Fundamental Tools Sheet 3: Joint distributions, conditioning, and generating functions

1. Suppose that X_1 and X_2 are independent and each have a standard exponential distribution (that is, their density is $\exp(-x)$, $x > 0$). Compute the density of X_1/X_2. Hence, or otherwise, find $\mathbb{P}(X_1 < X_2)$.

2. A point is chosen uniformly in a unit circle on $\mathbb{R}^2$, i.e. if (X, Y) are the co-ordinates of the point, the joint density of (X, Y) is, for some constant c, $f(x, y) = c$ when $x^2 + y^2 \leq 1$ and $f(x, y) = 0$ for $x^2 + y^2 > 1$. What is the correct value of c? Are X and Y independent? Let $D = \sqrt{X^2 + Y^2}$ be the distance of the point from the origin. Calculate the density of D.

3. Conditioning to be even. Suppose that $N \sim \text{Pois}(\lambda)$. Compute the conditional distribution of N given that it is even, i.e.

$$\mathbb{P}(N = n \mid N \text{ is even}).$$

Write your answer in terms of $\exp(\lambda)$. (**Hint:** recall the power series for $\exp(\lambda) + \exp(-\lambda)$ and $\exp(\lambda) - \exp(-\lambda)$.) Compute the conditional mean $\mathbb{E}[N|N \text{ is even}]$.

4. Encouraging to be even. If Z is a random variable taking values $0, 1, 2, 3, \ldots$ and

$$\mathbb{P}_\theta(Z = z) = \begin{cases} \frac{1}{C_\theta} \frac{\theta \lambda^z}{z!}, & \text{when } z \text{ is even;} \\ \frac{1}{C_\theta} \frac{\lambda^z}{z!}, & \text{when } z \text{ is odd.} \end{cases}$$

The normalising constant C_θ is determined by the condition $\sum_{z \geq 0} \mathbb{P}_\theta(Z = z) = 1$, and $\theta > 1$. Thus Z is like a Poisson variable but even numbers are favoured. When θ is very large, even numbers are highly favoured. Compute the value of C_θ in terms of θ and $\exp(\lambda)$. Show that

$$\mathbb{P}_\theta(Z = z) \to \mathbb{P}(N = n|N \text{ is even}) \text{ as } \theta \to \infty,$$

where $N \sim \text{Pois}(\lambda)$ (such that the RHS is the answer you obtained in the previous question).

5. Let X be a discrete random variable taking value on $\{-1, 0, 1\}$ with equal probability, and let $Y = |X|$.

(a) For the *mutually exclusive* events "$X = 0$" and "$Y \neq 0$", what can you say about the probability $\mathbb{P}(X = 0, Y \neq 0)$? Also find $\mathbb{P}(X = 0)\mathbb{P}(Y \neq 0)$. What can you conclude?

(b) Write down the joint probability mass function of X and Y (in form of a table, say). Also find the marginal probability mass functions of X and Y respectively.

(c) Check whether $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.

6. Find the probability generating function (p.g.f.) of the $X \sim \text{Bin}(n, p)$ random variable. Use this to derive the mean and variance of X. Now suppose $X \sim \text{Bin}(n, p)$ and $Y \sim \text{Bin}(m, p)$ are independent random variables. What is the p.g.f. of $X + Y$? What is the distribution of $X + Y$?

7. Suppose X is a random variable with p.g.f $g_X(t) = e^{\theta(t-1)}$. Deduce the probabilities $\mathbb{P}(X = k)$ for $k = 0, 1, 2, \ldots$.

8. Calculate the moment generating function of an $\text{Exp}(\lambda)$ random variable and use this to deduce its expectation and variance.

9. Compute the moment generating function of the random variable with distribution function

$$F(x) = \begin{cases} 1 - (1 + \lambda x)e^{-\lambda x}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

You should get $\left(\frac{\lambda}{\lambda - t}\right)^2$ for $t < \lambda$. Find the mean of this random variable.