BEHAVIOR OF THE SOLUTION OF A RANDOM SEMILINEAR HEAT EQUATION

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ABSTRACT. We consider a semilinear heat equation in one space dimension, with a random source at the origin. We study the solution, which describes the equilibrium of this system, and prove that, as the space variable tends to infinity, the solution becomes a.s. asymptotic to a steady state. We also study the fluctuations of the solution around the steady state.

1. INTRODUCTION.

One of the natural questions to ask in the study of random partial differential equations is the effect of randomness on the long term or asymptotic behavior. For linear equations, diffusions in a random environment is a typical case and the large time behavior has a long history. The case when the motion in the environment is either reversible or balanced it has been investigated in [5] and [6] among others. On the other hand even for the simplest of nonlinear equations the effect of random initial or boundary data, at locations far away from the random input is hard to analyze. Turbulence, where "far away" is in the frequency domain, has been a particularly challenging problem, with some recent progress. In [2], for the Navier-Stokes equation with random forcing in two space dimensions, the existence of a unique equilibrium is proved, though very little information is provided about its properties.

In this article we provide a detailed description of the equilibrium state of a randomly perturbed, nonlinear system. In particular, we consider a nonlinear heat equation in one space dimension, with a source at the origin that is random in time and with quadratic dissipation. There is a unique stochastic stationary state where there is equilibrium between the source and dissipation. The solution, which describes this state, decays far away from 0 in space and we are concerned here with the effect of randomness in the source at the origin on the decay at infinity of the solution.

We will consider the heat equation on $R \times R$, with a quadratic dissipative term and a stationary random source at the origin.

$$u_t + u_{xx} - u^2 + \lambda(t)\delta_0(x) = 0, \ -\infty < x < \infty, \ -\infty < t < \infty \tag{1.1}$$

where $\lambda(t) = \lambda(t,\omega) = \lambda(\theta_t\omega)$ is a stationary process represented through an ergodic measure-preserving action of the translation group $\omega \to \theta_t\omega$ acting on a probability

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space $(\Omega, \mathcal{F}, P)$. We assume, in order to simplify the analysis, that $\lambda(t)$ satisfies $0 < \lambda_1 \le \lambda(t) \le \lambda_2 < \infty$.

We can think of the solution of this equation as describing, under suitable rescaling, the density of a system of annihilating Brownian particles. The particles appear at the origin with an intensity determined by the source term $\lambda(t)$, performing subsequently independent Brownian motions and they are killed as a result of binary collisions resulting in the term $-u^2$. This fact has been made rigorous in the case of dimension greater or equal to two and in the absence of the source term in [7].

It is worth pointing out that similar methods apply to nonlinear equations of the form

$$u_t + u_{xx} - u^p + \lambda(t)\delta_0(x) = 0,\ -\infty < x < \infty,\ -\infty < t < \infty.$$

or even more generally to equations of the form

$$u_t + u_{xx} - f(u) + \lambda(t)\delta_0(x) = 0,\ -\infty < x < \infty,\ -\infty < t < \infty.$$

with $f(u) \simeq u^p$ as $u \to 0$.

It is not hard to show that there is a unique positive and bounded solution $u(t,x,\omega)$ of the above equation on $R \times R \times \Omega$, and this solution is covariant in the sense that $u(t+\tau, x, \omega) = u(t, x, \theta_\tau \omega)$. The sketch of this proof goes as follows (see [9] for details). Since the unique solution will be symmetric around $x = 0$, it is more convenient to consider it as a solution of

$$u_t + u_{xx} - u^2 = 0,\ 0 < x < \infty,\ -\infty < t < \infty \tag{1.2}$$

with Neumann boundary data i.e. $u_x(t,0) = -\frac{1}{2}\lambda(t)$. The solution is smooth for $x > 0$. To prove its existence we only need to notice that

a) If we prescribe terminal data, say $u(T,x) = 0$ at $t = T$, then the solutions $u^T(t,x) \uparrow$ as $T \uparrow \infty$ by the maximum principle.

b) By the maximum principle, the solutions $u^T(t,x)$ are all dominated by any positive solution of

$$u_t + u_{xx} - u^2 = 0; \quad u_x(t,0) = -\frac{1}{2}\lambda_2 \tag{1.3}$$

c) There are positive solutions of equation (1.3), that do not depend on t, given by

$$u_a(x) = \frac{6}{(x+a)^2}; \quad a > 0$$

satisfying

$$u_{xx} - u^2 = 0, \quad u_x(t,0) = -\frac{12}{a^3} \tag{3}$$

Therefore $u_a(x)$, with $a = (\frac{24}{\lambda_2})^{\frac{1}{3}}$ is a positive solution of equation (1.3) which provides the necessary bound to prove existence. Uniqueness within the class of bounded solutions is easily established. Again by the maximum principle it is not hard to see that if $\lambda(t)$ satisfies the bounds $0 < \lambda_1 \le \lambda(t) \le \lambda_2 < \infty$, then

$$\frac{6}{(x+a_1)^2} \le u(t,x) \le \frac{6}{(x+a_2)^2}$$

where $a_i = (\frac{24}{\lambda_i})^{\frac{1}{3}}$. We write

$$u(t,x) = \frac{6}{(x+a(t,x))^2}$$

The goal of this article is to establish the following two theorems.

Theorem 1.1. *If $\lambda(\cdot)$ is ergodic then there exists a constant $\bar{a}$ such that*

$$\lim_{x\to\infty} a(0,x) = \bar{a}$$

exists P almost surely.

For the second theorem we will assume that there exists $0 < A < \infty$, such that $\lambda(\cdot)$ has the property $\{\lambda(s) : s \leq a\}$ and $\{\lambda(s) : s \geq b\}$ are independent if $b-a \geq A$. We will use the standard notation $E^P[\ ]$ to denote expectation with repect to P. This condition is stronger than what we need, but we assume it in order to simplify the proofs.

Theorem 1.2. *If in addition $\lambda(\cdot)$, satisfies the mixing condition stated above, then as $x \to \infty$, $x(a(0,x) - \bar{a})$ has a limiting normal distribution with mean 0 and variance*

$$\sigma^2 = \lim_{x\to\infty} E^P[x^2(a(0,x)-\bar{a})^2]$$

We can show that $\sigma^2 > 0$ for a large class of examples.

Notice, that Theorems 1.1 and 1.2 imply that the solution $u(t,x)$ has, for every t, the following asymptotic expansion in x, for x tending to infinity :

$$u(t,x) \sim \frac{6}{x^2} - \frac{12\,\bar{a}}{x^3} + \frac{12\,G(t)}{x^4},$$

with $G(t)$ a gaussian random variable.

Idea of proof:

We first write the equation satisfied by a. After some calculation

$$u_t + u_{xx} - u^2 = 0$$

becomes

$$a_t + a_{xx} - \frac{6a_x}{(x+a)} - \frac{3a_x^2}{(x+a)} = 0 \tag{1.4}$$

The operator

$$a_{xx} - \frac{6a_x}{(x+a)} - \frac{3a_x^2}{(x+a)}$$

is viewed as a perturbation of the Bessel operator

$$a_{xx} - \frac{6}{x}a_x$$

by a Feynman-Kac term $c(t,x)$ and we write equation (1.4) as

$$a_t + a_{xx} - \frac{6}{x}a_x - c(t,x)a = 0 \tag{1.5}$$

where c is given by

$$c(t,x) = \frac{3a_x^2}{a(x+a)} - \frac{6a_x}{x(x+a)} \tag{1.6}$$

We will consider the equation (1.5), in a domain $(t,x) \in (-\infty,\infty)\times[\ell,\infty)$. If $\ell > 0$, there are no singularities. We will have to choose ℓ to be sufficiently large, and this choice will be made later on.

Let us consider the space time Bessel process $Q_{t,x}$ with generator

$$\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} - \frac{6}{x}\frac{\partial}{\partial x}$$

starting from (t,x) and the exit time τ_ℓ from (ℓ,∞),

$$\tau_\ell = \inf\{s : x(s) \le \ell\}.$$

Then

$$a(t,x,\omega) = E^{Q_{t,x}}\Big[a(\tau_\ell,\ell,\omega)R(t,\tau_\ell,\omega)\Big]$$

where

$$R(t,\sigma,\omega) = \exp[-r(t,\sigma,\omega)]$$

is the Feynman-Kac term, with

$$r(t,\sigma,\omega) = \int_t^\sigma c(s,x(s),\omega)ds.$$

We will in Theorem 6.2, prove a uniform bound of the form $\sup_{x,t,\omega}|xa_x| \le C$. This in turn will imply a uniform bound of the form $\sup_{x,t,\omega} x^3|c(t,x)| \le C$. We can then control $E^{Q_{t,x}}[R(t,\tau_\ell)]$ in terms of $E^{Q_{t,x}}[\exp[\int_t^{\tau_\ell} V(x(s))ds]$ with $V(x) = \frac{C}{x^3}$, which in turn can be controlled if $\ell = \ell(C)$ is large enough. The expectation is with respect to the Bessel process $Q_{t,x}$ and $R(t,\tau_\ell) = R(t,\tau_\ell,\omega)$ depends on the random boundary conditions $\lambda(t,\omega)$ at $x = 0$. If we define, for a fixed suitably large ℓ, the conditional expectation

$$g(t,x,\tau,\ell,\omega) = E^{Q_{t,x}}\big[R(t,\tau_\ell,\omega)\big|\tau_\ell = \tau\big] = E^{Q^{\tau,\ell}_{t,x}}\big[R(t,\tau,\omega)\big] \tag{1.7}$$

where $Q^{\tau,\ell}_{t,x}$ is the Bessel Bridge conditioned to exit from $[\ell,\infty)$ at time τ, then we can represent

$$a(t,x,\omega) = E^{Q_{t,x}}\big[a(\tau_\ell,\ell,\omega)g(t,x,\ell,\tau_\ell,\omega)\big]$$

We will show that, for fixed ℓ, as $x\to\infty$ and $t\to-\infty$ so long as $|t|\sim x^2$, $g(t,x,\tau,\ell,\omega)$ is nearly independent of t,x and can be approximated by a function $h(\tau,\ell,\omega)$ of τ which is covariant, i.e. $h(\tau,\ell,\omega) = h(0,\ell,\theta_\tau\omega)$. It will then follow that

$$a(0,x,\omega) \simeq E^{Q_{0,x}}\big[a(\tau_\ell,\ell,\omega)h(\tau_\ell,\ell,\omega)\big] = \int a(\tau,\ell,\omega)h(\tau,\ell,\omega)p(x,\ell,\tau)d\tau \tag{1.8}$$

where $p(x,\ell,\tau)$ is the density of the hitting time τ_ℓ of the Bessel process $Q_{0,x}$. This will give us a law of large numbers with

$$\lim_{x\to\infty} a(0,x,\omega) = \bar a = \int a(0,\ell,\omega)h(0,\ell,\omega)\,dP$$

and if the approximation in (1.8) is good enough then

$$x(a(0,x)-\bar a) \simeq x\int_0^\infty [(a(\tau,\ell,\omega)h(\tau,\ell,\omega)-\bar a)]p(x,\ell,\tau)d\tau$$

Since $p(x,\ell,\tau)$ the density of the distribution of the exit time τ_ℓ of the Bessel process $Q_{0,x}$, admits a diffusive scaling limit

$$p(x,\ell,\tau) \simeq \frac{1}{x^2} p(1,0,\frac{\tau}{x^2})$$

if $a(t,\ell,\omega)h(t,\omega)$ is sufficiently mixing in t, then a CLT can be proved with the required scaling. The fixed constant $\ell > 0$ will be chosen later. We consider the space-time Bessel process starting from $x > \ell$ at time $t < \tau$, and condition it so that the exit time from the interval (ℓ,∞) is τ, i.e. $\tau_\ell = \tau$ and denote the conditioned process (Bessel bridge) by $Q^{\tau,\ell}_{t,x}$. Since the Bessel process is not explicitly time dependent, $Q^{\tau,\ell}_{t,x}$ is covariant with respect to time translations i.e. $Q^{\tau,\ell}_{t,x}(A) = Q^{\tau-t,\ell}_{0,x}(\theta_t A)$. As t and x go to $-\infty$ and ∞ appropriately i.e. $t \sim x^2$ this process will have a limit on the space $C[(-\infty,0],[\ell,\infty)]$ that we will call $Q^{\tau,\ell}_*$. Then with

$$R(-\infty,\sigma,\omega,x(\cdot)) = \exp\big[-\int_{-\infty}^{\sigma} c(s,x(s),\omega)ds\big]$$

and

$$h(\tau,\ell,\omega) = E^{Q^{\tau,\ell}_*}[R(-\infty,\tau,\omega,x(\cdot))] = h(0,\ell,\theta_\tau\omega) \tag{1.9}$$

$g(t,x,\tau,\ell,\omega)$ will be approximated by $h(\tau,\ell,\omega) = h(0,\ell,\theta_\tau\omega)$. Therefore the solution $a(0,x,\omega)$ will be approximated by $\widehat{a}(0,x)$, defined by

$$\widehat{a}(0,x,\omega) = \int a(\tau,\ell,\omega)h(\tau,\ell,\omega)\, p(x,\ell,\tau)d\tau$$

For fixed ℓ, $a(\tau,\ell,\omega)$ and $h(\tau,\ell,\omega)$ are stationary processes in τ. While the density $p(x,\ell,\tau)$ of the distribution of τ_ℓ under $Q_{0,x}$ is not explicit, its Fourier transform is in closed form and in addition, it has the scaling property $x^2 p(x,\ell,x^2 t) = p(1,\frac{\ell}{x},t)$. Its behavior, as $x \to \infty$, is therefore easy to analyse. This asymptotic behavior is combined with the standard ergodic theorem to prove that

$$\lim_{x\to\infty} \widehat{a}(0,x,\omega) = \bar{a} = E^P[a(0,\ell,\omega)h(0,\ell,\omega)] \tag{1.10}$$

a.s. The estimate that plays the central role in our analysis is

$$\lim_{x\to\infty} x^2 E^P[|\widehat{a}(0,x,\omega) - a(0,x,\omega)|^2] = 0$$

This estimate reduces the CLT to proving a CLT for $x(\widehat{a}(0,x,\omega) - \bar{a})$. This will further require us to prove some mixing properties for $h(\tau,\ell,\omega)$ and $a(\tau,\ell,\omega)$ as processes in τ. We will show that they inherit these from similar properties of the source process $\lambda(\cdot,\omega)$

2. Bessel process.

We will now define the entrance process $Q^{\tau,\ell}_*$. Because of its covariant nature, we need only define $Q^{0,\ell}_*$.

Let us recall that $Q^{0,\ell}_{t,x}$ is the distribution of the Bessel process starting at time $t < 0$ from $x > \ell$ conditioned to exit from (ℓ,∞) at time 0, i.e. $\tau_\ell = 0$. This defines a Markov process on $C[[t,0];[\ell,\infty)]$ with transition probability densities, $s < \sigma < 0$ and $y,z > \ell$

$$q^{0,\ell}(s,y,\sigma,z) = \frac{q^D(\sigma-s,y,z)p(z,\ell,-\sigma)}{p(y,\ell,-s)}$$

where q^D is the transition density of the Bessel process with Dirichlet boundary condition at $x=\ell$.

The entrance process $Q_*^{0,\ell}$ is characterized by two properties. For every $x>\ell$, the distribution of the exit time τ_x from (x,∞) under $Q_*^{0,\ell}$ is given by the density $f^{0,\ell}(x,t)=p(x,\ell,-t)$ and the conditional distribution of $Q_*^{0,\ell}$ after time τ_x, given the past is given by $Q_{\tau_x,x}^{0,\ell}$. Uniqueness within the class of measures supported on paths $\{x(\cdot): x(t)\to\infty \text{ as } t\to-\infty\}$ is an easy consequence of $\tau_x\to-\infty$ as $x\to\infty$.

Existence will follow from Kolmogorov's consistency theorem provided $\{f^{0,\ell}(x,t)\}$ is consistent as a family of distributions of exit times. Under any $Q_{t,x}^{0,\ell}$ the successive hitting times $\{\tau_y\}$ of levels $x>y>\ell$ are Markovian with the transition density given by

$$p^{0,\ell}(y_1,\tau_1,y_2,\tau_2)=\frac{p(y_1,y_2,\tau_2-\tau_1)p(y_2,\ell,-\tau_2)}{p(y_1,\ell,-\tau_1)} \tag{2.1}$$

and

$$\begin{aligned}\int f^{0,\ell}(x,s)\,p^{0,\ell}(s,x,t,y)ds \int p(x,\ell,-s)\frac{p(x,y,t-s)p(y,\ell,-t)}{p(x,\ell,-s)}ds\\ =\int p(x,y,t-s)p(y,\ell,-t)ds=p(y,\ell,-t)\\ =f^{0,\ell}(y,t)\end{aligned}$$

which proves the necessary consistency.

If we condition the hitting time of level ℓ to be any arbitrary time τ then the corresponding $Q_*^{\tau,\ell}$ is just the time shift by τ of $Q_*^{0,\ell}$.

One of the quantities we will need an estimate on is

$$Q_*^{0,\ell}[x(-s)\le y]$$

Theorem 2.1. *For $y\ge\ell, s>0$*

$$Q_*^{0,\ell}[x(-s)\le y]\le c\int_{\frac{s}{y^2}}^{\infty} e^{-\frac{1}{4t}}t^{-\frac{9}{2}}dt$$

where

$$c^{-1}=\int_0^{\infty} e^{-\frac{1}{4t}}t^{-\frac{9}{2}}dt$$

In particular for any $r>0$ there is a constant C_r such that

$$E^{Q_*^{0,\ell}}[x(-s)^{-r}]\le C_r\, s^{-\frac{r}{2}}$$

For small s we can use the bound $x(-s)\ge\ell$. In addition we have the moment estimates

$$\int \tau^k p(x,\ell,\tau)\,d\tau\le C_k x^{2k} \tag{2.2}$$

for $k=1,2,3$.

Proof. Clearly

$$Q_*^{0,\ell}[x(-s) \le y] \le Q_*^{0,\ell}[\tau_y \le -s] = \int_s^\infty p(y,\ell,t)dt = Q_{0,y}[\tau_\ell \ge s] \le Q_{0,y}[\tau_0 \ge s]$$

and from the scaling properties of the Bessel process,

$$Q_{0,y}[\tau_0 \ge s] = Q_{0,1}[\tau_0 \ge \frac{s}{y^2}] = c\int_{\frac{s}{y^2}}^\infty e^{-\frac{1}{4t}} t^{-\frac{9}{2}} dt$$

Since $\tau_\ell \le \tau_0$, the estimates (2.2) on the moments of τ_ℓ follow from the scaling property of the distribution of τ_0 and the tail behavior of $p(1,0,t) = ce^{-\frac{1}{4t}}t^{-\frac{9}{2}}$. We can also bound

$$\begin{aligned}
E^{Q_*^{0,\ell}}[x(-s)^{-r}] &= \int_\ell^\infty y^{-r} dQ_*^{0,\ell}[x(-s) \le y] \\
&= r\int_\ell^\infty y^{-(r+1)} Q_*^{0,\ell}[x(-s) \le y]dy \\
&\le C\, r\int_\ell^\infty y^{-(r+1)}dy \int_{\frac{s}{y^2}}^\infty e^{-\frac{1}{4t}}t^{-\frac{9}{2}}dt \\
&\le C\, r\int_\ell^\infty y^{-(r+1)}(1+\frac{s}{y^2})^{-\frac{7}{2}}dy \\
&= C\, r\, s^{-\frac{r}{2}}\int_\ell^\infty \frac{y^{6-r}}{(y^2+1)^{\frac{7}{2}}}dy \\
&= C_r\, s^{-\frac{r}{2}}
\end{aligned}$$

□

We will need the following estimates on $p(x,\ell,\tau)$ the probability density of the exit time τ_ℓ, from (ℓ,∞) of the Bessel process starting from level $x > \ell$ at time 0.

Theorem 2.2. *There is a constant C such that for all $x \ge 2\ell$ and $\sigma \ge 0$,*

$$\int_0^\infty |p(x,\ell,\tau+\sigma) - p(x,\ell,\tau)|d\tau \le \frac{C\sigma}{x^2} \tag{2.3}$$

and for $x \ge 2y$, $y \ge \ell$,

$$\int_0^\infty |p(x,\ell,\tau) - p(x,y,\tau)|d\tau \le \frac{Cy^2}{x^2} \tag{2.4}$$

Proof. We note that the Laplace transform

$$\psi(x,0,\lambda) = \int e^{-\lambda\tau}p(x,0,\tau)d\tau = e^{-x\sqrt{\lambda}}[1 + x\sqrt{\lambda} + \frac{6x^2}{15}\lambda + \frac{x^3}{15}\lambda^{\frac{3}{2}}]$$

can be explicitly calculated as the solution of

$$\psi_{xx} - \frac{6}{x}\psi_x = \lambda\psi$$

with $\psi(0) = 1, \psi(\infty) = 0$. It is easy to see that for $0 \le a < b$,

$$\psi(b,a,\lambda) = \int e^{-\lambda\tau}p(b,a,\tau)d\tau = \frac{\psi(b,0,\lambda)}{\psi(a,0,\lambda)}$$

By analytic continuation, setting $\lambda = -i\xi$, the Fourier transform

$$\widehat{p}(1,a,\xi) = \int_0^\infty e^{i\,\xi\,\tau} p(1,a,\tau)d\tau = \frac{\psi(1,0,-i\xi)}{\psi(a,0,-i\xi)}$$

is seen to satisfy

$$\sup_{0\le a\le \frac{1}{2}} \int_{-\infty}^\infty [1+\xi^2]|D_\xi^r \widehat{p}(1,a,\xi)|^2 d\xi \le C$$

for $r = 1,2,3$. The only subtle point here is the differentiability of $\psi(1,0,\lambda)$ at $\lambda = 0$, which follows from an explicit expansion of the form $\psi(1,0,\lambda) = 1 + c_1\lambda + c_2\lambda^2 + c_3\lambda^3 + O(\lambda^{\frac{7}{2}})$ in powers of $\sqrt{\lambda}$. The Fourier transform $\widehat{p}$ therefore has three continuous derivatives at $\xi = 0$. This implies by Plancherel identity

$$\sup_{0\le a\le \frac{1}{2}} \int_0^\infty (1+\tau^2)^3 \, |p_\tau(1,a,\tau)|^2 d\tau \le C \tag{2.5}$$

Therefore

$$\sup_{0\le a\le \frac{1}{2}} \int_0^\infty (1+|\tau|) \, |p_\tau(1,a,\tau)| d\tau \le C \tag{2.6}$$

which in turn implies that for $0 \le a \le \frac{1}{2}$,

$$\int_0^\infty |p(1,a,\tau) - p(1,a,\tau+\sigma)| d\tau \le C\sigma$$

The scaling relation $p(x,\ell,\tau) = \frac{1}{x^2} p(1, \frac{\ell}{x}, \frac{\tau}{x^2})$ immediately establishes the first part of the theorem. As for the second part

$$p(x,\ell,\tau) - p(x,y,\tau) = \int_0^\infty [p(x,y,\tau-\sigma) - p(x,y,\tau)] p(y,\ell,\sigma) d\sigma$$

and using (2.2)

$$\begin{aligned}
\int_0^\infty |p(x,\ell,\tau) - p(x,y,\tau)| d\tau &\le \int_0^\infty \int_0^\infty |p(x,y,\tau-\sigma) - p(x,y,\tau)| p(y,\ell,\sigma) d\sigma d\tau \\
&= \frac{C}{x^2} \int \sigma p(y,\ell,\sigma) d\sigma \\
&\le \frac{Cy^2}{x^2} \qquad (2.7)
\end{aligned}$$

□

3. The Main Estimate.

We will now estimate the difference between $g(t,x,\tau,\ell,\omega)$ and $h(\tau,\ell,\omega)$. But before that we will need some preliminary estimates.

We start with Khasminski's lemma which we state and prove for completeness.

Lemma 3.1. *Let $\{P_x\}$ be a Markov family on a state space X with $x(t)$ as its trajectory. τ is the exit time from a set G and $V(x) \ge 0$ is a non-negative function on X. If*

$$\sup_{x\in G} E^{P_x}\Big[\int_0^\tau V(x(s))ds\Big] \le \theta$$

then for $n \geq 2$

$$E^{P_x}\Big[[\int_0^\tau V(x(s))ds\big]^n\Big] \leq n\theta E^{P_x}\Big[[\int_0^\tau V(x(s))ds\big]^{n-1}\Big] \tag{3.1}$$

Moreover if $\alpha > 0$ *satisfies* $\alpha\theta < 1$,

$$E^{P_x}\Big[\exp\big[\alpha\int_0^\tau V(x(s))ds\big]\Big] \leq \frac{1}{1-\alpha\theta} \tag{3.2}$$

Proof.

$$\begin{aligned} &E_x\Big[[\int_0^\tau V(x(s))ds]^n\Big] \\ &\qquad = n!E_x\Big[\int\cdots\int_{0\leq s_1\cdots<s_n<\tau} V(x(s_1))\cdots V(x(s_n))ds_1\cdots ds_n\Big] \\ &\qquad \leq n\theta E_x\Big[[\int_0^\tau V(x(s))ds]^{n-1}\Big] \end{aligned}$$

By induction

$$E_x\Big[[\int_0^\tau V(x(s))ds]^n\Big] \leq n!\theta^{n-1}E_x\Big[[\int_0^\tau V(x(s))ds]\Big] \leq n!\theta^n \tag{3.3}$$

If $\alpha\theta < 1$, then summing the exponential series we get

$$E^{P_x}\Big[\exp\big[\alpha\int_0^\tau V(x(s))ds\big]\Big] \leq \sum_{n=0}^\infty (\alpha\theta)^n = \frac{1}{1-\alpha\theta}$$

□

We now look at the space-time Bessel process with generator

$$\mathcal{A} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x^2} - \frac{6}{x}\frac{\partial}{\partial x}$$

in the domain $(-\infty,\infty)\times[\ell,\infty)$ stopped when it exits at $x=\ell$. For the function $u(x) = \frac{C}{4\ell} - \frac{C}{4x}$ we see that

$$(\mathcal{A}u)(x) = -\frac{C}{x^3}\,;\ u(\ell) = 0$$

and therefore with $V(x) = \frac{C}{x^3}$ and $\theta = \frac{C}{4\ell}$

$$\sup_{t,x\geq\ell} E^{Q_{t,x}}\big[\int_t^{\tau_\ell} V(x(s))ds\big] \leq \theta \tag{3.4}$$

Lemma 3.2. *There exists* ℓ *such that for all* $x > y \geq \ell$, *and* $n \geq 1$

$$\begin{aligned} &\sup_{\omega,x,t} y^n E^{Q_{t,x}}\Big[[\int_t^{\tau_y} |c(s,x(s),\omega)|ds\big]^n\Big] \\ &\qquad = \sup_{\omega,x,t} y^n\int E^{Q^{\tau,\ell}_{t,x}}\Big[[\int_t^{\tau_y} |c(s,x(s),\omega)|ds\big]^n\Big]p(x,\ell,\tau-t)d\tau \\ &\qquad \leq C_n \end{aligned} \tag{3.5}$$

If we replace the supremum over ω *by the mean value then*

(3.6)

$$\limsup_{y\to\infty}\sup_{x,t}\; y^n E^P\Big[E^{Q_{t,x}}\big[\int_t^{\tau_y}|c(s,x(s),\omega)|ds\big]^n\Big]$$
$$=\limsup_{y\to\infty}\sup_{x,t}\; y^n E^P\Big[\int E^{Q^{\tau,\ell}_{t,x}}\Big[\big[\int_t^{\tau_y}|c(s,x(s),\omega)|ds\big]^n\Big]p(x,\ell,\tau-t)d\tau\Big]$$
$$=0$$

The quantities really do not depend on t*, due to stationarity.*

Moreover given $\alpha>0$*, there exist* $\ell(\alpha)$ *and a constant* $C=C(\alpha)$ *such that if* $x>\ell\ge\ell(\alpha)$*, then*

$$\sup_{\omega,x,t} E^{Q_{t,x}}\big[\exp[\alpha\int_t^{\tau_\ell}|c(s,x(s),\omega)|ds]\big] \tag{3.7}$$
$$=\sup_{\omega,x,t}\int E^{Q^{\tau,\ell}_{t,x}}\big[\exp[\alpha\int_t^{\tau}|c(s,x(s),\omega)|ds]\big]p(x,\ell,\tau-t)d\tau$$
$$\le C(\alpha)$$

Proof. Since from (6.7) we have the uniform bound $|c(s,x,\omega)|\le\frac{C}{x^3}$, (3.3), (3.2) and (3.4) can be applied and this proves (3.5) and (3.7). Once we have (3.5) to prove (3.6) it is enough to prove it for $n=1$. For $n=1$, if we are allowed to average with respect to ω, then by stationarity, $E^P[|c(s,x)|]$ is independent of s. We use the uniform bound $\sup_x G(x,z)\le Cz$ on the Green's function

$$G(x,z)=\frac{1}{7}\,z^{-6}\left((\min\{x,z\})^7-y^7\right)$$

of the Bessel operator $D_{xx}-\frac{6}{x}D_x$ in the domain $(t,\infty)\times(y,\infty)$, to conclude that

$$E^P\big[E^{Q_{t,x}}[\int_t^{\tau_y}|c(s,x(s),\omega)|ds]\big]=E^{Q_{t,x}}\big[\int_t^{\tau_y}E^P[|c(s,x(s),\omega))|]ds\big]$$
$$\le C\int_y^\infty z\,E^P|c(s,z,\omega))|dz=o(\frac{1}{y})$$

in view of (6.9), and so (3.6) follows. We will need the estimates (3.7) for $\alpha\le 4$ and this will dictate the choice of ℓ. □

Lemma 3.3. *Estimates similar to Lemma 3.2. are valid for the entrance processes* $Q^{\tau,\ell}_*$*. There exists* ℓ *such that for* $y>\ell$*,*

$$\sup_{\omega,\tau}\; y^n E^{Q^{\tau,\ell}_*}\Big[\big[\int_{-\infty}^{\tau_y}|c(s,x(s),\omega)|ds\big]^n\Big]\le C_n \tag{3.8}$$

If we replace the supremum over ω *by the mean value then*

$$\limsup_{y\to\infty}\sup_{\tau}\; y^n E^P\Big[E^{Q^{\tau,\ell}_*}\big[\int_{-\infty}^{\tau_y}|c(s,x(s),\omega)|ds\big]^n\Big]=0 \tag{3.9}$$

The quantities are again independent of τ*.*

Moreover given $\alpha>0$*, there exist* $\ell=\ell(\alpha)$ *and a constant* $C=C(\alpha)$ *such that if* $x>\ell>\ell(\alpha)$*, then*

$$\sup_{\omega,\tau} E^{Q^{\tau,\ell}_*}\big[\exp[\alpha\int_{-\infty}^{\tau}|c(s,x(s),\omega)|ds]\big]\le C(\alpha) \tag{3.10}$$

Proof. Since the entrance process is covariant with respect to translations we can assume that $\tau = 0$. Let us estimate

$$y^n E^{Q_*^{0,\ell}}\Big[\big[\int_{\tau_x}^{\tau_y} |c(s,x(s),\omega)|ds\big]^n\Big]$$

uniformly in $x > y$. If we condition with respect to $\tau_x = t$, then using the bound of $V(x) = \frac{C}{x^3}$ for $|c(s,x,\omega)|$

$$\begin{aligned}
E^{Q_*^{0,\ell}}\Big[\big[\int_{\tau_x}^{\tau_y} |c(s,x(s),\omega)|ds\big]^n\Big] &\le \int E^{Q_{x,t}^{\ell,0}}\Big[\big[\int_t^{\tau_y} V(x(s))ds\big]^n\Big] f^{0,\ell}(x,t)dt \\
&= \int E^{Q_{x,0}^{\ell,-t}}\Big[\big[\int_0^{\tau_y} V(x(s))ds\big]^n\Big] f^{0,\ell}(x,t)dt \\
&= \int E^{Q_{x,0}^{\ell,t}}\Big[\big[\int_0^{\tau_y} V(x(s))ds\big]^n\Big] p(x,\ell,t)dt \\
&= E^{Q_{x,0}}\Big[\big[\int_0^{\tau_y} V(x(s))ds\big]^n\Big]
\end{aligned}$$

This proves (3.8) and (3.9). The proof of (3.10) is similar. □

We now turn to the proof of the main estimate of this section. For functions g of ω (and possibly other variables) we define the norm $\|g\|_{2,P}$ by

$$\|g\|_{2,P} = \Big[E^P|g(\omega)|^2\Big]^{\frac{1}{2}}$$

Theorem 3.4. *Let $g(0,x,\tau,\ell,\omega)$ and $h(\tau,\ell,\omega)$ be as defined earlier in (1.7) and (1.9). Then*

$$\lim_{x\to\infty} x\int_0^\infty \Big[E^P\big[|g(0,x,\tau,\ell,\omega) - h(\tau,\ell,\omega)|^2\big]\Big]^{\frac{1}{2}} p(x,\ell,\tau)d\tau = 0$$

Therefore

$$\lim_{x\to\infty} x\|\widehat{a}(0,x,\omega) - a(0,x,\omega)\|_{2,P} = 0$$

Proof. The proof consists of several steps. First we truncate $r(-\infty,\tau_\ell,\omega)$ and replace it with $r(\tau_x,\tau_\ell,\omega)$. For $j \ge 1$, we define

$$\sigma_j = \tau_{2^j\ell} = \inf\{t : x(t) \le 2^j\ell\}$$

be the hitting time of the level $y_j = 2^j\ell$. $R(\tau_a,\tau_\ell,\omega) = e^{-r(\tau_a,\tau_\ell,\omega)}$. Let n be the largest integer such that $2^n\ell \le x$. $R(\tau_x,\tau_\ell,\omega) = e^{-r(\tau_x,\tau_\ell,\omega)}$ will be successively replaced by $R(\sigma_j,\tau_\ell,\omega) = e^{-r(\sigma_j,\tau_\ell,\omega)}$ with $j = 1,\dots,n$.

STEP A. In order to estimate the successive differences we will need to estimate the size of the difference between $R(\tau_a,\tau_\ell,\omega) = e^{-r(\tau_a,\tau_\ell,\omega)}$ and $R(\tau_b,\tau_\ell,\omega) = e^{-r(\tau_b,\tau_\ell,\omega)}$ where $a > b > \ell$. More precisely if $Q_{0,x}^{\tau,\ell}$ is the Bessel Bridge, then we will need an estimate of

$$H(x,a,b) = \int \big\|E^{Q_{0,x}^{\tau,\ell}}\big[R(\tau_a,\tau_\ell) - R(\tau_b,\tau_\ell)\big]\big\|_{2,P}^2\, p(x,\ell,\tau)\,d\tau$$

First we use the bound $|e^x - e^y| \le e^{\max\{x,y\}}|x-y|$, to estimate

$$|R(\tau_a,\tau_\ell) - R(\tau_b,\tau_\ell)| \le e^{\xi}|r(\tau_a,\tau_b)|$$

where

$$\xi \leq \int_0^{\tau_\ell} |c(s,x(s),\omega)|ds \leq \int_0^{\tau_\ell} V(x(s))ds$$

and $V(x) = \frac{C}{x^3}$

$$\begin{aligned} E^{Q^{\tau,\ell}_{0,x}}\left[|R(\tau_a,\tau_\ell) - R(\tau_b,\tau_\ell)|^2\right] &\leq E^{Q^{\tau,\ell}_{0,x}}\left[e^{2\xi}|r(\tau_a,\tau_b)|^2\right] \\ &\leq \left[E^{Q^{\tau,\ell}_{0,x}}\left[e^{4\xi}\right]\right]^{\frac{1}{2}} \left[E^{Q^{\tau,\ell}_{0,x}}\left[|r(\tau_a,\tau_b)|^4\right]\right]^{\frac{1}{2}} \end{aligned}$$

Therefore

$$\begin{aligned} H(x,a,b) &\leq \left[E^P\left[\int \left[E^{Q^{\tau,\ell}_{0,x}}[e^{4\xi}]\right] p(x,\ell,\tau)\, d\tau\right]\right]^{\frac{1}{2}} \\ &\quad \times \left[E^P\left[\int \left[E^{Q^{\tau,\ell}_{0,x}}\left[|r(\tau_a,\tau_b)|^4\right] p(x,\ell,\tau)\, d\tau\right]\right]\right]^{\frac{1}{2}} \\ &= \left[E^P\left[\left[E^{Q_{0,x}}[e^{4\xi}]\right]\right]\right]^{\frac{1}{2}} \times \left[E^P\left[\left[E^{Q_{0,x}}\left[|r(\tau_a,\tau_b)|^4\right]\right]\right]\right]^{\frac{1}{2}} \\ &\leq \frac{\delta(b)}{b^2} \end{aligned} \tag{3.11}$$

where $\delta(b) \to 0$ as $b \to \infty$ in view of the estimate (3.6). In what follows we will use $\delta(b)$ to denote quantities that are $o(1)$ as $b \to \infty$. Since the estimate is uniform in x and a, we have the same estimate on

$$H(b) = \sup_\tau \left\|E^{Q^{\tau,\ell}_*}\left[R(-\infty,\tau_\ell) - R(\tau_b,\tau_\ell)\right]\right\|^2_{2,P} \leq \frac{\delta(b)}{b^2} \tag{3.12}$$

We now replace $R(-\infty,\tau_\ell)$ in the definition of

$$h(\tau,\ell,\omega) = E^{Q^{\tau,\ell}_*}[R(-\infty,\tau_\ell)]$$

by $R(\tau_x,\tau_\ell)$ and define

$$h_x(\tau,\ell,\omega) = E^{Q^{\tau,\ell}_*}[R(\tau_x,\tau)] = h_x(0,\ell,\theta_\tau\omega)$$

Then, from (3.12)

$$\sup_\tau \|h(\tau,\ell,\omega) - h_x(\tau,\ell,\omega)\|_{2,P} \leq \frac{\delta(x)}{x}$$

STEP B. Now we take up the difference between $h_x(\tau,\ell,\omega)$ and $g(t,x,\tau,\ell,\omega)$.

$$g(t,x,\tau,\ell,\omega) - h_x(\tau,\ell,\omega) = E^{Q^{\tau,\ell}_{0,x}}\left[R(\tau_x,\tau_\ell)\right] - E^{Q^{\tau,\ell}_*}\left[R(\tau_x,\tau_\ell)\right]$$

We represent

$$R(\tau_x,\tau_\ell) = 1 + \sum_{j=1}^n [R(\sigma_j,\tau_\ell) - R(\sigma_{j-1},\tau_\ell)] + [R(\tau_x,\tau_\ell) - R(\sigma_n,\tau_\ell)]$$

to obtain

$$\begin{aligned} g(0,x,\tau,\ell,\omega) - h_x(\tau,\ell,\omega) &= \sum_{j=1}^n [g_j(0,x,\tau,\ell,\omega) - h_j(\tau,\ell,\omega)] \\ &\quad + [g_{n+1}(0,x,\tau,\ell,\omega) - h_{x,n+1}(\tau,\ell,\omega)] \end{aligned}$$

where for $1 \le j \le n$

$$g_j(0,x,\tau,\ell,\omega) = E^{Q^{\tau,\ell}_{0,x}}\left[R(\sigma_j,\tau_\ell) - R(\sigma_{j-1},\tau_\ell)\right]$$
$$h_j(\tau,\ell,\omega)E^{Q^{\tau,\ell}_*}\left[R(\sigma_j,\tau_\ell) - R(\sigma_{j-1},\tau_\ell)\right]$$

and

$$g_{n+1}(0,x,\tau,\ell,\omega) = E^{Q^{\tau,\ell}_{0,x}}\left[R(0,\tau_\ell) - R(\sigma_n,\tau_\ell)\right]$$
$$h_{x,n+1}(\tau,\ell,\omega) = E^{Q^{\tau,\ell}_*}\left[R(\tau_x,\tau_\ell) - R(\sigma_n,\tau_\ell)\right]$$

STEP B1. First we estimate $\Delta(x,n+1)$, which is defined as

$$\begin{aligned}\Delta(x,n+1) &= \int \left\|g_{n+1}(0,x,\tau,\ell,\omega) - h_{x,n+1}(x,\tau,\ell,\omega)\right\|_{2,P} p(x.\ell,\tau)d\tau \\ &\le \int \left[\left\|g_{n+1}(0,x,\tau,\ell,\omega)\right\|_{2,P} + \left\|h_{x,n+1}(\tau,\ell,\omega)\right\|_{2,P}\right] p(x,\ell,\tau)d\tau\end{aligned}$$

Recalling that $y_j = 2^j\ell$ and that n is defined by $y_n \le x < 2y_n$.

$$\int \left\|g_{n+1}(0,x,\tau,\ell,\omega)\right\|_{2,P} p(x,\ell,\tau)d\tau = \sqrt{H(x,x,y_n)}$$

and

$$\int \left\|h_{x,n+1}(\tau,\ell,\omega)\right\|_{2,P} p(x,\ell,\tau)d\tau \le \sqrt{H(x)} + \sqrt{H(y_n)}$$

Therefore

$$\Delta(x,n+1) \le \sqrt{H(x,x,y_n)} + \sqrt{H(x)} + \sqrt{H(y_n)} \le 2\frac{\delta(y_n)}{y_n} + \frac{\delta(x)}{x} \le 5\frac{\delta(\frac{x}{2})}{x}$$

STEP B2. We will now estimate for $1 \le j \le n$, the quantity

$$\Delta(x,j) = \int \left\|g_j(0,x,\tau,\ell,\omega) - h_j(\tau,\ell,\omega)\right\|_{2,P} p(x,\ell,\tau)d\tau$$

By conditioning we can write

$$\begin{aligned}g_j(0,x,\tau,\ell,\omega) - h_j(\tau,\ell,\omega) \int &\left[E^{Q^{\tau,\ell}_{\sigma_j,y_j}}\left[R(\sigma_j,\tau_\ell) - R(\sigma_{j-1},\tau_\ell)\right]\right] \\ &\times \left[p^{\tau,\ell}(0,x,y_j,\sigma_j) - p(y_j,\ell,\tau-\sigma_j)\right]d\sigma_j\end{aligned}$$

and

$$\begin{aligned}\|g_j(0,x,\tau,\ell,\omega) - h_j(\tau,\ell,\omega)\|_{2,P} \le \int &\left|\left|E^{Q^{\tau,\ell}_{\sigma_j,y_j}}\left[R(\sigma_j,\tau_\ell) - R(\sigma_{j-1},\tau_\ell)\right]\right|\right|_{2,P} \\ &\times \left|p^{\tau,\ell}(0,x,y_j,\sigma_j) - p(y_j,\ell,\tau-\sigma_j)\right|d\sigma_j\end{aligned}$$

$$\begin{aligned}\Delta(x,j) \le \int\int &\left\|E^{Q^{\tau,\ell}_{\sigma_j,y_j}}\left[R(\sigma_j,\tau_\ell) - R(\sigma_{j-1},\tau_\ell)\right]\right\|_{2,P} \\ &\times \left|p^{\tau,\ell}(0,x,y_j,\sigma_j) - p(y_j,\ell,\tau-\sigma_j)\right|d\sigma_j\, p(x,\ell,\tau)d\tau\end{aligned}$$

Recall from (2.1) that

$$p^{\tau,\ell}(0,x,y_j,\sigma_j) = \frac{p(x,y_j,\sigma_j)p(y_j,\ell,\tau-\sigma_j)}{p(x,\ell,\tau)}$$

If we define for $a > b$

$$F(a,b,\tau) = \left\| E^{Q^{\tau,\ell}_{0,a}}\left[R(0,\tau_\ell) - R(\tau_b,\tau_\ell)\right]\right\|_{2,P}$$

then by stationarity

$$\begin{aligned}
&\Delta(x,j) \\
&\le \int\int F(y_j,y_{j-1},\tau-\sigma_j)|p^{\tau,\ell}(0,x,y_j,\sigma_j) - p(y_j,\ell,\tau-\sigma_j)|p(x,\ell,\tau)d\sigma_j\,d\tau \\
&= \int F(y_j,y_{j-1},\tau-\sigma_j)|p(x,y_j,\sigma_j) - p(x,\ell,\tau)|\,p(y_j,\ell,\tau-\sigma_j)\,d\sigma_j\,d\tau \\
&= \int F(y_j,y_{j-1},\tau)|p(x,y_j,\sigma_j) - p(x,\ell,\tau+\sigma_j)|\,p(y_j,\ell,\tau)\,d\sigma_j\,d\tau
\end{aligned}$$

We can use the estimates (2.3) and (2.4) to obtain

$$\int |p(x,y_j,\sigma_j) - p(x,\ell,\sigma_j+\tau)|d\sigma_j \le C\frac{(y_j^2+\tau)}{x^2}$$

and arrive at

$$\Delta(x,j) \le \frac{C}{x^2}\int F(y_j,y_{j-1},\tau)(y_j^2+\tau)p(y_j,\ell,\tau)d\tau$$

Moreover in view of (3.11)

$$\begin{aligned}
&\int |F(y_j,y_{j-1},\tau)|^2 p(y_j,\ell,\tau)d\tau \\
&\quad= \int E^P[\left|E^{Q^{\tau,\ell}_{0,y_j}}\left[R(0,\tau_\ell) - R(\tau_{y_{j-1}},\tau_\ell)\right]\right|^2]p(y_j,\ell,\tau)d\tau \\
&\quad\le E^P\int E^{Q^{\tau,\ell}_{0,y_j}}\Big[\left[R(0,\tau_\ell) - R(\tau_{y_{j-1}},\tau_\ell)\right]^2\Big]p(y_j,\ell,\tau)d\tau \\
&\quad= E^P\,E^{Q_{0,y_j}}\Big[\left[R(0,\tau_\ell) - R(\tau_{y_{j-1}},\tau_\ell)\right]^2\Big] \\
&\quad= H(y_j,y_j,y_{j-1}) \\
&\quad\le \frac{\delta(y_{j-1})}{y_{j-1}^2}
\end{aligned}$$

and from (2.2)

$$\int (y_j^2+\tau)^2 p(y_j,\ell,\tau)d\tau \le Cy_j^4$$

Applying now Schwartz's inequality,

$$\Delta(x,j) \le C\frac{y_j^2}{x^2}\frac{\delta(y_{j-1})}{y_{j-1}} \le 2C\frac{y_j\delta(y_{j-1})}{x^2}$$

we finally have have

$$\begin{aligned}
&\int \left\|g(0,x,\tau,\ell,\omega) - h(\tau,\ell,\omega)\right\|_{2,P} p(x,\ell,\tau)d\tau \\
&\qquad\qquad\le \frac{C}{x^2}\sum_{j=1}^n \delta(y_{j-1})\,y_j + \frac{\delta(\frac{x}{2})}{x} \\
&\qquad\qquad= o(\frac{1}{x})
\end{aligned}$$

□

4. Law of Large Numbers.

We will now provide a proof of the law of large numbers stated in Theorem 1.1. Our proof will rely on the main estimate of Theorem 3.4. Though, let us mention that the law of large numbers can be derived by much easier estimates, bypassing the much stronger one of Theorem 3.4. We prefer to give the proof by means of Theorem 3.4, because, on the one hand it provides an expression for the limit $\bar{a}$, (1.10), and on the other hand the representation that Theorem 3.4 provides will be crucial for the proof of the central limit theorem later on.

In view of the bound $|xa_x(t,x,\omega)| \leq C$, it is sufficient to prove

$$\lim_{n\to\infty} a(0,n,\omega) = \int a(\tau,\ell,\omega) g(0,n,\tau,\ell,\omega) p(\ell,n,\tau) d\tau = \bar{a}$$

with probability 1. Since

$$\widehat{a}(0,n,\omega) = \int a(\tau,\ell,\omega) h(\tau,\ell,\omega) p(\ell,n,\tau) d\tau$$

from the estimate of Theorem 3.4 it is clear that

$$E^P[|a(0,n,\omega) - \widehat{a}(0,n,\omega)|^2] = o(n^{-2})$$

and hence by an application of the Borel-Cantelli lemma, $|a(0,n,\omega) - \widehat{a}(0,n,\omega)| \to 0$ with probability 1. It is therefore sufficient to prove

$$\widehat{a}(0,n,\omega) = \int a(\tau,\ell,\omega) p(n,\ell,\tau) d\tau \to \bar{a}$$

a.e. P. If $x(t)$ is an arbitrary stationary stochastic process with finite first moment, and we wish to prove almost sure convergence to $E[x(t)]$ of averages of the form

$$X_n = \int x(t) f(n,t) dt$$

the following conditions on $\{f(n,t)\}$ are sufficient.

a) $f(n,t) \geq 0$ and $\int f(n,t)dt = 1$.

b) For every $h > 0$, $\int |f(n,t) - f(n,t+h)|dt \to 0$ as $n \to \infty$

c)

$$\sup_n \int (1+|t|) |\frac{\partial f(n,t)}{\partial t}| dt \leq C$$

The standard proof of the ergodic theorem proceeds by proving it for $x(t+h)-x(t)$, and then approximating $x(t)$, when $E[x(t)] = 0$ by linear combinations of these in L_1. Finally it all comes down to the maximal lemma, [8], Section 6.1. Since we know that the maximal lemma is valid for the standard averages, the problem reduces to estimating, for $x(t) \geq 0$, $\sup_n \int x(t)\, f(n,t) dt$ in terms of $\sup_\tau z(\tau)$ where

$$z(\tau) = \frac{1}{\tau} \int_0^\tau x(t)\, dt$$

$$\begin{aligned}\int x(t)\, f(n,t)dt &= \int \frac{d}{dt}[tz(t)]f(n,t)dt\\ &= -\int tz(t)\frac{df(n,t)}{dt}dt\\ &\leq \sup_\tau z(\tau)\int (1+|t|)|\frac{\partial f(n,t)}{\partial t}|dt\end{aligned}$$

We need a uniform bound

$$\int (1+|\tau|)\Big|\frac{\partial p(n,\ell,\tau)}{\partial \tau}\Big|\, d\tau \leq C$$

which follows from (2.6) by rescaling.

5. Central Limit Theorem.

Our goal is to prove a central limit theorem for the random variable

$$x\int [a(s,\ell,\omega)h(s,\ell,\omega)-\bar{a}]p(x,\ell,s)ds = x\int f(\theta_s\omega)p(x,\ell,s)ds$$

as $x\to\infty$ where $f(\omega)=[a(0,\ell,\omega)h(0,\ell,\omega)-\bar{a}]$. First we replace $p(x,\ell,s)$ by $p(x,0,s)$ which has the scaling property $x^2p(x,0,x^2s)=p(1,0,s)$ and the difference

$$\Delta_x = x\int [p(x,\ell,s)-p(x,0,s)]f(\theta_s\omega)ds$$

can be estimated easily with the help of (2.7).

$$\begin{aligned}\|\Delta_x\|_{2,P} &\leq x\|f(\omega)\|_{2,P}\int |p(x,\ell,s)-p(x,0,s)|ds\\ &\leq \frac{C\,\ell^2}{x}\,\|f(\omega)\|_{2,P}\end{aligned}$$

The next step is to reduce it to the standard type of central limit theorem. If

$$\xi(t)=\int_0^t f(\theta_s\omega)ds$$

we have the identity

$$\begin{aligned}x\int_0^\infty f(\theta_s\omega)p(x,0,s)ds =&x\int p(x,0,s)d\xi(s)\\ &= -x\int_0^\infty \xi(s)p_s(x,0,s)ds\\ &= -x^3\int_0^\infty \xi(x^2s)p_s(x,0,x^2s)ds\\ &= -\int_0^\infty x^{-1}\xi(x^2s)p_s(1,0,s)ds\end{aligned}$$

If $x^{-1}\xi(x^2t)$ converges to a Brownian motion with variance σ^2 and we have an estimate

$$E^P[|\xi(t)|^2]\leq Ct$$

then the tails

$$e_T(\omega)=\int_T^\infty x^{-1}\xi(x^2s)p_s(1,0,s)ds$$

can be controlled in $L_2(P)$ uniformly in x. It is now easy to derive from a central limit theorem for $x^{-1}\xi(x^2t)$, similar theorems for $x\int f(\theta_s\omega)p(x,0,s)ds$ and $x\int f(\theta_s\omega)p(x,\ell,s)ds$. The limiting variance will now be equal to

$$\sigma^2\int_0^\infty |p(1,0,s)|^2 ds$$

We summarize all of this as

Theorem 5.1. *Let $f(\omega)$ be a function in $L_2(P)$ with mean 0 such that for some constant C,*

$$E^P[|\int_0^t f(\theta_s\omega)ds\big|^2]\le Ct \tag{5.1}$$

and

$$x^{-1}\int_0^{x^2t} f(\theta_s\omega)ds$$

satisfies a central limit theorem with variance σ^2. Then a central limit theorem holds for

$$x\int_0^\infty f(\theta_s\omega)p(x,\ell,s)ds$$

with limiting variance

$$\sigma^2\int_0^\infty |p(1,0,s)|^2 ds$$

We assume now that that our source process has the property that $\{\lambda(s,\omega):s\le a\}$ and $\{\lambda(s,\omega):s\ge b\}$ are independent if $b-a\ge A$ for some $A<\infty$. We will prove the CLT under this condition. First we establish a general theorem.

Let $(\Omega,\mathcal{F},P)$ be a probability space and θ_t a one parameter family of measure preserving of transformations. Assume that $\{\mathcal{F}_t^s; s\le t\}$ is a family of sub σ-fields (corresponding to events observable during time $[s,t]$) satisfying $\mathcal{F}_t^{s'}\subset\mathcal{F}_t^s$ if $t\ge s'\ge s$, $\mathcal{F}_{t'}^s\subset\mathcal{F}_t^s$ if $s\le t'\le t$ and $\theta_\tau\mathcal{F}_t^s=\mathcal{F}_{t+\tau}^{s+\tau}$. Let $\mathcal{F}_t=\vee_{s\le t}\mathcal{F}_t^s$ and $\mathcal{F}^s=\vee_{t\ge s}\mathcal{F}_t^s$. Finally $\mathcal{F}=\vee_s\mathcal{F}^s=\vee_t\mathcal{F}_t$. Assume that for some finite A, $\mathcal{F}^t$ and $\mathcal{F}_s$ are independent under P provided $t>s+A$. It is not very hard to show that if $f(\omega)$ is measurable with respect to some $\mathcal{F}_b^a$ with $\int f(\omega)dP=0$ and $\int f^2(\omega)dP<\infty$ then

$$x^{-1}\xi(x^2t)=x^{-1}\int_0^{tx^2} f(\theta_s\omega)ds$$

satisfies a central limit theorem as $x\to\infty$ and converges to a Brownian motion with a limiting covariance variance $\sigma^2\, s\wedge t$ where σ^2 is given by

$$\begin{aligned}\sigma^2&=\int_{-\infty}^{\infty}[\int f(\omega)f(\theta_t\omega)dP]\,dt\\&=\int_{-A-(b-a)}^{A+(b-a)}[\int f(\omega)f(\theta_t\omega)dP]\,dt\le 2(A+b-a)\int f^2(\omega)dP\end{aligned}$$

If f is only $\mathcal{F}$ measurable, then it needs to be well approximated by functions measurable with respect to $\mathcal{F}_b^a$.

First we obtain an estimate on the variance of $\xi(t)=\int_0^t f(\theta_s\omega)ds$.

Theorem 5.2. *Assume that* $f_k = E^P[f(\omega)|\mathcal{F}_k^{-k}]$ *satisfies* $E^P[|f - f_k|^2] \le Ck^{-\alpha}$ *for* $k \ge 1$ *with* $\alpha > 0$*. Then*

$$\|\xi(t)\|_{2,P} \le \begin{cases} Ct^{1-\frac{\alpha}{2}} & \text{if } 0 < \alpha < 1 \\ C\sqrt{t}\log t & \text{if } \alpha = 1 \\ C\sqrt{t} & \text{if } \alpha > 1 \end{cases}$$

Proof. Let us write

$$f = f_1 + \sum_{j=1}^{\infty}[f_{2^j} - f_{2^{j-1}}]$$

with the corresponding integrals

$$\xi(t) = \sum_{j=0}^{\infty}\eta_j(t)$$

with

$$\eta_j(t) = \int_0^t [f_{2^j}(\theta_s\omega) - f_{2^{j-1}}(\theta_s\omega)]ds \tag{5.2}$$

for $j \ge 1$, and

$$\eta_0(t) = \int_0^t f_1(\theta_s\omega)ds$$

Then

$$\|\xi(t)\|_{2,P} \le \sum_{j=0}^{\infty}\|\eta_j(t)\|_{2,P}$$

Since the correlations $E^P[f_{2^j}(s)f_{2^j}(s+t)]$ vanish if $|t| \ge A + 2^j$, it is easy to obtain the bound

$$\begin{aligned}\|\eta_j(t)\|_{2,P}^2 &\le Ct\min\{t, A+2^j\}\|f_{2^j} - f_{2^{j-1}}\|_{2,P}^2 \\ &\le Ct\min\{t, A+2^j\}2^{-j\alpha} \\ &\le Ct\min\{t, 2^j\}2^{-j\alpha}\end{aligned}$$

where C is now a constant that depends on A. An easy estimation of the sum

$$\sum_j[\min\{t, 2^j\}2^{-j\alpha}]^{\frac{1}{2}}$$

proves the the thorem. □

We note that if $\alpha > 1$, we have the estimate

$$\min\{t, 2^j\}2^{-j\alpha} \le C\,2^{-j(\alpha-1)}$$

and can use it to prove the central limit theorem for $\xi(t)$.

Theorem 5.3. *Let* $f \in L_2(P)$*. Assume that* $f_k(\omega) = E^P[f(\omega)|\mathcal{F}_k^{-k}]$ *satisfies*

$$E^P[|f - f_k|^2] \le Ck^{-\alpha}$$

for some $\alpha > 1$*. Then the central limit theorem is valid for*

$$x^{-1}\int_0^{tx^2} f(\theta_s\omega)ds$$

with the limiting Brownian Motion having covariance $\sigma^2\, s\wedge t$, where

$$\sigma^2 = \lim_{k\to\infty}\int_{-\infty}^{\infty} E^P[f_{2^k}(\omega)f_{2^k}(\theta_s\omega)]ds = \int_{-\infty}^{\infty} E^P[f(\omega)f(\theta_s\omega)]ds$$

Proof. If we write

$$f = f_1 + \sum_{j\geq 1}[f_{2^j} - f_{2^{j-1}}]$$

since each finite sum statisfies the central limit theorem ([3]), it is clearly sufficient to show that $\eta_j(t)$ defined in equation (5.2) satisfies

$$\|\eta_j(t)\|_{2,P} \leq \delta_j\sqrt{t}$$

for some δ_j satisfying $\sum_j \delta_j < \infty$. Clearly $\delta_j = 2^{-j\frac{1-\alpha}{2}}$ works. □

Remark 5.4. *Because of Theorem 5.1 the CLT holds for*

$$x\int[a(s,\ell,\omega)h(s,\ell,\omega) - \bar{a}]p(x,\ell,s)ds$$

provided we get an estimate of the type used in Lemma 5.1 with $\alpha > 1$, for $f(\omega) = a(0,\ell,\omega)h(0,\ell,\omega)$.

Since a and h are bounded functions of ω, it is sufficient to prove the estimate seperately on a and h.

Theorem 5.5. *Let $f = a(0,\ell,\omega)$ and $f_k = E^P[f|\mathcal{F}_k^{-k}]$. Then*

$$E^P[|f - f_k|^2] \leq Ck^{-4}$$

Proof. Since $u(t,\ell) = \frac{6}{(\ell + a(t,\ell))^2}$, and we have uniform bounds on u, $a(t,\ell)$ depends in a Lipschitz manner on $u(t,\ell)$ with a Lipschitz constant that depends on ℓ. For a fixed ℓ, we can therefore take $f = u(0,\ell,\omega)$ instead of $a(0,\ell,\omega)$. According to Theorem 6.4, the total variation of $u(0,\ell,\omega)$ if we vary $\lambda(\cdot)$ arbitrarily outside $[-k,k]$ is dominated by $C\,\ell\,k^{-2}$. The theorem is now an easy consequence. □

A corresponding estimate for $f(\omega) = h(0,\ell,\omega)$ is more complicated. We do it in several steps, with each step formulated as a lemma.

Starting from the measure P on the space Ω of functions $\lambda(t)$ on $(-\infty,\infty)$, we define a measure P^k on the subset of $\Omega\times\Omega$ consisting of functions $(\omega_1,\omega_2) = (\lambda_1(t),\lambda_2(t))$ such that $\lambda_1(t) = \lambda_2(t)$ if $|t|\leq k$. The measure P^k is uniquely defined by the property that the distribution of the process $\lambda(t) = \lambda_1(t) = \lambda_2(t)$ in the interval $[-k,k]$ is the same as under P and for $|t|\geq k$, $\lambda_1(t),\lambda_2(t)$ are conditionally independent given $\{\lambda(t) : |t|\leq k\}$, each component having the same conditional distribution as that of $\{\lambda(t) : |t|\geq k\}$ given $\{\lambda(t) : |t|\leq k\}$ under P. This gurantees that under P^k the marginal distributions of both $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ are P, the two components are almost surely identical for $|t|\leq k$, and are conditionally indpendent given their common values on $[k,k]$.

Then it is easy to see that

$$E^P[|f(\omega) - E[f|\mathcal{F}_k^{-k}]|^2] = \frac{1}{2}E^{P^k}[|f(\omega^1) - f(\omega^2)|^2]$$

Lemma 5.6. *Let $c(t,x,\omega)$ be as in equation (1.6) and $f(\omega)=h(0,\ell,\omega)$. Then for any $\beta>0$,*

$$
\begin{aligned}
E^{P^k}[|f(\omega^1)-f(\omega^2)|^2] &\le C E^{P^k} E^{Q_*^{0,\ell}}[\big|\int_{-\infty}^0 [c(t,x(t),\omega^1)-c(t,x(t),\omega^2)]dt\big|^2 \\
&\le C_\beta E^{P^k} E^{Q_*^{0,\ell}}[\int_{-\infty}^0 (1+|t|)^{1+\beta}|c(t,x(t),\omega^1)-c(t,x(t),\omega^2)|^2 dt]
\end{aligned}
$$

Proof. We start with the formula

$$f(\omega^i)=E^{Q_*^{0,\ell}}\big[\exp[-\int_{-\infty}^0 c(t,x(t),\omega^i)dt]\big]$$

and use the inequality

$$|e^x-e^y|\le e^{\max\{x,y\}}|x-y|.$$

From (3.10) we have the bounds

$$\sup_\omega E^{Q_*^{0,\ell}}\big[\exp[2\int_{-\infty}^0 |c(t,x(t),\omega^i)|dt]\big]\le C$$

Hence we obtain,

$$
\begin{aligned}
|f(\omega^1)-f(\omega^2)|^2 &\le C E^{Q_*^{0,\ell}}[\big|\int_{-\infty}^0 [c(t,x(t),\omega^1)-c(t,x(t),\omega^2)]dt\big|^2] \\
&\le C_\beta E^{Q_*^{0,\ell}}\big[\int_{-\infty}^0 (1+|t|)^{1+\beta}\big|c(t,x(t),\omega^1)-c(t,x(t),\omega^2)\big|^2 dt\big]
\end{aligned}
$$

□

We use the estimates on the difference of two solutions provided in Theorem 6.5 to continue with our estimation. We get a preliminary estimate which is not good enough, but will bootstrap from it to a better one.

Lemma 5.7. *For any $\beta>0$, there exists a constant C_β such that*

$$E^{P^k}[|f(\omega^1)-f(\omega^2)|^2]\le C_\beta k^{-1+\beta}$$

Proof. On the difference $|c(t,x,\omega^1)-c(t,x,\omega^2)|$ we have two different estimates. For $|t|\le\frac{k}{2}$, from Theorem 6.5

$$|c(t,x,\omega^1)-c(t,x,\omega^2)|\le\frac{C}{kx}$$

and, from the bounds (6.7) on $c(t,x,\omega)$

$$|c(t,x,\omega^1)-c(t,x,\omega^2)|\le\frac{C}{x^3}$$

We now use the estimates from Theorem 2.1, and the fact that $x\ge\ell$, to conclude that, for $|t|\le\frac{k}{2}$

$$E^{Q_*^{0,\ell}}\big[|c(t,x(t),\omega^1)-c(t,x(t),\omega^2)|^2\big]\le\frac{C}{k^2(1+|t|)} \tag{5.3}$$

and for all t,

$$E^{Q_*^{0,\ell}}\big[|c(t,x(t),\omega^1)-c(t,x(t),\omega^2)|^2\big]\le\frac{C}{(1+|t|)^3} \tag{5.4}$$

If we now use the estimates (5.3) for $t \le k$ and (5.4) for $t \ge k$, it is easy to see that

$$E^{Q_*^{0,\ell}}\Big[\int_{-\infty}^{0}(1+|t|)^{1+\beta}\big|c(t,x(t),\omega^1)-c(t,x(t),\omega^2)\big|^2dt\Big] \le C_\beta k^{-1+\beta}$$

□

Lemma 5.8. *Given any $\gamma < 2$, there is a constant C_γ such that*

$$E^P[|a(0,x,\omega)-\bar{a}|^2] \le C_\gamma x^{-\gamma}$$

Proof. Applying the second part of Theorem 5.2 to $f(\omega) = a(0,\ell,\omega)h(0,\ell,\omega)$, we see that the lemma is valid with $\gamma = 2(1-\beta)$. □

Lemma 5.9. *There exists $\alpha > 1$ and a constant C such that*

$$E^{P^k}[|f(\omega^1)-f(\omega^2)|^2] \le Ck^{-\alpha}$$

Proof. Now we go back and improve on the basic estimate

$$\sup_{t,\omega}|c(t,x,\omega)| \le \frac{C}{x^3}$$

in the mean by getting a better estimate in the mean on $a_x(t,x)$. From

$$E^P[|a(t,x,\omega)-\bar{a}|^2] \le C_\beta x^{-2(1-\beta)}$$

one sees, using Theorem 6.1, that

$$E^P[|a_x(t,x,\omega)|^2] \le C_\beta x^{-4+2\beta}$$

which in turn improves the estimate on $c(t,x,\omega)$ to

$$E^P[|c(t,x,\omega)|^2] \le x^{-8+2\beta}$$

Now we return to the proof of Lemma 5.6 and use (5.3) for $|t| \le k^\delta$ where $\delta < 1$. The new estimate is

$$E^{P^k}[|f(\omega^1)-f(\omega^2)|^2] \le C_\beta k^{\delta(1+\beta)-2} + C_\beta k^{-\delta(4-2\beta)}$$

One can pick $\delta < 1$ and $\beta > 0$ such that this works out to

$$E^{P^k}[|f(\omega^1)-f(\omega^2)|^2] \le Ck^{-\alpha}$$

for some $\alpha > 1$. □

This completes the proof of the central limit theorem. We finally establish the positivity of the variance in a special class of examples.

Let $y(t)$ be a stationary Gaussian process with mean 0 and covariance $\rho(t-s) = E^P[y(s)y(t)]$. We assume that $\rho(\cdot) \ge 0$, has compact support and therefore satisfies our mixing condition. Let $\phi(y)$ be a smooth function on $\mathbf{R}$ satisfying $\lambda_1 \le \phi(y) \le \lambda_2$. Assume further that $\phi(y)$ is monotone and $\phi'(y) > 0$. Use the ϕ to construct the random source as $\lambda(t) = \phi(y(t))$. Then the limiting variance

$$\sigma^2 = \lim_{x\to\infty} x^2 E^P[|a(0,x,\omega)-\bar{a}|^2] > 0$$

The proof proceeds by a perturbation argument. Suppose for each $x > 0$, $f(x,t) \ge 0$ is a function with compact support in t. We denote by P^x the Gaussian process with mean

$$m(x,t) = \int \rho(t-s)f(x,s)ds$$

and covariance $\rho(t-s)$. Then the Radon-Nikodym derivative $R(x,\omega)$ of P^x with respect to P is given by

$$\frac{dP^x}{dP} = \exp\big[\int f(x,t)y(t)dt - \frac{1}{2}\int\int \rho(t-s)f(x,s)f(x,t)dsdt\big]$$

and

$$\int [R(x,\omega)]^2 dP = e^{H(x)}$$

where

$$H(x) = \int\int \rho(t-s)f(x,s)f(x,t)dsdt$$

The resulting source process can also be represented as as

$$\lambda^x(t) = \phi(y(t) + m(x,t))$$

where $y(\cdot)$ has P for its distribution. Because

$$E^{P^x}[x|a(0,x,\omega)-\bar{a}|] \le e^{\frac{1}{2}H(x)]}\big[E^P[x^2|a(0,x,\omega)-\bar{a}|^2]\big]^{\frac{1}{2}}$$

if $\sigma^2 = 0$ and $f(x,t)$ is chosen so that $H(x)$ remains bounded as $x \to \infty$, then

$$E^{P^x}[x|a(x,0,\omega)-\bar{a}|] \to 0$$

On the other hand we can solve

$$u_t + u_{xx} - u^2 = 0$$

with different Neumann boundary data corresponding to $\lambda(t) = \phi(y(t))$ and $\lambda^x(t) = \phi(y(t) + m(x,t))$ and get two solutions u_1 and u_2. Since $m(x,t) \ge 0$ and $\phi(\cdot)$ is monotone it follows that $u_2 \ge u_1$ and the difference v will satisfy

$$v_t + v_{xx} - (u_1+u_2)v = 0; v_x(t,0) = -\frac{1}{2}[\phi(y(t)+m(x,t)) - \phi(y(t)]$$

From the relation $u(t,x) = \frac{6}{(x+a(t,x))^2}$, and the lower bounds $u(t,x) \ge \frac{6}{(x+a_1)^2}$ it is not hard to obtain an estimate of the form (see (6.13))

$$a_1(t,x) - a_2(t,x) \ge c_2\, x^3(u_2(x) - u_1(x)) = c_2 x^3 v(t,x)$$

If $\sigma^2 = 0$, both $E^{P^x}[x|a(x,0,\omega)-\bar{a}|]$ and $E^P[x|a(x,0,\omega)-\bar{a}|] \to 0$ and this implies

$$E^P[x^4 v(0,x,\omega)] \le C\, E^P[x[a_1(0,x,\omega) - a_2(0,x,\omega)]] \to 0$$

as $x \to \infty$ and we will show that does not happen for a suitable choice of $f(t,x)$.

We have an upper bound of $g = \frac{12}{(x+a_2)^2}$ on $u_1 + u_2$. This will provide a lower bound on v,

$$v(0,x,\omega) \ge \int [\phi(y(t)+m(x,t)) - \phi(y(t))]q_g(0,x,t)dt$$

where q_g is as in the proof of Theorem 6.4. We pick $f(x,t) = \frac{1}{x}f(\frac{t}{x^2})$ where $f > 0$ is compactly supported and equal to 1 on $[-T,T]$. It is easy to verify that

$$\sup_{x\ge 1} H(x) = \sup_{x\ge 1} x^{-2}\int\int \rho(t-s)f(\frac{t}{x^2})f(\frac{s}{x^2})dtds < \infty$$

On the other hand we have

$$
\begin{aligned}
E^P[x^4 v(0,x,\omega)] &\geq x^4 \int \left[E^P[\phi(y(t)+\frac{1}{x}\int \rho(t-s)f(\frac{s}{x^2})ds)-\phi(y(t))]\right]q_g(0,x,t)dt \\
&\simeq x^3 \int E^P[\phi'(y(t))]q_g(0,x,t)dt \int \rho(t-s)f(\frac{s}{x^2})ds
\end{aligned}
$$

From (6.11) it follows that for large x,

$$
\begin{aligned}
\int q_g(0,x,t)dt &\geq \frac{C_1}{x^3} \\
\int t^2 q_g(0,x,t)dt &\leq C_2\, x
\end{aligned}
$$

This implies that if T is chosen large enough, then for all $x \geq 1$,

$$
\int_0^{x^2T} q_g(0,x,t)\,dt \geq \frac{C_1}{x^3} - \frac{C_2 x}{x^4 T^2} \geq \frac{C_3}{x^3}
$$

By stationarity, $E^P[\phi'(y(t))]$ is a positive constant and

$$
\inf_{x\geq 1} \inf_{0\leq t\leq x^2T} \int \rho(t-s)f(\frac{s}{x^2})\,ds \geq \int_0^{x^2T} \rho(s)ds > 0
$$

establishing the required contradiction.

6. Appendix.

Since our goal is to control the Feynman-Kac term we will need some estimates on $c(t,x,\omega)$. Some of them will be uniform in ω and some will be only in the mean. In addition we will need some estimates on the transition probabilities of the Bessel process. We will collect all of these estimates in this section.

We will need the following estimate from the theory of parabolic partial differential equation.

Theorem 6.1. *For any t_0 and $x_0 > 0$, let $D^r_{t_0,x_0} = \{(t,x) : |x-x_0| \leq r, t_0 \leq t \leq t_0 + r^2\}$. If $v(t,x)$ is a solution of an equation of the form*

$$
v_t + v_{xx} + b(t,x)v_x + q(t,x)v = 0 \tag{6.1}
$$

in $G \subset (-\infty,\infty)\times[0,\infty)$, and $D^1_{t_0,x_0} \subset D^2_{t_0,x_0} \subset G$. Assume that b and q are bounded by a constant B. Then

$$
\sup_{(s,y)\in D^1_{t_0,x_0}} |v_x(s,y)| \leq C\|v(t,x)\|_{L_2[D^2_{t_0,x_0}]} \leq 4C \sup_{(s,y)\in D^2_{t_0,x_0}} |v(s,y)|
$$

where C depends only on B. In particular if $v(t,x)$ is solution of an equation of the form (6.1), with $b(t,x)$ and $q(t,x)$ satisfying

$$
\begin{aligned}
\sup_{t,x} x|b(t,x)| &\leq C \\
\sup_{t,x} x^2|q(t,x)| &\leq C \\
\sup_{t,x} x^\alpha|v(t,x)| &\leq C
\end{aligned}
$$

then

$$\sup_{t,x} x^{\alpha+1}|v_x(t,x)| \leq C$$

Proof. A proof of the first half can be found in [4]. To see the second half, let v be solution of (6.1) in $(-\infty,\infty)\times[0,\infty)$. If we define $v^k(t,x)=\frac{1}{k}v(k^2t,kx)$, then $v^k(t,x)$ satisfies

$$v_t^k(t,x)+v_{xx}^k+b^k(t,x)v_x^k(t,x)+q^k(t,x)v^k(t,x)=0$$

where $b^k(t,x)=kb(k^2t,kx)$ and $q^k(t,x)=k^2q(k^2t,kx)$. In particular if we assume that $x|b(t,x)|\leq C$ and $x^2|q(t,x)|\leq C$ for $x\geq 4$, then $b^k(t,x)$ and $q^k(t,x)$ are uniformly bounded on $[1,\infty)\times(-\infty,\infty)$ for $k\geq 1$ and $D_{t,x}^r\subset(-\infty,\infty)\times(4,\infty)$ for $r=1,2$. We can now get a bound on v_x

$$|v_x(k^2t,kx)| \leq \frac{C}{k}\|v(k^2s,ky)\|_{2,D_{t,x}^2} \leq \frac{C}{k}\sup_{(s,y)\in D_{t,x}^2}|v(k^2s,ky)|$$

□

Theorem 6.2. *Consider a smooth solution u of*

$$u_t+u_{xx}-u^2=0, \quad x>0,\ -\infty<t<\infty \tag{6.2}$$

with $u_x(t,0)=-\frac{1}{2}\lambda(t)$. Let $a(t,x)$ be the corresponding solution

$$a(t,x)=\sqrt{\frac{6}{u(t,x)}}-x \tag{6.3}$$

of

$$a_t+a_{xx}-\frac{6a_x}{(x+a)}-\frac{3a_x^2}{(x+a)}=0 \tag{6.4}$$

on $G=[-\infty,\infty)\times(0,\infty)$. The following bounds are valid.

$$0<a_2\leq a(t,x)\leq a_1<\infty \tag{6.5}$$

for all t,x and ω.

$$\sup_{t,x,\omega}|xa_x(t,x,\omega)|\leq C<\infty \tag{6.6}$$

In particular the function $c(t,x,\omega)$ defined in (1.6) satisfies the uniform bound

$$\sup_{t,\omega}|c(t,x,\omega)|\leq\frac{C}{x^3} \tag{6.7}$$

Proof. We saw already that (6.5) is valid by maximum principle. We will establish (6.6). We start with equation (6.2) and differentiate it with respect to x. Then for the derivative $v=u_x$, satisfies

$$v_t+v_{xx}-2uv=0\,;\ |v(t,0)|=\frac{1}{2}\lambda(t)\leq\frac{\lambda_2}{2}$$

We also have an estimate on u of the form

$$u(t,x)\geq\frac{6}{(x+a_2)^2}$$

and $w = \frac{1}{(x+a_2)^3}$ solves the equation

$$w_t + w_{xx} - \frac{12}{(x+a_2)^2} w = 0\,;\; w(t,0) = \frac{1}{a_2^3}$$

Standard comparison using the maximum principle gives the bound.

To derive (6.6) we rewrite the equation for a in the form

$$a_t + a_{xx} - b(t,x)a_x = 0 \tag{6.8}$$

where

$$b(t,x) = \frac{6 + 3a_x(t,x)}{(x+a)}$$

satisfies $|xb(t,x)| \le C$. It follows from Theorem 6.1 that

$$\sup_{t,x,\omega} |xa_x(t,x,\omega)| \le C$$

which proves (6.6). (6.7) is now an easy consequence of the formula (1.6) for $c(t,x,\omega)$. □

We now improve slightly on this estimate but in the mean with respect to ω.

Theorem 6.3. *There is a finite constant C such that*

$$E^P\left[\int_0^\infty x|a_x(0,x,\omega)|^2 dx\right] \le C < \infty \tag{6.9}$$

Proof. Multiply the equation (6.4) for a by xa and take expectations.

$$E^P[xaa_{xx}] = E^P[\frac{6xaa_x}{(x+a)}] + E^P[\frac{3xaa_x^2}{(x+a)}]$$

Integrate with respect to x from y_1 to y_2 and integrate by parts the trem on the left.

$$E^P[xaa_x]|_{y_1}^{y_2} - \int_{y_1}^{y_2} E^P[xa_x^2]dx - \int_{y_1}^{y_2} E^P[aa_x]dx = \int_{y_1}^{y_2} E^P[\frac{6xaa_x}{(x+a)}]dx + \int_{y_1}^{y_2} E^P[\frac{3xaa_x^2}{(x+a)}]dx$$

Let us split the term

$$\frac{6xaa_x}{(x+a)} = 6aa_x - \frac{6a^2a_x}{(x+a)}$$

and note that

$$\int_{y_1}^{y_2} aa_x dx = \frac{1}{2}[a^2(y_2) - a^2(y_1)]$$

All the terms except $\int_{y_1}^{y_2} E^P[xa_x^2]dx$ are seen to remain bounded as $y_2 \to \infty$. We finally get

$$\int_0^\infty E^P[xa_x^2]dx \le C$$

□

We now begin estimating the difference between two solutions of (6.2).

Theorem 6.4. *Let $v(t,x) = u^1(t,x) - u^2(t,x)$ be the difference of two solutions corresponding to two different initial values $\lambda^1(\cdot)$ and $\lambda^2(\cdot)$, that agree on $\{t : |t| \le k\}$. Then, for $x > 0$ and $|t| \le \frac{k}{2}$, v satisfies*

$$|v(t,x)| \le \frac{Cx}{k^2}$$
$$|v(t,x)| \le \frac{C}{kx}$$
$$|v_x(t,x)| \le \frac{C}{kx^2}$$

Proof. Let $v = u^1 - u^2$ be the difference of two solutions with boundary data $-\frac{1}{2}\lambda^1(\cdot)$ and $-\frac{1}{2}\lambda^2(\cdot)$ that agree on $\{t : |t| \le k\}$. Then with $\sigma(t) = \lambda_1(t) - \lambda_2(t)$

$$v_t + v_{xx} - (u^1 + u^2)v = 0 : v_x(t,0) = -\frac{1}{2}\sigma(t)$$

Any bounded solution $w(t,x)$ of an equation of the form

$$w_t + w_{xx} - g\,w = 0 : w_x(t,0) = -\frac{1}{2}\sigma(t)$$

with a positive g has, by the maximum principle, a representation

$$w(t,x) = \int_t^\infty \sigma(\tau) q_g(t,x,\tau)d\tau \tag{6.10}$$

with a nonnegative q_g. A formula for q_g can be written down in terms of a Feynman-Kac formula

$$q_g(t,x,\tau) = \frac{1}{2\sqrt{\pi(\tau - t)}} e^{-\frac{x^2}{4(\tau-t)}} E_{x,t,\tau}[\exp[-\int_t^\tau g(s,x(s))ds]$$

where $E_{x,t,\tau}$ is expectation with respect to Brownian motion (with variance 2) starting from $x > 0$ at time t and conditioned to exit $(0,\infty)$ at time τ. In particular our solution v is given by

$$v(t,x) = \int_t^\infty \sigma(\tau) q_{u_1+u_2}(t,x,\tau)d\tau$$

Since we have a lower bound of the form $u_i \ge \frac{6}{(x+a_1)^2}$, we can bound $q_{u_1+u_2}$ by q_g, corresponding to $g = \frac{12}{(x+a_1)^2}$. We estimate q_g by exhibiting special solutions of the form

$$w(t,x) = \frac{a(t)}{(x+a_1)^3} + \frac{b(t)}{(x+a_1)} + c(t)(x+a_1) \tag{6.11}$$

w will satisfy

$$w_t + w_{xx} - \frac{12}{(x+a_1)^2} w = 0$$

provided $a'(t) = 10b(t)$, $b'(t) = 12c(t)$ and $c'(t) = 0$. This yields the estimates

$$\int_t^\infty (\tau - t)^2 q_{u_1+u_2}(t,x,\tau)d\tau \le Cx$$
$$\int_t^\infty q_{u_1+u_2}(t,x,\tau)d\tau \le C\,x^{-3}$$

for $x \geq \ell$. By interpolation we also have

$$\int_t^\infty |\tau - t| q_{u_1+u_2}(t,x,\tau) d\tau \leq C\, x^{-1}$$

This takes care of the estimates. □

We now translate these into estimates on $a^1(t,x) - a^2(t,x)$

Let $\lambda^i(\cdot)$ be two different initial conditions at $x = 0$ that agree in $|t| \leq k$. Let $u^i(t,x)$ be the corresponding solutions

$$u^i_t(t,x) + u^i_{xx}(t,x) - u^i(t,x)^2 = 0;\ \ u^i(t,0) = \lambda^i(t)$$

with

$$a^i(t,x) = \sqrt{\frac{6}{u^i(t,x)}} - x$$

and

$$c^i(t,x) = \frac{3a^i_x(t,x)^2}{a^i(t,x)(x + a^i(t,x))} - \frac{6a^i_x(t,x)}{x(x + a^i(t,x))} \tag{6.12}$$

We have the following estimates.

Theorem 6.5. *For $x \geq \ell$ and $|t| \leq \frac{k}{2}$*

$$|a^1(t,x) - a^2(t,x)| \leq \frac{C\, x^2}{k}$$
$$|a^1_x(t,x) - a^2_x(t,x)| \leq \frac{C\, x}{k}$$
$$|c^1(t,x) - c^2(t,x)| \leq \frac{C}{kx}$$

Proof. Since

$$a^1(t,x) - a^2(t,x) = \frac{\sqrt{6}(u^2(t,x) - u^1(t,x))}{\sqrt{u^1(t,x)}\sqrt{u^2(t,x)}(\sqrt{u^1(t,x)} + \sqrt{u^2(t,x)})} \tag{6.13}$$

and from Theorem 6.4, for $v = u_1 - u_2$ we have the estimates

$$|v(t,x)| \leq \frac{C}{xk}$$

and from equation (6.13)

$$|a^1(t,x) - a^2(t,x)| \leq Cx^3 |u^1(t,x) - u^2(t,x)|$$

which proves the first estimate. From Theorem 6.1 it follows that

$$|u^1_x(t,x) - u^2_x(t,x)| \leq \frac{C}{x^2 k}$$

and this in turn implies the second estimate. The last estimate follows from equation (6.12) and the first two estimates of the theorem. □

References

[1] Bricmont, J.; Kupiainen, A. Random walks in asymmetric random environments. Comm. Math. Phys. 142 (1991), no. 2, 345–420.

[2] Hairer, M.; Mattingly, J.C. Ergodicity of the 2D Navier-Stokes Equations with Degenerate Stochastic Forcing. To appear in the Annals of Maths. (2005)

[3] Hall, P.; Heyde, C. C.: Martingale Limit Theory and its Application. Probability and Mathematical Statistics. Academic Press, Inc.

[4] Lieberman, Gary M.: Second Order Parabolic Differential Equations. World Scientific Publishing Co., Inc., River Edge, NJ

[5] Papanicolaou, George C. Diffusions and random walks in random media. The mathematics and physics of disordered media (Minneapolis, Minn., 1983), 391–399, Lecture Notes in Math., 1035, Springer, Berlin, 1983.

[6] Papanicolaou, George C.; Varadhan, S. R. S. Diffusions with random coefficients. Statistics and probability: essays in honor of C. R. Rao, pp. 547–552, North-Holland, Amsterdam, 1982.

[7] Sznitman, A.S. Propagation of chaos for a system of annihilating Brownian spheres. Comm. Pure and Appl. Math. (1987) 663-690.

[8] Varadhan, S.R.S. Probability Theory, Courant Lecture Notes; 7 (2001)

[9] Zygouras, N. *Limit theorems for a randomly or periodically forced semilinear equation.* PhD Thesis, NYU, New York, 2004

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