LYAPOUNOV NORMS FOR RANDOM WALKS IN LOW DISORDER AND DIMENSION GREATER THAN THREE.

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ABSTRACT. We consider a simple random walk on Z^d, $d > 3$. We also consider a collection of i.i.d. positive and bounded random variables $(\,V_\omega(x)\,)_{x\in Z^d}$, which will serve as a random potential. We study the annealed and quenched cost to perform long crossing in the random potential $-(\lambda + \beta V_\omega(x)\,)$, where λ is positive constant and $\beta > 0$ is small enough . These costs are measured by the Lyapounov norms. We prove the equality of the annealed and the quenched norm.

1. INTRODUCTION

Consider a simple random walk $(S_n)_{n\geq 0}$ on $\mathbb{Z}^d$, $d > 3$ and denote by P_x its distribution when it starts from $x \in \mathbb{Z}^d$. Consider also $\lambda, \beta > 0$ and a collection of i.i.d. random variables $(\,V_\omega(x))_{x\in\mathbb{Z}^d}$, independent of the walk. We denote by $\mathbb{P}$ the distribution of this collection. We assume that V_ω is nonnegative and bounded. We think of S_n as a random walk in the random potential $(-\lambda - \beta\, V_\omega(x)\,)_{x\in\mathbb{Z}^d}$.

One of the fundamental quantities in the study of random walks is the Green's function , which is defined as

$$G_\lambda(x, y, \omega) := \sum_{N=1}^{\infty} E_x \left[e^{-\sum_{n=1}^{N}(\lambda+\beta V_\omega(S_n))};\, S_N = y \right]$$

We may think of the Green's function as the expected number of visits to $y \in \mathbb{Z}^d$ of a random walk starting at $x \in \mathbb{Z}^d$, before it gets killed by the potential $-(\lambda+\beta V_\omega(\cdot))$. It is known [11] that when $|x - y|$ tends to infinity, self-averaging phenomena take place, that result to an almost sure asymptotic exponential decay of the Green's function.

In particular, it is shown by Zerner [11] that there exists a nondegenerate norm $\alpha_\lambda(\cdot)$ on $\mathbb{R}^d$ such that for $\mathbb{P} - a.e.$ disorder ω

$$\lim_{x\to\infty} -\frac{\log G_\lambda(0, [x], \omega)}{\alpha_\lambda(x)} = 1. \tag{1.1}$$

The norm $\alpha_\lambda(x)$ is called the quenched Lyapounov norm. It was first introduced in the continuous setting of Brownian motion in a Poissonian potential by Sznitman. Equation (1.1) is the discrete analogue of his shape theorem [9].

The Lyapounov norm is a measure of the cost of the random walk to perform long crossings, in the potential $-(\lambda+\beta V_\omega(x))$. To make this more clear we consider

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the quantity

$$e_\lambda(x,y,\omega) := E_x\left[e^{-\sum_{n=1}^{T_y}(\lambda+\beta V_\omega(S_n))}\right], \tag{1.2}$$

where $T_y = \inf\{n\colon S_n = y\}$ is the hitting time of the site y. One can think of $e_\lambda(x,y,\omega)$ as the probability of the random walk, which starts at x to hit the site y before it gets killed by the potential.

The Lyapounov norm can be now defined as follows. For any $x \in \mathbb{Z}^d$

$$\alpha_\lambda(x) = \lim_{n\to\infty} -\frac{1}{n}\log e_\lambda(0,[nx],\omega), \qquad \mathbb{P}\text{-a.s.} \tag{1.3}$$

The $\mathbb{P}-a.s$ existence of the limit is guaranteed by the following supermultiplicative property

$$e_\lambda(0,x+y,\omega) \geq e_\lambda(0,x,\omega)\, e_\lambda(x,x+y,\omega). \tag{1.4}$$

It is easy to conclude from (1.3) and (1.4) that

$$\begin{aligned} \alpha_\lambda(nx) &= n\alpha_\lambda(x), && x\in\mathbb{Z}^d,\ n\in\mathbb{N}\\ \alpha_\lambda(x+y) &\leq \alpha(x)+\alpha(y), && x,y\in\mathbb{Z}^d, \end{aligned}$$

and one can use these properties to extend α_λ as a norm in $\mathbb{R}^d$.

Besides the $\mathbb{P}-a.s.$ asymptotic exponential decay of the Green's function, one is interested to know the averaged or annealed asymptotic exponential decay of it. It turns out that this is governed by the annealed Lyapounov norm $\beta_\lambda(x)$, $x\in\mathbb{Z}^d$ and in analogy with (1.1) we have that

$$\lim_{x\to\infty} -\frac{\log \mathbb{E}\, G_\lambda(0,[x],\omega)}{\beta_\lambda(x)} = 1. \tag{1.5}$$

The annealed Lyapounov norm β_λ is defined for $x\in\mathbb{Z}^d$ by

$$\beta_\lambda(x) = \lim_{n\to\infty} -\frac{1}{n}\log \mathbb{E}\, e_\lambda(0,[nx],\omega) \tag{1.6}$$

and in analogy with the quenched case, it can be extended as a norm on $\mathbb{R}^d$. The existence of the above limit is guaranteed once again by the subadditivity of $-\log\mathbb{E}\, e_\lambda(0,[nx],\omega)$, as this follows from (1.4).

Another very important property of the Lyapounov norms is that they govern the large deviations properties of the random walk S_n in the random potential $-\beta V_\omega(x)$, $x\in\mathbb{Z}^d$. This fact was first established in the deep work of A.-S. Sznitman [9], and was later extended in the discrete case by Zerner and Flury [11],[6]. In fact, it was the need to describe these large deviations, that led to the introduction of the Lyapounov norms. To be more precise, consider the measure

$$dQ_{0,\omega} := \frac{1}{Z_{N,\omega}}\, e^{-\beta\sum_{n=1}^N V_\omega(S_n)}\, dP_0, \tag{1.7}$$

where $Z_{N,\omega} = E_0\left[e^{-\beta\sum_{n=1}^N V_\omega(S_n)}\right]$ is the normalisation constant. Then it turns out that

$$Q_{0,\omega}\Big(\frac{S_n}{n}\simeq x\Big) \simeq e^{-n\,I(x)}, \tag{1.8}$$

where $I(x) = \sup_{\lambda>0}\big(\alpha_\lambda(x)-\lambda\big)$. Similar large deviations principle holds also for the annealed measures, with the annealed rate function $J(x) = \sup_{\lambda>0}\big(\beta_\lambda(x)-\lambda\big)$.

It was conjectured in [9] that in high dimensions and low disorder the annealed and quenched norms coincide. We will prove in this paper that, for $d > 3$ and β small enough, $\alpha_\lambda \equiv \beta_\lambda$, thus verifying the conjecture. More precisely we have that

Theorem 1.1. *For any $\lambda > 0$, and $d > 3$ there exists a $\beta_*(\lambda) > 0$, that depends on λ, such that if $0 < \beta < \beta_*$, then $\alpha_\lambda(x) = \beta_\lambda(x)$, for every $x \in \mathbb{R}^d$.*

This belief was based on analogies with the situation of directed polymers [2],[4]. In this case one considers a space-time potential $\beta\, V_\omega(x, n)$, $(x, n) \in \mathbb{Z}^{d-1} \times \mathbb{N}$, as a collection of i.i.d. random variables, and a simple $(d-1)-$dimensional random walk $(X_n)_{n\geq 0}$. It has been proved among many other things, that when $d \geq 4$ and the disorder is low, i.e. β is small enough, then the fraction

$$\frac{E_0\left[e^{-\beta\sum_{n=1}^{N} V_\omega(X_n,n)}\right]}{\mathbb{E}E_0\left[e^{-\beta\sum_{n=1}^{N} V_\omega(X_n,n)}\right]},$$

converges $\mathbb{P} - a.s$, as $N \to \infty$, to a strictly positive random variable W_∞. In fact, the convergence to a strictly positive limit can be shown to be an equivalent characterization of the low disorder regime.

The belief that the annealed and the quenched Lyapounov norms are equal, was further reinforced by the very nice work of M.Flury. In [5] it was established that if $(X_n)_{n\geq 0}$ is a random walk on $\mathbb{Z}^d$, $d > 3$, with a drift in the coordinate direction $\hat{e}_1$, then, as N tends to infinity

$$-\frac{1}{N}\log E_0^h\left[e^{-\beta\sum_{n=1}^{N} V_\omega(X_n)}\right] \sim -\frac{1}{N}\log \mathbb{E}\, E_0^h\left[e^{-\beta\sum_{n=1}^{N} V_\omega(X_n)}\right], \tag{1.9}$$

for β small enough. Here E^h is the expectation of the random with drift in direction $\hat{x}_1$.

In our setting, the presence of $\lambda > 0$ in (1.2) penalises the walks that move very slowly towards the target site y, thus imposing an effective drift on the random walks S_n. In other words our situation parallels in a sense the directed case, and at the same time is a generalisation of [5].

The method we follow to prove Theorem 1.1 uses ideas of [5], which are also present in the work of Bolthausen and Sznitman [3] in the context or random walks in random environments. Central in our work, as well as in [5],[3], is a second-to-first moment estimate, which, among other things, depends on what is known as a mass gap estimate (see section 3). The mass gap estimate appears in [5], but it can traced back to works related to the behavior of self avoiding walks, see for example [7]. It states roughly that the annealed cost of a walk to move from a hyperplane to a hyperplane at distance L, restricted to move only in between the hyperplanes and such that the graph of the walk cannot be splitted into two nonintersecting sets, is exponentially faster, in L, than the cost of the walk, with just the restriction to move in between the hyperplanes. It has in some sense the same flavor as the exponential moment estimate on the displacement , up to a regeneration time, of a random walk in a random environment [8]. The mass gap estimate is proved in [5],[7] using a rather involved multiscale argument. This argument is also difficult to extend when the walk's drift is not along a coordinate axis, which is essentially the framework we will be working. In section 3 we provide a simple proof independent of the direction, in the case that β is small. Moreover, in the case that the direction coincides with a coordinate axis, we provide a proof for arbitrary β, which significantly simplifies the already existing ones.

The proof of Theorem 1.1 proceeds as follows. In Section 2 we show how the point-to-point Lyapounov norms are related to the point-to-hyperplane Lyapounov norms. Moreover, we relate the presence of λ in (1.2) to the presence of an effective drift for the walk in the random potential and state the second-to-first moment condition. In Section 3 we prove the mass gap estimate. In Section 4 we built a Markovian Structure in our model, in such a way to parallel the situation in directed polymers. In Section 5 we proceed to the estimate of the second-to-first moment. Finally, in Section 6 we show some consequences of the equality of the two norms.

2. Some Auxiliary Results

In this section we prove some auxiliary results, that lead to the statement of the main estimate in Proposition 2.16. A notational convention, that we follow through out the paper, is that we refrain denoting in the expectations of the random walks the starting position, if this is 0. In this case, will just write P, E, as opposed to P_0, E_0.

2.1. **Dual Norms.** We define the dual to the quenched Lyapounov norm as

$$\alpha_\lambda^*(\ell) := \sup_{\{x\in\mathbb{R}^d\,:\,\ell\cdot x=1\}} \frac{1}{\alpha_\lambda(x)} \qquad ;\ell\in\mathbb{R}^d \tag{2.1}$$

and the dual to the annealed Lyapounov norm as

$$\beta_\lambda^*(\ell) := \sup_{\{x\in\mathbb{R}^d\,:\,\ell\cdot x=1\}} \frac{1}{\beta_\lambda(x)} \qquad ;\ell\in\mathbb{R}^d \tag{2.2}$$

.

The dual norms $\alpha_\lambda^*(\ell)$ and $\beta_\lambda^*(\ell)$ are in fact norms and they govern the cost for the walk to perform crossings from a point to a hyperplane. This is described in the following proposition, the proof of which can be found in [6]. Reference [6] contains further properties of these norms. The continuous analogue of the next proposition was proven in [8].

Proposition 2.1. *Let $\ell\in\mathbb{R}^d$ and $T_{\ell,L}=\inf\{n\colon S_n\cdot\ell\geq L\}$. Then we have that*

$$\lim_{L\to\infty} -\frac{1}{L}\log E\left[e^{-\sum_{n=1}^{T_{\ell,L}}(\lambda+\beta V_\omega(S_n))}\right] = \frac{1}{\alpha_\lambda^*(\ell)},$$

and

$$\lim_{L\to\infty} -\frac{1}{L}\log \mathbb{E}E\left[e^{-\sum_{n=1}^{T_{\ell,L}}(\lambda+\beta V_\omega(S_n))}\right] = \frac{1}{\beta_\lambda^*(\ell)},$$

The next proposition justifies the characterisation as dual norms and it will lead to the important observation of Corollary 2.3

Proposition 2.2. *Consider the quenched and the annealed dual norms $\alpha_\lambda^*(\ell)$ and $\beta_\lambda^*(\ell)$. Then*

$$(i)\qquad \alpha_\lambda(x) = \sup_{\{\ell\in\mathbb{R}^d\,:\,\ell\cdot x=1\}} \frac{1}{\alpha_\lambda^*(\ell)} \tag{2.3}$$

$$(ii)\qquad \beta_\lambda(x) = \sup_{\{\ell\in\mathbb{R}^d\,:\,\ell\cdot x=1\}} \frac{1}{\beta_\lambda^*(\ell)}. \tag{2.4}$$

Proof. We will only prove (i), the proof of (ii) being identical. Let us denote by $\alpha_\lambda^{**}(x)$ the right hand side of (2.3). By the definition of the dual norm α_λ^* we have that, for every $\ell \in \mathbb{R}^d$, $\alpha_\lambda^*(\ell) \geq \frac{1}{\alpha_\lambda(x)}$, for every $x \in \mathbb{R}^d$ such that $\ell \cdot x = 1$. Thus,

$$\alpha_\lambda^{**}(x) := \sup_{\{\ell\colon\, \ell\cdot x=1\}} \frac{1}{\alpha_\lambda^*(\ell)} \leq \alpha_\lambda(x).$$

On the other hand,

$$\begin{aligned}
\alpha_\lambda^{**}(x) \quad &:= \quad \sup_{\{\ell\colon\, \ell\cdot x=1\}} \frac{1}{\alpha_\lambda^*(\ell)} = \sup_{\{\ell\colon\, \ell\cdot x=1\}} \frac{1}{\sup_{\{y\colon\, \ell\cdot y=1\}} \frac{1}{\alpha_\lambda(y)}} \\
&= \quad \sup_{\{\ell\colon\, \ell\cdot x=1\}} \inf_{\{y\colon\, \ell\cdot y=1\}} \alpha_\lambda(y) = \sup_{\{\ell\colon\, \ell\cdot x=1\}} \inf_{\{y\colon\, \ell\cdot y=1\}} \big(\, \alpha_\lambda(y) + \ell\cdot(x-y) \,\big) \\
&= \quad \sup_{\{\ell\colon\, \ell\cdot x=1\}} \inf_{\{y\colon\, \ell\cdot y=1\}} \big(\, \alpha_\lambda(y) - \alpha_\lambda(x) + \ell\cdot(x-y) \,\big) + \alpha_\lambda(x).
\end{aligned}$$

By the convexity of $\alpha_\lambda(\cdot)$ it follows that there exists an $\ell \in \mathbb{R}^d$ such that for every $y \in \mathbb{R}^d$, $\alpha_\lambda(y) - \alpha_\lambda(x) + \ell\cdot(x-y) \geq 0$, and so $\alpha_\lambda^{**}(x) \geq \alpha_\lambda(x)$. □

Corollary 2.3. *If $\alpha_\lambda^*(\hat\ell) = \beta_\lambda^*(\hat\ell)$, for every unit vector $\hat\ell \in \mathbb{R}^d$, then the quenched and annealed Lyapounov norms are equal, i.e. $\alpha_\lambda \equiv \beta_\lambda$.*

Proof. It follows immediately from the fact that α_λ^* and β_λ^* are norms and Proposition 2.2. □

2.2. **A Change Of Measure.** In this paragraph we show how the presence of a positive λ in the potential gives rise to an effective drift for the walk.

Let $\hat\ell \in \mathbb{R}^d$ be an arbitrary unit vector, $\ell = |\ell|\,\hat\ell$ and denote by P^ℓ the random walk with transition probabilities

$$\pi_\ell(x,y) = \begin{cases} \frac{e^{(y-x)\cdot\ell}}{Z_\ell} & \text{, if} \quad |x-y| = 1, \\ 0 & \text{, if} \quad |x-y| \neq 1. \end{cases} \tag{2.5}$$

$Z_\ell = 2\sum_{i=1}^d \cosh(\hat e_i\cdot\ell)$, where $(\hat e_i)_{i=1}^d$ denote the canonical unit vectors. Notice that the random walk P^ℓ has a drift such that $E^\ell[(S_1 - S_0)\cdot\hat\ell] > 0$. The Radon-Nikodym derivative of the biased random walk P^ℓ with respect to the simple random walk P on the σ-algebra $\mathcal{F}_n = \sigma\{S_i\colon 0 \leq i \leq n\}$ is easily computed to be

$$\frac{dP^\ell}{dP}\Big|_{\mathcal{F}_n} = \left(\frac{2d}{Z_\ell}\right)^n e^{\sum_{i=1}^n (S_i - S_{i-1})\cdot\ell}.$$

Let us now compute

$$\begin{aligned}
E\left[e^{-\sum_{i=1}^{T_{\hat\ell,L}} (\lambda+\beta V_\omega(S_i))}\right] &= E^\ell\left[\frac{dP}{dP^\ell}\Big|_{\mathcal{F}_{T_{\hat\ell,L}}} e^{-\lambda T_{\hat\ell,L} - \sum_{i=1}^{T_{\hat\ell,L}} \beta V_\omega(S_i)}\right] \\
&= E^\ell\left[e^{-\sum_{i=1}^{T_{\hat\ell,L}} (S_i - S_{i-1})\cdot\ell - T_{\hat\ell,L}\log(2d/Z_\ell)}\, e^{-\lambda T_{\hat\ell,L} - \sum_{i=1}^{T_{\hat\ell,L}} \beta V_\omega(S_i)}\right].
\end{aligned} \tag{2.6}$$

We will now choose $|\ell|$, such that

$$\log\left(\frac{2d}{Z_\ell}\right) + \lambda = 0. \tag{2.7}$$

That is, we will choose $|\ell|$, such that $Z_\ell = 2d\,e^\lambda$, or $\sum_{i=1}^d \cosh(\hat{e_i}\cdot\ell) = d\,e^\lambda$. Notice that if $|\ell| = 0$, then the left hand side of the last equation is equal to d, while for

$|\ell| \to \infty$, it tends to infinity. Thus, there will be a $|\ell|$, depending on λ, such that (2.7) is satisfied. For this $|\ell|$, (2.6) is equal to

$$E^\ell \left[e^{-S(T_{\hat{\ell},L})\cdot \ell} e^{-\sum_{i=1}^{T_{\hat{\ell},L}} \beta V_\omega(S_i)} \right].$$

Notice $|\ell|\, L \le S(T_{\hat{\ell},L}) \cdot \ell \le |\ell|\, L + |\ell|$, since $T_{\hat{\ell},L}$ is defined as the first time that the random walk enters the half space $\{x \in \mathbb{Z}^d \colon x \cdot \hat{\ell} \ge L\}$. We then have that

$$\begin{aligned} e^{-|\ell|\, L - |\ell|}\, E^\ell \left[e^{-\sum_{i=1}^{T_{\hat{\ell},L}} \beta V_\omega(S_i)} \right] &\le E^\ell \left[e^{-S_{T_{\hat{\ell},L}}\cdot \ell} e^{-\sum_{i=1}^{T_{\hat{\ell},L}} \beta V_\omega(S_i)} \right] \\ &\le e^{-|\ell|\, L}\, E^\ell \left[e^{-\sum_{i=1}^{T_{\hat{\ell},L}} \beta V_\omega(S_i)} \right]. \end{aligned}$$

Thus we have proven that

Proposition 2.4. *Let $\hat{\ell} \in \mathbb{R}^d$ an arbitrary unit vector and choose $\ell = |\ell|\, \hat{\ell}$, so that it satisfies (2.7), or equivalently $\sum_{i=1}^d \cosh(\hat{e_i} \cdot \ell) = de^\lambda$, then*

$$\begin{aligned} &\lim_{L\to\infty} \frac{1}{L} \log E \left[e^{-\sum_{i=1}^{T_{\hat{\ell},L}} (\lambda + \beta V_\omega(S_i))} \right] \\ &\qquad = \lim_{L\to\infty} \frac{1}{L} \log E^\ell \left[e^{-\sum_{i=1}^{T_{\hat{\ell},L}} \beta V_\omega(S_i)} \right] - |\ell|. \end{aligned}$$

Clearly, the analogue of Proposition 2.4 for the annealed measures is also valid. We can combine Propositions 2.1, 2.4 and Corollary 2.3 to arrive at

Corollary 2.5. *If for every unit vector $\hat{\ell} \in \mathbb{R}^d$*

$$\begin{aligned} &\lim_{L\to\infty} \frac{1}{L} \log E^\ell \left[e^{-\sum_{i=1}^{T_{\hat{\ell},L}} \beta V_\omega(S_i)} \right] \\ &\qquad = \lim_{L\to\infty} \frac{1}{L} \log \mathbb{E}\, E^\ell \left[e^{-\sum_{i=1}^{T_{\hat{\ell},L}} \beta V_\omega(S_i)} \right], \end{aligned} \tag{2.8}$$

where $\ell \in \mathbb{R}^d$ is chosen as in Proposition 2.4, E^ℓ is defined by (2.5) and $T_{\hat{\ell},L} = \inf\{n \colon S_n \cdot \hat{\ell} \ge L\}$, then $\alpha_\lambda \equiv \beta_\lambda$.

Our focus will therefore be to verify the assumption of the last corollary, when β is small enough. From now on, $\ell \in \mathbb{R}^d$ will be an arbitrary, fixed vector with rational coordinates and $\hat{\ell}$ the corresponding unit vector $\ell/|\ell|$. P^ℓ denotes the distribution of the walk with transition probabilities as in (2.5), corresponding to the chosen $\hat{\ell}$. We denote by

$$h := E_x^\ell \left[(S_1 - S_0) \cdot \hat{\ell} \right] = \frac{\sum_{i=1}^d \hat{e}_i \cdot \hat{\ell} \sinh(\hat{e}_i \cdot \ell)}{\sum_{i=1}^d \cosh(\, \hat{e}_i \cdot \ell)} > 0, \tag{2.9}$$

the lentgh of the projection of the local drift on the direction $\hat{\ell}$.

The reason we have chosen the vector $\hat{\ell}$ to have rational coordinates is to be able to construct renewal structures, through which we obtain our estimates. One way to see the difficulties that would arise if one considers an $\hat{\ell}$ with irrational coordinates is to notice that in this case, except from trivial situations, the hyperplane $z \cdot \hat{\ell} = 0$ includes no points of the lattice, other than 0.

Let us also assume, without loss of generality, that $\hat{\ell} \cdot \hat{e}_i > 0$, for $i = 1, \ldots, d$. Let

$$l_1 := \hat{e}_1 \cdot \hat{\ell}, \tag{2.10}$$

the distance of the hyperplane with normal vector $\hat{\ell}$, which contains 0, from the corresponding hyperplane which contains $(1, 0, \ldots, 0)$. Due to the fact that ℓ has rational coordinates, there will only be a finite number of hyperplanes in between the above mentioned ones, which are normal to $\hat{\ell}$ and contain lattice points.

We denote by r the number of hyperplanes with normal vector $\hat{\ell}$, that are needed to exhaust the lattice points $\{z \in \mathbb{Z}^d : 0 < z \cdot \hat{\ell} \leq l_1\}$. Since $\hat{\ell}$ has rational coordinates, it follows that r is finite. It is easy to see, that the closest hyperplane with normal vector $\hat{\ell}$, which contains lattice points is at distance l_1/r from the hyperplane $\{z \in \mathbb{R}^d : z \cdot \hat{\ell} = 0\}$.

In the rest of the paper we will work in the framework set in these last paragraphs. In particular, we will prove the equality of the point to hyperplane dual norms for vectors with rational coordinates. The equality at arbitrary vectors will then follow by the continuity of $\alpha^*_\lambda(\cdot)$ and $\beta^*_\lambda(\cdot)$ - recall the fact that they are norms.

2.3. **Bridges and Irreducible Bridges.** It will be convenient for our analysis to make one more reduction. Namely to reduce the evaluation of the point to hyperplane dual norms to the evaluations of *masses of bridges*. We will explain the terminology in the sequel.

Definition 2.6. *Let us define the local time at $x \in \mathbb{Z}^d$ as $\mathcal{L}_{(M,N)} = \sum_{n=M+1}^{N} 1_x(S_n)$.*

Definition 2.7. *Let ω^i, $i = 1, \ldots, p$, $p \geq 1$ independent copies of $\omega \in \Omega$ and $\mathcal{L}^i_{(M^i,N^i)}$, $i = 1, \ldots, p$ the local times for p random walk trajectories, $M^i < N^i$. Then we define*

$$\begin{aligned} \Phi^{(p)}_\beta(M^1, \ldots, M^p; N^1, \ldots, N^p) &= -\log \mathbb{E}\left[e^{-\beta \sum_{i=1}^p \sum_x V_{\omega^i}(x) \mathcal{L}^i_{(M^i,N^i)}(x)} \right] \\ \tilde{\Phi}^{(p)}_\beta(M^1, \ldots, M^p; N^1, \ldots, N^p) &= -\log \mathbb{E}\left[e^{-\beta \sum_{i=1}^p \sum_x V_\omega(x) \mathcal{L}^i_{(M^i,N^i)}(x)} \right] \end{aligned}$$

Notice that in $\Phi^{(p)}_\beta$ we consider random walk trajectories in independent potentials, V_{ω^i}, while in $\tilde{\Phi}^{(p)}_\beta$ we consider the trajectories in the same potential V_ω.

In order to lighten the notation we will often write the above functions as $\Phi^{(p)}_\beta(M^i; N^i)$ and $\tilde{\Phi}^{(p)}_\beta(M^i; N^i)$. In the case that $M^i = 0$, for $i = 1, \ldots, p$, we will write $\Phi^{(p)}_\beta(N^i)$ and $\tilde{\Phi}^{(p)}_\beta(N^i)$ instead. Finally, when $p = 1$ we will write Φ_β, instead of $\Phi^{(1)}_\beta$.

The next proposition collects some properties of the function Φ_β, which are easy to verify

Proposition 2.8. *(i) For M, N integers we have that*

$$\Phi_\beta(M + N) \leq \Phi_\beta(N) + \Phi_\beta(M).$$

(ii) Let $N_1 < N_2 < N$, then

$$\Phi_\beta(N) \geq \Phi_\beta([0, N_1] \cup [N_2, N]).$$

(iii) If $(S(n))_{0 \leq n \leq N_1} \cap (S(n))_{N_1 \leq n \leq N_1 + N_2} = \emptyset$, then

$$\Phi_\beta(N_1 + N_2) = \Phi_\beta(N_1) + \Phi_\beta(N_2)$$

The notation used on the right hand side of (ii) means that in the evaluation of $\Phi_\beta([0,N_1]\cup[N_2,N])$ we consider the local time $\mathcal{L}_{[0,N_1]\cup[N_2,N]} := \mathcal{L}_{N_1}+\mathcal{L}_{N_2,N}$. The proof of (ii) makes use of the monotonicity $\mathcal{L}_N \geq \mathcal{L}_{[0,N_1]\cup[N_2,N]}$, while the proof of (iii) the independence of the potentials seen by the two parts of the walk. Finally, the proof of (i) makes an easy use of Hölder's inequality. More precisely, we have

$$\mathbb{E}\left[e^{-\beta V_\omega(x)\mathcal{L}_{M+N}(x)}\right]^{\frac{\mathcal{L}_M(x)}{\mathcal{L}_{M+N}(x)}} \geq \mathbb{E}\left[e^{-\beta V_\omega(x)\mathcal{L}_M(x)}\right],$$

and

$$\mathbb{E}\left[e^{-\beta V_\omega(x)\mathcal{L}_{M+N}(x)}\right]^{\frac{\mathcal{L}_{(M,M+N)}(x)}{\mathcal{L}_{M+N}(x)}} \geq \mathbb{E}\left[e^{-\beta V_\omega(x)\mathcal{L}_{(M,M+N)}(x)}\right],$$

and it only remains to multiply the above inequalities, take the product over the sites x and use the independence of the potentials.

Definition 2.9. *We define the entrance times* $T_L := \inf\{n\colon S(n)\cdot\hat{\ell} \geq L\}$, *and* $\tilde{T}_L := \inf\{n\colon S(n)\cdot\hat{\ell}\leq L\}$

Notice that T_L coincides with $T_{\hat{\ell},l}$, which appears below (2.8). For simplicity we will be using the notation T_L instead.

Definition 2.10. *(i) Consider the walk* $(\,S(n)\,)_{M\leq n\leq N}$. *We will say that the walk forms a bridge of span* L, *and denote it by* $Br(M,N;L)$, *if*

$$S(M)\cdot\hat{\ell} \leq S(n)\cdot\hat{\ell} < S(N)\cdot\hat{\ell},$$

for $M\leq n<N$, *and* $(S(N)-S(M))\cdot\hat{\ell}=L$. *When* $M=0$, *we will write* $Br(N;L)$ *instead.*

(ii) Let us denote

$$B(L) = E^\ell\left[e^{-\Phi_\beta(T_L)};\,Br(T_L;L)\right] = \sum_{N=0}^\infty E^\ell\left[e^{-\Phi_\beta(N)};\,Br(N;L)\right]. \tag{2.11}$$

Definition 2.11. *Consider the random walk* $(S(n))_{M\leq n\leq N}$. *We will say that the random walk has a break point at level* L, *if there exists an* $M<n<N$ *such that* $S(n)\cdot\hat{\ell}=L$ *and*

$$S(n_1)\cdot\hat{\ell} < S(n)\cdot\hat{\ell}\leq S(n_2)\cdot\hat{\ell},$$

for $n_1<n\leq n_2$.

Definition 2.12. *(i) Consider the random walk* $(\,S(n)\,)_{M\leq n\leq N}$. *We will say that the random walk forms an irreducible bridge of span* L, *and we denote it by* $Ir(M,N;L)$, *if it forms a bridge of span* L *with no break points. When* $M=0$ *we will write* $Ir(N;L)$ *instead.*

(ii) Let us denote

$$I_\beta(L) = E^\ell\left[e^{-\Phi_\beta(T_L)};\,Ir(T_L;L)\right] = \sum_{N=0}^\infty E^\ell\left[e^{-\Phi_\beta(N)};\,Ir(N;L)\right]. \tag{2.12}$$

Definition 2.13. *Let us define the (annealed) mass for bridges by*

$$\begin{aligned} m_B(\beta) &= \lim_{L\to\infty} -\frac{1}{L}\log\mathbb{E}E^\ell\left[e^{-\sum_{n=0}^{T_L}\beta V_\omega(S_n)};Br(T_L;L)\right] \\ &= \lim_{L\to\infty} -\frac{1}{L}\log B(L). \end{aligned} \tag{2.13}$$

Notice, that $m_B(\beta)$ depends on the direction $\hat{\ell}$, that we choose. We will refrain, though, from denoting this explicitly.

Proposition 2.14. *There exists a constant $C<1$, such that for every L,*

$$Ce^{-m_B(\beta)L} \leq B(L) \leq e^{-m_B(\beta)L}.$$

Proof. Regarding the rightmost inequality we have that, for every L_1, L_2

$$\begin{aligned}
\mathrm{B}(L_1+L_2) &= E^\ell\left[e^{-\Phi_\beta(T_{L_1+L_2})}; \mathrm{Br}(T_{L_1+L_2}; L_1+L_2)\right] \\
&\geq E^\ell\left[e^{-\Phi_\beta(T_{L_1+L_2})}; \mathrm{Br}(T_{L_1}; L_1) \cap \mathrm{Br}(T_{L_1}, T_{L_2}; L_2)\right] \\
&= \mathrm{Br}(L_1)\mathrm{B}(L_2),
\end{aligned}$$

where in the last equality we used Proposition 2.8 (iii). The right hand side of the desired inequality follows now by the supermultiplicativity.

Regarding the leftmost inequality, we will obtain a submultiplicative inequality. This will be done as follows. We will observe a path in $\mathrm{Br}(T_{L_1+L_2}; L_1+L_2)$ until the first time it crosses level L_1, the contribution of which to the partition function is essentially $B(L_1)$, and then from the last time that the path lies below level L_1 until the first time it lies on level L_1+L_2. This contribution is essentially $B(L_2)$. Let $\overline{S}_L := \sup\{n : S_n \cdot \hat{\ell} \leq L\}$. In more detail, using again Proposition 2.8 (ii) and (iii), we have

$$\begin{aligned}
&\mathrm{B}(L_1+L_2) \leq E^\ell\left[e^{-\Phi_\beta(T_{L_1}-1)-\Phi_\beta(\overline{S}_{L_1}, T_{L_1+L_2})}\,;\, \mathrm{Br}(T_{L_1+L_2}; L_1+L_2)\right] \\
&= \sum_{M,x,y} E^\ell\left[e^{-\Phi_\beta(T_{L_1}-1)-\Phi_\beta(\overline{S}_{L_1}, T_{L_1+L_2})}; \overline{S}_{L_1}=M,\, S_M=x,\, S_{T_{L_1}-1}=y,\, \mathrm{Br}(T_{L_1+L_2}; L_1+L_2)\right] \\
&= \sum_{M,x,y} E^\ell\left[e^{-\Phi_\beta(T_{L_1}-1)};\, S_{T_{L_1}-1}=y,\, 0<\inf_{n\leq T_{L_1}} S_n\cdot\hat{\ell}\right]\cdot P_y^\ell(\,S_M=x\,)1_{L_1-l_1<x\cdot\hat{\ell}\leq L_1} \\
&\qquad \cdot E_x^\ell\left[\,e^{-\Phi_\beta(T_{L_1+L_2})};\, \mathrm{Br}(T_{L_1+L_2}; L_1+L_2-x\cdot\hat{\ell}),\, \inf_{1\leq n<T_{L_1+L_2}} S_n\cdot\hat{\ell}>L_1\,\right].
\end{aligned}$$

We now want to make use of the fact, that the above expectations and probabilities, as functions of the intial point, really depend on which hyperplane the initial point belongs to, and not on the point itself. To make use of this, let us denote, for any point x, by $[x]$ to be a representative lattice point of the hyperplane, that x belongs to. With a slight abuse of notation we will use the notation $[x]$ for the corresponding hyperplane, as well. Then, using the translation invariance of the last expectation, we have, that the above is estimated by

$$\begin{aligned}
&\sum_{[x],y} E^\ell\left[e^{-\Phi_\beta(T_{L_1}-1)};\, S_{T_{L_1}-1}=y,\, 0<\inf_{n\leq T_{L_1}} S_n\cdot\hat{\ell}\right]\cdot \sum_M P_y^\ell(\,S_M\in[x]\,)1_{L_1-l_1<[x]\cdot\hat{\ell}\leq L_1} \\
&\qquad \cdot E_{[x]}^\ell\left[\,e^{-\Phi_\beta(T_{L_1+L_2})};\, \mathrm{Br}(T_{L_1+L_2}; L_1+L_2-[x]\cdot\hat{\ell}),\, \inf_{1\leq n<T_{L_1+L_2}} S_n\cdot\hat{\ell}>L_1\,\right],
\end{aligned}$$

and since

$$\sum_M P_y^\ell(\,S_M\in[x]\,)1_{L_1-l_1<[x]\cdot\hat{\ell}\leq L_1} = \sum_M P_{[y]}^\ell(\,S_M\in[x]\,)1_{L_1-l_1<[x]\cdot\hat{\ell}\leq L_1},$$

the last is equal to

$$\sum_{[x],[y]} E^{\ell}\left[e^{-\Phi_\beta(T_{L_1}-1)};\, S_{T_{L_1}-1}\in[y],\, 0<\inf_{n\le T_{L_1}} S_n\cdot\hat{\ell}\right]\cdot\sum_M P^{\ell}_{[y]}(\,S_M\in[x]\,)1_{L_1-l_1<[x]\cdot\hat{\ell}\le L_1}$$

$$\cdot E^{\ell}_{[x]}\left[e^{-\Phi_\beta(T_{L_1+L_2})};\,\mathrm{Br}(T_{L_1+L_2};L_1+L_2-[x]\cdot\hat{\ell}),\, \inf_{1\le n<T_{L_1+L_2}} S_n\cdot\hat{\ell}>L_1\right]$$

$$\le\sum_{[x]} E^{\ell}\left[e^{-\Phi_\beta(T_{L_1}-1)};\; 0<\inf_{n\le T_{L_1}} S_n\cdot\hat{\ell}\right]\cdot$$

$$\cdot\sup_{L_1-l_1\le[y]\cdot\hat{\ell}<L_1}\sum_M P^{\ell}_{[y]}(\,S_M\in[x]\,)1_{L_1-l_1\le[x]\cdot\hat{\ell}\le L_1}$$

$$\cdot\; E^{\ell}_{[x]}\left[e^{-\Phi_\beta(T_{L_1+L_2})};\,\mathrm{Br}(T_{L_1+L_2};L_1+L_2-[x]\cdot\hat{\ell}),\, \inf_{1\le n<T_{L_1+L_2}} S_n\cdot\hat{\ell}>L_1\right]$$

It is easy to see, that there exist constants C_1, C_2, such that, for every $[x]$ with $L_1-l_1\le[x]\cdot\hat{\ell}\le L_1$,

$$E^{\ell}_{[x]}\left[e^{-\Phi_\beta(T_{L_1+L_2})};\,\mathrm{Br}(T_{L_1+L_2};L_1+L_2-[x]\cdot\hat{\ell}),\, \inf_{1\le n<T_{L_1+L_2}} S_n\cdot\hat{\ell}>L_1\right] \le C_1 B(L_2)$$

and

$$E^{\ell}\left[e^{-\Phi_\beta(T_{L_1}-1)};\; 0<\inf_{n\le T_{L_1}} S_n\cdot\hat{\ell}\right]\le C_2 B(L_1)$$

The only difference between the above expectations and the corresponding values $B(L_1)$ and $B(L_2)$ is that either the initial of final point of the trajectories in the corresponding expectations might not lie on the beginning or ending hyperplane, that determine the bridges. Nevertheless, they lie nearby, and so we could change the beginning or ending of these paths in a deterministic way, so that they become typical bridges of spans L_1 and L_2, respectively. The cost for these alterations is clearly uniformly, over L_1, L_2, finite (it depends, though, on β and $\|V_\omega\|$). We, therefore, have that

$$B(L_1+L_2)\le C_1C_2\sup_{L_1-l_1\le[y]\cdot\hat{\ell}\le L_1}\sum_M P^{\ell}_{[y]}(\,L_1-l_1\le S_M\cdot\hat{\ell}\le L_1\,)\;B(L_1)B(L_2).$$

Since the random walk P^{ℓ} has a drift, the constant

$$C_1C_2\sup_{L_1-l_1\le[y]\cdot\hat{\ell}\le L_1}\sum_M P^{\ell}_{[y]}(\,L_1-l_1\le S_M\cdot\hat{\ell}\le L_1\,)$$

is finite, uniformly in L_1. From this, the leftmost inequality of the proposition follows by submultiplicativity. □

Finally, we show that as $\beta\to 0$ the annealed mass converges to 0.

Proposition 2.15. *The annealed mass is continous at $\beta=0$, and therefore*

$$\lim_{\beta\to 0} m_B(\beta)=0.$$

Proof. Clearly, $m_B(0) = 0$ and so we only need to prove the continuity at 0. But this follows immediately from the fact that

$$\begin{aligned} 0 \le m_B(\beta) &\le \lim_{L\to\infty} -\frac{1}{L}\log E^\ell\left[e^{-\beta\|V_\omega\|T_L}\right] \\ &\le \beta\|V_\omega\| \lim_{L\to\infty}\frac{E^\ell[T_L]}{L}, \end{aligned}$$

where we used Jensen's inequality at the last step, and the fact that, due to the drift of E^ℓ, the last limit is finite. □

2.4. **The Second To First Moment Condition.** The next proposition highlights the main estimate. The validity of (2.14) or equivalently (2.16) implies the equality of the Lyapounov norms. In the rest of the paper we will be working towards the proof of (2.16).

Proposition 2.16. *Let $P^\ell_{y_1,y_2}$ denote the joint distribution of two independent random walks with distribution $P^\ell_{y_1}$ and $P^\ell_{y_2}$. Let also $\tilde\Phi^{(2)}_\beta$ and $\Phi^{(2)}_\beta$ as defined in definition 2.7. If*

$$\sup_L \frac{\mathbb{E}E^\ell\left[e^{-\sum_{i=1}^{T_L^1}\beta V_\omega(S_i^1)-\sum_{i=1}^{T_L^2}\beta V_\omega(S_i^2)}\right]}{\mathbb{E}E^\ell\left[e^{-\sum_{i=1}^{T_L^1}\beta V_{\omega^1}(S_i^1)-\sum_{i=1}^{T_L^2}\beta V_{\omega^2}(S_i^2)}\right]} < \infty \tag{2.14}$$

then

$$\lim_{L\to\infty} -\frac{1}{L}\log E^\ell\left[e^{-\sum_{i=1}^{T_L}\beta V_\omega(S_i)}\right] = \lim_{L\to\infty} -\frac{1}{L}\log \mathbb{E}E^\ell\left[e^{-\sum_{i=1}^{T_L}\beta V_\omega(S_i)}\right]. \tag{2.15}$$

If (2.14) is valid for every vector $\ell \in \mathbb{R}^d$ with rational coordinates (or, in other words, for the corresponding to it unit vector $\hat\ell = \ell/|\ell|$), then the annealed and quenched Lyapounov norms are equal.

Proof. The left hand side of (2.15) is greater or equal than its right hand side. This follows by Jensen's inequality, since

$$\lim_{L\to\infty} -\frac{1}{L}\mathbb{E}\log E^\ell\left[e^{-\sum_{i=1}^{T_L}\beta V_\omega(S_i)}\right] \ge \lim_{L\to\infty} -\frac{1}{L}\log \mathbb{E}E^\ell\left[e^{-\sum_{i=1}^{T_L}\beta V_\omega(S_i)}\right].$$

and by dominated convergence the left hand side of the last inequality is equal to the left hand side of (2.15). Suppose, now, that this inequality is strict. Consider, then, the function

$$U_\omega(L) = \frac{E^\ell\left[e^{-\sum_{i=1}^{T_L}\beta V_\omega(S_i)}\right]}{\mathbb{E}E^\ell\left[e^{-\sum_{i=1}^{T_L}\beta V_\omega(S_i)}\right]},$$

and observe, that, in the case of strict inequality, a.s. $U_\omega(L) \to 0$ as $L \to \infty$, since then the numerator will decay exponentially faster than the denominator. A straightforward computation shows that the left hand side of (2.14) is equal to $\sup_L \mathbb{E}\, U_\omega^2$, and so (2.14) impies that U_ω is uniformly bounded in $L^2(\mathbb{P})$. It therefore follows, that $\mathbb{E}U_\omega(L) \to 0$, as $L \to \infty$. On the other hand, this is a contradiction, since $\mathbb{E}U_\omega(L) = 1$, for every L.

Finally, we can combine Corollary 2.5 with the continuity of the dual norms to obtain the equality of the Lyapounov norms. □

Proposition 2.17. *The estimate* (2.14) *is valid, if the following estimate is valid*

$$\sup_L \sup_{y_1,y_2} \frac{\sum_{N^1,N^2} E^{\ell}_{y^1,y^2}\left[e^{-\tilde{\Phi}^{(2)}_{\beta}(N^1,N^2)};\ Br(N^1;L)\cap Br(N^2;L)\right]}{\sum_{N^1,N^2} E^{\ell}_{y^1,y^2}\left[e^{-\Phi^{(2)}_{\beta}(N^1,N^2)};\ Br(N^1;L)\cap Br(N^2;L)\right]} < \infty. \tag{2.16}$$

Proof. We will establish, that the left side of (2.16) bounds, up to constants, the left side of (2.14). To this end, we restrict the expectation in the denominator of (2.14) to the paths, that belong to $\mathrm{Br}(T^1_L;L)\cap\mathrm{Br}(T^2_L;L)$, to get that this denominator is bounded below by

$$\sum_{N^1,N^2} E^{\ell}\left[e^{-\Phi^{(2)}_{\beta}(N^1,N^2)};\ \mathrm{Br}(N^1;L)\cap\mathrm{Br}(N^2;L)\right]. \tag{2.17}$$

Regarding the numerator in (2.14) we have that it is bounded above by

$$\begin{aligned}
&\sum_{M^1,M^2}\sum_{y^1,y^2}\mathbb{E}E^{\ell}\left[e^{-\sum_{j=1,2}\sum_{i=M^j+1}^{T^j_L}\beta V_\omega(S^j_i)};\ \overline{S}^j_0=M^j,\ S_{M^j}=y^j, \text{for } j=1,2\right]\\
=&\sum_{M^1,M^2}\ \sum_{-l_1<y^1\cdot\hat{\ell},y^2\cdot\hat{\ell}\le 0} P^{\ell}\left(S^j_{M^j}=y^j\ ,j=1,2\right)\cdot\\
&\qquad\cdot\mathbb{E}E^{\ell}_{y^1,y^2}\left[e^{-\sum_{j=1,2}\sum_{i=1}^{T^j_L}\beta V_\omega(S^j_i)};\ \inf_{n^j<T^j_L} S^j(n^j)\cdot\hat{\ell}>0\right]\\
\le&\ C_3\sum_{M^1,M^2}\ \sum_{-l_1<y^1\cdot\hat{\ell},y^2\cdot\hat{\ell}\le 0} P^{\ell}\left(S^j_{M^j}=y^j\ ,j=1,2\right)\cdot\\
&\qquad\cdot\sup_{-l_1<y^1\cdot\hat{\ell},y^2\cdot\hat{\ell}\le 0}\ \sum_{N^1,N^2} E^{\ell}_{y^1,y^2}\left[e^{-\tilde{\Phi}^{(2)}_{\beta}(N^1,N^2)};\ \mathrm{Br}(N^1;L)\cap\mathrm{Br}(N^2;L)\right].
\end{aligned} \tag{2.18}$$

The last inequality is obtained in a similar fashion as the corresponding estimates at the end of Proposition 2.14: one needs to change in a deterministic fashion the end points of the trajectories in the last expectation, in order to obtain bridges of span L. The resulting constant C_3 depends on β and $\|V_\omega\|$. Since, due to the drift of P^ℓ

$$\sum_{M^1,M^2}\ \sum_{-l_1<y^1\cdot\hat{\ell},y^2\cdot\hat{\ell}\le 0} P^{\ell}\left(S^j_{M^j}=y^j\ ,j=1,2\right)$$

is finite, we can combine the estimates of (2.17) (notice that the expectation in this relation is independent of the starting point) and (2.18) to obtain that the left hand side of (2.16) dominates (up to constants) the left hand side of (2.14). □

3. Mass Gap Estimate

Let us define define the following random variables

$$\begin{aligned}
&M_0 := 0, \qquad \eta_0 := 0\\
&\eta_1 := \inf\{n : S_n\cdot\hat{\ell} > S_0\cdot\hat{\ell}\}\\
&D := \inf\{n\colon S_n\cdot\hat{\ell} < S_0\cdot\hat{\ell}\}\\
&\overline{M} := \sup\{S_n\cdot\hat{\ell}\colon n\le D\}\\
&M_1 := \sup\{S_n\cdot\hat{\ell}\colon \eta_1\le n\le D\circ\theta_{\eta_1}\}
\end{aligned}$$

and inductively

$$\eta_i := \inf\{n > \eta_{i-1} \colon S_n \cdot \hat{\ell} > M_{i-1}\}$$
$$M_i := \sup\{S_n \cdot \hat{\ell} \colon \eta_i < n < D \circ \theta_{\eta_i}\}.$$

Proposition 3.1. *Let β small enough. Then there exists $\rho = \rho(\lambda) > 0$, such that*

$$\sum_{L \colon L \in \frac{l_1}{r}\mathbb{N}}^{\infty} e^{(m_B(\beta)+\rho)L} I_\beta(L) < \infty. \tag{3.1}$$

Proof. Let us first prove the result for the case, that $\beta = 0$. Let $\rho_0 > 0$ to be specified later.

$$\begin{aligned}
\sum_{L \colon L \in \frac{l_1}{r}\mathbb{N}}^{\infty} e^{\rho_0 L} I_0(L) &= \sum_{L \colon L \in \frac{l_1}{r}\mathbb{N}}^{\infty} \sum_{k=1}^{\infty} e^{\rho_0 L} P^\ell\big(\mathrm{Ir}(T_L; L);\, T_L = \eta_k\big) \\
&= \sum_{L \colon L \in \frac{l_1}{r}\mathbb{N}}^{\infty} \sum_{k=1}^{\infty} E^\ell\Big[e^{\rho_0 \sum_{i=1}^{k-1}(M_i - S_{\eta_i})}\, e^{\rho_0 \sum_{i=1}^{k}(S_{\eta_i} - M_{i-1})}; \\
&\qquad D \circ \theta_{\eta_1} < \cdots < D \circ \theta_{\eta_k} < \eta_{k-1} = T_L\Big] \\
&\le e^{\rho_0} \sum_{L \colon L \in \frac{l_1}{r}\mathbb{N}}^{\infty} \sum_{k=1}^{\infty} E^\ell\Big[e^{\rho_0 \sum_{i=1}^{k-1}(M_i - S_{\eta_i}+1)}; \\
&\qquad D \circ \theta_{\eta_1} < \cdots < D \circ \theta_{\eta_{k-1}} < \eta_k = T_L\Big] \\
&= e^{\rho_0} \sum_{k=1}^{\infty} E^\ell\Big[\, e^{\rho_0 \sum_{i=1}^{k-1}(M_i - S_{\eta_i}+1)}; \\
&\qquad D \circ \theta_{\eta_1} < \cdots < D \circ \theta_{\eta_{k-1}} < \eta_k < \infty\Big] \\
&= e^{\rho_0} \sum_{k=1}^{\infty} E^\ell\Big[e^{\rho_0(1+\overline{M})};\, D < \infty\Big]^{k-1}. \qquad (3.2)
\end{aligned}$$

We now need to show that, for ρ_0 small enough, $E^\ell\left[e^{\rho_0(1+\overline{M})};\, D < \infty\right] < 1$. Since for $\rho_0 = 0$ the last quantity equals $P^\ell\left[\, D < \infty\right] < 1$, it is enough to show that for ρ_0 small enough $E^\ell\left[e^{\rho_0 \overline{M}};\, D < \infty\right] < \infty$. To this end, we estimate the tails $P^\ell\left(\overline{M} > x;\, D < \infty\right)$. Notice, that this event implies that the walk can backtrack below 0, only after it goes beyond level x, that is only after time $[\,x/(\hat{e}_1 \cdot \hat{\ell})\,]$. Therefore, we have

$$\begin{aligned}
P^\ell\big(\overline{M} > x;\, D < \infty\big) &\le \sum_{n > [\,x/(\,\hat{e}_1 \cdot \hat{\ell}\,)\,]} P^\ell(\,D = n) \\
&\le \sum_{n > [\,x/(\,\hat{e}_1 \cdot \hat{\ell}\,)\,]} P^\ell(S_n \cdot \hat{\ell} < 0) < e^{-Cx}, \qquad (3.3)
\end{aligned}$$

where the last inequality, for some $C > 0$, follows from standard large deviation results. This proves that (3.2) is finite, and thus the mass gap estimate for $\beta = 0$. To prove (3.1) for arbitrary small β, we pick $\rho = \rho_0/2$. By Proposition 2.15, we have that for β small enough, $m_B(\beta)+\rho<\rho_0$. Using also the fact that $\mathrm{I}_\beta<\mathrm{I}_0$, we have that (3.1) writes as

$$\sum_{L:\, L\frac{r}{l_1}=1}^{\infty} e^{(m_B(\beta)+\rho)L}\mathrm{I}_\beta(L) < \sum_{L:\, L\frac{r}{l_1}=1}^{\infty} e^{\rho_0 L} I_0(L) < \infty,$$

by the first part of the proof. □

The next proposition proves the mass gap estimate for any arbitrary β. We state it in the case that $\hat{\ell}=\hat{e}_1$.

Proposition 3.2. *Assume, that $\hat{\ell}=\hat{e}_1$. Then, for any $\beta>0$, there exists a $\rho=\rho(\lambda)>0$, such that*

$$\sum_L e^{(\rho+m_B(\beta))\,L} I_\beta(L) < \infty. \tag{3.4}$$

Proof. For the sake of this proof only, P^ℓ_k, k an integer, will denote the distribution of the random walk P^ℓ starting from level k. That is, starting from the hyperplane $\{x\colon x\cdot\hat{\ell}=k\}$. Let us observe the walk that forms the $\mathrm{Ir}(L)$, until the first time it hits level $L-1$. This part of the walk, $(S_n)_{n\le T_{L-1}}$, might have break points, and let $k\in[1,L-1]$ be its first break point. In order to have an $\mathrm{Ir}(L)$, the walk needs to backtrack, in order to cover that break point. Based on this observation, we can write the following renewal equation

$$\mathrm{I}_\beta(L) \le \sum_{k=1}^{L-1} \mathrm{I}_\beta(k)\, P^\ell_k\big(T_{L-1}<D<T_L\big)\mathrm{B}_\beta(L-k+1).$$

We now multiply by $e^{(\rho+m_B(\beta))\,L}$ and sum up the inequality in L.

$$\begin{aligned}\sum_{L=2}^{\infty} e^{(\rho+m_B(\beta))L}\mathrm{I}_\beta(L) &\le \sum_{L=2}^{\infty}\sum_{k=1}^{L-1} e^{(\rho+m_B(\beta))k}\mathrm{I}_\beta(k)\, e^{\rho(L-k)} P^\ell_k\big(T_{L-1}<D<T_L\big)\\ &\qquad\qquad e^{m_B(\beta)(L-k)}\mathrm{B}_\beta(L-k)\\ &= \sum_{k=1}^{\infty} e^{(\rho+m_B(\beta))k}\mathrm{I}_\beta(k) \sum_{L=k}^{\infty} e^{\rho(L-k+1)} P^\ell_k\big(T_L<D<T_{L+1}\big)\\ &\qquad\qquad e^{m_B(\beta)\,(L-k+1)}\mathrm{B}_\beta(L-k+1).\end{aligned}$$

We can now use the inequality $e^{m_B(\beta)\,(L-k+1)}\mathrm{B}_\beta(L-k+1)\le 1$, from proposition 2.14, to get the bound

$$\begin{aligned}\sum_{L=2}^{\infty} e^{(\rho+m_B(\beta))L}\mathrm{I}_\beta(L) &\le \sum_{k=1}^{\infty} e^{(\rho+m_B(\beta))k}\mathrm{I}_\beta(k) \sum_{L=k}^{\infty} e^{\rho(L-k+1)} P^\ell_k\big(T_L<D<T_{L+1}\big)\\ &= \sum_{k=1}^{\infty} e^{(\rho+m_B(\beta))k}\mathrm{I}_\beta(k)\\ &\qquad\qquad \sum_{L=k}^{\infty} e^{\rho(L-k+1)} P^\ell\big(\overline{M}=L-k;\, D<\infty\big)\end{aligned}$$

and as in (3.3), we have that for ρ small enough $\sum_{L=k}^{\infty} e^{\rho(L-k+1)} P^{\ell}\big(\overline{M} = L-k;\ D < \infty\big) := \delta < 1$. Therefore

$$(1-\delta)\sum_{L=2}^{\infty} e^{(\rho+m_B(\beta))L}\mathrm{I}_\beta(L) < \delta e^{(\rho+m_B(\beta))}\mathrm{I}_\beta(1),$$

thus proving our claim. □

4. Markovian Structure

In this section we built a Markovian structure with the purpose to formalise the notion of *direction* that underlies our model, and therefore make the analogy with dierected polymers [4] more transparent. The notion of direction is based on the following regeneration structure. Due to the presence of a drift, or equivalently the presence of a positive λ, the path of the walk includes points, such that after the walk hits them, it does not go below the hyperplane they belong to. In other words the trajectory of the walk consists of a union of irreducible bridges. We formalise this as follows

Definition 4.1. *The measure P^β denotes the distribution of the Markov process $(S(\tau_i), S(\tau_i)\cdot\hat{\ell}, \tau_i)$, with transition probabilities*

$$\begin{aligned}&p^\beta(y_{i+1}, L_{i+1}, n_{i+1};\, y_i, L_i, n_i) := e^{m_B(\beta)(L_{i+1}-L_i\,)}\,\cdot\\ &\quad\cdot E^{\ell}_{y_i}\left[e^{-\Phi_\beta(n_{i+1}-n_i)};\ Ir(n_{i+1}-n_i, L_{i+1}-L_i),\ S(n_{i+1}-n_i) = y_{i+1}\right].\end{aligned}$$

Let us mention, that the notation in the above above definition would had been lighter, if we had considered, equivallently, the Markov process $(S(\tau_i), \tau_i))$. The reason we insist in the above notation is to highlight the levels where the renewals take place. This will make things more transparent later on.

The next proposition shows, that the above kernel is indeed a probability kernel.

Proposition 4.2. *For any β we have that*

$$\sum_{L:\ L\in\frac{l_1}{r}\mathbb{N}}^{\infty}\ \sum_{N=0}^{\infty} e^{m_B(\beta)L}E^{\ell}\left[e^{-\Phi_\beta(N)};\ Ir(N;L)\right] = 1.$$

Proof. Consider $B(L) = \sum_{N=0}^{\infty} E^{\ell}\left[e^{-\Phi_\beta(N)};\ \mathrm{Br}(N;L)\right]$, and decompose the expectation according to the level of the first break point. We then obtain the following renewal equation

$$\mathrm{B}(L) = \sum_{1\le k\le L} \mathrm{I}(k)\,\mathrm{B}(L-k).$$

Define $\mathrm{b}(s) := \sum_{L\ge 0} s^L e^{m_B(\beta)L} B(L)$ and $\mathrm{i}(s) := \sum_{L\ge 1} s^L e^{m_B(\beta)L} I(L)$, for $0 < s < 1$. Then the previous equation can be transformed to the equation

$$\mathrm{b}(s) = 1 + \mathrm{b}(s)\,\mathrm{i}(s). \tag{4.1}$$

By proposition 2.14 we have that $\mathrm{b}(1) = \infty$, and so (4.1) implies that $\mathrm{i}(1) = 1$, or that $\sum_{L:\ L\in\frac{l_1}{r}\mathbb{N}}^{\infty}\sum_{N=0}^{\infty} e^{m_B(\beta)L}E^{\ell}\left[e^{-\Phi(N)};\ \mathrm{Ir}(N;L)\right] = 1$ □

The mass gap shows that $S(\tau_1)\cdot\hat{\ell}$ has exponential moments under P^β. The next proposition shows that τ_1 has also exponential moments. The proof follows the lines of the moment estimates of the regeneration times of random walks in random environment [3].

Proposition 4.3. *For β small enough, there exists a constant c_1 such that*

$$P^\beta\left(\tau_1 > u\right) \le e^{-c_1 u}.$$

Proof. Consider h as defined in (2.9), then

$$P^\beta\left(\tau_1 > u\right) \le P^\beta\left(S(\tau_1)\cdot\hat{\ell} > \frac{h}{2}u\right) + P^\beta\left(\tau_1 > u,\, S(\tau_1)\cdot\hat{\ell} \le \frac{h}{2}u\right).$$

Regarding the first term we have that

$$P^\beta\left(S(\tau_1)\cdot\hat{\ell} > \frac{h}{2}u\right) \le e^{-\rho\frac{h}{2}}E^\beta\left[e^{\rho S(\tau_1)\cdot\hat{\ell}}\right] \le Ce^{-\rho\frac{h}{2}u},$$

by the exponential mass gap, Proposition 3.1.

Regarding the second term we have that

$$\begin{aligned} P^\beta\left(\tau_1 > u,\, S(\tau_1)\cdot\hat{\ell} \le \frac{h}{2}u\right) &= \sum_L\sum_{N=1}^\infty 1_{N>u,\, L\le\frac{h}{2}u}\, e^{m_B(\beta)L}E^\ell\big[e^{-\Phi_\beta(N)};\, \mathrm{Ir}(N;L)\big] \\ &\le e^{\frac{h}{2}m_B(\beta)u}\sum_{L\le\frac{h}{2}u}\sum_{N>u}P^\ell\big(\mathrm{Br}(N,L)\big) \\ &\le e^{\frac{h}{2}m_B(\beta)u}P^\ell\big(S(u)\cdot\hat{\ell}\le\frac{h}{2}u\big) \end{aligned}$$

On the other hand $N_n\cdot\hat{\ell} := S(n)\cdot\hat{\ell} - S(0)\cdot\hat{\ell} - \sum_{i=1}^{n-1}E^\ell_{S_{i-1}}(S_i - S_{i-1})\cdot\hat{\ell}$ is a P^ℓ-martingale. By (2.9) we have that increments of $N_n\cdot\hat{\ell}$ satisfy $\big|N_n\cdot\hat{\ell} - N_{n-1}\cdot\hat{\ell}\big| \le 1+h$ - recall that under P^ℓ, $S(0)=0$. Moreover, on the set $\{S(u)\cdot\hat{\ell}\le\frac{h}{2}u\}$ we have that $N_u\cdot\hat{\ell}\le-\frac{h}{2}u$.

By Azuma's inequality [1] we have that

$$e^{\frac{h}{2}m_B(\beta)u}P^\ell\big(S(u)\cdot\hat{\ell}\le\frac{h}{2}u\big) \le e^{\frac{h}{2}m_B(\beta)u}P^\ell\big(N_u\cdot\hat{\ell}\le-\frac{h}{2}u\big) \le e^{\frac{h}{2}m_B(\beta)u}e^{-\frac{h^2}{32(1+h)}\frac{(u-2)^2}{u}},$$

and this implies the result for β small enough, since $m_B(\beta)$ tends to 0, as β tends to 0. □

The following corollary generalises the mass gap estimate and it will also be usefull.

Corollary 4.4. *For β and θ small enough, we have that*

$$E^\beta\left[e^{\theta|S(\tau_1)|}\right] < \infty.$$

Proof. The proof follows easily from the previous proposition and the observation that $|S(\tau_1)|\le\tau_1$. □

5. Second To First Moment Estimate

We are now moving towards the proof of the main estimate (2.16). We will need to consider two independent copies of the walks with distributions $P^\ell_{y_1}$, $P^\ell_{y_2}$. We will denote the two paths by (S^1_n) and (S^2_n), respectively. We denote their joint distribution by $P^\ell_{y_1,y_2}$. These two copies will then naturally give rise to two independent copies of the Markovian process defined in Definition 4.1. We will denote this joint distribution by $P^\beta_{y_1,y_2}$. Let τ^1_i and τ^2_i as defined in Definition 4.1, corresponding to the two independent Markovian copies. We then define

$$\begin{aligned} \iota^1(N_1) &:= \inf\{n\colon \sum_{i=1}^n \tau^1_i \geq N_1\} \\ \iota^2(N_2) &:= \inf\{n\colon \sum_{i=1}^n \tau^2_i \geq N_2\}. \end{aligned} \tag{5.1}$$

and

$$\zeta^1_i := 1_{\{\exists j\colon \mathrm{Ir}(\tau^1_i;\ell_i)\cap \mathrm{Ir}(\tau^2_j;\ell'_j)\neq\emptyset\}}. \tag{5.2}$$

The random variable ζ^1_i indicates whether the two copies intersect within an irreducible bridge. $\mathrm{Ir}(\tau^1_i;\ell^1_i)$ will denote the $i-th$ irreducible bridge for the first walk, and similarly $\mathrm{Ir}(\tau^2_j;\ell^2_j)$ the $j-th$ irreducible bridge for the second walk.

Lemma 5.1. *Consider two walks $(S^1_n)_{n\leq N_1}$ and $(S^2_n)_{n\leq N_2}$. Let $\tilde\Phi^{(2)}(N_1,N_2)$ and $\Phi^{(2)}(N_1,N_2)$ as defined in Definition 2.7. Then if $\alpha(\beta) := 2\beta\|V_\omega\|$, we have*

$$-\tilde\Phi^{(2)}(N_1,N_2) \leq \alpha(\beta)\sum_{i=1}^{\iota(N_1)} \tau^1_i\,\zeta^1_i - \Phi^{(2)}(N_1,N_2)$$

Proof. Let $\mathcal{L}^1_{N_1}(\cdot)$ and $\mathcal{L}^2_{N_2}(\cdot)$ be the local times of the paths $(S^1_n)_{n\leq N_1}$ and $(S^2_n)_{n\leq N_2}$, respectively. Let also $(V_\omega(x))_{x\in\mathbb{Z}^d}$ and $(V'_\omega(x))_{x\in\mathbb{Z}^d}$ two independent copies of the disorder. We have by Definition 2.7 that

$$-\tilde\Phi^{(2)}(N_1,N_2) = \sum_{x\in\mathbb{Z}^d} \log \mathbb{E}\,\exp\Big(-\beta V_\omega(x)\big(\mathcal{L}^1_{N_1}(x)+\mathcal{L}^2_{N_2}(x)\big)\Big).$$

Moreover, by adding and subtracting the term $\beta V'_\omega(x)\mathcal{L}^1_{N_1}(x)$, we have that

$$\begin{aligned} \log \mathbb{E}\,\exp\Big(&-\beta V_\omega(x)\big(\mathcal{L}^1_{N_1}(x)+\mathcal{L}^2_{N_2}(x)\big)\Big) \\ &= \log\mathbb{E}\,\exp\Big(-\beta\,(V_\omega(x)-V'_\omega(x))\mathcal{L}^1_{N_1}(x)\Big)\cdot \\ &\qquad \exp\Big(-\beta V'_\omega(x)\mathcal{L}^1_{N_1}(x)-\beta V_\omega(x)\mathcal{L}^2_{N_2}(x)\Big) \\ &\leq \log e^{\alpha(\beta)\mathcal{L}^1_{N_1}(x)}\mathbb{E}e^{-\beta\,V_\omega(x)\mathcal{L}^1_{N_1}(x)}\mathbb{E}e^{-\beta\,V_\omega(x)\mathcal{L}^2_{N_2}(x)} \\ &= \alpha(\beta)\mathcal{L}^1_{N_1}(x) + \log\mathbb{E}e^{-\beta\,V_\omega(x)\mathcal{L}^1_{N_1}(x)} + \log\mathbb{E}e^{-\beta\,V_\omega(x)\mathcal{L}^2_{N_2}(x)}, \end{aligned}$$

where $\alpha(\beta) = 2\beta\|V_\omega\|$. Repeating the same calculation with N_1 and N_2 interchanged we obtain that

$$-\tilde\Phi^{(2)}(N_1,N_2) \leq \alpha(\beta)\sum_{x\in\mathbb{Z}^d}\mathcal{L}^1_{N_1}(x)\wedge\mathcal{L}^2_{N_2}(x) - \Phi(N_1) - \Phi(N_2) \tag{5.3}$$

Let τ_i^1 and τ_i^2 be the time durations of the i^{th} irreducible bridge for the walks S^1 and S^2, respectively.

Then we have that

$$\sum_{x\in\mathbb{Z}^d} \mathcal{L}^1_{N_1}(x) \wedge \mathcal{L}^2_{N_2}(x) \le \sum_{i=1}^{\iota(N_1)} \tau_i^1\, \zeta_i^1, \tag{5.4}$$

that is, the total amount of time, that the first walk spends on sites visited also by the second walk, is bounded by the total time of the irreducible bridges of the first walk, inside which, it intersects with the second walk. The result now follows from (5.3) and (5.4). □

Let us point out that if ζ_i^1, was equal to zero for every $i \ge 1$, then we see from the previous proposition, that the estimate (2.16) holds trivially. This, of course, would not be the case, and so the main point is to be able control the frequency of the event $\zeta_i^1 > 0$ and the exponential moments of the duration of the corresponding irreducible bridges. This is summarised in proposition 5.3, which follows.

We denote first by

$$\sigma_1^1 = \inf\{i \ge 1 \colon \zeta_i^1 > 0\}, \tag{5.5}$$

and in a similar way we define σ_1^2.

We will also need the following general estimate on Green's function, which is proven in [3]

Proposition 5.2. *([3]) Let $p(x)$, $x \in \mathbb{Z}^d$ a probability distribution on $\mathbb{Z}^d$, with covariance matrix Σ_p. Let p_n denote the n-fold convolution of it and*

$$G(z) := \sum_{i,j\ge 0} \sum_{x\in\mathbb{Z}^d} p_i(x) p_j(x+z).$$

If $d \ge 4$ and there are constants $\gamma_1, \gamma_2, \gamma_3 > 0$, such that

$$\sum_{x\in\mathbb{Z}^d} p(x) e^{\gamma_1 |x|} < \infty, \qquad \Sigma_p \ge \gamma_2 Id$$

$$|\sum_{x\in\mathbb{Z}^d} x p(x)| \ge \gamma_3$$

then there exist constants $K_1(d,\gamma_1,\gamma_2)$, $K_2(d,\gamma_1,\gamma_2,\gamma_3)$, such that

$$\sup_{x\in\mathbb{Z}^d} p_n(x) < K_2\, n^{-d/2}$$

$$\sup_{x\in\mathbb{Z}^d} (1+|x|^{\frac{d-3}{2}}) G(x) < K_3$$

It is clear that the distribution $p(x) := P^\beta(S(\tau_1) = x)$ is nondegenerate, so the covariance matrix satisfies $\Sigma_p \ge \gamma_2\, Id$ (Id is the identity matrix). Also , that $|\sum_{x\in\mathbb{Z}^d} xp(x)| \ge \gamma_3$ for appropriate γ_3, and by Corollary 4.4, $\sum_{x\in\mathbb{Z}^d} p(x) e^{\gamma_1|x|} < \infty$. Therefore, the previous proposition is applicable, and this will be usefull in the following proposition.

Proposition 5.3. *For β small enough it holds that*

$$\sup_{y^1,y^2} E^\beta_{y^1,y^2}\left[e^{\alpha(\beta)\tau^1_{\sigma_1^1}};\, \sigma_1^1 < \infty \right] < 1. \tag{5.6}$$

Proof. We decompose the expectation with respect to the positions of the random walks at the beginning of the irreducible bridges, inside which they intersect. Using also the Markov property we have that

$$E^{\beta}_{y^1,y^2}\left[e^{\alpha(\beta)\tau^1_{\sigma^1_1}};\, \sigma^1_1<\infty\right]=$$

$$=\sum_{m^1,m^2=1}^{\infty} E^{\beta}_{y^1,y^2}\left[e^{\alpha(\beta)\tau^1_{m^1}};\, \sigma^1_1=m^1,\sigma^2_1=m^2\right]$$

$$=\sum_{m^1,m^2=1}^{\infty}\sum_{x^1,x^2} P^{\beta}_{y^1,y^2}\big(\, S^1(m^1)=x^1, S^2(m^2)=x^2\,\big)\, E^{\beta}_{x^1,x^2}\left[e^{\alpha(\beta)\tau^1_1};\, \zeta^1_1>0\right]. \tag{5.7}$$

We now use the fact, that the event $\zeta^1_1>0$ implies that $\tau^1_1+\tau^2_1\geq |x^1-x^2|$. In other words if the walks starting at x^1,x^2 intersect inside their first irreducible bridges then the total duration of these bridges has to be greater than their initial distance. Then (5.7) is estimated by -we also use the fact that τ^1_1 and τ^2_1 have the same distribution-

$$\sum_{m^1,m^2=1}^{\infty}\sum_{x^1,x^2} P^{\beta}_{y^1,y^2}\big(\, S^1(m^1)=x^1, S^2(m^2)=x^2\,\big)\, e^{-\alpha(\beta)\,|x^1-x^2|}$$
$$E^{\beta}_{x^1,x^2}\left[e^{\alpha(\beta)\tau^1_1}\, e^{\alpha(\beta)(\tau^1_1+\tau^2_1)};\, \zeta^1_1>0\right]$$
$$\leq \sum_{m^1,m^2=1}^{\infty}\sum_{x^1,x^2} P^{\beta}_{y^1,y^2}\big(\, S^1(m^1)=x^1, S^2(m^2)=x^2\,\big)\, e^{-\alpha(\beta)\,|x^1-x^2|}$$
$$E^{\beta}_{x^1,x^2}\left[e^{4\alpha(\beta)\tau^1_1};\, \zeta^1_1>0\right]$$
$$=\sum_{m^1,m^2=1}^{\infty}\sum_{x,z} P^{\beta}_{y^1,y^2}\big(\, S^1(m^1)=x, S^2(m^2)=x+z\,\big)\, e^{-\alpha(\beta)\,|z|}$$
$$E^{\beta}_{x,x+z}\left[e^{4\alpha(\beta)\tau^1_1};\, \zeta^1_1>0\right]$$

By Proposition 4.3, for β small enough, we can bound the above by

$$C\sum_{m^1,m^2=1}^{\infty}\sum_{x,z} P^{\beta}_{y^1,y^2}\big(\, S^1(m^1)=x, S^2(m^2)=x+z\,\big)\, e^{-\alpha(\beta)\,|z|}.$$

Define

$$G(z;y_1,y_2):=\sum_{m^1,m^2=1}^{\infty}\sum_{x} P^{\beta}_{y^1,y^2}\big(\, S^1(m^1)=x, S^2(m^2)=x+z\,\big).$$

Thus, we have obtained that

$$E^{\beta}_{y^1,y^2}\left[e^{\alpha(\beta)\tau_{\sigma^1_1}};\, \sigma^1_1<\infty\right]\leq \sum_{z} e^{-\alpha(\beta)\,|z|}G(z;y_1,y_2)<\infty,$$

by Proposition 5.2. Therefore we can apply the dominated convergence in (5.7) to obtain that

$$\lim_{\beta\to 0} E^{\beta}_{y^1,y^2}\left[e^{\alpha(\beta)\tau_{\sigma^1_1}};\, \sigma^1_1<\infty\right]$$
$$=P^{0}_{y^1,y^2}\left[\,\sigma^1_1<\infty\right]=1-P^{\ell}_{y^1,y^2}\big(\text{the walks do not intersect}\big)<1,$$

unifromly in y^1, y^2, since $d > 3$. This clearly implies that for β small enough (5.6) is valid. ☐

We are finally ready for the proof of the main estimate.

Proposition 5.4. *Uniformly on $y^1, y^2 \in \mathbb{Z}^d$ and β small enough, we have that*

$$\sup_L \frac{\sum_{N^1,N^2} E^{\ell}_{y^1,y^2}\left[e^{-\tilde{\Phi}^{(2)}_{\beta}(N^1,N^2)};\, Br(N^1;L)\cap Br(N^2;L)\right]}{\sum_{N^1,N^2} E^{\ell}_{y^1,y^2}\left[e^{-\Phi^{(2)}_{\beta}(N^1,N^2)};\, Br(N^1;L)\cap Br(N^2;L)\right]} < \infty. \tag{5.8}$$

Proof. Proposition 2.14 implies that the left hand side of (5.8) is bounded by

$$Ce^{2m_B(\beta)L} \sum_{N^1,N^2=1}^{\infty} E^{\ell}_{y^1,y^2}\left[e^{-\tilde{\Phi}^2_{\beta}(N^1,N^2)};\, \mathrm{Br}(N^1;L)\cap \mathrm{Br}(N^2;L)\right] \tag{5.9}$$

Moreover, we have the decomposition

$$\mathrm{Br}(N^j;L) = \bigcup_{k=1}^{\infty} \bigcup_{\substack{(\ell^j_i)_{i\le k}\\ \ell_1+\cdots+\ell_k=L}} \bigcup_{(n^j_i)_{i\le k}} \bigcap_{i=1}^{k} \mathrm{Ir}(n^j_i;\ell^j_i) \tag{5.10}$$

for $j = 1, 2$. Using this fact, Lemma 5.1 and Proposition 2.8 (iii) we have that (5.9) (we drop the constant C) can be estimated by

$$\sup_L \sum_{k^1,k^2=1}^{\infty} \sum_{\substack{(\ell^j_i)_{i\le k^j}\\ \ell^j_1+\cdots+\ell^j_{k^j}=L}} \sum_{N_1,N_2} \sum_{\substack{(n^j_i)_{i\le k^j}\\ n^j_1+\cdots+n^j_{k^j}=N^j}} E^{\ell}_{y_1,y_2}\Big[e^{\alpha(\beta)\sum_i^{k^1} n^1_i\,\zeta^1_i}$$
$$\prod_{j=1,2}\prod_{i=1}^{k^j} e^{m_B(\beta)\ell^j_i-\Phi_\beta(n^j_i)}\,\mathrm{Ir}(n^j_i;\ell^j_i)\Big]$$

$$= \sup_L \sum_{k^1,k^2=1}^{\infty} E^{\beta}_{y_1,y_2}\Big[e^{\alpha(\beta)\sum_i^{k^1}\tau^1_i\,\zeta^1_i};\, \sum_{i=1}^{k^j}\ell^j_i = L,\, j=1,2\Big] \tag{5.11}$$

Denote by $\sigma^j_1 = \inf\{i\colon \zeta^j_i > 0\}$. In the case that $\sigma^1_1 > k^1$, the last expectation is equal to

$$P^{\beta}_{y_1,y_2}\Big[\sigma^j_1 > k^j;\, \sum_{i=1}^{k^j}\ell^j_i = L,\, j=1,2\Big] \le 1.$$

So it follows that (5.11) is bounded by

$$1 + \sup_L \sum_{k^1,k^2=1}^{\infty} \sum_{s^j\le k^j,\, j=1,2} E^{\beta}_{y_1,y_2}\Big[e^{\alpha(\beta)\sum_i^{k^1}\tau^1_i\,\zeta_i};\, \sigma^j_1 = s^j,\, j=1,2;\, \sum_{i=1}^{k^j}\ell^j_i = L,\, j=1,2\Big]$$

For any $y_1, y_2 \in \mathbb{Z}^d$, denote by

$$\Psi(y_1,y_2) := \sup_L \sum_{k^1,k^2=1}^{\infty} E^{\beta}_{y_1,y_2}\Big[e^{\alpha(\beta)\sum_i^{k^1}\tau^1_i\,\zeta_i};\, \sum_{i=1}^{k^j}\ell^j_i = L,\, j=1,2\Big] \tag{5.12}$$

and $\|\Psi\| := \sup_{y_1,y_2\in\mathbb{Z}^d} \Psi(y_1,y_2)$. Use the Markov property of the coupled measure $P^\beta_{y_1,y_2}$ at (s^1,s^2) to write the expectation as

$$1+\sup_L \sum_{k^1,k^2=1}^{\infty} \sum_{s^j\le k^j} E^\beta_{y_1,y_2}\Big[e^{\alpha(\beta)\tau^1_{s^1}};\ \sigma^j_1=s^j,\ j=1,2$$

$$E^\beta_{S^1(\tau^1_{s^1}),S^2(\tau^2_{s^2})}\big[e^{\alpha(\beta)\sum_{i=s^1+1}^{k^1}\tau^1_i\,\zeta_i};\ \sum_{i=s^j+1}^{k^j}\ell^j_i = L-\sum_{i=1}^{s^j}\ell^j_i\,\big]\Big]$$

$$\le 1+\|\Psi\|\sum_{s^j;\,j=1,2} E^\beta_{y_1,y_2}\Big[e^{\alpha(\beta)\tau^1_{s^1}};\ \sigma^j_1=s^j,\ j=1,2\Big]$$

$$=1+\|\Psi\|E^\beta_{y_1,y_2}\Big[e^{\alpha(\beta)\tau_{\sigma^j_1}};\ \sigma^j_1<\infty,\ j=1,2\Big],$$

where we used the fact that

$$\sup_L\sum_{k^j\ge s^j} E^\beta_{z^1,z^2}\big[e^{\alpha(\beta)\sum_{i=s^1+1}^{k^1}\tau^1_i\,\zeta_i};\ \sum_{i=s^j+1}^{k^j}\ell^j_i=L-\sum_{i=1}^{s^j}\ell^j_i\,\big]=\Psi(z^1,z^2).$$

Thus we have obtained the estimate

$$\Psi(y_1,y_2)\le 1+\|\Psi\|E^\beta_{y_1,y_2}\Big[e^{\alpha(\beta)\tau_{\sigma^j_1}};\ \sigma^j_1<\infty,\ j=1,2\Big].$$

Since y_1,y_2 are arbitrary we have that

$$\|\Psi\|\le 1+\|\Psi\|\sup_{y_1,y_2} E^\beta_{y_1,y_2}\Big[e^{\alpha(\beta)\tau_{\sigma^j_1}};\ \sigma^j_1<\infty,\ j=1,2\Big].$$

and by Proposition 5.3, it follows that $\|\Psi\|<\infty$ for small enough β. This concludes the proof. ☐

Proof of Theorem 1.1. It follows from Proposition 5.4, Proposition 2.17 and Proposition 2.16.

6. Some Consequences

Let us mention in this paragraph some consequences of Theorem 1.1.

It is in general difficult to obtain asymptotic properties of Green's function for random walks in a potential - not necessarily random. As far as we know, the only case that this has been successful is the case when the potential is constant [11]. In this case it has been computed that

Theorem 6.1. *(Zerner) Suppose that the potential V_ω is identically equal to* 0 *and let $\alpha_\lambda(\cdot)$, $\lambda>0$ the corresponding Lyapounov norm. Then for $x=(x_1,\dots,x_d)\in\mathbb{R}^d$ we have that*

$$\alpha_\lambda(x)=\sum_{i=1}^d x_i\sinh^{-1}(x_i s),$$

where $s>0$ solves the equation

$$e^\lambda d=\sum_{i=1}^d\sqrt{1+(x_i s)^2}.$$

The situation where the potential is inhomogeneous becomes much more involved. In the annealed case, one expects that the random walk behaves as if it was in a constant, averaged potential. Still, though, this correspondence is non trivial. In the low disorder regime, W.M. Wang [10] has obtained an asymptotic expansion, with respect to the disorder β, of the annealed Lyapounov norm. Using supersymmetric methods, she obtained that in the low disorder regime the annealed Lyapounov norm becomes asymptotic to the Lyapounov norm of a walk in a constant potential. Let $G_\lambda(x,y)$ denote the Green's function corresponding to a constant potential $-\lambda$. The translation of the main result in [10] is the following

Theorem 6.2. *(Wang) For every $\lambda > 0$ there exists a β_0, such that for $0 < \beta < \beta_0$ and every $x, y \in \mathbb{Z}^d$,*

$$\log \mathbb{E}G_\lambda(x,y,\omega) = \log G_{\tilde{\lambda}}(x,y) + O(\gamma^4(\beta))(|x-y|+1),$$

where

$$\tilde{\lambda} = \log\left(\frac{1}{2d}\tilde{e}\right),$$

with

$$\tilde{e} := 2de^\lambda - \gamma(\beta)\mathbb{E}[v] - \gamma^2(\beta)\mathbb{E}[(v-\mathbb{E}v)^2]\, G^2_{\lambda_1}(x,y) - \gamma^3(\beta)\mathbb{E}[(v-\mathbb{E}v)^3]\, G^3_{\lambda_1}(x,y)$$

$$\lambda_1 := \log\frac{1}{2d}(2d\, e^\lambda - \gamma(\beta)\mathbb{E}v)$$

and $\gamma(\beta)$ and v defined by the relation

$$\beta V(x) := -\log(2de^\lambda) + \log(2d\, e^\lambda - \gamma(\beta)\, v(x)), \quad x \in \mathbb{Z}^d$$

Notice that $\gamma(\beta) \sim 0$, as $\beta \sim 0$. Moreover, the importance of the above theorem is that the expansion is uniform as $|x-y| \to \infty$.

Our Theorem 1.1 can be combined with Theorems 6.1 and 6.2 to provide an asymptotic expression of the quenched Lyapounov norms $\alpha_\lambda(\cdot)$ in the low disorder regime.

Corollary 6.3. *For every $\lambda > 0$, there exists a β_*, such that for every $0 < \beta < \beta_*$*

$$\alpha_\lambda(x) = \beta_\lambda(x) = \hat{\alpha}_{\tilde{\lambda}}(x) + O(\gamma^4(\beta)),$$

where

$$\hat{\alpha}_{\tilde{\lambda}}(x) := \lim_{N\to\infty}\frac{1}{N}\log G_{\tilde{\lambda}}(0, Nx).$$

Finally, Corollary 6.3 in combination with Corollary of Theorem 7.2 in [10], shows that the quenched Lyapounov norm α_λ can be extended as an analytic function in λ in the right half plane of the complex plane. This answers, in the case when $d > 3$ and β small enough, another question posed in [9].

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