QUESTION 1

PART D

In part c established LRT $\phi(\underline{X})$

$$\phi(\underline{X}) = \begin{cases} 1, & \text{if } |T(\underline{x})| > \frac{z_{1-\alpha/2}}{\sqrt{n}} \\ 0, & \text{otherwise} \end{cases}$$

$$\iff \phi(\underline{X}) = \begin{cases} 1, & W(\underline{X}) > \exp\left(\sqrt{n}\, z_{1-\frac{\alpha}{2}} - \frac{n}{2}\right) \\ 0, & \text{otherwise} \end{cases}$$

where when $Z \sim N(0,1)$

$$\mathbb{P}(Z < z_\alpha) = \alpha$$

Is $\phi(\underline{X})$ a Uniformly most powerful test when $\Theta_0 = \{0\}$, $\Theta_1 = \{-1, 1\}$?

Power ← probability that the test statistic lies in the rejection region

Uniformly Most Powerful ← most powerful test of size α regardless of which element of the alternative hyp. space is the truth.

Thus $\phi(\underline{X})$ is not UMP as $\exists$ a more powerful test depending on $\mu \in \{-1, 1\}$, $\tilde{\phi}(\underline{X})$

Investigate $\mu = 1$

$$\tilde{\phi}(\underline{X}) = \begin{cases} 1, & \text{if } \frac{Z_{1-\alpha}}{\sqrt{n}} < T(\underline{X}) \\ 0, & \text{otherwise} \end{cases}$$

describes a test of size α

i.e. $\mathbb{E}_{\mu=0}\left[\tilde{\phi}(\underline{X})\right] = \alpha$ where

$$\mathbb{E}_{\mu=1}\left[\tilde{\phi}(\underline{X})\right] \geq \mathbb{E}_{\mu=1}\left[\phi(\underline{X})\right]$$

To find $\mathbb{E}_{\mu=1}\left[\phi(\underline{X})\right]$

Recall $\underset{\mu=1}{T}(\underline{X}) \sim \mathcal{N}\left(1, \frac{1}{n}\right)$

$$\Rightarrow Y(\underline{x}) = \sqrt{n}\left(T(\underline{X}) - 1\right) \sim \mathcal{N}(0,1)$$

Note: $|T(\underline{X})| \geq \frac{z_{1-\frac{\alpha}{2}}}{\sqrt{n}}$

$$\Longleftrightarrow T(\underline{X}) > \frac{z_{1-\frac{\alpha}{2}}}{\sqrt{n}} \quad \underline{\text{or}} \quad T(\underline{X}) < \frac{z_{\frac{\alpha}{2}}}{\sqrt{n}}$$

$$\Longleftrightarrow T(\underline{X}) - 1 > \frac{z_{1-\frac{\alpha}{2}}}{\sqrt{n}} - 1 \quad \underline{\text{or}} \quad T(\underline{X}) - 1 < \frac{z_{\alpha/2}}{\sqrt{n}} - 1$$

$$\Longleftrightarrow \sqrt{n}\left(T(\underline{X}) - 1\right) > z_{1-\frac{\alpha}{2}} - \sqrt{n} \quad \underline{\text{or}} \quad \sqrt{n}\left(T(\underline{X}) - 1\right) < z_{\alpha/2} - \sqrt{n}$$

$$\Longleftrightarrow \underbrace{Y(\underline{x}) > z_{1-\frac{\alpha}{2}} - \sqrt{n} \quad \underline{\text{or}} \quad Y(\underline{x}) < z_{\frac{\alpha}{2}} - \sqrt{n}}$$

Re-express $\phi(\underline{X}) = \begin{cases} 1, & \\ 0, & \text{otherwise} \end{cases}$

Note $Y(\underline{X}) = Z \sim \mathcal{N}(0,1)$

Therefore

$$\mathbb{E}_{\mu=1}\left[\phi(\underline{X})\right] = \mathbb{P}\left(Z > z_{1-\frac{\alpha}{2}} - \sqrt{n}\right) + \mathbb{P}\left(Z < z_{\frac{\alpha}{2}} - \sqrt{n}\right)$$

Similarly it can be shown

$$\mathbb{E}_{\mu=1}\left[\tilde{\phi}(\underline{X})\right] = \mathbb{P}\left(Z > z_{1-\alpha} - \sqrt{n}\right)$$

When $n = 1$

$$\mathbb{E}_{\mu=1}\left[\phi(\underline{X})\right] = 0.168 + 0.0015$$

$$\mathbb{E}_{\mu=1}\left[\tilde{\phi}(\underline{X})\right] = 0.261$$

Thus $\tilde{\phi}(\underline{X})$ is more powerful than $\phi(\underline{X})$ when $\mu = 1$, so $\phi(\underline{X})$ cannot be UMP.

PART E As $n \to \infty$

$$\mathbb{E}_{\mu=1}\left[\phi(\underline{X})\right] \to \mathbb{P}(Z > -\infty) + \mathbb{P}(Z < -\infty) = 1$$

$$\mathbb{E}_{\mu=1}\left[\tilde{\phi}(\underline{X})\right] \to \mathbb{P}(Z > -\infty) = 1$$

Thus as $n \to \infty$ $\phi(\underline{X})$ is UMP