Fundamental Tools - Probability Theory III

MSc Financial Mathematics

The University of Warwick

September 26, 2018

Joint distribution function

For two random variables X and Y, we define their joint distribution function by

$$F_{XY}(x,y) = \mathbb{P}(X \leqslant x, Y \leqslant y), \quad -\infty < x, y < \infty.$$

Two important classes:

- X and Y are jointly discrete if both X and Y are discrete. It is convenient to work with their joint probability mass function
$$p_{XY}(x,y) = \mathbb{P}(X = x, Y = y).$$
- X and Y are jointly continuous if there exists a non-negative function $f_{XY} : \mathbb{R}^2 \to \mathbb{R}$ such that
$$\mathbb{P}(X \leqslant x, Y \leqslant y) = F_{XY}(x,y) = \int_{u=-\infty}^{u=x} \int_{v=-\infty}^{v=y} f_{XY}(u,v) dv du.$$
We call f_{XY} the joint probability density function of X and Y.

Recovering marginal distribution

Given a joint probability distribution/mass/density function of (X, Y), we can recover the the corresponding marginal characteristics of X as follows:

- Marginal distribution function

$$F_X(x) = \mathbb{P}(X \leqslant x) = \mathbb{P}(X \leqslant x, Y < \infty) = F_{XY}(x, \infty).$$

- Marginal probability mass function (if (X, Y) are jointly discrete)

$$p_X(x) = \mathbb{P}(X = x) = \mathbb{P}(X = x, Y \in S_Y) = \sum_y p_{XY}(x, y).$$

- Marginal probability density function (if (X, Y) are jointly continuous)

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} F_{XY}(x, \infty) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy.$$

Working with jointly continuous random variables

From definition, the joint distribution and density function of (X, Y) are related via

$$f_{XY}(x, y) = \frac{d^2}{dxdy} F_{XY}(x, y).$$

In univariate case, we compute probabilities involving a continuous random variable via simple integration:

$$\mathbb{P}(X \in A) = \int_A f(x)dx.$$

In bivariate case, probabilities are computed via double integration

$$\mathbb{P}((X, Y) \in A) = \iint_A f_{XY}(x, y)dxdy.$$

and the calculation is not necessarily straightforward.

Example 1

Let (X, Y) be a pair of jointly continuous random variables with joint density function $f(x, y) = 1$ on $(x, y) \in [0, 1]^2$ (and is zero elsewhere). Find $\mathbb{P}(X - Y < 1/2)$.

Example 2

Let (X, Y) be a pair of jointly continuous random variables with joint density function $f(x, y) = e^{-y}$ on $0 < x < y < \infty$ (and is zero elsewhere). Verify that the given f is a well-defined joint density function. Find the marginal density function of X.

Expected value involving joint distribution

Let (X, Y) be a pair of random variables. For a given function $g(\cdot, \cdot)$, the expected value of the random variable $g(X, Y)$ is given by

$$\mathbb{E}(g(X,Y)) = \begin{cases} \sum_{x,y} g(x,y)p_{XY}(x,y); \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{XY}(x,y)dxdy. \end{cases}$$

Two important specifications of $g(\cdot, \cdot)$:

- Set $g(x, y) = x + y$. One could obtain $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$.
- Set $g(x, y) = xy$. This leads to computation of the covariance measure between X and Y defined by

  $$Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

  and correlation measure defined by

  $$corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{\text{var}(X)\,\text{var}(Y)}}.$$

Conditional distributions

If (X, Y) are jointly discrete, the conditional probability mass function of X given $Y = y$ is

$$p_{X|Y}(x|y) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)} = \frac{p_{XY}(x, y)}{p_Y(y)}.$$

If (X, Y) are jointly continuous, the conditional probability density function of X given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

which can be interpreted as follows:

$$\begin{aligned}\mathbb{P}(x \leqslant X \leqslant x + \delta x | y \leqslant Y \leqslant y + \delta y) &= \frac{\mathbb{P}(x \leqslant X \leqslant x + \delta x, y \leqslant Y \leqslant y + \delta y)}{\mathbb{P}(y \leqslant Y \leqslant y + \delta y)} \\ &\approx \frac{f_{XY}(x, y)\delta x \delta y}{f_Y(y)\delta y} = f_{X|Y}(x|y)\delta x.\end{aligned}$$

Conditional probability and expectation

With conditional probability mass/density function, we can work out the conditional probability and expectation as follows:

- Conditional probability:

$$\mathbb{P}(X \in A | Y = y) = \begin{cases} \sum_{x \in A} p_{X|Y}(x|y); \\ \int_A f_{X|Y}(x|y) dx. \end{cases}$$

- Conditional expectation:

$$\mathbb{E}(X|Y = y) = \begin{cases} \sum_x x p_{X|Y}(x|y); \\ \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx. \end{cases}$$

Example 3

Let the joint density function of X and Y be $f_{XY}(x,y) = \frac{e^{-x/y}e^{-y}}{y}$ on $0 < x, y < \infty$ (and zero elsewhere). Find the conditional density $f_{X|Y}$, and compute $\mathbb{P}(X > 1|Y = y)$ and $\mathbb{E}(X|Y = y)$.

Independent random variables

We say that X and Y are independent random variables if

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(X \in B)$$

for any Borel set $A, B \in \mathcal{B}(\mathbb{R})$.

The following are equivalent conditions for X and Y being independent:

- $\mathbb{E}(f(X)g(Y)) = \mathbb{E}(f(X))\mathbb{E}(g(Y))$ for all functions f, g.
- $p_{XY}(x, y) = p_X(x)p_Y(y)$ or equivalently $p_{X|Y}(x|y) = p_X(x)$ in case (X, Y) are jointly discrete.
- $f_{XY}(x, y) = f_X(x)f_Y(y)$ or equivalently $f_{X|Y}(x|y) = f_X(x)$ in case (X, Y) are jointly continuous.

Independence and zero correlation

If X and Y are independent, then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$. This leads to:

1. $Cov(X, Y) = 0$, which also implies the correlation between X and Y is zero.
2. $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$.

The reverse is not true in general. The most importantly, zero correlation does not imply independence. See problem sheet.

Sum of independent random variables

We are interested in the following question: Suppose X and Y are two independent random variables. What is the distribution of $X + Y$ then?

The procedure is easier in the discrete case

Example: Suppose $X \sim Poi(\lambda_1)$ and $Y \sim Poi(\lambda_2)$.

Sum of independent random variables

Assume (X, Y) are jointly continuous and let $Z = X + Y$. Then the CDF of Z is

$$F_Z(z) = \mathbb{P}(Z \leqslant z) = \mathbb{P}(X + Y \leqslant z) = \iint_{x+y \leqslant z} f_X(x) f_Y(y) dxdy$$

$$= \int_{y=-\infty}^{y=\infty} \left(\int_{x=-\infty}^{x=z-y} f_X(x) dx \right) f_Y(y) dy$$

$$= \int_{y=-\infty}^{y=\infty} F_X(z-y) f_Y(y) dy.$$

Differentiation w.r.t z gives the density of Z as

$$f_Z(z) = \int_{y=-\infty}^{y=\infty} f_X(z-y) f_Y(y) dy.$$

Example

Let X and Y be two independent $U[0,1]$ random variables. Find the density function of $Z = X + Y$.

Probability generating function

Probability generating function is only defined for a discrete random variable X taking values in non-negative integers $\{0, 1, 2, ...\}$. It is defined as

$$G_X(s) = \mathbb{E}(s^X) = \sum_{k=0}^{\infty} s^k p_X(k).$$

- View G_X as a Taylor's expansion in s:

  $$G_X(s) = p_X(0) + p_X(1)s + p_X(2)s^2 + \cdots$$

  We could then deduce $p_X(n) = \frac{G_X^{(n)}(0)}{n!}$, i.e. G_X uniquely determines the pmf of X. In other words, if the probability generating functions of X and Y are equal, then X and Y have the same distribution.
- If X and Y are independent,

  $$G_{X+Y}(s) = \mathbb{E}(s^X s^Y) = \mathbb{E}(s^X)\mathbb{E}(s^Y) = G_X(s)G_Y(s).$$

  Hence we can study the distribution of $X + Y$ via $G_X(s)G_Y(s)$.

Moments calculation from probability generating function

Given G_X, we can derive the moments of X.

$$\begin{aligned}
G_X^{(1)}(s) &= \mathbb{E}(\frac{d}{ds}s^X) = \mathbb{E}(Xs^{X-1}) \\
&\implies \mathbb{E}(X) = G_X^{(1)}(1) \\
G_X^{(2)}(s) &= \mathbb{E}(\frac{d^2}{ds^2}s^X) = \mathbb{E}(X(X-1)s^{X-2}) \\
&\implies \mathbb{E}(X(X-1)) = G_X^{(2)}(1) \\
G_X^{(3)}(s) &= \mathbb{E}(\frac{d^3}{ds^3}s^X) = \mathbb{E}(X(X-1)(X-2)s^{X-3}) \\
&\implies \mathbb{E}(X(X-1)(X-2)) = G_X^{(3)}(1)
\end{aligned}$$

Example

Find the pgf of $Poi(\lambda)$. If $X \sim Poi(\lambda_1)$ and $Y \sim Poi(\lambda_2)$, what is the distribution of $X + Y$?

Moment generating function (mgf)

Moment generating function (mgf) can be defined for general random variable via

$$m_X(t) = \mathbb{E}(e^{tX}) = \begin{cases} \sum_x e^{tx} p_X(x), & \text{if } X \text{ is discrete;} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x) dx, & \text{if } X \text{ is continuous.} \end{cases}$$

Consider the n-th derivative of $m_X(t)$:

$$m_X^{(n)}(t) = \frac{d^n}{dt^n}\mathbb{E}(e^{tX}) = \mathbb{E}(\frac{d^n}{dt^n}e^{tX}) = \mathbb{E}(X^n e^{tX})$$

from which we obtain $\mathbb{E}(X^n) = m_X^{(n)}(0)$ on letting $t = 0$.

A mgf also uniquely determines the underlying distribution:

- If X and Y have the same mgf, then they must have the same distribution.
- Suppose X and Y are independent, the mgf of $X + Y$ is

  $$m_{X+Y}(t) = \mathbb{E}(e^{tX}e^{tY}) = \mathbb{E}(e^{tX})\mathbb{E}(e^{tY}) = m_X(t)m_Y(t).$$

  Hence we can study the distribution of $X + Y$ via $m_X(t)m_Y(t)$, just like pgf.

Example

Find the moment generating function of $N(\mu, \sigma^2)$. If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$, what is the distribution of $X + Y$?