though of spaces of distributions that are derivatives of functions in $\mathscr{C}^{1-\alpha}$ (assuming that $\alpha \in (0, 1)$). Rigorously, Besov spaces can be defined through the norm

$$\|\varphi\|_{\mathscr{C}^\alpha} = \sup_{j\in\mathbb{N}} 2^{\alpha j} \|\Delta_j \varphi\|_{L^\infty} \ ,$$

there $\Delta_j\varphi$ is the j-th Paley block $\Delta_j\varphi = \mathscr{F}^{-1}(\varrho_j \cdot \mathscr{F}(\varphi))$, defined in terms of a dyadic partition of the unity (see [BCD11] for the complete definition), which can be thought of roughly as a projection on frequencies of order 2^j, namely $\varrho_j(k) \simeq 1_{\{[2^{j-1}, 2^j)\}}(|k|)$).

Lemma 3.11 *In dimension $d = 2$ the process X_t and any of its Wick powers $:X^n_t:$ satisfies for any $\alpha > 0$*

$$\mathbb{E} \sup_{0\leqslant t\leqslant T} \| :X^n_t: \|_{\mathscr{C}^\alpha} < C(n, \alpha, T) < \infty \ .$$

Lemma 3.12 (Schauder estimates) *For any $\alpha \in \mathbb{R}$ and $\beta > 0$ we can estimate uniformly over $\varphi \in \mathcal{S}'(\mathbf{T}^d)$:*

$$\|P_t\varphi\|_{\mathscr{C}^{\alpha+\beta}} = t^{-\frac{\beta}{2}} \|\varphi\|_{\mathscr{C}^\alpha} \ .$$

The final ingredient is a rule for the product of distributions, which is allowed if the sum of the regularities of the two distributions is strictly positive (so at least one must be a sufficiently smooth function).

Lemma 3.13 *For any $\alpha, \beta \in \mathbb{R}$ such that $\alpha+\beta>0$, one can estimate uniformly over $\varphi, \psi \in \mathcal{S}'(\mathbf{T}^d)$*

$$\|\varphi \cdot \psi\|_{\mathscr{C}^{\alpha\wedge\beta}} \lesssim \|\varphi\|_\alpha \|\psi\|_\beta \;.$$

We are now ready to prove well-posedness of (3.8). For simplicity we will assume that $\varphi_0 = X_0$.

Theorem 3.14 *Assume that $\varphi_0 = X_0$. Then almost surely there exists a unique mild solution ψ to (3.8) in the space $L^\infty([0,T];\mathscr{C}^\alpha(\mathbf{T}^d))$, for any $\alpha \in (0,2)$ and $T>0$.*

Proof. By mild solution we mean that ψ solves the following fixed point problem

$$\psi_t = -\int_0^t P_{t-s}\big[\psi^3 + 3X\psi^2 + 3 :X^2: \psi + :X^3:\big]\,\mathrm{d}s \;.$$

We will shot that the map $\mathcal{I}$ defined by

$$\mathcal{I}(f)_t = -\int_0^t P_{t-s}\big[f^3 + 3Xf^2 + 3 :X^2: f + :X^3:\big]\,\mathrm{d}s$$

is well defined $\mathcal{I}\colon L^\infty([0,T];\mathscr{C}^\alpha) \to L^\infty([0,T];\mathscr{C}^\alpha)$ and is a contraction for sufficiently small $T>0$. Extending the solution to all $T>0$ is then more involved (and requires a-priori estimates that use the negative sign appearing in front of the non-linearity).

As for the contraction property over small times, we can estimate

$$\begin{aligned}\|\mathcal{I}(f)_t\|_{\mathscr{C}^\alpha} \lesssim \int_0^t &\|f_s\|^3_{\mathscr{C}^\alpha} + s^{-\frac{\alpha+\varepsilon}{2}}\|X\|_{\mathscr{C}^{-\varepsilon}}\|f\|^2_{\mathscr{C}^\alpha} \\ &+ s^{-\frac{\alpha+\varepsilon}{2}}\|:X^2:\|_{\mathscr{C}^{-\varepsilon}}\|f\|_{\mathscr{C}^\alpha} + s^{-\frac{\alpha+\varepsilon}{2}}\|:X^3:\|_{\mathscr{C}^{-\varepsilon}}\,\mathrm{d}s \;.\end{aligned}$$

This holds for any $\varepsilon>0$, but let us choose ε sufficiently small such that $\varepsilon+\alpha \in (0,2)$, which is possible since $\alpha \in (0,2)$. Then we can estimate

$$|\!|\!|\mathcal{I}(f)|\!|\!|_T \lesssim T^{1-\frac{\alpha+\varepsilon}{2}}(M + |\!|\!|f|\!|\!|_T^3)\;,$$

where $|\!|\!|f|\!|\!|_T = \sup_{0\leqslant s\leqslant T}\|f_s\|_{\mathscr{C}^\alpha}$ and M is a random constant such that

$$\sup_{0\leqslant s\leqslant T}\big\{\|X_s\|_{\mathscr{C}^{-\varepsilon}} + \|:X_s^2:\|_{\mathscr{C}^{-\varepsilon}} + \|:X_s^3:\|_{\mathscr{C}^{-\varepsilon}}\big\} \leqslant M\;.$$

Therefore, if we denote with $\mathcal{X}$ the space (and $\mathcal{X}_R$ the ball):

$$\mathcal{X} = L^\infty([0,T];\mathscr{C}^\alpha)\;,\qquad \mathcal{X}_R = \{f \in \mathcal{X} \,:\, |\!|\!|f|\!|\!|_T \leqslant R\}\;,$$

we find that for $T \leqslant T_\star(R)$, for some $T_\star(R)>0$ sufficiently small, $\mathcal{I}:\mathcal{X}_R \to \mathcal{X}_R$, and moreover, by the same estimates as above

$$|\!|\!|\mathcal{I}(f)-\mathcal{I}(g)|\!|\!|_T \lesssim T^{1-\frac{\alpha+\varepsilon}{2}}(1+R^2)(1+M)|\!|\!|f-g|\!|\!|_T\;,$$

so that (again provided $T_\star(R)>0$ is sufficiently small), leads to $\mathcal{I}$ being a contraction on $\mathcal{X}_R$. Therefore, there exists a unique fixed point ψ, which is the local solution of the theorem. □

4 Construction of the Φ^4_2 measure

In this last section we address the study of long-time properties of the Φ^4 model (although many results extend to other SPDEs). In the case of Φ^4, the invariant measure can be somewhat easily guessed, in analogy to finite-dimensional Langevin dynamics:

$$\mathrm{d}Y_t = -\nabla\Psi(Y_t)\,\mathrm{d}t + \mathrm{d}B_t \;, \tag{4.1}$$

where Ψ is some for of potential. In the case of (4.1), the invariant probability measure is given by

$$p(\mathrm{d}y) = \frac{1}{Z} e^{-\Psi(y)}\,\mathrm{d}y \;,$$

provided that the right hand-side is integrable, otherwise the invariant measure would not be a probability measure. In the case of $\Psi(y) = y^4$, this would lead to $Z^{-1}\exp(-y^4)\,\mathrm{d}y$. We observe that in finite dimensions there is a natural reference measure, which is the Lebesgue measure. In the case of Φ^4_d, the closest possible analogue of the Lebesgue is the Gaussian measure associated to the free field $(X_t)_{t\geqslant 0}$. The invariant measure $\mathbb{P}^\Phi$ to (3.1) in the case $m=0$ is then formally given through the following density with respect to the invariant measure $\mathbb{P}^X$ of $(X_t)_{t\geqslant 0}$:

$$\frac{\mathrm{d}\mathbb{P}^\Phi}{\mathrm{d}\mathbb{P}^X} = \frac{1}{Z}\exp\left(-\int_{\mathbf{T}^d} : X^4 : (x)\,\mathrm{d}x\right) \,. \tag{4.2}$$

One way to construct the Φ^4_d measure is by proving ergodicity of (3.1) (this is possible in dimension $d\leqslant 3$). In dimension $d=2$ a different construction is possible, which builds solely on the analysis of Wick powers (this is known as Nelson's construction [Nel66], see also [Hai21, Chapter 9]). The key difficulty in constructing (4.2) is to prove that the right hand-side is integrable. Here we must make use of the negative sign that is appearing in front of the fourth power, yet this is not entirely obvious since $: X^4 :$ is not positive (in fact, it has mean zero).

Nonetheless, for finite $u\geqslant 1$ we can still find an upper bound, as

$$: X^4_u := c_u^2(0) H^4(X_u/\sqrt{c_u(0)}) \gtrsim -c_u^2(0) \simeq -\log(u)^2 \;. \tag{4.3}$$

Then we can write

$$: X^4 :=: X^4_u : + Y_u \;, \qquad Y_u = 4\int_u^\infty : X^3_r : \,\mathrm{d}X_r \;.$$

We now want to control, for $K>0$ large, the tail probability

$$\mathbb{P}\big(\langle : X^4 :, 1\rangle \leqslant -K\big) = \mathbb{P}\big(\langle : X^4_u : + Y_u, 1\rangle \leqslant -K\big) \;. \tag{4.4}$$

The key idea of the proof is to suitably divide between high and low frequencies. For the low frequency component we want to use (4.3), while for the high frequency component we want to use a moment estimate on Y. Here we find that

$$\mathbb{E}[Y_u(x)Y_u(y)] = 4!\big(\mathscr{G}^4(x-y) - c_u^4(x-y)\big) \;.$$

It follows therefore that for a suitable φ_u (which is in $L^2(\mathbf{T}^d)$ uniformly over u)

$$\mathbb{E}\langle Y_u, 1\rangle^2 \simeq \int_{(\mathbf{T}^d)^2} \mathscr{G}^4(x-y) - c_u^4(x-y)\,\mathrm{d}x\,\mathrm{d}y$$

$$\simeq \int_{\mathbf{T}^d} (\mathscr{G}(x) - c_u(x))\varphi(x)\,\mathrm{d}x \lesssim \|\mathscr{G} - c_u\|_{L^2} \;.$$

Now for the last term, $\mathscr{G} - c_u$ acts like a projection on frequencies $\gtrsim u$. Therefore, we find for any $\varepsilon > 0$

$$\|\mathscr{G} - c_u\|_{L^2} \lesssim_\varepsilon u^{-2+\varepsilon} \;.$$

Now we can go back to (4.4), to find that if we choose u such that $K - c\log^2(u) \in [1, 2]$, then

$$\begin{aligned}\mathbb{P}\big(\langle : X^4 :, 1\rangle \leqslant -K\big) &\leqslant \mathbb{P}(|\langle Y_u, 1\rangle| \geqslant K - c\log^2(u)) \\ &\leqslant \mathbb{E}|Y_u|^{2p} \leqslant Cp^p u^{(-2+\varepsilon)p} \;.\end{aligned}$$

Now recall that

$$u \simeq \exp(\sqrt{K}\,\widetilde{c}) \;.$$

Therefore, we have found that

$$\mathbb{P}\big(\langle : X^4 :, 1\rangle \leqslant -K\big) \leqslant C(pe^{-c_1\sqrt{K}})^p \;.$$

Choosing $p = e^{c_2\sqrt{K}}$ for $c_2 < c_1$ leads to

$$\mathbb{P}\big(\langle : X^4 :, 1\rangle \leqslant -K\big) \lesssim \exp(-c_3 \exp(c_2\sqrt{K})) \;.$$

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