Regularity structures reading group

The group follows the lecture notes of A. Chandra and H. Weber, *Stochastic PDEs, regularity structures, and interacting particle systems*, Ann. Fac. Sci. Toulouse Math. (6) **26** (2017), no. 4, 847–909, arXiv:1508.03616. References of the form Theorem 2.7, Lemma 2.11 or §2.3.1 are to these notes.

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First session 07/10

Overview: the equations, white noise, scaling, and why renormalization is needed

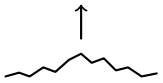
Speaker: Theodoros Galanis

1 Motivating examples

Two model equations, both driven by a space-time white noise ξ.

- **KPZ** — height h of a randomly growing interface:

$$\partial_t h = \Delta h + \tfrac{1}{2}\,|\nabla h|^2 + \xi$$



- Φ^4_d — comes from constructive quantum field theory (CQFT):

$$\partial_t \varphi = \Delta \varphi - \varphi^3 - \varphi + \xi.$$

  Formally, this dynamics leaves invariant the measure

$$\mu(d\varphi) \;\propto\; \exp\Big(-\int \tfrac{1}{2}\varphi^4 + \varphi^2\Big)\, \underbrace{\exp\Big(-\int |\nabla\varphi|^2\Big)\, D\varphi}_{d\nu \,\sim\, \mathrm{GFF}},$$

  where the Gaussian part ν is the Gaussian free field.

Type of equation. Linear smoothing operator + spatial non-linearity + white noise.

Goal. Give meaning to these equations and prove *local well-posedness*: existence and uniqueness, on a compact domain, up to a short random time.

2 Space-time white noise

2.1 Definition

The noise ξ is a random distribution, $\xi \in \mathcal{S}'(\mathbb{R}\times\mathbb{R}^d)$, which is centered Gaussian with covariance

$$\mathbb{E}\big[(\xi,\eta_1)(\xi,\eta_2)\big] = \int_{\mathbb{R}\times\mathbb{R}^d} \eta_1\,\eta_2\; dt\, dx, \qquad \eta_1,\eta_2 \in C_c^\infty.$$

Formally, this says that ξ is uncorrelated at different points:

$$\text{“ } \mathbb{E}\big[\xi(t,x)\,\xi(t',x')\big] = \delta(t-t')\,\delta(x-x') \text{ ”}.$$

Example: for white noise in one variable (time only), testing against indicators of intervals,

$$\mathbb{E}\big[(\xi,\mathbf{1}_{[0,t]})(\xi,\mathbf{1}_{[0,s]})\big] = \int_{\mathbb{R}} \mathbf{1}_{[0,\,t\wedge s]} = t\wedge s,$$

which is the covariance of Brownian motion.

2.2 Scaling of the noise

For $\tau, \lambda > 0$, rescale time by τ and space by λ. On test functions set

$$(S^{\tau,\lambda}\eta)(t,x) := \tau^{-1}\lambda^{-d}\,\eta(\tau^{-1}t, \lambda^{-1}x),$$

and define the rescaled noise by duality:

$$(\xi_{\tau,\lambda}, \eta) := (\xi,\, S^{\tau,\lambda}\eta).$$

This is the rigorous version of $\xi_{\tau,\lambda}(t,x) = \xi(\tau t, \lambda x)$, since formally, by a change of variables,

$$(\xi_{\tau,\lambda}, \eta) = \int \xi(\tau t, \lambda x)\,\eta(t,x)\,dt\,dx.$$

Its variance is

$$\mathbb{E}\,(\xi_{\tau,\lambda}, \eta)^2 = \int \big(S^{\tau,\lambda}\eta\big)^2\,dt\,dx = \tau^{-2}\lambda^{-2d}\int \big(\eta(\tau^{-1}t, \lambda^{-1}x)\big)^2\,dt\,dx = \tau^{-1}\lambda^{-d}\int \eta^2.$$

Both sides are centered Gaussian, so comparing variances gives

$$\xi_{\tau,\lambda} \overset{\text{law}}{=} \tau^{-1/2}\lambda^{-d/2}\,\xi.$$

Parabolic scaling. With $\tau = \lambda^2$, the scaling natural for the heat operator:

$$\xi_{\lambda^2,\lambda} \overset{\text{law}}{=} \lambda^{-\frac{d+2}{2}}\,\xi.$$

3 Scaling of the equations

3.1 The linear equation

Start with the stochastic heat equation (SHE), the linear part of both examples:

$$\partial_t Z = \Delta Z + \xi.$$

Scale invariance. We look for exponents α, β, γ such that the rescaled solution

$$\hat{Z}(t,x) = \lambda^{\alpha}\,Z(\lambda^{\beta}t, \lambda^{\gamma}x)$$

solves the same equation. Differentiating,

$$\begin{aligned}
\partial_t\hat{Z} &= \lambda^{\alpha+\beta}\,(\partial_t Z)(\lambda^{\beta}t, \lambda^{\gamma}x) = \lambda^{\alpha+\beta}\,\big[\Delta Z + \xi\big](\lambda^{\beta}t, \lambda^{\gamma}x),\\
\Delta\hat{Z} &= \lambda^{\alpha+2\gamma}\,(\Delta Z)(\lambda^{\beta}t, \lambda^{\gamma}x),
\end{aligned}$$

and therefore

$$\partial_t\hat{Z} = \lambda^{\beta-2\gamma}\,\Delta\hat{Z} + \lambda^{\alpha+\frac{\beta}{2}-\frac{d\gamma}{2}}\,\hat{\xi}, \qquad \text{where} \qquad \hat{\xi} = \lambda^{\frac{\beta}{2}+\frac{d\gamma}{2}}\,\xi_{\lambda^{\beta},\lambda^{\gamma}} \overset{\text{law}}{=} \xi$$

is again a white noise, by the scaling of Section 2.2. So $\hat{Z}$ solves SHE when both exponents of λ vanish:

$$\beta = 2\gamma, \qquad \alpha = \frac{d\gamma}{2} - \frac{\beta}{2}.$$

Normalizing $\gamma = 1$:

$$\gamma = 1, \qquad \beta = 2, \qquad \alpha = \frac{d}{2} - 1.$$

3.2 The non-linear equations

Apply the same scaling to the two examples, with $\hat{\varphi}$ and $\hat{h}$ defined like $\hat{Z}$:

$$\partial_t \hat{\varphi} = \Delta \hat{\varphi} - \lambda^{4-d}\, \hat{\varphi}^3 - \lambda^2\, \hat{\varphi} + \hat{\xi},$$

$$\partial_t \hat{h} = \Delta \hat{h} + \tfrac{1}{2}\, \lambda^{1-\frac{d}{2}}\, |\nabla \hat{h}|^2 + \hat{\xi}.$$

The non-linear terms now carry a positive power of λ (when $d < 4$, resp. $d < 2$), so at small scales, $\lambda \to 0$, the non-linearity is negligible:

$$\hat{\varphi},\ \hat{h}\ \approx\ Z \qquad \text{at small scales.}$$

3.3 Subcriticality

More generally, an SPDE of the form

$$\partial_t u = \Delta u + F(u, \nabla u) + \xi \tag{$\star$}$$

is called *subcritical* if, under the scaling α, β, γ above, the non-linearity disappears at small scales:

$$F(u, \nabla u)\ \rightsquigarrow\ F_\lambda(u, \nabla u) \longrightarrow 0 \qquad \text{as } \lambda \to 0.$$

- At large scales, $\lambda \to +\infty$, the relevant exponents α, β, γ are different.

Meta-theorem. For subcritical ($\star$), and u_0 having the same regularity as we expect for u: there exists a natural notion of solution, on compact sets and for a short random time, and such solutions are unique.

4 Renormalization

4.1 Why classical theories fail

The obstacle is the multiplication of distributions:

$$\text{the product } C^\alpha \times C^\beta \to C^{\alpha \wedge \beta} \text{ is well defined iff } \alpha + \beta > 0.$$

The solutions are too rough for this ($C^{\alpha-}$ means $C^{\alpha-\kappa}$ for every $\kappa > 0$):

- $d = 1$: $Z \in C^{\frac{1}{2}-}$. At small scales $h \approx Z$, so $h \in C^{\frac{1}{2}-}$ and "$|\partial_x h|^2$" is not well defined.
- $d = 2$: $Z \in C^{0-}$, so already $Z \cdot Z$ fails. Since $\varphi \approx Z$, the term φ^3 would require "Z^3".

So classical solution theories fail.

4.2 Regularizing the noise

Take a mollifier ρ, rescale it parabolically, and smooth the noise:

$$\rho_\delta(t, x) = \delta^{-(d+2)}\, \rho(\delta^{-2} t, \delta^{-1} x), \qquad \xi_\delta := \xi * \rho_\delta.$$

With the smooth noise ξ_δ the equation makes sense classically:

$$\partial_t \varphi_\delta = \Delta \varphi_\delta - \varphi_\delta^3 + \xi_\delta.$$

But the limit is trivial: in $d = 2$, on the torus, $\varphi_\delta \to 0$ as $\delta \to 0$.

4.3 Renormalized equations

Instead, subtract counter-terms with constants c_δ that diverge as $\delta \to 0$:

$$\partial_t \varphi_\delta = \Delta \varphi_\delta - \left(\varphi_\delta^3 - 3c_\delta\, \varphi_\delta\right) + \xi_\delta,$$
$$\partial_t h_\delta = \partial_x^2 h_\delta + \tfrac{1}{2}\left(\partial_x h_\delta\right)^2 - c_\delta + \xi_\delta.$$

These are the *renormalized equations*, and the solutions are defined as the limits

$$h = \lim_{\delta \to 0} h_\delta, \qquad \varphi = \lim_{\delta \to 0} \varphi_\delta.$$

- A different renormalization scheme gives a different family of renormalized equations (for example, shifting by a constant).
- The form of the non-linearity forces specific counter-terms: its shape dictates the shape of the term.
- We are free to choose the finite part of the diverging constant.

4.4 Expanding around the linear solution

The counter-terms have this form because one expands around the solution Z of the linear equation, and stops after finitely many steps.

First step. Writing $\varphi = Z + v$, the remainder v solves

$$\partial_t v = \Delta v - v^3 - 3Zv^2 - 3\,{:}Z^2{:}\,v - {:}Z^3{:},$$

where the powers of Z are replaced by renormalized (Wick) powers:

$${:}Z^2{:} = \text{“}Z^2 - \infty\text{”}, \qquad {:}Z^3{:} = \text{“}Z^3 - 3\infty Z\text{”}.$$

Second step. Subtract the next most singular term as well, with K the heat kernel:

$$\varphi = Z - K * \left({:}Z^3{:}\right) + \tilde{v}.$$

In $d = 2$ the first step is already enough. In mild form (up to the initial condition) the equation for v reads

$$v = -K * \left(v^3 + 3Zv^2 + 3\,{:}Z^2{:}\,v + {:}Z^3{:}\right),$$

and the Banach fixed point theorem closes: since Z, ${:}Z^2{:}$, ${:}Z^3{:} \in C^{0-}$, the worst-behaved term is $K * {:}Z^3{:} \in C^{2-}$, so $v \in C^{2-}$ and the products Zv^2 and ${:}Z^2{:}\,v$ are of type $C^{2-} \times C^{0-}$, hence well defined.

In $d = 3$ these products are no longer defined, and even continuing the expansion does not yield any improvement of the regularity.

Second session 14/10

Parabolic Hölder–Besov spaces and a Kolmogorov theorem

Speaker: Haokai Xie

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1 Overview: the fixed point problem

Take the Φ^4_d equation with the term $-\varphi$ dropped, and read the non-linearity and the noise together as a force f:

$$\partial_t \varphi = \Delta \varphi \overbrace{- \varphi^3 + \xi}^{f}.$$

How do we solve the heat equation with force f?

1.1 Variation of constants

Formally, with the heat kernel

$$K(t,x) = (4\pi t)^{-d/2} \exp\Big(- \frac{|x|^2}{4t} \Big),$$

the solution is

$$\begin{aligned} \varphi &= K * f + \underbrace{(\text{initial } \ldots)}_{=0 \text{ (assumed)}} \\ &= -K * (\varphi^3) + \underbrace{K * \xi}_{=: Z \text{ (SHE)}}. \end{aligned}$$

1.2 Banach fixed point theorem

So φ should be a fixed point of the map

$$\Psi \colon (\mathcal{X}, d) \longrightarrow (\mathcal{X}, d), \qquad \psi \longmapsto -K * (\psi^3) + Z.$$

Banach fixed point theorem. If Ψ is a contraction, i.e. there exists $q \in (0,1)$ such that

$$d\big(\Psi(\psi_1), \Psi(\psi_2)\big) \le q\, d(\psi_1, \psi_2),$$

then there exists a unique $\psi^* \in (\mathcal{X}, d)$ such that $\psi^* = \Psi(\psi^*)$, and for any $\psi_0 \in \mathcal{X}$,

$$\psi^* = \lim_{n \to \infty} \psi_n \qquad \text{where} \qquad \psi_n := \Psi(\psi_{n-1}) \quad \forall\, n \in \mathbb{N}.$$

Implement it. Start the iteration from $\varphi_0 = 0$:

$$\begin{aligned} \varphi_1 &= \Psi(\varphi_0) = -K * (0^3) + Z = Z, \\ \varphi_2 &= \Psi(\varphi_1) = -K * (Z^3) + Z. \end{aligned}$$

Is Z^3 well defined?

1.3 Dimension matters a lot

"FACT": $Z \in \mathcal{C}_{\mathfrak{s}}^{(1-\frac{d}{2})-}$, the parabolic Hölder–Besov space of Section 3 (for the exponent see Section 4.1 of the first session). The answer depends on the dimension:

	$d=1$	$d=2$	$d=3$
is Z a function? (§2.2)	$\frac{1}{2}-$	$0-$	$(-\frac{1}{2})-$
is Z^3 defined?	✓	×	×
how to fix?	☺	Wick powers (§2.3.1)	Wick powers and renormalization (§2.3.2)
does the fixed point close?	☺	Da Prato–Debussche (§2.1)	Regularity structures!

Goal for today. To prepare for $Z \in \mathcal{C}_{\mathfrak{s}}^{(1-\frac{d}{2})-}$, we need to

1. define $\mathcal{C}_{\mathfrak{s}}^{\alpha}$ for $\alpha \in \mathbb{R}$;
2. check that $\xi \in \mathcal{C}_{\mathfrak{s}}^{-\frac{d}{2}-1-}$ a.s.

(And Schauder estimates tell you: integrating in time raises the regularity by 2.)

2 Parabolic scaling

The heat equation $\partial_t u = \Delta u$ is invariant under

$$\hat{u}(t,x) = u(\lambda^2 t, \lambda x),$$

i.e. shrink time by λ^2 when shrinking space by λ. Another way to see the same scaling is Brownian motion:

$$B_{\lambda t} \overset{\text{law}}{=} \lambda^{1/2} B_t \sim N(0, \lambda t).$$

2.1 Parabolic distance

For $z = (t,x)$ and $z' = (t',x')$ in $\mathbb{R} \times \mathbb{R}^d$ set

$$\|z' - z\|_{\mathfrak{s}} := |t'-t|^{1/2} + \sum_{j=1}^{d} |x'_j - x_j|,$$

and define the parabolic dimension

$$d_{\mathfrak{s}} := d + 2,$$

since time accounts for 2.

2.2 Parabolic degree of a polynomial

For a multi-index $k = (k_0, k_1, \dots, k_d) \in \mathbb{N}^{d+1}$ and $z = (t,x) \in \mathbb{R} \times \mathbb{R}^d$, write

$$z^k = t^{k_0} x_1^{k_1} \cdots x_d^{k_d}, \qquad \deg(z^k) = |k|_{\mathfrak{s}} := 2k_0 + \sum_{j=1}^{d} k_j,$$

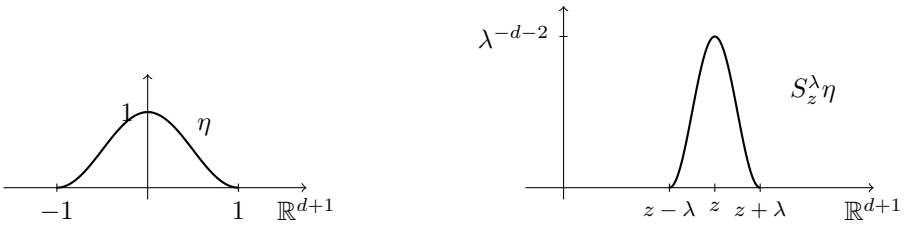
and for a polynomial P, $\deg P(z)$ is the maximum over its monomials.

2.3 Rescaled test functions

For a test function η, a point $z = (t, x)$ and $\lambda > 0$, let

$$S_z^\lambda \eta(s, y) = \lambda^{-d-2}\, \eta\Big(\frac{s-t}{\lambda^2}, \frac{y-x}{\lambda}\Big) :$$

η centered at z and zoomed in. The factor λ^{-d-2} comes from the change of variables and keeps the area unchanged; the result is supported in $\mathbb{B}_{\mathfrak{s}}(z, \lambda)$, the ball of radius λ around z for the parabolic distance $\|\cdot\|_{\mathfrak{s}}$. In words: shift and amplify.



Claim. $\int S_z^\lambda \eta = \int \eta$. Indeed, substituting $\tau = \frac{s-t}{\lambda^2}$, $ds = \lambda^2\, d\tau$ and $\zeta = \frac{y-x}{\lambda}$, $dy = \lambda^d\, d\zeta$,

$$\begin{aligned}\int_{\mathbb{R}^{d+1}} S_z^\lambda \eta(s, y)\, ds\, dy &= \int_{\mathbb{R}^{d+1}} \lambda^{-d-2}\, \eta\Big(\frac{s-t}{\lambda^2}, \frac{y-x}{\lambda}\Big)\, ds\, dy \\ &= \int_{\mathbb{R}^{d+1}} \lambda^{-d-2}\, \eta(\tau, \zeta)\, \lambda^{2+d}\, d\tau\, d\zeta = \int \eta.\end{aligned}$$

3 Parabolic Hölder–Besov spaces

3.1 Parabolic Hölder space, $\alpha \in (0, 1)$

$u \in \mathcal{C}_{\mathfrak{s}}^\alpha(\mathbb{R} \times \mathbb{R}^d)$ if $u\colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is continuous and for every compact $\mathfrak{K} \subseteq \mathbb{R}^{d+1}$ one has

$$\sup_{\substack{z, z' \in \mathfrak{K} \\ z \neq z'}} \frac{|u(z) - u(z')|}{\|z - z'\|_{\mathfrak{s}}^\alpha} < \infty.$$

So u is locally uniformly α-Hölder. We focus on compact sets because of the SPDE: finite time, compact space.

3.2 Parabolic Besov space, $\alpha < 0$

$u \in \mathcal{C}_{\mathfrak{s}}^\alpha$ if $u \in \mathcal{S}'(\mathbb{R}^{d+1})$ and for every compact $\mathfrak{K} \subseteq \mathbb{R}^{d+1}$,

$$\|u\|_{\mathcal{C}_{\mathfrak{s}}^\alpha(\mathfrak{K})} := \sup_{z \in \mathfrak{K}} \sup_{\substack{\eta \in B^r \\ \lambda \in (0,1]}} \left| \frac{\langle u, S_z^\lambda \eta \rangle}{\lambda^\alpha} \right| < \infty, \qquad r := \lceil -\alpha \rceil.$$

The set B^r, $r > 0$. It consists of the $\eta\colon \mathbb{R}^{d+1} \to \mathbb{R}$ such that

1) η is smooth;
2) $\operatorname{supp}(\eta) \subseteq \mathbb{B}_{\mathfrak{s}}(0, 1)$;
3) $\|\eta\|_{C^r} \leq 1$, where

$$\|\eta\|_{C^r} := \sup_{|k| \leq r} \sup_{x \in \mathbb{R}^{d+1}} |D^k \eta(x)|,$$

with $k = (k_0, k_1, \dots, k_d) \in \mathbb{N}^{d+1}$, $|k| = \sum_{j=0}^d k_j$ and $D^k f(x) = \partial_{x_0}^{k_0} \cdots \partial_{x_d}^{k_d} f(x)$.

3.3 Examples

The Dirac mass: $\delta_0 \in \mathcal{C}_{\mathfrak{s}}^{-d-2}$. Indeed, for $z=(t,x)$,

$$\langle \delta_0, S_z^\lambda \eta\rangle = (S_z^\lambda \eta)(0) = \lambda^{-d-2}\,\eta\Big(\frac{-t}{\lambda^2}, \frac{-x}{\lambda}\Big) \asymp \lambda^{-d-2} \quad \text{as } \lambda \downarrow 0$$

(for $z=0$).

A power of the parabolic distance: $z \mapsto \|z\|_{\mathfrak{s}}^\alpha \in \mathcal{C}_{\mathfrak{s}}^\alpha$ if $0 < -\alpha < d+2$. Let $u(z) = \|z\|_{\mathfrak{s}}^\alpha$ and test at $z=0$. With $\lambda^2\,ds = dt$ and $\lambda^d\,dy = dx$,

$$\begin{aligned}
\langle u, S_0^\lambda \eta\rangle &= \int_{\mathbb{R}^{d+1}} u(z)\,\lambda^{-d-2}\,\eta\big(t/\lambda^2, x/\lambda\big)\,dt\,dx \\
&= \lambda^{-d-2}\int_{\mathbb{R}^{d+1}} \Big(|t|^{1/2} + \sum_{j=1}^d |x_j|\Big)^\alpha \eta\big(t/\lambda^2, x/\lambda\big)\,dt\,dx \\
&= \int_{\mathbb{R}^{d+1}} \Big[(\lambda^2|s|)^{1/2} + \lambda\sum_{j=1}^d |y_j|\Big]^\alpha \eta(s,y)\,ds\,dy = \lambda^\alpha \int_{\mathbb{R}^{d+1}} u(s,y)\,\eta(s,y)\,ds\,dy.
\end{aligned}$$

Since η is supported in the unit ball, the last integral is over $\mathbb{B}_{\mathfrak{s}}(0,1)$, and with the Jacobian of the change to the parabolic radius r,

$$\int_{\mathbb{B}_{\mathfrak{s}}(0,1)} u(s,y)\,\eta(s,y)\,ds\,dy \;\lesssim\; \int_0^1 r^\alpha\, r^{d_{\mathfrak{s}}-1}\,dr = \int_0^1 r^{\alpha+d+1}\,dr < \infty \quad \text{iff} \quad \alpha + d + 1 > -1$$

$$\text{iff} \quad -\alpha < d+2.$$

3.4 The case $\alpha \geq 0$, $\alpha \notin \mathbb{N}$

Let $u \in \mathcal{S}'(\mathbb{R}^{d+1})$. We say $u \in \mathcal{C}_{\mathfrak{s}}^\alpha$ if there exist polynomials $\{P_z\}_{z\in\mathbb{R}^{d+1}}$ with $\deg_{\text{para}}(P_z) < \alpha$ such that for any compact $\mathfrak{K} \subseteq \mathbb{R}\times\mathbb{R}^d$ one has

$$\|u\|_{\mathcal{C}_{\mathfrak{s}}^\alpha(\mathfrak{K})} := \sup_{z\in\mathfrak{K}} \sup_{\substack{\eta\in B^0 \\ \lambda\in(0,1]}} \left|\frac{\langle u - P_z, S_z^\lambda\eta\rangle}{\lambda^\alpha}\right| < \infty.$$

Connection with the old definition ($\alpha \in (0,1)$). With $\tau = \frac{s-t}{\lambda^2}$ and $\zeta = \frac{y-x}{\lambda}$,

$$\begin{aligned}
\frac{\langle u - P_z, S_z^\lambda\eta\rangle}{\lambda^\alpha} &= \frac{1}{\lambda^\alpha}\int_{\mathbb{R}^{d+1}} \big(u(s,y) - P_z(s,y)\big)\,\lambda^{-d-2}\,\eta\Big(\frac{s-t}{\lambda^2}, \frac{y-x}{\lambda}\Big)\,ds\,dy \\
&= \int_{\mathbb{B}_{\mathfrak{s}}(0,1)} \frac{u(t+\lambda^2\tau, x+\lambda\zeta) - P_z(t+\lambda^2\tau, x+\lambda\zeta)}{\lambda^\alpha}\,\eta(\tau,\zeta)\,d\tau\,d\zeta,
\end{aligned}$$

and if $\alpha \in (0,1)$ the polynomial P_z is the constant $u(t,x)$, so the quotient under the integral is

$$\frac{u(t+\lambda^2\tau, x+\lambda\zeta) - u(t,x)}{\lambda^\alpha},$$

the Hölder quotient of Section 3.1.

4 Kolmogorov continuity theorem

4.1 Classical statement

Kolmogorov continuity theorem (classical). Let $X = (X_t)_{t\in\mathbb{R}}$ be a random process taking values in a complete metric space $(\mathcal{X}, d)$. If there exist $\alpha \in (0,1]$ and $p > 1/\alpha$ such that

$$\mathbb{E}\big[d(X_t, X_s)^p\big] \lesssim |t-s|^{\alpha p} \qquad \forall\, s,t \geq 0,$$

i.e. $\|X_t - X_s\|_{L^p} \lesssim |t-s|^\alpha$, then there exists a continuous $\tilde{X}$ such that
1) $\tilde{X} = X$ a.s.;

2) $\tilde{X}$ is locally α'-Hölder continuous for all $\alpha' \in \big(0, \alpha - \frac{1}{p}\big)$.

- Remark: if $X = (X_t)_{t\in\mathbb{R}^d}$ and $p > d/\alpha$, then $\alpha' \in \big(0, \alpha - \frac{d}{p}\big)$.

4.2 Version for distributions

Theorem 2.7 (Kolmogorov continuity theorem for distributions). Let ξ be a random distribution, i.e. $\xi\colon \mathcal{S}(\mathbb{R}^{d+1}) \to \mathbb{R}$ linear. If there exist $\alpha < 0$ and $p \geq 1$ such that

$$\sup_{z\in\mathbb{R}^{d+1}} \sup_{\eta} \mathbb{E}\big|\xi(\eta_z^\lambda)\big|^p \lesssim \lambda^{\alpha p} \qquad \forall\, \lambda \in (0,1],$$

where $\eta_z^\lambda = S_z^\lambda \eta$ and the inner supremum runs over $\eta \in \mathcal{S}(\mathbb{R}^{d+1})$ with $\operatorname{supp}(\eta) \subseteq \mathbb{B}_{\mathfrak{s}}(0,1)$ and $\|\eta\|_\infty \leq 1$, then there exists $\tilde{\xi} \in \mathcal{S}'(\mathbb{R}^{d+1})$ such that
1) $\tilde{\xi}(\eta) = \xi(\eta)$ a.s.;

2) for any compact $\mathfrak{K} \subseteq \mathbb{R}\times\mathbb{R}^d$,

$$\mathbb{E}\Big[\|\tilde{\xi}\|^p_{\mathcal{C}_{\mathfrak{s}}^{\alpha'}(\mathfrak{K})}\Big] < \infty \qquad \forall\, \alpha' < \alpha - \frac{d_{\mathfrak{s}}}{p} = \alpha - \frac{d+2}{p}.$$

4.3 Sketch of proof

Classical case: why $\alpha' < \alpha - \frac{1}{p}$? It suffices to consider $[-1,1]$. The Hölder constant is controlled by the increments on dyadic scales:

$$\sup_{\substack{s\neq t\\ s,t\in[-1,1]}} \frac{|X_s - X_t|}{|s-t|^{\alpha'}} \lesssim \sup_{k\geq 0} \sup_{s\in 2^{-k}\mathbb{Z}\cap[-1,1]} 2^{k\alpha'} \big|X_{s+2^{-k}} - X_s\big|. \tag{$*$}$$

Taking p-th moments and bounding the supremum by the sum,

$$\begin{aligned}
\mathbb{E}\big[(\dots)^p\big] &\leq \sum_{k\geq 0} \sum_{s\in 2^{-k}\mathbb{Z}\cap[-1,1]} 2^{kp\alpha'}\, \mathbb{E}\Big[\big|X_{s+2^{-k}} - X_s\big|^p\Big] \\
&= \sum_{k\geq 0} \sum_{s\in 2^{-k}\mathbb{Z}\cap[-1,1]} 2^{kp\alpha' - k\alpha p} \underbrace{\mathbb{E}\left[\frac{|X_{s+2^{-k}} - X_s|^p}{2^{-k\alpha p}}\right]}_{\text{uniformly bounded, by assumption}} \\
&\lesssim \sum_{k=0}^{\infty} \sum_{s\in 2^{-k}\mathbb{Z}\cap[-1,1]} 2^{kp\alpha' - k\alpha p} = \sum_{k=0}^{\infty} 2^k\, 2^{kp\alpha' - k\alpha p} = \sum_{k=0}^{\infty} 2^{-k(\alpha p - 1 - \alpha' p)} < \infty
\end{aligned}$$

iff $\alpha p - 1 - \alpha' p > 0$, that is iff $\alpha' < \alpha - \frac{1}{p}$. (After the bound by assumption the summand no longer depends on s, and there are about 2^k points s.)

Distributions. "$(*)$" holds as well:

$$\|\xi\|_{\mathcal{C}^{\alpha'}_{\mathfrak{s}}(\bar{\mathfrak{K}})} \le \sup_{k\ge 0} \sup_{z} 2^{k\alpha'} \left|\left(\xi, S^{2^{-k}}_z \eta\right)\right|, \qquad z \in \left(\frac{\mathbb{Z}}{2^{2k}} \times \frac{\mathbb{Z}^d}{2^k}\right) \cap \bar{\mathfrak{K}},$$

where $\bar{\mathfrak{K}}$ is another compact set containing $\mathfrak{K}$, and a careful choice of η is needed. The rest follows.

5 Regularity of white noise

5.1 White noise is in $\mathcal{C}_{\mathfrak{s}}^{-\frac{d+2}{2}-}$

Scale. By the covariance of ξ and the substitution of Section 2.3,

$$\mathbb{E}\Big[\langle \xi, S^\lambda_z \eta\rangle^2\Big] = \int \Big(\lambda^{-d-2}\, \eta\Big(\frac{s-t}{\lambda^2}, \frac{y-x}{\lambda}\Big)\Big)^2 \, ds\, dy = \int_{\mathbb{B}_{\mathfrak{s}}(0,1)} \lambda^{-d-2}\, \eta(\tau,\zeta)^2 \, d\tau\, d\zeta \;\lesssim\; \lambda^{-d-2}.$$

Use all p-th moments. Since $\langle \xi, S^\lambda_z \eta\rangle$ is Gaussian,

$$\mathbb{E}\Big[\langle \xi, S^\lambda_z \eta\rangle^p\Big] \;\lesssim\; \mathbb{E}\Big[\langle \xi, S^\lambda_z \eta\rangle^2\Big]^{p/2} \;\lesssim\; \lambda^{-\frac{d+2}{2}\cdot p},$$

so by Theorem 2.7 (Section 4.2)

$$\xi \in \mathcal{C}_{\mathfrak{s}}^{-\frac{d+2}{2}-\frac{d+2}{p}} \quad \forall p \ge 1 \qquad \Longrightarrow \qquad \xi \in \mathcal{C}_{\mathfrak{s}}^{-\frac{d+2}{2}-}.$$

5.2 The same argument for Brownian motion

Use that all p-th moments are finite for Gaussians, together with the Kolmogorov continuity theorem. With $p = 4$ this gives γ-Hölder continuity for

$$\left(\tfrac{1}{2} - \tfrac{1}{4}\right)- \;=\; \tfrac{1}{4} - .$$

In general,

$$\mathbb{E}\big[|B_t - B_s|^p\big] \lesssim \mathbb{E}\big[|B_t - B_s|^2\big]^{p/2} = |t-s|^{\frac{1}{2}\cdot p},$$

thus B is γ-Hölder for $\gamma \in \left(0, \frac{1}{2} - \frac{1}{p}\right)$, for all $p > 0$, and therefore for all $\gamma \in \left(0, \frac{1}{2}\right)$.

Third session 21/10

Duhamel's formula, Schauder estimates and the graphical notation

Speaker: Juan Pablo Chávez Ochoa

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1 Reminder of the classical Duhamel principle (variation of constants)

1.1 ODEs

For the linear equation

$$u'(t) = a\,u(t) + f(t), \qquad u(0) = u_0,$$

a simple manipulation (multiply by e^{-at} and integrate) gives the solution

$$u(t) = e^{at}u_0 + \int_0^t e^{a(t-s)}\,f(s)\,ds.$$

1.2 PDEs: the heat equation

The analogue is the heat equation with a force f and zero initial condition,

$$\partial_t u = \Delta u + f, \qquad u(0,\cdot) = 0. \tag{HE}$$

First consider the homogeneous equation

$$(\partial_t - \Delta)u = 0. \tag{1}$$

Equation (1) is invariant under the scaling $x \mapsto \lambda x$, $t \mapsto \lambda^2 t$. This suggests a solution that spreads on scale $\sqrt{t}$ in space:

$$\Phi(t,x) := t^{-\alpha}\,v\Big(\frac{x}{\sqrt{t}}\Big).$$

With $z = x/\sqrt{t}$,

$$\int_{\mathbb{R}^d} \Phi(t,x)\,dx = t^{d/2-\alpha}\int_{\mathbb{R}^d} v(z)\,dz.$$

Since we want constant mass, take $\alpha = d/2$. Substituting into (1):

$$\Delta v + \tfrac{1}{2}\langle z, \nabla v\rangle + \tfrac{d}{2}\,v = 0.$$

Guess a radial solution, $v(z) = \omega(|z|)$, $r = |z|$:

$$\alpha\,\omega(r) + \tfrac{1}{2}\,r\,\omega'(r) + \omega''(r) + \frac{d-1}{r}\,\omega'(r) = 0 \quad\Longrightarrow\quad v(z) = C\,e^{-|z|^2/4}.$$

Normalizing its mass, we conclude that

$$\Phi(t,x) = \frac{1}{(4\pi t)^{d/2}}\exp\Big(-\frac{|x|^2}{4t}\Big),$$

and

$$\begin{cases} (\partial_t - \Delta)\Phi = 0 & \text{on } \mathbb{R}^d \times (0,\infty), \\ \Phi = \delta_0 & \text{on } \mathbb{R}^d \times \{t = 0\}. \end{cases}$$

(This Φ is the heat kernel written K in Section 1 of the second session; in this session K denotes its periodization, defined in Section 1.3.)

To solve

$$\begin{cases} (\partial_t - \Delta)u = f & \text{on } \mathbb{R}^d \times (0,\infty), \\ u(0,\cdot) = 0 & \text{on } \mathbb{R}^d \times \{t=0\}, \end{cases} \tag{P1}$$

observe that (under regularity conditions)

$$(\partial_t - \Delta)(\Phi * f) = \big((\partial_t - \Delta)\Phi\big) * f = \delta_0 * f = f, \qquad (\Phi * f)(0,\cdot) = 0,$$

therefore

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} \Phi(t-s, x-y)\, f(s,y)\, dy\, ds = \Phi * f\,(t,x).$$

1.3 Periodic boundary conditions

We may ignore boundary conditions by working with periodic boundary conditions:

$$\begin{cases} \text{spatial variable:} & x \in \mathbb{T}^d \cong [-\pi,\pi]^d, \\ \text{time variable:} & t \in \mathbb{R}. \end{cases} \tag{PBC}$$

On the torus, we may define

$$K(t,x) := \sum_{k \in 2\pi\mathbb{Z}^d} \frac{1}{(4\pi t)^{d/2}} \exp\Big(-\frac{|x-k|^2}{4t}\Big), \qquad t>0,\ x \in \mathbb{T}^d.$$

Under general regularity assumptions, the solution to (HE) is

$$u(t,x) = \int_0^t \int_{\mathbb{T}^d} K(t-s, x-y)\, f(s,y)\, dy\, ds \tag{2}$$

(with zero initial condition). We may extend $K \equiv 0$ on $\big((-\infty,0] \times \mathbb{T}^d\big) \setminus \{(0,0)\}$ and regard $K \in C^\infty\big(\mathbb{R} \times \mathbb{T}^d \setminus \{(0,0)\}\big)$. The Duhamel principle in this case shows that

$$(\partial_t - \Delta)K = \delta_{(0,0)}$$

in the distributional sense. Equivalently, $u = K * f$ is a solution of (HE) in the weak sense.

2 The stochastic heat equation and the Schauder estimate

2.1 SPDEs

Consider

$$\partial_t u = \Delta u + \xi \qquad \text{under (PBC)}.$$

We use (2) to provide a solution in the weak sense. Following the same reasoning as before, we may rewrite

$$(\partial_t - \Delta)u = \xi,$$

so we want to make sense of the expression

$$u(t,x) = (K * \xi)(t,x).$$

To justify these operations, we appeal to the Schauder estimate. Let

$$\Lambda_t = (0,t) \times \mathbb{T}^d, \qquad \Lambda = (0,\infty) \times \mathbb{T}^d.$$

2.2 The Schauder estimate

Theorem 2.8 (Schauder estimate). For all $f \in \mathcal{C}_{\mathfrak{s}}^{\alpha}(\Lambda)$, define

$$u(s,x) := \int_{\Lambda_s} K(s-r, x-y)\, f(r,y)\, dy\, dr,$$

interpreted in the distributional sense for $\alpha < 0$. Then, if $\alpha \notin \mathbb{Z}$, we have

$$\|u\|_{\mathcal{C}_{\mathfrak{s}}^{\alpha+2}(\Lambda_t)} \lesssim \|f\|_{\mathcal{C}_{\mathfrak{s}}^{\alpha}(\Lambda_t)}.$$

Here $\mathcal{C}_{\mathfrak{s}}^{\alpha}$ is the parabolic Hölder–Besov space of Section 3 of the second session.

2.3 Regularity of the solution

Hence, we may apply Duhamel's principle and propose the following as a solution to the SHE:

$$Z(t,x) = K * \xi\,(t,x) := \int_{\Lambda_t} K(t-s, x-y)\, \xi(s,y)\, dy\, ds.$$

We may define Z deterministically for each realization of white noise. From Theorems 2.7 (Section 4.2 of the second session, applied to ξ in Section 5 there) and 2.8, we have for all $\kappa > 0$:

$$\xi \in \mathcal{C}_{\mathfrak{s}}^{-\frac{d}{2}-1-\kappa}(\mathbb{R}\times\mathbb{T}^d) \quad \Longrightarrow \quad Z = K * \xi \in \mathcal{C}_{\mathfrak{s}}^{1-\frac{d}{2}-\kappa}(\mathbb{R}\times\mathbb{T}^d).$$

Observe that

$$\begin{cases} d = 1: & Z \text{ is a continuous function;} \\ d \geq 2: & Z \text{ should be a distribution.} \end{cases}$$

3 Graphical notation

We will introduce a new notation that allows us to handle more complex calculations with K and ξ.

3.1 Trees

We denote Z by $\uparrow$:

- vertex: single instance of ξ;
- edge: convolution with the heat kernel K.

Since Z in general need not be a function, we work with the general object

$$\big(\uparrow, S_z^{\lambda}\eta\big) := \int_{\Lambda}\int_{\Lambda} S_z^{\lambda}\eta(z_1)\, K(z_1 - z_2)\, dz_1\, \xi(dz_2),$$

with $S_z^{\lambda}\eta$ the rescaled test function of Section 2.3 of the second session. The covariance structure of ξ then yields

$$\mathbb{E}\Big[\big(\uparrow, S_z^{\lambda}\eta\big)^2\Big] = \int_{\Lambda^3} S_z^{\lambda}\eta(z_1)\, S_z^{\lambda}\eta(\bar{z}_1)\, K(z_1 - w)\, K(\bar{z}_1 - w)\, dw\, dz_1\, d\bar{z}_1. \tag{3}$$

We may represent this integral graphically with the following concept.

3.2 Kernels of a given order

Definition. For $\zeta \in \mathbb{R}$, $G\colon \mathbb{R}^{d+1} \setminus \{0\} \to \mathbb{R}$ is a *kernel of order* ζ if

$$\|G\|_\zeta := \sup_{0<\|z\|_{\mathfrak{s}}<1} \frac{|G(z)|}{\|z\|_{\mathfrak{s}}^\zeta} < \infty.$$

Lemma 2.10. The heat kernel $K(z)$ is of order $-(d_{\mathfrak{s}} - 2)$, where $d_{\mathfrak{s}} = d + 2$.

We may represent (3) as

$$\mathbb{E}\Big[\big(\uparrow, S_z^\lambda \eta\big)^2\Big] = \quad \underset{-d_{\mathfrak{s}}+2}{\circ} \!\!-\!\!\blacksquare\!\!-\!\! \underset{-d_{\mathfrak{s}}+2}{\circ}$$

- black square: integration over Λ;
- gray vertices: integration against $S_z^\lambda \eta$.

Graphically,

$$\mathbb{E}\big[\,\uparrow^2\,\big] = \overbrace{\circ\!\!-\!\!\blacksquare\!\!-\!\!\circ}^{\text{glue } \uparrow \text{ with } \uparrow}.$$

The resulting order is

$$\underbrace{(-d_{\mathfrak{s}}+2)}_{\circ\!-} + \underbrace{(-d_{\mathfrak{s}}+2)}_{-\!\circ} + \underbrace{d_{\mathfrak{s}}}_{\blacksquare} = -d_{\mathfrak{s}} + 4.$$

3.3 Reducing graphs

We may deal with more complex integrals at the level of graphs by recursively reducing the complexity of the graph. We use the following.

Lemma 2.11. Let K_1, K_2 be kernels of orders $\zeta_1, \zeta_2 \in \mathbb{R}$, with compact support. Then

$$\|K_1 K_2\|_{\zeta_1+\zeta_2} \le \|K_1\|_{\zeta_1} \|K_2\|_{\zeta_2},$$

so $K_1 K_2$ is of order $\zeta_1 + \zeta_2$. Furthermore, if

$$\zeta_1 \wedge \zeta_2 > -d_{\mathfrak{s}} \qquad \text{and} \qquad \zeta := \zeta_1 + \zeta_2 + d_{\mathfrak{s}} < 0,$$

then

$$\|K_1 * K_2\|_\zeta \lesssim \|K_1\|_{\zeta_1} \|K_2\|_{\zeta_2}.$$

3.4 The regularity of Z by dimension

The case $d \ge 3$. We apply this to $\mathbb{E}\big[(\uparrow, S_z^\lambda \eta)^2\big]$ when $d \ge 3$ (so that $d_{\mathfrak{s}} \ge 5$): Lemma 2.11 replaces the two kernels, integrated against each other over the black square, by one kernel of order $-d_{\mathfrak{s}} + 4$,

$$\underset{-d_{\mathfrak{s}}+2}{\circ}\!\!-\!\!\blacksquare\!\!-\!\!\underset{-d_{\mathfrak{s}}+2}{\circ} \lesssim \quad \overbrace{\underset{-d_{\mathfrak{s}}+4}{\circ\!\!-\!\!\circ}}^{\text{the two kernels convolved into one}}$$

It remains to test this kernel against the two copies of $S_z^\lambda \eta$: for η with compact support and $\zeta > -d_{\mathfrak{s}}$,

$$\int_\Lambda \int_\Lambda S_z^\lambda \eta(z_1)\, S_z^\lambda \eta(\bar{z}_1)\, \|z_1 - \bar{z}_1\|_{\mathfrak{s}}^\zeta \, dz_1 \, d\bar{z}_1 \;\lesssim\; \lambda^\zeta,$$

so $\mathbb{E}\big[(\text{ı}, S_z^\lambda \eta)^2\big] \lesssim \lambda^{-d_{\mathfrak{s}}+4} = \lambda^{2-d}$ (and all moments, since the pairing is Gaussian), and Theorem 2.7 gives $\text{ı} \in \mathcal{C}_{\mathfrak{s}}^{1-\frac{d}{2}-\kappa}$ again. In cases where $\zeta \geq 0$, $(\text{ı}, S_z^\lambda \eta)$ is not the right object to work with.

The case $d = 2$. Critical case, $-d_{\mathfrak{s}} + 4 = 0$. The inequality holds if we consider the right-hand side as a logarithmically diverging kernel:

$$\mathbb{E}\Big[\big(\text{ı}, S_z^\lambda \eta\big)^2\Big] \lesssim 1 + |\log \lambda|.$$

Since log grows slower than any polynomial, $\text{ı} \in \mathcal{C}_{\mathfrak{s}}^{-\kappa}$ for all $\kappa > 0$.

The case $d = 1$. We need to subtract a suitable polynomial approximation. We work with $\big(\text{ı} - \text{ı}(z), S_z^\lambda \eta\big)$:

$$\mathbb{E}\Big[\big(\text{ı} - \text{ı}(z), S_z^\lambda \eta\big)^2\Big] \lesssim \lambda \quad \Longrightarrow \quad \text{ı} \in \mathcal{C}_{\mathfrak{s}}^{\frac{1}{2}-\kappa}.$$

A negative exponent describes a power singularity; zero is the logarithmic case; a positive exponent describes regularity only after subtracting the appropriate local approximation.

Fourth session

28/10

Wick powers and a primer on iterated integrals

Speaker: Maria Galani

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1 The non-linear equation as a fixed point problem

So far we have treated the linear case, using the variation of constants formula, or Duhamel's principle, to get a closed form solution, and Schauder estimates to guarantee that our solution makes sense regularity-wise. Now we want to introduce a non-linearity and try to figure out what the solution would be then.

1.1 Duhamel's principle for Φ^4_1

Consider the one-dimensional case,

$$\partial_t \varphi = \Delta \varphi - \varphi^3 \; (- m^2 \varphi) + \xi, \qquad \varphi(0, x) = 0. \tag{Φ^4_1}$$

Duhamel's principle gives

$$\varphi(t,x) = -\int_0^t \int_{\mathbb{T}^1} K(t-s, x-y)\, \varphi(s,y)^3 \, dy \, ds + \int_0^t \int_{\mathbb{T}^1} K(t-s, x-y)\, \xi(s,y) \, dy \, ds. \tag{1}$$

Here we removed the linear term for simplicity. In particular this is not a closed form solution, since we have φ on both sides: if we wanted to find φ, we would have to solve this fixed point problem. For this let us study the space a bit more closely.

Recall from Section 2.3 of the third session that

$$\int_0^t \int_{\mathbb{T}^1} K(t-s, x-y)\, \xi(s,y) \, dy \, ds \;\in\; \mathcal{C}_{\mathfrak{s}}^{\frac{1}{2}-\kappa}(\mathbb{R}_+ \times \mathbb{T}^1) \qquad \forall\, \kappa > 0,$$

while

$$\int_0^t \int_{\mathbb{T}^1} K(t-s, x-y)\, \varphi(s,y)^3 \, dy \, ds \;\in\; ?$$

In which space does the second integral term lie?

1.2 Multiplication of Hölder–Besov distributions

Theorem. Let $\alpha + \beta > 0$. Then there exists a bilinear form $B \colon \mathcal{C}_{\mathfrak{s}}^{\alpha} \times \mathcal{C}_{\mathfrak{s}}^{\beta} \to \mathcal{C}_{\mathfrak{s}}^{\alpha \wedge \beta}$ such that

(1) for smooth functions f, g: $B(f,g)$ is the pointwise product;

(2) for $f \in \mathcal{C}_{\mathfrak{s}}^{\alpha}$, $g \in \mathcal{C}_{\mathfrak{s}}^{\beta}$:

$$\|B(f,g)\|_{\mathcal{C}_{\mathfrak{s}}^{\alpha \wedge \beta}} \lesssim \|f\|_{\mathcal{C}_{\mathfrak{s}}^{\alpha}} \|g\|_{\mathcal{C}_{\mathfrak{s}}^{\beta}}.$$

If $\alpha + \beta \leq 0$, there is no bilinear form $B \colon \mathcal{C}_{\mathfrak{s}}^{\alpha} \times \mathcal{C}_{\mathfrak{s}}^{\beta} \to \mathcal{C}_{\mathfrak{s}}^{\alpha \wedge \beta}$ satisfying (1) and (2).

So in particular, if the sum of the regularities of the elements of the parabolic Hölder–Besov spaces is positive, then there is a canonical way of defining the product between them (function with distribution, function with function, etc.).

1.3 The contraction in $d = 1$

In particular we see that, with 0 as our initial data, φ^3 is defined without difficulty in this space (the theorem applied twice, the regularities $\frac{1}{2} - \kappa$ having positive sum): for every $\omega \in \Omega$ there exists $T > 0$ such that

$$\Psi\colon\ \varphi \longmapsto \int_0^{\cdot}\int_{\mathbb{T}^1} K(\cdot - s, \cdot - y)\,\xi(s,y)\,dy\,ds - \int_0^{\cdot}\int_{\mathbb{T}^1} K(\cdot - s, \cdot - y)\,\varphi^3(s,y)\,dy\,ds$$

is a contraction on bounded balls of $\mathcal{C}_{\mathfrak{s}}^{\frac{1}{2}-\kappa}([0,T]\times\mathbb{T}^1)$.

1.4 The Picard iteration for $d \geq 2$

For $d \geq 2$ (with $\mathbb{T}^1$ replaced by $\mathbb{T}^d$),

$$\int_0^{\cdot}\int_{\mathbb{T}^d} K(\cdot - s, \cdot - y)\,\xi(s,y)\,dy\,ds \ \in\ \mathcal{C}_{\mathfrak{s}}^{\frac{2-d}{2}-\kappa},$$

a distribution of negative regularity. Start the Picard iteration from $\varphi_0 = 0$:

$$\varphi_1 = \Psi(\varphi_0) = \text{\Large\textupsilon},$$
$$\varphi_2 = \text{\Large\textupsilon} - \int_0^{\cdot}\int_{\mathbb{T}^d} K(\cdot - s, \cdot - y)\,\big(\text{\Large\textupsilon}\big)^3(s,y)\,ds\,dy.$$

How does $\big(\text{\Large\textupsilon}\big)^3$ make sense?

2 Wick powers

To make sense of $\big(\text{\Large\textupsilon}\big)^3$, a power of a distribution, we can mollify it. Let

$$\rho_\delta(t,x) = \delta^{-(2+d)}\,\rho(\delta^{-2}t, \delta^{-1}x),$$

with ρ a smooth function integrating to 1, be our mollifier (as in Section 4.2 of the first session). Define

$$\text{\Large\textupsilon}_\delta(z) := \text{\Large\textupsilon} * \rho_\delta(z) = \int_{(0,\infty)\times\mathbb{T}^d} K * \rho_\delta(z - \bar{z})\,\xi(d\bar{z}).$$

For every $\delta > 0$, $\text{\Large\textupsilon}_\delta$ is smooth, so $\big(\text{\Large\textupsilon}_\delta\big)^3$ can be defined! The integral against $\xi(d\bar{z})$ is a stochastic integral; the next section makes this notation precise.

3 Stochastic iterated integrals

We would like to develop an integral formalism for integrating against space-time white noise. We will follow a more general setting than $\mathbb{R}\times\mathbb{T}^d$.

3.1 White noise on a measure space

Let $(E, \mathcal{E}, \mu)$ be a measure space with μ a σ-finite, non-atomic measure. We want to extend our original setting beyond smooth functions, to $\varphi \in L^2(E)$, as hinted at by the fact that the pairing is centered Gaussian. Consider a centered Gaussian family of random variables (ξ, φ), $\varphi \in L^2(E,\mu)$, with

$$\mathbb{E}\big[\langle\xi,\varphi_1\rangle\langle\xi,\varphi_2\rangle\big] = \int_E \varphi_1(x)\,\varphi_2(x)\,d\mu(x)$$

(before, μ was the Lebesgue measure on $\mathbb{R} \times \mathbb{T}^d$).

Properties. For $A \in \mathcal{E}$ with $\mu(A) < \infty$ write $\xi(A) := \xi(\mathbf{1}_A) = (\xi, \mathbf{1}_A)$. Then

- $\mathbb{E}[\xi(A)] = 0$ and $\mathbb{E}[\xi(A)^2] = \mu(A)$;
- $A_1, A_2 \in \mathcal{E}$ with $A_1 \cap A_2 = \emptyset \implies \xi(A_1)$, $\xi(A_2)$ independent;
- $(A_i)_{i\in\mathbb{N}} \subseteq \mathcal{E}$ pairwise disjoint $\implies \xi\big(\biguplus_i A_i\big) = \sum_i \xi(A_i)$ in $L^2(\Omega, \mathbb{P})$.

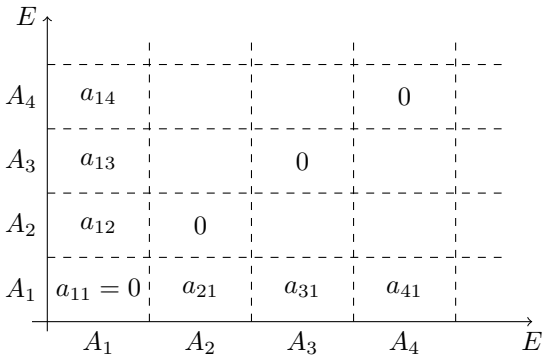
Although it kind of looks like ξ could be interpreted as a random signed measure, it does not have such a modification (cf. the Besov regularity of ξ, Section 5 of the second session).

3.2 Elementary functions and the iterated integral for $n = 2$

We construct "$\int_{E^n} f(z_1, \dots, z_n)\, \xi(dz_1) \cdots \xi(dz_n)$" for $f \in L^2(E^n, \mu^{\otimes n})$. Take $n = 2$. First let f be elementary, i.e. $f\colon E \times E \to \mathbb{R}$,

$$f = \sum_{\substack{j,k=1 \\ j\neq k}}^{N} a_{j,k}\, \mathbf{1}_{A_j \times A_k},$$

for A_j pairwise disjoint with $\mu(A_j) < \infty$ and $a_{j,k} \in \mathbb{R}$:



Note: $f(z, z) = 0$ for all $z \in E$. Define

$$\int_{E\times E} f(z_1, z_2)\, \xi(dz_1)\, \xi(dz_2) := \sum_{\substack{j,k=1 \\ j\neq k}}^{N} a_{j,k}\, \xi(A_j)\, \xi(A_k).$$

3.3 The extended Itô isometry

For such an f,

$$\mathbb{E}\Big[\Big(\int_{E\times E} f(z_1, z_2)\, \xi(dz_1)\, \xi(dz_2)\Big)^2\Big] = \mathbb{E}\Big[\Big(\sum_{\substack{i,j=1 \\ i\neq j}}^{N} a_{ij}\, \xi(A_i)\, \xi(A_j)\Big)^2\Big]$$

$$= \sum_{\substack{i,j,k,l=1 \\ i\neq j,\ k\neq l}}^{N} a_{ij}\, a_{kl}\, \mathbb{E}\big[\xi(A_i)\, \xi(A_j)\, \xi(A_k)\, \xi(A_l)\big].$$

The $\xi(A_i)$ are jointly Gaussian with

$$\mathbb{E}\big[\xi(A_i)\,\xi(A_j)\big] = \begin{cases} 0 & i \neq j, \\ \mu(A_i) & i = j, \end{cases}$$

so the fourth moment splits into pairings, of which the pairing of i with j vanishes since $i \neq j$:

$$\begin{aligned}
&= \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \sum_{\substack{k,l=1 \\ k\neq l}}^{N} a_{ij}\,a_{kl}\,\big[\delta_{ik}\,\mu(A_i)\,\delta_{jl}\,\mu(A_j) + \delta_{il}\,\mu(A_i)\,\delta_{jk}\,\mu(A_j)\big] \\
&= \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \big(a_{ij}^2\,\mu(A_i)\,\mu(A_j) + a_{ij}\,a_{ji}\,\mu(A_i)\,\mu(A_j)\big) \\
&= \int_{E\times E} f(z_1,z_2)^2\,\mu(dz_1)\,\mu(dz_2) + \int_{E\times E} f(z_1,z_2)\,f(z_2,z_1)\,\mu(dz_1)\,\mu(dz_2) \\
&= 2\int_{E\times E} \Big(\tfrac{1}{2}\big(f(z_1,z_2)+f(z_2,z_1)\big)\Big)^2\,d\mu(z_1)\,d\mu(z_2) \qquad \text{(symmetrization)} \qquad (*) \\
&\le 2\int_{E\times E} f(z_1,z_2)^2\,d\mu(z_1)\,d\mu(z_2).
\end{aligned}$$

(Indeed, $\big(\frac{1}{2}(f(z_1,z_2)+f(z_2,z_1))\big)^2 = \frac{f(z_1,z_2)^2+f(z_2,z_1)^2}{4} + \frac{f(z_1,z_2)\,f(z_2,z_1)}{2}$, and by Cauchy–Schwarz the integral of the last term is $\le \int_{E\times E} \frac{f(z_1,z_2)^2}{2}\,d\mu(z_1)\,d\mu(z_2)$, so the whole integral is $\le \int_{E\times E} f(z_1,z_2)^2\,d\mu(z_1)\,d\mu(z_2)$.)

3.4 The square of a stochastic integral

Since μ is non-atomic, the off-diagonal simple functions are dense in $L^2(E\times E, \mu\otimes\mu)$, so by $(*)$ the definition extends to all $f \in L^2(E\times E, \mu\otimes\mu)$. Likewise every $g \in L^2(E,\mu)$ is approximated by simple functions,

$$g \approx \sum_{i=1}^{N} a_i\,\mathbf{1}_{A_i}.$$

Then

$$\Big(\int_E g(x)\,\xi(dx)\Big)^2 \approx \underbrace{\sum_{\substack{i,j=1 \\ i\neq j}}^{N} a_i\,a_j\,\xi(A_i)\,\xi(A_j)}_{\to\ \int_{E\times E} g(x_1)\,g(x_2)\,\xi(dx_1)\,\xi(dx_2)\ \text{(finer partition)}} + \sum_{i=1}^{N} a_i^2\,\xi(A_i)^2.$$

For the diagonal term,

$$\mathbb{E}\Big[\sum_{i=1}^{N} a_i^2\,\xi(A_i)^2\Big] = \sum_{i=1}^{N} a_i^2\,\mu(A_i) \;\approx\; \int_E g(x)^2\,\mu(dx),$$

$$\mathbb{E}\Big[\Big(\sum_{i=1}^{N} a_i^2\,\xi(A_i)^2\Big)^2\Big] = \sum_{i,j=1}^{N} a_i^2\,a_j^2\,\mathbb{E}\big[\xi(A_i)^2\,\xi(A_j)^2\big] = \Big(\sum_{i=1}^{N} a_i^2\,\mu(A_i)\Big)^2 + 2\sum_{i=1}^{N} a_i^4\,\mu(A_i)^2,$$

so its variance, $2\sum_i a_i^4\,\mu(A_i)^2$, vanishes as the partition gets finer. Hence

$$\Big(\int_E g(x)\,\xi(dx)\Big)^2 = \int_{E\times E} g(x)\,g(y)\,\xi(dx)\,\xi(dy) + \int_E g(x)^2\,\mu(dx). \qquad (2)$$

3.5 The one-dimensional case: Itô's formula

Take $E = \mathbb{R}$, $\mu = \lambda$ the Lebesgue measure and $f = \mathbf{1}_{[0,t]}$. Then

$$\Big(\int_{\mathbb{R}} \mathbf{1}_{[0,t]}(s)\, \xi(ds) \Big)^2 = \int_{\mathbb{R}\times\mathbb{R}} \mathbf{1}_{[0,t]}(s_1)\, \mathbf{1}_{[0,t]}(s_2)\, \xi(ds_1)\, \xi(ds_2) + \int_{\mathbb{R}} \mathbf{1}_{[0,t]}(s)^2\, ds,$$

that is,

$$W_t^2 = 2 \int_0^t \int_0^{s_2} dW_{s_1}\, dW_{s_2} + t,$$

Itô's formula for W_t^2. Similarly for the third power,

$$\Big(\int_E g(z)\, \xi(dz) \Big)^3 = \int_{E^3} g(z_1)\, g(z_2)\, g(z_3)\, \xi(dz_1)\, \xi(dz_2)\, \xi(dz_3) + 3 \int_E g(z)^2\, \mu(dz) \int_E g(z)\, \xi(dz). \tag{3}$$

3.6 Higher powers and Hermite polynomials

For general n: by this construction one can relate the n-th power of the stochastic integral to the n-fold iterated integral through a Hermite polynomial,

$$\int \cdots \int \prod_{j=1}^n f(z_j)\, \xi(dz_j) = H_n\Big(\int f(z)\, \xi(dz),\ \|f\|_{L^2(\mu)} \Big),$$

with H_n the Hermite polynomials:

$$H_0(Z,\sigma) = 1, \qquad H_1(Z,\sigma) = Z, \qquad H_2(Z,\sigma) = Z^2 - \sigma^2, \qquad H_3(Z,\sigma) = Z^3 - 3\sigma^2 Z.$$

3.7 Nelson's estimate

$$\mathbb{E}\Big[\Big| \int f(z_1, \dots, z_n)\, \xi(dz_1) \cdots \xi(dz_n) \Big|^p \Big] \le C_{n,p} \Big(\int f(z_1, \dots, z_n)^2\, \mu(dz_1) \cdots \mu(dz_n) \Big)^{p/2}. \tag{$\circledast$}$$

This is a consequence of hypercontractivity.

4 The cube of Z_δ as an iterated integral

In particular we interpret $\big(\uparrow_\delta\big)^3$ as an iterated stochastic integral and use the previous identity (3). It holds

$$\big(\uparrow_\delta(y)\big)^3 = \int \mathcal{W}_\delta^{(3)}(y; z_1, z_2, z_3)\, \xi(dz_1)\, \xi(dz_2)\, \xi(dz_3) + 3 \int \mathcal{W}_\delta^{(1)}(y; z_1)\, \xi(dz_1),$$

with

$$\mathcal{W}_\delta^{(3)}(y; z_1, z_2, z_3) = \prod_{i=1}^3 K * \rho_\delta(y - z_i), \qquad \mathcal{W}_\delta^{(1)}(y; z_1) = K * \rho_\delta(y - z_1) \cdot \int \big(K * \rho_\delta(y - z)\big)^2\, dz.$$

With graphical notation,

$$\big(\uparrow_\delta\big)^3 = \Psi_\delta + 3 \ \bigcirc_\delta \uparrow_\delta,$$

where Ψ_δ is integrated against space-time white noise (three noise vertices), and the black square of the loop is a dummy variable, which is integrated out.

Here ρ regularizes K at its scale, i.e.

$$|K * \rho_\delta(z)| \lesssim \frac{1}{(\|z\|_{\mathfrak{s}} + \delta)^{d_{\mathfrak{s}} - 2}}, \qquad d_{\mathfrak{s}} = d + 2 \text{ the scaled dimension.}$$

Fifth session

04/11

Wick powers (continuation), the renormalized Picard iteration and the Da Prato–Debussche argument

Speaker: Gianmarco Del Pino

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1 The story leading up to the convergence of $:Z_\delta^3:$

1.1 The problem: Φ_d^4 as a fixed point

The problem is Φ_d^4,

$$\partial_t \varphi = \Delta \varphi - \varphi^3 + \xi \qquad \text{on } \mathbb{R}_+ \times \mathbb{T}^d$$

($d = 2$ and $d = 3$ are those of interest: for $d = 2$ the Da Prato–Debussche argument works; for $d = 3$ it fails and one needs regularity structures). It is solved as a fixed point, $\Psi(\varphi) = \varphi$, where

$$\Psi\colon\ \varphi \longmapsto \int_0^{\cdot} \int_{\mathbb{T}^d} K(\cdot - s, \cdot - y)\, \xi(dy, ds) - \int_0^{\cdot} \int_{\mathbb{T}^d} K(\cdot - s, \cdot - y)\, \varphi^3(s, y)\, dy\, ds.$$

This arises from Duhamel's formula as follows. Duhamel is

$$u(t, x) = \int_0^t \int_{\mathbb{T}^d} K(t - s, x - y)\, f(s, y)\, dy\, ds + \int_{\mathbb{T}^d} K(t, x - y)\, u_0(y)\, dy,$$

arising from the problem

$$\begin{cases} \partial_t u = \Delta u + f, \\ u(0, \cdot) = u_0. \end{cases}$$

Replacing $f = -\varphi^3 + \xi$ and $u_0 \equiv 0$,

$$\varphi(t, x) = \int_0^t \int_{\mathbb{T}^d} K(t - s, x - y)\, \xi(dy, ds) - \int_0^t \int_{\mathbb{T}^d} K(t - s, x - y)\, \varphi^3(s, y)\, dy\, ds,$$

i.e.

$$\varphi(t, x) = Z(t, x) - \int_0^t \int_{\mathbb{T}^d} K(t - s, x - y)\, \varphi^3(s, y)\, dy\, ds \qquad (Z = \text{↑}).$$

1.2 The case $d = 1$

In $d = 1$ the map Ψ is a contraction on $\mathcal{C}_{\mathfrak{s}}^{\frac{1}{2}-\kappa}([0, T] \times \mathbb{T}^1)$ for all κ (Section 1.3 of the fourth session). Moreover,

$$\varphi(t, x) - Z(t, x) = -\int_0^t \int_{\mathbb{T}^d} K(t - s, x - y)\, \varphi^3(s, y)\, dy\, ds$$

is more regular than φ, since by the Schauder estimate, for the Picard iterates φ_n,

$$\varphi_{n-1}^3 \in \mathcal{C}_{\mathfrak{s}}^{\frac{1}{2}-\kappa}(\cdots) \quad \Longrightarrow \quad -\int_0^t \int_{\mathbb{T}^d} K(t - s, x - y)\, \varphi_{n-1}^3(s, y)\, dy\, ds \in \mathcal{C}_{\mathfrak{s}}^{\frac{5}{2}-\kappa}(\cdots).$$

Why is $\varphi_{n-1}^3 \in \mathcal{C}_{\mathfrak{s}}^{\frac{1}{2}-\kappa}$? Because in $d = 1$, $\varphi_{n-1} \in \mathcal{C}_{\mathfrak{s}}^{\frac{1}{2}-\kappa}$ on $\Lambda_T \Longrightarrow \varphi_{n-1}^3 \in \mathcal{C}_{\mathfrak{s}}^{\frac{1}{2}-\kappa}$ on Λ_T. So

$$\varphi_n(t, x) - Z(t, x) \in \mathcal{C}_{\mathfrak{s}}^{\frac{5}{2}-\kappa} \quad \text{on } \Lambda_T \text{ in } d = 1.$$

Idea. $\varphi_0 = Z \in \mathcal{C}_{\mathfrak{s}}^{\frac{1}{2}-\kappa} \Longrightarrow \varphi_1 - Z \in \mathcal{C}_{\mathfrak{s}}^{\frac{5}{2}-\kappa} \Longrightarrow \varphi_1 \in \mathcal{C}_{\mathfrak{s}}^{\frac{1}{2}-\kappa} \Longrightarrow \varphi_1^3 \in \mathcal{C}_{\mathfrak{s}}^{\frac{1}{2}-\kappa}$.

However, in $d \geq 2$ the fixed point becomes hard to solve, and the Picard iteration

$$\begin{cases} \varphi_{n+1} = \Psi(\varphi_n), \\ \varphi_1 = Z \end{cases}$$

doesn't work. Instead, we regularize, observe the divergence and renormalize.

1.3 Why does Picard fail in $d \geq 2$?

In $d \geq 2$, $Z \in \mathcal{C}_{\mathfrak{s}}^{1-\frac{d}{2}-\kappa}$. The first iterate is $\varphi_1 = Z$. The second, $\varphi_2 = \Psi(Z)$, needs Z^3, which cannot be made sense of yet $\Longrightarrow$ the fixed point analysis breaks down.

- Why isn't there a notion of Z^3, Z^2, when Z is a distribution?

1.4 Regularization step

Consider $\xi_\delta = \xi * \rho_\delta$, where $\rho_\delta = \delta^{-(2+d)}\rho(\delta^{-2}t, \delta^{-1}x)$. One might be tempted to consider the PDE

$$\partial_t \varphi_\delta = \Delta \varphi_\delta - \varphi_\delta^3 + \xi_\delta,$$

but $\varphi_\delta \to 0$ as $\delta \downarrow 0$. (The cube is to be replaced by $\varphi_\delta^3 - 3C_\delta \varphi_\delta$, Section 5. Heuristically: $\partial_t \varphi_\delta = -3C_\delta \varphi_\delta$ gives $\varphi_\delta = e^{-3C_\delta t}$, and $C_\delta \to \infty$.) Instead, we try to define $\text{\Large\textupsilon}^3$ by approximation.

2 Iterated integrals

2.1 The Hermite identity

Goal. Give meaning to $\big(\text{\Large\textupsilon}_\delta\big)^3$:

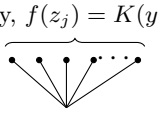
$$\big(\text{\Large\textupsilon}_\delta\big)^3(y) = (Z * \rho_\delta)^3(y) = \Big(\int (K * \rho_\delta)(y - z)\, \xi(dz) \Big)^3.$$

Here μ is the Lebesgue measure (not called λ, because λ is already used in $S_z^\lambda \eta$). As ξ is not a random measure, the above does not in general equal

$$\iiint (K * \rho_\delta)(y - z_1)\, (K * \rho_\delta)(y - z_2)\, (K * \rho_\delta)(y - z_3)\, \xi(dz_1)\, \xi(dz_2)\, \xi(dz_3).$$

Instead, the Hermite identities provide the right formula:

$$H_n\Big(\int f(z)\, \xi(dz),\ \|f\|_{L^2(\mu)} \Big) = \int \cdots \int \prod_{j=1}^{n} f(z_j)\, \xi(dz_1) \cdots \xi(dz_n)$$



We prove the case $n = 3$ and write it in graphical notation (general n is similar).

2.2 Proof of the Hermite identity for $n = 3$

Let $(A_j)_{j \in S}$ be a partition of space-time $= E$. Then consider

$$f(x) = \sum_{j \in S} \alpha_j\, \mathbf{1}_{A_j}(x).$$

Then we see that $\big(\int f(x)\,\xi(dx)\big)^3$ results in a series of sums, which one may re-organize as follows:

$$\begin{aligned}
\Big(\int f\,d\xi\Big)^3 &= \sum_{j,k,\ell\in S} \alpha_j\alpha_k\alpha_\ell\,\xi(A_j)\,\xi(A_k)\,\xi(A_\ell) \\
&= \sum_{\substack{j,k,\ell\in S\\ \text{distinct}}} \alpha_j\alpha_k\alpha_\ell\,\xi(A_j)\,\xi(A_k)\,\xi(A_\ell) && \text{all indices distinct,} \\
&\quad + 3\sum_{\substack{j,\ell\in S\\ j\neq\ell}} \alpha_j^2\alpha_\ell\,\xi(A_j)^2\,\xi(A_\ell) && \text{two indices match, } \tbinom{3}{2}=3 \text{ combos,} \\
&\quad + \sum_{j\in S} \alpha_j^3\,\xi(A_j)^3 && \text{all indices matching.}
\end{aligned}$$

Middle sum. The logic is to split $\xi(A_j)^2$ into rest and mean parts. That is, we write

$$\xi(A_j)^2 = \big(\xi(A_j)^2 - \mu(A_j)\big) + \mu(A_j), \qquad \text{where } \mu(A_j) = \mathbb{E}\big[\xi(A_j)^2\big]$$

(μ would be Lebesgue measure if $(E,\mu) = (\mathbb{R}\times\mathbb{T}^d, \mathrm{Leb})$). Then the middle sum equals

$$3\Big(\sum_{\substack{j,\ell\in S\\ j\neq\ell}} \alpha_j^2\alpha_\ell\,\mu(A_j)\,\xi(A_\ell)\Big) + 3\Big(\sum_{\substack{j,\ell\in S\\ j\neq\ell}} \alpha_j^2\alpha_\ell\,\big(\xi(A_j)^2-\mu(A_j)\big)\,\xi(A_\ell)\Big),$$

the mean part and the rest. Expanding the mean part (adding and subtracting the terms with $j=\ell$),

$$\begin{aligned}
3\Big(\sum_{\substack{j,\ell\in S\\ j\neq\ell}} \alpha_j^2\alpha_\ell\,\mu(A_j)\,\xi(A_\ell)\Big) &= 3\Big(\sum_{j\in S}\alpha_j^2\,\mu(A_j)\Big)\Big(\sum_{\ell\in S}\alpha_\ell\,\xi(A_\ell)\Big) - 3\sum_{\ell\in S}\alpha_\ell^3\,\mu(A_\ell)\,\xi(A_\ell) \\
&= 3\Big(\int f(x)^2\,\mu(dx)\Big)\Big(\int f\,d\xi\Big) - 3\sum_{\ell\in S}\alpha_\ell^3\,\mu(A_\ell)\,\xi(A_\ell).
\end{aligned}$$

Only the middle sum needs to be re-adjusted, for a reason we are about to elucidate:

$$\begin{aligned}
\Big(\int f\,d\xi\Big)^3 &= \sum_{\substack{j,k,\ell\in S\\ \text{distinct}}} \alpha_j\alpha_k\alpha_\ell\,\xi(A_j)\,\xi(A_k)\,\xi(A_\ell) && \text{(A)} \\
&\quad + 3\Big(\int f(x)^2\,\mu(dx)\Big)\Big(\int f(x)\,\xi(dx)\Big) && \text{(B)} \\
&\quad + 3\Big(\sum_{\substack{j,\ell\in S\\ j\neq\ell}} \alpha_j^2\alpha_\ell\,\big(\xi(A_j)^2-\mu(A_j)\big)\,\xi(A_\ell)\Big) && \text{(C)} \\
&\quad + \sum_{j\in S}\alpha_j^3\,\big(\xi(A_j)^3 - 3\mu(A_j)\,\xi(A_j)\big). && \text{(D)}
\end{aligned}$$

The expected values of (C) and (D) are zero, and as the partition gets finer one sees that the variances of these sums tend to 0:

- (C), (D) $\to 0$ in $L^2(\Omega)$ as the partition is refined, i.e. $\max_j \mu\big(A_j^{(k)}\big) \to 0$ as $k\to\infty$;
- (A) $\to \iiint f(x_1)\,f(x_2)\,f(x_3)\,\xi(dx_1)\,\xi(dx_2)\,\xi(dx_3)$.

So that, with $f(x_i) = (K * \rho_\delta)(y - x_i)$ in mind,

$$\Big(\int f\, d\xi\Big)^3 = \iiint f(x_1)\, f(x_2)\, f(x_3)\, \xi(dx_1)\, \xi(dx_2)\, \xi(dx_3) + 3 \underbrace{\Big(\int f^2(x)\, \mu(dx)\Big)}_{C_\delta} \Big(\int f(x)\, \xi(dx)\Big).$$

The law of large numbers for (C). Only diagonal terms survive:

$$\mathbb{E}\Big[\Big(\sum_{\substack{j,\ell\in S\\ j\neq\ell}} \alpha_j^2\alpha_\ell \left(\xi(A_j)^2 - \mu(A_j)\right) \xi(A_\ell)\Big)^2\Big] = \sum_{j\neq\ell} 2\alpha_j^4\, \mu(A_j)^2\, \alpha_\ell^2\, \mu(A_\ell)$$

$$\leq 2\Big(\max_j \alpha_j^2\, \mu(A_j)\Big)\Big(\int f^2\, d\mu\Big)^2 \longrightarrow 0$$

as the partition is refined.

2.3 Application to the kernel of Z_δ

Application to $f(x) = K * \rho_\delta(y - x)$, after fixing y:

$$\iiint K * \rho_\delta(y - x_1)\, K * \rho_\delta(y - x_2)\, K * \rho_\delta(y - x_3)\, \xi(dx_1)\, \xi(dx_2)\, \xi(dx_3)$$
$$= \underbrace{\Big(\int K * \rho_\delta(y - x)\, \xi(dx)\Big)^3}_{(1)} - 3 \underbrace{\Big(\int \big(K * \rho_\delta(y - x)\big)^2\, dx\Big)\Big(\int K * \rho_\delta(y - x)\, \xi(dx)\Big)}_{(2)}.$$

Right-hand side. (1) $= (Z_\delta(y))^3$; (2) $= C_\delta\, Z_\delta(y)$.

Remark. C_δ is independent of y by change of variables, translation invariance of $\mathbb{R} \times \mathbb{T}^d$ and the periodic boundary conditions of K.

Left-hand side. $= :Z_\delta^3:(y)$. Hence $(Z_\delta(y))^3 = :Z_\delta^3:(y) + 3C_\delta\, Z_\delta(y)$.

Connection to graphical notation. $\big(\uparrow_\delta\big)^3(y) = (Z_\delta(y))^3$; $\Psi_\delta = :Z_\delta^3:$; $C_\delta = \bigcirc_\delta$. In other words,

$$\big(\uparrow_\delta\big)^3(y) = \Psi_\delta(y) + 3C_\delta\, \uparrow_\delta(y).$$

3 Why are those the right symbols?

The subscript δ on a graph means "convolve with K_δ", where $K_\delta := K * \rho_\delta$.

- $\big(\uparrow_\delta\big)^3$ is obvious.
- $C_\delta = \bigcirc_\delta = \int K_\delta(y - z)^2\, dz = \int \underbrace{K_\delta(y - z)}_{\text{loop 1}}\, \underbrace{K_\delta(y - z)}_{\text{loop 2}}\, dz$: the black square is the variable z, which is integrated out, and the two halves of the loop are the two kernels.
- $\Psi_\delta(y) = \iiint \prod_{i=1}^3 K_\delta(y - z_i)\, \xi(dz_1)\, \xi(dz_2)\, \xi(dz_3)$: joining at the bottom means the kernels are within a big integral.

4 Convergence of the Wick cube for $d = 2$

$d = 2$, $d_{\mathfrak{s}} = 4$: we show $:Z^3_\delta: \to :Z^3:$ in $L^p(\Omega; \mathcal{C}^{-\kappa}_{\mathfrak{s}}(\mathfrak{K}))$.

The norm. Recall (Section 3.2 of the second session) that for $u \in \mathcal{S}'(\mathbb{R}^{d+1})$ and $\mathfrak{K} \subset\subset \mathbb{R}^{d+1}$ compact,

$$\|u\|_{\mathcal{C}^\alpha_{\mathfrak{s}}(\mathfrak{K})} = \sup_{z \in \mathfrak{K}} \sup_{\substack{\eta \in B^r \\ \lambda \in (0,1]}} \left| \frac{\langle u, S^\lambda_z \eta \rangle}{\lambda^\alpha} \right| < \infty, \qquad r = \lceil -\alpha \rceil,$$

$$B^r := \Big\{ \eta \colon \mathbb{R}^{d+1} \to \mathbb{R} : \ \operatorname{supp}(\eta) \subseteq \mathbb{B}_{\mathfrak{s}}(0,1), \ \sup_{k \le r} \|D^k \eta\|_\infty \le 1 \Big\}.$$

Strategy. Fix $\delta, \delta' > 0$. By the extended Itô isometry (Section 3.3 of the fourth session, with 3! in place of the factor 2),

$$\mathbb{E}\Big[\big(:Z^3_\delta: - :Z^3_{\delta'}:, \, S^\lambda_z \eta \big)^2 \Big] = 3! \, \big\| F_{\delta,\lambda} - F_{\delta',\lambda} \big\|^2_{L^2(E^3)},$$

where

$$F_{\delta,\lambda}(z_1, z_2, z_3) := \int_E S^\lambda_z \eta(y) \prod_{i=1}^3 K_\delta(y - z_i) \, dy.$$

It may be shown that, for $\theta \in (0,1)$,

$$\big\| F_{\delta',\lambda} - F_{\delta,\lambda} \big\|^2_{L^2(E^3)} \begin{cases} \lesssim (\delta \vee \delta')^{2\theta} \, \lambda^{-3-2\theta}, & d = 3 \quad \text{(regularity structures)}, \\ \lesssim (\delta \vee \delta')^{2\theta} \, \lambda^{-2\theta} \big(1 + |\log \lambda| \big)^2, & d = 2 \quad \text{(Da Prato–Debussche)}. \end{cases}$$

Hence, for $d = 2$,

$$\mathbb{E}\Big[\big(:Z^3_\delta: - :Z^3_{\delta'}:, \, S^\lambda_z \eta \big)^2 \Big] \lesssim (\delta \vee \delta')^{2\theta} \, \lambda^{-2\theta} \big(1 + |\log \lambda| \big)^2.$$

Now, for every $\kappa' > 0$, $\lambda \mapsto \lambda^{2\kappa'} (1 + |\log \lambda|)^2$ is bounded on $(0,1]$, so

$$\mathbb{E}\left[\Big(\frac{:Z^3_\delta: - :Z^3_{\delta'}:}{(\delta \vee \delta')^\theta}, \, S^\lambda_z \eta \Big)^2 \right] \lesssim \lambda^{-2\theta - 2\kappa'}.$$

Since $:Z^3_\delta: - :Z^3_{\delta'}:$ lies in the third chaos, Nelson's estimate (Section 3.7 of the fourth session) applies, and so

$$\mathbb{E}\left[\left| \Big(\frac{:Z^3_\delta: - :Z^3_{\delta'}:}{(\delta \vee \delta')^\theta}, \, S^\lambda_z \eta \Big) \right|^p \right] \lesssim \lambda^{-\theta p - \kappa' p}.$$

By Kolmogorov (Theorem 2.7),

$$\mathbb{E}\Big[\big\| :Z^3_\delta: - :Z^3_{\delta'}: \big\|^p_{\mathcal{C}^{\alpha'}_{\mathfrak{s}}} \Big] \lesssim (\delta \vee \delta')^{\theta p} \qquad \forall \, \alpha' < -\theta - \kappa' - \frac{4}{p}.$$

Here $\theta \in (0,1)$ is free, $\kappa' > 0$ is free, and by equivalence of moments p is irrelevant. So $\big(:Z^3_\delta: \big)_{\delta > 0}$ is Cauchy in $L^p(\Omega; \mathcal{C}^{-\kappa}_{\mathfrak{s}}(\mathfrak{K}))$ for all $p < \infty$ and $\kappa > 0$, and we write $:Z^3:$ for the limit:

$$\Psi := \lim_{\delta \downarrow 0} \big(\Psi^3_\delta - 3 C_\delta \Psi_\delta \big), \qquad C_\delta \sim \begin{cases} \log(\delta^{-1}), & d = 2, \\ \delta^{-1}, & d = 3. \end{cases}$$

5 Picard and the Da Prato–Debussche argument

5.1 The naive equation and the Wick substitution

Start with the naive equation

$$\partial_t \varphi_\delta = \Delta \varphi_\delta - \varphi_\delta^3 + \xi_\delta. \tag{$*$}$$

We may write $\varphi_\delta = Z_\delta + V_\delta$, where $\partial_t Z_\delta = \Delta Z_\delta + \xi_\delta$, and so

$$\partial_t V_\delta = \Delta V_\delta - \big(Z_\delta^3 + 3Z_\delta^2 V_\delta + 3Z_\delta V_\delta^2 + V_\delta^3\big).$$

With the Wick substitution $Z_\delta^3 = :Z_\delta^3: + 3C_\delta Z_\delta$ and $Z_\delta^2 = :Z_\delta^2: + C_\delta$ (pointwise identities),

$$\partial_t V_\delta = \Delta V_\delta - \underbrace{\big[:Z_\delta^3: + 3\,:Z_\delta^2: V_\delta + 3Z_\delta V_\delta^2 + V_\delta^3\big]}_{\text{converges as } \delta\to 0} - \underbrace{3C_\delta\,\varphi_\delta}_{\text{diverges, as } C_\delta\to\infty}.$$

So the naive approach fails.

5.2 The renormalized equation and the fixed point for the remainder

Solution. Add $3C_\delta\varphi_\delta$ to $(*)$:

$$\partial_t \varphi_\delta = \Delta \varphi_\delta - \big(\varphi_\delta^3 - 3C_\delta\varphi_\delta\big) + \xi_\delta.$$

The split is then $\varphi_\delta = \varv_\delta + V_\delta$, where $\partial_t \varv_\delta = \Delta \varv_\delta + \xi_\delta$ and

$$\begin{aligned}\partial_t V_\delta &= \Delta V_\delta - \big[:Z_\delta^3: + 3\,:Z_\delta^2: V_\delta + 3Z_\delta V_\delta^2 + V_\delta^3\big] \\ &= \Delta V_\delta - V_\delta^3 - 3\,\varv_\delta V_\delta^2 - 3\big(\varv_\delta^2 - C_\delta\big)V_\delta - \big(\varv_\delta^3 - 3C_\delta\,\varv_\delta\big).\end{aligned}$$

Passing to the limit $\delta \downarrow 0$ (in what sense?), Picard suggests that we should look for a solution to

$$V = K * \big[- V^3 - 3\,\varv\, V^2 - 3\,\varvee\, V - \varveebar\big], \tag{3.3}$$

with $\varv, \varvee, \varveebar \in \mathcal{C}_{\mathfrak{s}}^{-\kappa}$ for all $\kappa > 0$ small in $d = 2$. (The tag (3.3) is the number of this equation in Chandra–Weber.)

5.3 Identifying the space for the contraction

We would like to apply the Banach fixed point theorem to the operator $M\colon X \to X$ given by

$$M(V) = K * \big[- V^3 - 3\,\varv\, V^2 - 3\,\varvee\, V - \varveebar\big].$$

How may one identify X? Suppose $V \in \mathcal{C}_{\mathfrak{s}}^{\beta}$, β some number. First, in order for the products $:Z^2:V$, $Z\,V^2$ to make sense, we make use of Theorem 2.13 (Section 1.2 of the fourth session):

> "If $\alpha + \beta > 0$ with $f \in \mathcal{C}_{\mathfrak{s}}^{\alpha}$, $g \in \mathcal{C}_{\mathfrak{s}}^{\beta}$, then $\|fg\|_{\mathcal{C}_{\mathfrak{s}}^{\alpha\wedge\beta}} \lesssim \|f\|_{\mathcal{C}_{\mathfrak{s}}^{\alpha}} \|g\|_{\mathcal{C}_{\mathfrak{s}}^{\beta}}$."

In our case $:Z^2: \in \mathcal{C}_{\mathfrak{s}}^{-\kappa}$, so β should be at least κ for the products to be well defined. ($d = 3$ fails here: there $:Z^2: \in \mathcal{C}_{\mathfrak{s}}^{-1-\kappa}$ and $:Z^3: \in \mathcal{C}_{\mathfrak{s}}^{-\frac{3}{2}-\kappa}$, so V is no better than $\mathcal{C}_{\mathfrak{s}}^{\frac{1}{2}-\kappa}$ and the product $:Z^2:V$ is not covered by the theorem.)

So, by Theorem 2.13:

term	regularity
V^3	$2-\kappa$
$\varv V^2$	$-\kappa$
$\varv V$	$-\kappa$
$\varv$	$-\kappa$

$\Longrightarrow$ the overall regularity of $-V^3 - 3\varv V^2 - 3\varv V - \varv$ is $-\kappa$. By Schauder, $K * (\,\cdot\,) \in \mathcal{C}_{\mathfrak{s}}^{2-\kappa}$, so we would expect $\beta = 2-\kappa$, and so the contraction is employed on $X = \mathcal{C}_{\mathfrak{s}}^{2-\kappa}$.

5.4 The contraction and the convergence of φ_δ

Indeed, one may show that for $V_1, V_2 \in \mathcal{C}_{\mathfrak{s}}^{2-\kappa}$,

$$\|M(V_1) - M(V_2)\|_{\mathcal{C}_{\mathfrak{s}}^{2-\kappa}} \leq \tfrac{1}{2}\,\|V_1 - V_2\|_{\mathcal{C}_{\mathfrak{s}}^{2-\kappa}}.$$

So there exists a unique $V \in \mathcal{C}_{\mathfrak{s}}^{2-\kappa}$ satisfying the fixed point.

Going back, we now discuss the convergence of $Z_\delta \to Z$ and $V_\delta \to V$.

1) $Z_\delta \to Z$ in $L^q(\Omega; \mathcal{C}_{\mathfrak{s}}^{-\kappa})$ for all $q < \infty$, by a Kolmogorov-type argument.
2) $:Z_\delta^2: \to :Z^2:$ and $:Z_\delta^3: \to :Z^3:$ in $L^q(\Omega; \mathcal{C}_{\mathfrak{s}}^{-\kappa})$.

Then one may show that

$$\begin{aligned}\|V_\delta - V\|_{\mathcal{C}_{\mathfrak{s}}^{2-2\kappa}} &= \big\|K * [\text{Wick terms}]\big\|_{\mathcal{C}_{\mathfrak{s}}^{2-2\kappa}} \\ &\lesssim \|Z_\delta - Z\|_{\mathcal{C}_{\mathfrak{s}}^{-\kappa}} + \big\|:Z_\delta^3: - :Z^3:\big\|_{\mathcal{C}_{\mathfrak{s}}^{-\kappa}} + \big\|:Z_\delta^2: - :Z^2:\big\|_{\mathcal{C}_{\mathfrak{s}}^{-\kappa}},\end{aligned}$$

but we do *not* have $V_\delta \to V$ in $L^p(\Omega; \mathcal{C}_{\mathfrak{s}}^{-\kappa})$. As a result, for all $\varepsilon > 0$,

$$\lim_{\delta \downarrow 0} \mathbb{P}\big(\|V_\delta - V\|_{\mathcal{C}_{\mathfrak{s}}^{2-2\kappa}} > \varepsilon\big) = 0 \quad \Longrightarrow \quad \lim_{\delta \downarrow 0} \mathbb{P}\big(\|V_\delta - V\|_{\mathcal{C}_{\mathfrak{s}}^{-\kappa}} > \varepsilon\big) = 0.$$

(Moreover, L^q-convergence implies convergence in probability, so $\lim_{\delta\downarrow 0} \mathbb{P}(\|Z_\delta - Z\|_{\mathcal{C}_{\mathfrak{s}}^{-\kappa}} > \varepsilon) = 0$.) So

$$\mathbb{P}\big(\|\varphi_\delta - \varphi\|_{\mathcal{C}_{\mathfrak{s}}^{-\kappa}} > \varepsilon\big) \longrightarrow 0.$$

The solution to Φ_2^4 is then $\varphi = V + Z$, $\varphi \in \mathcal{C}_{\mathfrak{s}}^{-\kappa}$ in the distributional sense. That is, for all $f \in C_c^\infty(\mathbb{R}_+ \times \mathbb{T}^2)$,

$$\langle \varphi, -\partial_t f - \Delta f\rangle = -\langle :\varphi^3:, f\rangle + \langle \xi, f\rangle,$$

where $\langle \cdot, \cdot \rangle$ is the space-time pairing.

Sixth session 11/11

The Da Prato–Debussche argument (continuation), the abstract framework of regularity structures

Speaker: Alastair Crossley

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1 The Φ^4_d equation as a fixed point problem

Consider the stochastic partial differential equation

$$\partial_t \varphi = \Delta \varphi - \varphi^3 + \xi, \qquad \varphi(0,x) = 0, \tag{1}$$

where ξ is space-time white noise. Our goal is to establish existence and uniqueness of a suitable solution.

Recall that for the linear heat equation with forcing,

$$\partial_t u = \Delta u + f, \qquad u(0,\cdot) = 0,$$

Duhamel's formula gives $u = K * f$, where K is the heat kernel. Formally, equation (1) therefore becomes

$$\varphi = K * \xi - K * \varphi^3. \tag{2}$$

Both terms raise questions about regularity and the meaning of the product. Set $Z := K * \xi$ and define

$$\Psi(\varphi) := Z - K * \varphi^3.$$

If Ψ is a contraction on a suitable complete ball, the Banach fixed point theorem produces a unique fixed point satisfying (2).

1.1 The one-dimensional case

The Schauder estimate (Theorem 2.8, for $\alpha \notin \mathbb{Z}$) provides the regularizing property

$$K*\colon\ \mathcal{C}^\alpha_{\mathfrak{s}} \longrightarrow \mathcal{C}^{\alpha+2}_{\mathfrak{s}}.$$

White noise has regularity $\xi \in \mathcal{C}^{-\frac{d+2}{2}-\kappa}_{\mathfrak{s}}$, so

$$Z = K * \xi \in \mathcal{C}^{1-\frac{d}{2}-\kappa}_{\mathfrak{s}}.$$

For $d = 1$ this gives $Z \in \mathcal{C}^{\frac{1}{2}-\kappa}_{\mathfrak{s}}$: in particular, Z is a continuous function (locally, for small $\kappa > 0$). Products such as φ^3 can be defined, and on a sufficiently short time interval the fixed point argument closes in a suitable space of positive regularity.

1.2 The two-dimensional obstruction

For $d = 2$ we instead have

$$\xi \in \mathcal{C}^{-2-\kappa}_{\mathfrak{s}}, \qquad Z \in \mathcal{C}^{-\kappa}_{\mathfrak{s}}.$$

Thus Z has negative regularity and need not be a function. In particular, Z^2 and Z^3 are not canonically defined by the usual multiplication theory.

Introduce a smooth mollification $\xi_\delta = \xi * \rho_\delta$ and set $Z_\delta = K * \xi_\delta$. For each $\delta > 0$, Z_δ^2 and Z_δ^3 are well defined, but they do not converge as $\delta \downarrow 0$ without renormalization. For example, the variance counter-term

$$C_\delta := \mathbb{E}\big[Z_\delta(z)^2\big] \sim \log \frac{1}{\delta}$$

diverges logarithmically (up to constants and the precise setting). By the Itô isometry it is the constant $C_\delta = \int K_\delta(y-z)^2\,dz$ of Section 3 of the fifth session.

2 Wick powers and the Da Prato–Debussche decomposition

To remove these divergences, introduce the Wick powers

$$:Z^2: := \lim_{\delta\downarrow 0} \big(Z_\delta^2 - C_\delta\big), \tag{3}$$

$$:Z^3: := \lim_{\delta\downarrow 0} \big(Z_\delta^3 - 3C_\delta Z_\delta\big). \tag{4}$$

In dimension two, these renormalized random distributions have regularity $\mathcal{C}_{\mathfrak{s}}^{-\kappa}$ (for arbitrary sufficiently small $\kappa > 0$).

2.1 Separating the singular and regular parts

Write

$$\varphi_\delta = Z_\delta + v_\delta$$

and consider the renormalized equation

$$\partial_t \varphi_\delta = \Delta \varphi_\delta - \big(\varphi_\delta^3 - 3C_\delta \varphi_\delta\big) + \xi_\delta. \tag{5}$$

Since $\partial_t Z_\delta = \Delta Z_\delta + \xi_\delta$, subtraction gives

$$\begin{aligned}\partial_t v_\delta &= \Delta v_\delta - \big[(Z_\delta + v_\delta)^3 - 3C_\delta(Z_\delta + v_\delta)\big] \\ &= \Delta v_\delta - v_\delta^3 - 3Z_\delta v_\delta^2 - 3\big(Z_\delta^2 - C_\delta\big)v_\delta - \big(Z_\delta^3 - 3C_\delta Z_\delta\big).\end{aligned}$$

Passing formally to the limit yields

$$\partial_t v = \Delta v - v^3 - 3Zv^2 - 3\,:Z^2:\, v - :Z^3:, \tag{6}$$

with mild form

$$v = -K * \big(v^3 + 3Zv^2 + 3\,:Z^2:\, v + :Z^3:\big). \tag{7}$$

(This is the fixed point equation (3.3) of Section 5.2 of the fifth session, where the remainder was called V.) The Schauder estimate suggests $v \in \mathcal{C}_{\mathfrak{s}}^{2-\kappa}$. The mixed products are then meaningful, since a distribution of regularity $-\kappa$ can be multiplied by a function of regularity $2-\kappa$ when the sum is positive (Theorem 2.13). This is the key to the Da Prato–Debussche argument in $d = 2$.

Consequently $\varphi = Z + v$ is a distributional solution of the renormalized Φ_2^4 equation. The fixed point step, together with suitable stochastic estimates for the Wick powers, establishes the desired well-posedness.

3 Why the same argument fails in dimension three

For $d = 3$,

$$\xi \in \mathcal{C}_{\mathfrak{s}}^{-\frac{5}{2}-\kappa}, \qquad Z \in \mathcal{C}_{\mathfrak{s}}^{-\frac{1}{2}-\kappa},$$

and the corresponding renormalized powers have heuristic regularities

$$:Z^2: \in \mathcal{C}_{\mathfrak{s}}^{-1-\kappa}, \qquad :Z^3: \in \mathcal{C}_{\mathfrak{s}}^{-\frac{3}{2}-\kappa}.$$

The forcing $:Z^3:$ suggests $v \in \mathcal{C}_{\mathfrak{s}}^{\frac{1}{2}-\kappa}$ after convolution with K. However, the product $:Z^2:v$ would combine regularities

$$(-1-\kappa) + \left(\tfrac{1}{2}-\kappa\right) = -\tfrac{1}{2} - 2\kappa < 0.$$

The standard multiplication theorem (Theorem 2.13) therefore does not define this product.

3.1 Why subtracting further singular terms is not enough

One might remove the most singular contribution by defining

$$Y := K * :Z^3: \in \mathcal{C}_{\mathfrak{s}}^{\frac{1}{2}-\kappa}, \qquad \varphi = Z - Y + v.$$

Expanding the cube gives

$$\begin{aligned}(Z-Y+v)^3 = Z^3 - 3Z^2Y + 3Z^2v + 3ZY^2 - 6ZYv + 3Zv^2 \\ - Y^3 + 3Y^2v - 3Yv^2 + v^3.\end{aligned}$$

Even after suitable renormalization, the troublesome term $:Z^2:v$ remains: its regularities still add to a negative number. (If the product were defined, it would lie in $\mathcal{C}_{\mathfrak{s}}^{-1-\kappa}$, so at best $v \in \mathcal{C}_{\mathfrak{s}}^{1-\kappa}$, and $(-1-\kappa)+(1-\kappa) < 0$.) Continuing this finite expansion does not resolve the underlying difficulty.

4 From Taylor expansions to regularity structures

The key insight is that a classical Taylor expansion encodes local behavior using basis elements with known scaling. For a smooth function,

$$f(y) = f(x) + f'(x)(y-x) + \tfrac{1}{2}f''(x)(y-x)^2 + \cdots.$$

Under the spatial rescaling $y - x \mapsto \lambda(y-x)$, the monomial $(y-x)^k$ scales like λ^k. Its degree (or *homogeneity*; the *order* in Chandra–Weber) is therefore k.

Under parabolic scaling, time has degree 2 and each spatial coordinate degree 1. For a multi-index $k = (k_0, k_1, \ldots, k_d)$, the monomial X^k has homogeneity

$$|X^k| = |k|_{\mathfrak{s}} = 2k_0 + \sum_{j=1}^{d} k_j,$$

the parabolic degree of Section 2.2 of the second session. The stochastic object Z has negative homogeneity, heuristically $|Z| = -\frac{1}{2} - \kappa$ in $d = 3$. A regularity structure enlarges the polynomial basis to include abstract symbols representing singular stochastic objects and their combinations. Instead of requiring a solution to have a classical Taylor expansion, one describes it locally using these additional symbols.

4.1 Definition: a regularity structure

A *regularity structure* is a triple (A, T, G), where:

1. $A \subset \mathbb{R}$ is an indexing set of homogeneities (in the standard definition, locally finite and bounded below).
2. T is a graded vector space,
$$T = \bigoplus_{\alpha \in A} T_\alpha,$$
where each T_α is a Banach space (finite-dimensional in the polynomial examples). For $\tau \in T_\alpha$, write $|\tau| = \alpha$.
3. G is a group of continuous linear transformations on T such that, for every $\Gamma \in G$ and $\tau \in T_\alpha$,
$$\Gamma\tau - \tau \in \bigoplus_{\beta < \alpha} T_\beta.$$

The maps in G are called *structure group* elements; they describe changes of local expansion center. (This is Definition 3.2 of Chandra–Weber, who take every T_α finite-dimensional.)

The polynomial sector illustrates this group action. If $\mathbf{1}$ is the constant symbol and X a spatial coordinate symbol, translation by h acts by

$$\Gamma_h \mathbf{1} = \mathbf{1}, \qquad \Gamma_h X = X + h\mathbf{1},$$

and hence

$$\Gamma_h X^2 = X^2 + 2hX + h^2\mathbf{1}.$$

For instance, changing the center of expansion from x to a uses $h = a - x$ and

$$(y - x)^2 = (y - a)^2 + 2(a - x)(y - a) + (a - x)^2.$$

These transformations compose as $\Gamma_h \Gamma_k = \Gamma_{h+k}$, and $\Gamma_h^{-1} = \Gamma_{-h}$, so they form a group. (Chandra–Weber use the opposite sign, $\Gamma_h X^k := (X - h)^k$: their Γ_h is the Γ_{-h} above.)

What comes next. The abstract symbols alone do not specify distributions. A *model* assigns concrete distributions to symbols and makes their changes of base point consistent. Modelled distributions then play the role of locally varying Taylor coefficients, and reconstruction recovers a genuine distribution from such local data.

Seventh session 18/11

The abstract framework of regularity structures (continuation).
Models and modelled distributions

Speaker: Kevin Mancini

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1 From the Da Prato–Debussche argument to abstract expansions

We talked about Φ^4_d,

$$\partial_t \varphi = \Delta\varphi - \varphi^3 + \xi \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d, \qquad \mathrel{\mathsf{I}} = \int_{\mathbb{R}_+ \times \mathbb{T}^d} K(t-s, x-y)\, \xi(s,y)\, dy\, ds,$$

and saw that we are able to get a local solution in $d = 2$ through the Da Prato–Debussche argument.

- If $d = 2$: $\varphi = \mathrel{\mathsf{I}} + v$, where $\mathrel{\mathsf{I}}$ is less regular and v more regular $\Longrightarrow$ solve a fixed point problem.
- If $d = 3$: we fail, even if we decompose further,

$$\varphi = \mathrel{\mathsf{I}} - \mathrel{\mathsf{Y}} + \cdots + v, \qquad \mathrel{\mathsf{Y}} = K * \underbrace{:\mathrel{\mathsf{I}}^3:}_{\mathrel{\mathsf{Y}}}$$

  (the Y of the sixth session).

Intuition. We need a more complex expansion, with "coefficients" which change over space:

$$\Phi(z) = \Phi_{\mathrel{\mathsf{I}}}(z)\, \mathrel{\mathsf{I}} + \Phi_{\mathrel{\mathsf{Y}}}(z)\, \mathrel{\mathsf{Y}} + \cdots + \Phi_{\mathbf{1}}(z)\, \mathbf{1}.$$

Note. The $\mathrel{\mathsf{I}}$ and $\mathrel{\mathsf{Y}}$ of this expansion will actually be different objects: abstract symbols (colored blue in the slides). We solve a fixed point problem in an abstract setting to find this family of functions $(\Phi_{\mathrel{\mathsf{I}}}, \Phi_{\mathrel{\mathsf{Y}}}, \dots, \Phi_{\mathbf{1}})$, not just v!

2 Regularity structures

2.1 Definition

A *regularity structure* is a triple (A, T, G).

- $A \subseteq \mathbb{R}$ is an indexing set, bounded from below and with no accumulation points.
- $T = \bigoplus_{\alpha \in A} T_\alpha$, where each T_α is a finite-dimensional vector space coming with a basis and a norm.
- G is a family of linear transformations $\Gamma\colon T \to T$ such that for all $\alpha \in A$ and all $\tau \in T_\alpha$,

$$\Gamma\tau - \tau \in T_\alpha^-, \qquad T_\alpha^- := \bigoplus_{\beta < \alpha} T_\beta;$$

  also we want that G is a group under composition.

(This is Definition 3.2 as in Chandra–Weber; Section 4.1 of the sixth session stated it with Banach spaces T_α.)

2.2 Intuitions

1) A lists the "orders" of the objects which will appear in the expansion. Negative orders refer to objects which blow up at the base point, positive ones to objects which vanish there; $0 \in A$ by assumption, and it refers to constants.

   $\rightsquigarrow$ We want a most singular object (A is bounded from below).

   $\rightsquigarrow$ T_γ^- is always finite-dimensional (A has no accumulation points).

2) For $\alpha \in A$, an element $\tau \in T_\alpha$ is an abstract symbol which represents an object of order α; τ is called *homogeneous* and $|\tau| = \alpha$. The norm $\|\cdot\|_\alpha$ on T_α does not need to be fixed. For $\tau \in T$,
$$\|\tau\|_\alpha := \|\mathcal{Q}_\alpha \tau\|_\alpha, \qquad \text{where } \mathcal{Q}_\alpha \colon T \to T_\alpha \text{ is the projection.}$$

3) Γ adds terms of order $< \alpha$ and will be used to implement the changes of base point.

Idea. $\uparrow$ and Ψ will be abstract symbols, not distributions: they belong to T and are homogeneous, with specific orders $\alpha \in A$. G will be used later to introduce a concept of regularity for objects in T!

3 The example: polynomials

Idea. The expansion of the Da Prato–Debussche argument will become a "jet", namely a collection of "Taylor expansions" indexed by space-time base points.

3.1 Hölder functions and Taylor polynomials

Proposition. Let $f \colon \mathbb{R}^{d+1} \to \mathbb{R}$ and $\alpha > 0$. Then $f \in \mathcal{C}^\alpha_{\mathfrak{s}}(\mathbb{R}^{d+1})$ if and only if

(i) for all multi-indices j with $|j|_{\mathfrak{s}} = 2j_0 + \sum_{i=1}^d j_i < \alpha$, the derivative $D^j f$ exists and is continuous;

(ii) for every $z \in \mathbb{R}^{d+1}$ we have
$$\sup_{\bar z \colon 0 < \|\bar z - z\|_{\mathfrak{s}} \le 1} \frac{\left| f(\bar z) - \sum_{|k|_{\mathfrak{s}} < \alpha} \frac{1}{k!} D^k f(z) \, (\bar z - z)^k \right|}{\|\bar z - z\|_{\mathfrak{s}}^\alpha} \le C$$
(with C uniform over z in compact sets).

(Not hard to prove.)

Corollary. In the same setting, $f \in \mathcal{C}^\alpha_{\mathfrak{s}}(\mathbb{R}^{d+1})$ if and only if (i) holds and, for every multi-index j with $|j|_{\mathfrak{s}} < \alpha$ and every $z \in \mathbb{R}^{d+1}$,
$$\sup_{\bar z \colon 0 < \|\bar z - z\|_{\mathfrak{s}} \le 1} \left| D^j f(\bar z) - \sum_{|k|_{\mathfrak{s}} < \alpha - |j|_{\mathfrak{s}}} \frac{1}{k!} D^{j+k} f(z) \, (\bar z - z)^k \right| \le C \, \|\bar z - z\|_{\mathfrak{s}}^{\alpha - |j|_{\mathfrak{s}}}. \tag{3.9}$$

Proof. Just use the Proposition with $D^j f \in \mathcal{C}^{\alpha - |j|_{\mathfrak{s}}}_{\mathfrak{s}}(\mathbb{R}^{d+1})$. (The tag (3.9) is the number of this bound in Chandra–Weber.)

Intuition. For f there exists a family of polynomials which can describe f locally!

3.2 Abstract polynomials

We now start! The regularity structure is $\bar{\mathcal{T}} = (A, T, G)$ with $A = \mathbb{N}$, the possible parabolic degrees.

Definition (abstract polynomial). A polynomial $P(z) = \sum_k \tau_k z^k$ in $\mathcal{P}(\mathbb{R}^{d+1})$ is identified with the family of its coefficients $\tau = (\tau_k)_{k\in\mathbb{N}^{d+1}}$, written

$$\tau = \sum_{k\in\mathbb{N}^{d+1}} \tau_k X^k \quad \text{(a finite sum)}, \qquad \tau_k \in \mathbb{R},$$

where k is a multi-index with $d+1$ components and $X^k = (0, \dots, 0, 1, 0, \dots)$ has its 1 in the k-th place; in particular $X^0 = \mathbf{1}$. This τ is the abstract polynomial. We set

$$T := \bigoplus_{\ell\in\mathbb{N}} T_\ell, \qquad T_\ell := \operatorname{span}\{X^k : |k|_{\mathfrak{s}} = \ell\},$$

and $\tau_k = \langle \tau, X^k \rangle$ are the coefficients of the expansion of τ.

Note. Do not confuse $X^0 = \mathbf{1}$, of order zero, with $X_0 = X^{(1,0,\dots,0)}$, of order 2. Here

$$X_i := X^{e_i} = X^{(0,\dots,1,\dots,0)}, \qquad X^k = X_0^{k_0} \cdot X_1^{k_1} \cdot X_2^{k_2} \cdots X_d^{k_d}.$$

3.3 The lift of a Hölder function and the maps Π_z

Definition. To $f \in \mathcal{C}^\alpha_{\mathfrak{s}}$ we associate the expansion at the space-time point z,

$$F\colon \mathbb{R}^{d+1} \to T, \qquad F(z) = \sum_{|k|_{\mathfrak{s}}<\alpha} \frac{1}{k!} D^k f(z)\, X^k$$

(by the Proposition it is well defined).

Remark. In every point this expansion has more information than $f(z)$, which is $f(z) = \langle F(z), \mathbf{1}\rangle$: in fact, it has local information!

Definition. We define a linear map $\Pi_z\colon T \to \mathcal{S}'(\mathbb{R}^{d+1})$, which sends τ to a distribution (for now, a polynomial), by

$$\big(\Pi_z X^k\big)(\bar{z}) := (\bar{z} - z)^k$$

and extend by linearity.

Remarks.

- The map Π_z recovers polynomials from abstract polynomials.
- The Taylor approximation for f with base point z is $\big(\Pi_z F(z)\big)(\cdot)$.

4 Comparing expansions at different base points

4.1 The structure group

Problem. In this theory we already defined a notion of smoothness for $f \in \mathcal{C}^\alpha_{\mathfrak{s}}$ (including distributions, $\alpha < 0$); now we want to introduce a way to measure smoothness (boundedness) in terms of abstract expansions. Indeed, comparing $F(x) - F(y)$ makes little sense, as these expansions have different base points. The idea is an analogue of parallel transport:

$$F(x) - \Gamma_{xy} F(y).$$

Definition. For $h \in \mathbb{R}^{d+1}$ the linear map $\Gamma_h \colon T \to T$ is defined by

$$\Gamma_h X^k := (X - h\mathbf{1})^k = \prod_{i=0}^{d} (X_i - h_i \mathbf{1})^{k_i}$$

and extended by linearity. We set $G := \{\Gamma_h\}_{h \in \mathbb{R}^{d+1}}$; one can easily check that it satisfies the conditions. (This is the sign of Chandra–Weber: the Γ_h of Section 4.1 of the sixth session is the Γ_{-h} here.)

Proposition. If $\Gamma_{xy} := \Gamma_{y-x}$, then

$$\Pi_y \tau = \Pi_x \Gamma_{xy} \tau \qquad \forall\, \tau \in T,\ x, y \in \mathbb{R}^{d+1}.$$

Proof. Verify it for X^k; the rest is linearity:

$$\big(\Pi_y X^k\big)(z) = (z-y)^k, \qquad \Big(\Pi_x \big(X - (y-x)\mathbf{1}\big)^k\Big)(z) = \big((z-x) - (y-x)\big)^k = (z-y)^k. \qquad \square$$

4.2 A new characterization of $\mathcal{C}^\alpha_{\mathfrak{s}}$

Now we can introduce a new characterization of $\mathcal{C}^\alpha_{\mathfrak{s}}$, $\alpha \geq 0$.

Theorem 3.3. Let $\bar{\mathcal{T}} = (A, T, G)$ be the regularity structure of abstract polynomials in $d+1$ components, let $\alpha > 0$ and $f \colon \mathbb{R}^{d+1} \to \mathbb{R}$. Then $f \in \mathcal{C}^\alpha_{\mathfrak{s}}(\mathbb{R}^{d+1})$ if and only if there exists $F \colon \mathbb{R}^{d+1} \to T^-_\alpha$ such that $\langle F(z), \mathbf{1} \rangle = f(z)$ and, for every compact $\mathfrak{K} \subseteq \mathbb{R}^{d+1}$ and every $\beta \in A$ with $\beta < \alpha$,

$$\sup_{x \in \mathfrak{K}} \|F(x)\|_\beta + \sup_{\substack{x,y \in \mathfrak{K} \\ x \neq y}} \frac{\|F(x) - \Gamma_{xy} F(y)\|_\beta}{\|x - y\|_{\mathfrak{s}}^{\alpha - \beta}} < \infty.$$

That is: locally, F is uniformly bounded and has a smooth characterization.

Proof. ($\Longrightarrow$) By the Proposition there is the F of Section 3.3. First we consider a general $\beta < \alpha$:

$$\begin{aligned}
\Gamma_{xy} F(y) &= \sum_{|k|_{\mathfrak{s}} < \alpha} \frac{D^k f(y)}{k!} \big(X - (y-x)\mathbf{1}\big)^k \\
&= \sum_{|k|_{\mathfrak{s}} < \alpha} \sum_{j \leq k} \frac{D^k f(y)}{k!} \frac{k!}{(k-j)!\, j!} X^j (x-y)^{k-j} = \sum_j \Big[\sum_{\substack{k \geq j \\ |k|_{\mathfrak{s}} < \alpha}} \cdots \Big] X^j.
\end{aligned}$$

Apply $\mathcal{Q}_\beta$: only the j with $|j|_{\mathfrak{s}} = \beta$ remain. Update the indices: $k = j + m$, where $j \leq k$ gives $m \geq 0$ and $|j+m|_{\mathfrak{s}} < \alpha$ gives $|m|_{\mathfrak{s}} < \alpha - |j|_{\mathfrak{s}}$. The coefficient of X^j is

$$\langle \Gamma_{xy} F(y), X^j \rangle = \sum_{|m|_{\mathfrak{s}} < \alpha - \beta} \frac{D^{j+m} f(y)}{j!\, m!} (x-y)^m.$$

We now consider, for $|j|_{\mathfrak{s}} = \beta$, $\langle F(x), X^j \rangle = \frac{D^j f(x)}{j!}$, so that

$$\langle F(x) - \Gamma_{xy} F(y), X^j \rangle = \frac{1}{j!} \underbrace{\Big(D^j f(x) - \sum_{|m|_{\mathfrak{s}} < \alpha - \beta} \frac{D^{j+m} f(y)\, (x-y)^m}{m!} \Big)}_{=: R_j(x,y)}.$$

Using the ℓ^1 norm on T_β,

$$\frac{\|F(x) - \Gamma_{xy}F(y)\|_\beta}{\|x-y\|_{\mathfrak{s}}^{\alpha-\beta}} = \sum_{|j|_{\mathfrak{s}}=\beta} \frac{|R_j(x,y)|}{j!\,\|x-y\|_{\mathfrak{s}}^{\alpha-\beta}}.$$

Take the supremum over $\mathfrak{K}$.

1. If $\|x-y\|_{\mathfrak{s}} \le 1$, use (3.9), with $z = y$ and $\bar{z} = x$.
2. Otherwise the denominator is at least 1. Let M bound the derivatives $D^i f$, $|i|_{\mathfrak{s}} < \alpha$, on $\mathfrak{K}$ (they are continuous, hence bounded on compacts). Since $|(x-y)^m| \le \|x-y\|_{\mathfrak{s}}^{|m|_{\mathfrak{s}}} \le \operatorname{diam}_{\mathfrak{s}}(\mathfrak{K})^{|m|_{\mathfrak{s}}}$,

$$\begin{aligned}\frac{|R_j(x,y)|}{j!\,\|x-y\|_{\mathfrak{s}}^{\alpha-\beta}} &\le |D^j f(x)| + \sum_{|m|_{\mathfrak{s}}<\alpha-\beta} \frac{\left|D^{j+m} f(y)\,(x-y)^m\right|}{m!} \\ &\le M\left(1 + \sum_{|m|_{\mathfrak{s}}<\alpha-\beta} \frac{\operatorname{diam}_{\mathfrak{s}}(\mathfrak{K})^{|m|_{\mathfrak{s}}}}{m!}\right),\end{aligned}$$

   which is finite.

Finally,

$$\|F(x)\|_\beta = \sum_{|j|_{\mathfrak{s}}=\beta} \frac{1}{j!}\,|D^j f(x)|,$$

which is bounded on compacts, since the derivatives are continuous.

($\Longleftarrow$) For the other implication the proof is easier.

4.3 An example

Let $d = 1$, $f(t,x) = e^x + tx$ and $\alpha = \frac{5}{2}$. The expansion includes the multi-indices up to

$$|k|_{\mathfrak{s}} < \tfrac{5}{2} \iff 2k_0 + k_1 < \tfrac{5}{2},$$

that is

X_0,	X_1,	X_1^2,	$\mathbf{1}$
$(1,0)$	$(0,1)$	$(0,2)$	$(0,0)$

but not X_0X_1: order 3! With

$$\partial_x f = e^x + t, \qquad \partial_t f = x, \qquad \tfrac{1}{2}\,\partial_x^2 f = \tfrac{1}{2}\,e^x,$$

we can define the expansion

$$F(t,x) = (e^x + tx)\,\mathbf{1} + (e^x + t)\,X_1 + \tfrac{1}{2}\,e^x\,X_1^2 + x\,X_0.$$

Moreover,

$$\big(\Pi_{(t,x)}F(t,x)\big)(\bar{t},\bar{x}) = (e^x + tx) + (e^x + t)(\bar{x} - x) + \tfrac{1}{2}\,e^x(\bar{x}-x)^2 + x\,(\bar{t} - t),$$

the Taylor expansion truncated at parabolic degree!