Fundamental Tools Sheet 2: Random variables and probability distributions

1. Let $\Omega = \{1, 2, ..., 6\}$, $\mathcal{F} = \{\Omega, \emptyset, \{1, 2, 3\}, \{4, 5, 6\}\}$. Define the random variables $X(\omega) = \sqrt{\omega}$ and $Y(\omega) = 1_{(\omega \in \{1,2,3\})}$. Are X and Y measurable w.r.t $\mathcal{F}$? Explain briefly if any of them is not.

2. Suppose that $\Omega = \{H, T\}^2$ represents the outcome of tossing two coins. Define a random variable by
$$X(\omega) = \begin{cases} 1, & \text{if both tosses are heads,} \\ -1, & \text{if both are tails,} \\ 0, & \text{otherwise.} \end{cases}$$
Write down $\sigma(X)$, the smallest σ-algebra which X is measurable with respect to.

3. Let $p_X(k) = \exp(-\lambda)\lambda^k/k!$, for $k = 0, 1, 2, \ldots$ Verify that p_X defines a probability mass function of some random variable X. Compute the mean and variance of X.

4. Let $X \sim Poi(\lambda)$ and define $Y = \frac{1}{1+X}$. Find the expected value of Y.

5. You flip a (biased) coin n times. The coin has probability $p \in (0, 1)$ of landing heads, and each flip is independent. Let H denote the total number of heads. Write down the mass function for H, i.e. $\mathbb{P}(H = k)$ for $k = 0, 1, 2, \ldots, n$. What is the name of this distribution? Compute the mean and variance of H. (Hint: use the fact that $H = \mathbb{1}_1^H + \ldots + \mathbb{1}_n^H$, where $\mathbb{1}_i^H$ is the indicator for the i^{th} toss being a head, and then linearity of expectation).

6. **Memoryless property.** A light bulb in use has a probability $p \in (0, 1)$ of burning out each day, independent of all the other days. Let N be the random variable defined such that the light bulb burns out on the N-th day since it has been in use. What distribution can you use to model N? If you are told that a working light bulb has already been in use for 10 days, what is the probability that it will survive for another 5 days? Does this probability change if you are told instead that the light bulb has already been in use for 100 days?

7. Find the mean and variance of $X \sim \text{Exp}(\lambda)$ which has a probability density function $f(x) = \lambda e^{-\lambda x}$ on $x > 0$ (and zero elsewhere).

8. For which value of c does the following function define a probability density function?
$$f(x) = \begin{cases} c(1 + x)^{-3}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$
Compute the expectation of the random variable with density f and write down the cumulative distribution function.

9. Suppose Y is a non-negative continuous random variable. Show that $\mathbb{E}[Y] = \int_0^\infty \mathbb{P}(Y > y)\,\mathrm{d}y$.

10. The random variable $Y = e^X$ is said to have a *lognormal* distribution if $X \sim \mathrm{N}(\mu, \sigma^2)$. Find the probability density function of Y. Show also that the expectation of Y is $e^{(\mu+\sigma^2/2)}$.

11. The lifetime of a light bulb is an $\text{Exp}(\lambda)$ random variable. The policy is to replace the light bulb upon its failure or upon its cumulative working time has reached a fixed constant C, whichever occurs first. Let T denotes the replacement time of the lightbulb. Find and sketch the CDF of T. Is T a continuous random variable? Is T a discrete random variable?