Fokker–Plank (p.34). Proof of $\dfrac{\mathrm{d}f_t}{\mathrm{d}t} = L^\star f_t$:
We have

$$L^\star(x,y) = \frac{\pi(y)}{\pi(x)} L(y,x) \qquad \text{and} \qquad f_t(x) = \frac{\mu P_t(x)}{\pi(x)} = \frac{1}{\pi(x)} \sum_y \mu(y) P_t(y,x).$$

Then, on the one hand,

$$\begin{aligned} L^\star f_t(x) &= \sum_y L^\star(x,y) f_t(y) \\ &= \sum_y \frac{\pi(y)}{\pi(x)} L(y,x) \frac{1}{\pi(y)} \sum_z \mu(z) P_t(z,y) = \frac{1}{\pi(x)} \sum_{y,z} \mu(z) P_t(z,y) L(y,x), \end{aligned}$$

and on the other hand,

$$\frac{\mathrm{d}f_t}{\mathrm{d}t} = \frac{1}{\pi(x)} \sum_y \mu(y) \frac{\mathrm{d}}{\mathrm{d}t} P_t(y,x) = \frac{1}{\pi(x)} \sum_{y,z} \mu(y) \sum_z P_t(y,z) L(z,x).$$

Dirichlet form: definition (p.35).

$$\begin{aligned} \mathcal{E}(f,g) = \langle f, -Lg \rangle &= \langle f, (\mathrm{Id} - T) g \rangle \\ &= \sum_x \pi(x) f(x) g(x) - \sum_{x,y} \pi(x) f(x) T(x,y) g(y) \\ &= \sum_{x,y} \pi(x) T(x,y) f(x) (g(x) - g(y)). \end{aligned} \tag{1}$$

Dirichlet form: symmetrised version for T reversible (p.35). Note that in the sum (1), the diagonal terms $x = y$ vanish. Fix distinct x and y. Then, using detailed balance,

$$\begin{aligned} &\pi(x)T(x,y)f(x)(g(x) - g(y)) + \pi(y)T(y,x)f(y)(g(y) - g(x)) \\ &= \pi(x)T(x,y)f(x)(g(x) - g(y)) + \pi(x)T(x,y)f(y)(g(y) - g(x)) \\ &= \pi(x)T(x,y)(f(x) - f(y))(g(x) - g(y)). \end{aligned}$$

By symmetry, we also get

$$\begin{aligned} &\pi(x)T(x,y)f(x)(g(x) - g(y)) + \pi(y)T(y,x)f(y)(g(y) - g(x)) \\ &= \pi(y)T(y,x)(f(y) - f(x))(g(y) - g(x)), \end{aligned}$$

so

$$\begin{aligned} &\pi(x)T(x,y)f(x)(g(x) - g(y)) + \pi(y)T(y,x)f(y)(g(y) - g(x)) \\ &= \tfrac{1}{2}\pi(x)T(x,y)(f(x) - f(y))(g(x) - g(y)) \\ &\quad + \tfrac{1}{2}\pi(y)T(y,x)(f(y) - f(x))(g(y) - g(x)), \end{aligned}$$

so it easily follows that

$$\mathcal{E}(f,g) = \frac{1}{2}\sum_{x,y} \pi(x)T(x,y)(f(x)-f(y))(g(x)-g(y)).$$

Dirichlet form: symmetrised version for $f = g$ (p.35).

$$\begin{aligned}\mathcal{E}(f,f) &= \sum_{x,y} \pi(x)T(x,y)f(x)(f(x)-f(y)) \\ &= \left(\frac{1}{2}+\frac{1}{2}\right)\sum_{x,y}\pi(x)T(x,y)f(x)^2 - \sum_{x,y}\pi(x)T(x,y)f(x)f(y). \qquad (2)\end{aligned}$$

Note that

$$\begin{aligned}\sum_{x,y}\pi(x)T(x,y)f(x)^2 &= \sum_x \pi(x)f(x)^2 \\ &= \sum_{x,y}\pi(y)T(y,x)f(x)^2 = \sum_{x,y}\pi(x)T(x,y)f(y)^2.\end{aligned}$$

Hence, the expression in (2) equals

$$\begin{aligned}&\frac{1}{2}\sum_{x,y}\pi(x)T(x,y)f(x)^2 + \frac{1}{2}\sum_{x,y}\pi(x)T(x,y)f(y)^2 - \sum_{x,y}\pi(x)T(x,y)f(x)f(y) \\ &= \frac{1}{2}\sum_{x,y}\pi(x)T(x,y)(f(x)-f(y))^2.\end{aligned}$$

Computation of $\nabla \mathrm{Ent}(f)$ (p.38).

$$\begin{aligned}\frac{\partial}{\partial f_z}\mathrm{Ent}(f) &= \frac{\partial}{\partial f_z}\left(\sum_x \pi(x)f(x)\log f(x)\right) \\ &\quad - \frac{\partial}{\partial f_z}\left(\left(\sum_x \pi(x)f(x)\right)\cdot \log\left(\sum_x \pi(x)f(x)\right)\right) \\ &= \pi(z)(\log f(z)+1) - (\pi(z)\log \mathbb{E}[f] + \pi(z)) \\ &= \pi(z)\log\frac{f(z)}{\mathbb{E}[f]}.\end{aligned}$$

To get $\nabla\mathrm{Ent}(f)$, we need to divide each entry $\frac{\partial}{\partial f_z}\mathrm{Ent}(f)$ by $\pi(z)$, because the inner product in the expression

$$\Phi(f) = \Phi(f_0) + \langle \nabla\Phi(f_0), f - f_0\rangle + o(\|f-f_0\|_2)$$

is not the canonical inner product of $\mathbb{R}^d$, but rather, $\langle f,g\rangle = \sum_x \pi(x)f(x)g(x)$. This shows that

$$\nabla\mathrm{Ent}(f) = \log\frac{f}{\mathbb{E}[f]},$$

or, if f is a density,

$$\nabla \mathrm{Ent}(f) = \log f.$$

Computation of $\lim_{\theta\to 0} \Lambda(\theta)$ (p.41).

$$\begin{aligned}\Lambda(\theta) = \frac{1}{\theta}\log \mathbb{E}\left[\exp\left(\theta f - \frac{\theta^2}{2\alpha}\right)\right] &\approx \frac{1}{\theta}\log \mathbb{E}\left[1 + \theta f - \frac{\theta^2}{2\alpha} + o(\theta^2)\right] \\ &= \frac{1}{\theta}\log\left(1 + \theta\mathbb{E}[f] - \frac{\theta^2}{2\alpha} + o(\theta^2)\right) \xrightarrow{\theta\to 0} \mathbb{E}[f].\end{aligned}$$

Computation of $\Lambda'(\theta)$ (p. 41)

$$\begin{aligned}\Lambda'(\theta) &= \left(\frac{\log \mathbb{E}[F_\theta]}{\theta}\right)' \\ &= \frac{1}{\theta^2}\cdot\left(\frac{(\mathbb{E}[F_\theta])')}{\mathbb{E}[F_\theta]}\cdot\theta - \log\mathbb{E}[F_\theta]\right) = \frac{(\mathbb{E}[F_\theta])'\cdot\theta - \mathbb{E}[F_\theta]\cdot\log\mathbb{E}[F_\theta])}{\theta^2\mathbb{E}[F_\theta]}. \quad (3)\end{aligned}$$

We also compute

$$\begin{aligned}(\mathbb{E}[F_\theta])'\cdot\theta &= \theta\cdot\left(\sum_x \pi(x)e^{\theta f(x) - \frac{\theta^2}{2\alpha}}\right)' \\ &= \sum_x \pi(x)\cdot\left(\theta f(x) - \frac{\theta^2}{2\alpha}\right)\cdot e^{\theta f(x)-\frac{\theta^2}{2\alpha}} \\ &= \sum_x \pi(x)\cdot\left(\theta f(x) - \frac{\theta^2}{2\alpha}\right)\cdot e^{\theta f(x)-\frac{\theta^2}{2\alpha}} - \sum_x \pi(x)\frac{\theta^2}{2\alpha}e^{\theta f(x)-\frac{\theta^2}{2\alpha}} \\ &= \mathbb{E}[F_\theta\log F_\theta] - \frac{\theta^2}{2\alpha}\mathbb{E}[F_\theta].\end{aligned}$$

Plugging this back in (3) gives the desired expression.

Computation of $\frac{\mathrm{d}\|f_t\|_{q_t}}{\mathrm{d}t}$ (p.43) We will use the differentiation rule

$$\frac{\mathrm{d}}{\mathrm{d}t}(a(t)^{b(t)}) = \left(b'(t)\log a(t) + \frac{a'(t)b(t)}{a(t)}\right)\cdot a(t)^{b(t)}.$$

Let $A_t := \|f_t\|_{q_t}^{q_t} = \mathbb{E}[f_t^{q_t}]$. Then,

$$\begin{aligned}(\|f_t\|_{q_t})' = (A_t^{1/q_t})' &= \left(\left(\frac{1}{q_t}\right)'\cdot\log A_t + \frac{A_t'}{A_t q_t}A_t^{1/q_t}\right) \\ &= \left(-\frac{q_t'}{q_t^2}\cdot\log A_t\cdot A_t + \frac{1}{q_t^2}\cdot q_t\cdot A_t'\right)\cdot\|f_t\|_{q_t}^{1-q_t}. \quad (4)\end{aligned}$$

We also compute

$$\begin{aligned}
q_t A_t' &= q_t \sum_x \pi(x)(f_t(x)^{q_t})' \\
&= q_t \sum_x \pi(x)\left(q_t' \log f_t(x) + \frac{f_t'(x) q_t}{f_t(x)}\right) \cdot f_t(x)^{q_t} \\
&= q_t' \sum_x \pi(x) \log(f_t(x)^{q_t}) \cdot f_t(x)^{q_t} + q_t^2 \sum_x \pi(x) f_t'(x) \cdot f_t(x)^{q_t - 1}.
\end{aligned}$$

Using this, the expression in parentheses in (4) equals

$$\begin{aligned}
&- \frac{q_t'}{q_t^2} \cdot \log A_t \cdot A_t + \frac{q_t'}{q_t^2} \sum_x \pi(x) \log(f_t(x)^{q_t}) \cdot f_t(x)^{q_t} + \sum_x \pi(x) f_t'(x) \cdot f_t(x)^{q_t - 1} \\
&= \frac{q_t'}{q_t^2} \mathrm{Ent}(f_t^{q_t}) - \mathscr{E}(f_t, f_t^{q_t - 1}).
\end{aligned}$$

Proof of inequality $(u-v)(u^{q-1}-v^{q-1}) \geq \frac{4(q-1)}{q^2}(u^{q/2}-v^{q/2})$ **(p.43).** There is no loss in assuming $u > v$. Dividing both sides of the desired inequality by v^q, it becomes

$$\left(\frac{u}{v} - 1\right)\left(\left(\frac{u}{v}\right)^{q-1} - 1\right) \geq \frac{4(q-1)}{q^2}\left(\left(\frac{u}{v}\right)^{q/2} - 1\right)^2$$

Letting $t := (\frac{u}{v})^{q/2} > 1$, $\varphi := \frac{2}{q}$ and $\psi := \frac{2(q-1)}{q}$, this becomes

$$(t^{\varphi} - 1)(t^{\psi} - 1) \geq \varphi\psi(t-1)^2.$$

To prove this, we note that $\varphi + \psi = 2$, and write

$$\begin{aligned}
(t^{\varphi} - 1)(t^{\psi} - 1) &= \varphi \int_1^t s^{\varphi - 1}\, \mathrm{d}s \cdot \psi \int_1^t s^{\psi - 1}\, \mathrm{d}s \\
&\geq \varphi\psi \left(\int_1^t s^{\frac{\varphi - 1}{2}} \cdot s^{\frac{\psi - 1}{2}}\, \mathrm{d}s\right)^2 = \varphi\psi(t-1)^2.
\end{aligned}$$

Equality $(f(x)-f(y))\log\frac{f(x)}{f(y)} = (\sqrt{f(x)} - \sqrt{f(y)})^2 \phi\left(\log\frac{f(x)}{f(y)}\right)$ **(p. 46).**

$$\begin{aligned}
\phi\left(\log\frac{f(x)}{f(y)}\right) &= \log\frac{f(x)}{f(y)} \cdot \frac{\sqrt{f(x)/f(y)} + 1}{\sqrt{f(x)/f(y)} - 1} \\
&= \log\frac{f(x)}{f(y)} \cdot \frac{\sqrt{f(x)} + \sqrt{f(y)}}{\sqrt{f(x)} - \sqrt{f(y)}} \\
&= \log\frac{f(x)}{f(y)} \cdot \frac{f(x) + f(y)}{(\sqrt{f(x)} - \sqrt{f(y)})^2}.
\end{aligned}$$

Computation of $\mathscr{E}(\sqrt{f^x}, \sqrt{f^x})$ (p.45).

$$\mathscr{E}(\sqrt{f^x}, \sqrt{f^x}) = \frac{1}{2} \sum_{w,z} \pi(w) T(w,z) \left(\sqrt{f^x(w)} - \sqrt{f^x(z)} \right)^2 .$$

In the double sum above, the terms where $x \notin \{w, z\}$ vanish, as well as the term where $w = z = x$. By reversibility, for any $y \neq x$, the term where $w = x, z = y$ equals the term where $w = y, z = x$. This gives

$$\begin{aligned} \mathscr{E}(\sqrt{f^x}, \sqrt{f^x}) &= \sum_{y : y \neq x} \pi(x) T(x,y) \left(\sqrt{f^x(x)} - \sqrt{f^x(y)} \right)^2 \\ &= \sum_{y : y \neq x} \pi(x) T(x,y) \cdot \frac{1}{\pi(x)} = 1. \end{aligned}$$

Verification that ϕ is increasing on $(0, \infty)$ (p. 46). We compute

$$\begin{aligned} \phi'(r) &= \frac{e^{r/2}+1}{e^{r/2}-1} + r \left(\frac{e^{r/2}+1}{e^{r/2}-1} \right)' \\ &= \frac{e^{r/2}+1}{e^{r/2}-1} + r \cdot \left(\frac{\frac{1}{2} e^{r/2}(e^{r/2}-1) - \frac{1}{2} e^{r/2}(e^{r/2}+1)}{(e^{r/2}-1)^2} \right) \\ &= \frac{e^{r/2}+1}{e^{r/2}-1} + r \cdot \left(- \frac{e^{r/2}}{(e^{r/2}-1)^2} \right) = \frac{e^r - 1 - r e^{r/2}}{(e^{r/2}-1)^2}. \end{aligned}$$

Hence, we need to check that $e^r \geq 1 + r e^{r/2}$ for $r > 0$. This comes from writing the power series

$$\begin{aligned} 1 + r e^{r/2} &= 1 + r \left(1 + \frac{r}{2} + \frac{r^2}{2^2 2!} + \frac{r^3}{2^3 3!} + \cdots \right) \\ &= 1 + r + \frac{r^2}{2!} + \frac{r^3}{2^2 2!} + \frac{r^4}{2^3 3!} + \cdots \end{aligned}$$

Since $2^n n! \geq (n+1)!$ for $n \geq 2$, the above is smaller than e^r.