Sketch of solutions to Sheet 1

September 24, 2018

- Try not to consult these before you have tried the questions thoroughly.
- Very likely the solutions outlined below only represent a tiny subset of all possible ways of solving the problems. You are highly encouraged to explore alternative approaches!

1. A σ-algebra $\mathcal{F}$ is a collection of subsets of Ω satisfying: (i) $\Omega \in \mathcal{F}$, (ii) $A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F}$ and (iii) $A_i \in \mathcal{F}$ for $i = 1, 2, 3, ... \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

   The stated $\mathcal{F}$ is not a σ-algebra on Ω since a number of the aforementioned properties fail to hold. For example:

   - Ω and $\Omega^C = \emptyset$ are not in $\mathcal{F}$;
   - $\{1\} \in \mathcal{F}$ but $\{1\}^C = \{2, 3\} \notin \mathcal{F}$;
   - $\{1\} \in \mathcal{F}$ and $\{2\} \in \mathcal{F}$ but $\{1\} \cup \{2\} = \{1, 2\} \notin \mathcal{F}$.

2. The required σ-algebra $\mathcal{F}$ must contain B_1, B_2 and B_3. By the basic properties of a σ-algebra, it also has to contain $\emptyset$ and Ω. All we need to do is to add the suitable complements and unions of B_i's. The answer is:

$$\mathcal{F} = \{\emptyset, \Omega, B_1, B_2, B_3, B_1 \cup B_2, B_1 \cup B_3, B_2 \cup B_3\}.$$

3. Here $\Omega = \{HH, HT, TH, TT\}$. We would like to be able to confirm the occurrence of the event "the tosses give the same outcome", which is represented by the subset $\{HH, TT\}$. Thus we would like to find the smallest σ-algebra containing the event $\{HH, TT\}$. By adding the suitable elements to fulfil the necessary constraints of a σ-algebra, the answer can be written down as $\mathcal{F} = \{\Omega, \emptyset, \{HH, TT\}, \{HT, TH\}\}$.

4. Let H denote head and T denote tail. This experiment can be modelled by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where:

$$\Omega = \{HH, HT, TH, TT\}, \quad \mathcal{F} = 2^{\Omega}, \quad \mathbb{P}(A) = \frac{|A|}{|\Omega|}.$$

   The events of interested can be written as:

   $E_H = \{HH, HT, TH\}$, $E_T = \{HT, TH, TT\}$, $E_H \cap E_T = \{HT, TH\}$.

   Then, $\mathbb{P}(E_H) = \frac{3}{4}$, $\mathbb{P}(E_T) = \frac{3}{4}$ and $\mathbb{P}(E_H \cap E_T) = \frac{2}{4} = \frac{1}{2}$. We see that $\mathbb{P}(E_H \cap E_T) \neq \mathbb{P}(E_H)\mathbb{P}(E_T)$ which implies E_H and E_T are not independent.

5. (i) Clearly A and A^C are disjoint, and $A \cup A^C = \Omega$. Hence $1 = \mathbb{P}(\Omega) = \mathbb{P}(A \cup A^C) = \mathbb{P}(A) + \mathbb{P}(A^C)$, or $\mathbb{P}(A^C) = 1 - \mathbb{P}(A)$. (ii) Define $G := F_2 \backslash F_1 := F_2 \cap F_1^C$ such that F_1 and G are disjoint. If $F_1 \subseteq F_2$, we also have $F_1 \cup G = F_2$. Then $\mathbb{P}(F_2) = \mathbb{P}(F_1 \cup G) = \mathbb{P}(F_1) + \mathbb{P}(G)$, from which we get $\mathbb{P}(G) = \mathbb{P}(F_2) - \mathbb{P}(F_1)$. The result follows by noticing that $\mathbb{P}(G) \geqslant 0$. (Remark: probability of any event is always lying between 0 and 1 because $\mathbb{P}$ maps to $[0, 1]$ by definition.)

6. $\mathbb{P}(\text{the dice show different numbers}) = \frac{36-6}{36} = 5/6$, since there are 6 ways of getting the same numbers among the 6×6 possibilities.

$$\begin{aligned}
&\mathbb{P}(\text{at least one die shows six AND the dice show different numbers}) \\
&= \mathbb{P}(\text{One die shows six and another one is not a six}) \\
&= \frac{36 - 25 - 1}{36} \\
&= 5/18
\end{aligned}$$

(there are 5×5 cases where none of the dice gives six so there are $36 - 25 = 11$ ways of having "at least one die gives six". Among these 11 cases, remove the case where both dice are six).

The required conditional probability is then $\frac{5}{18}/\frac{5}{6} = 1/3$.

7. Let C (resp. B) be the event that the randomly selected Jaguar car is manufactured in Coventry (resp. Birmingham). Let F be the event that the car is faulty. From the given information, we have $\mathbb{P}(C) = 0.7$, $\mathbb{P}(B) = 1 - 0.7 = 0.3$, $\mathbb{P}(F|C) = 0.2$, $\mathbb{P}(F|B) = 0.1$.

(a) $\mathbb{P}(F \cap C) = \mathbb{P}(F|C)\mathbb{P}(C) = 0.2 \times 0.7 = 0.14$.

(b) $\mathbb{P}(F) = \mathbb{P}(F \cap (C \cup B)) = \mathbb{P}(F \cap C) + \mathbb{P}(F \cap B) = \mathbb{P}(F|C)\mathbb{P}(C) + \mathbb{P}(F|B)\mathbb{P}(B) = 0.2 \times 0.7 + 0.1 \times 0.3 = 0.17$.

(c) $\mathbb{P}(C|F) = \frac{\mathbb{P}(F \cap C)}{\mathbb{P}(F)} = 0.14/0.17 = 0.8235$.

(Remark: a tree diagram might be a helpful way to summarise the given information in this type of question.)

8. Refer to slide 14 on the day 1 handout for an example (of course, you should go through the calculations to check the pairwise-independence and joint-dependence properties!)

9. We are choosing 5 cards from a deck of 52 cards where the ordering is not important. Thus there are C_5^{52} different hands.

Firstly notice that there are C_4^{13} different combinations of 4 spades. The remaining card is chosen from 39 non-spade cards, which has 39 (or C_1^{39}) possibilities. Hence the total number of hands with exactly 4 spades is $C_4^{13} \times 39$.

10. Argument similar to the previous question. There are C_5^{52} different hands, and $C_3^{13} \times C_2^{13}$ hands with 3 spades and 2 hearts. Required probability is $\frac{C_3^{13} \times C_2^{13}}{C_5^{52}}$.

11. For clarity of exposition, let's assume $n = 7$ and $r = 9$. An example of allocation can be represented as below:

$$[\circ \circ | \circ | \circ \circ | \quad | \circ | \circ \circ \circ | \quad]$$

In this case, we have 2 balls in box 1, 1 ball in box 2, 2 balls in box 3, 0 ball in box 4, 1 ball in box 5, 3 balls in box 6 and 0 ball in box 7. A typical outcome can thus be represented by an ordering of 6 '|' and 9 '$\circ$' in a line, and there are $C_6^{9+6} = C_6^{15}$ distinguishable arrangements. More generally when we have n boxes and r balls, the answer is C_{n-1}^{n+r-1}.

12. Again for clarity of exposition, let's assume $r = 4$ and $b = 3$ (such that $r > b - 1$ is satisfied). Imagine we have all the red balls lying on a row as shown below. The underscore "_" represents the space around to each red ball.

_ R _ R _ R _ R _

Now we are trying to complete the ordering by adding the blue balls into the queue. If we do not want any blue balls lying next to each other, then in each spot marked by a "_", there can at most be one blue ball. The operation now becomes selecting 3 spots from the 5 spots available, and put a blue ball in each of the 3 selected spots. How many possible selections can be made? It is simply C^5_3.

Apply this logic to the arbitrary number of r and b, we will see that the number of orderings is given by C^{r+1}_b.

13. Consider the deck as a collection of two types of card only: 13 indistinguishable hearts and 39 indistinguishable non-hearts. Using the result from the previous question, there are $C^{39+1}_{13} = C^{40}_{13}$ cases where no two hearts are next to each other. Since there are C^{52}_{13} distinguishable orderings in the well-shuffled deck (remember we consider the hearts are indistinguishable, so as the non-hearts), the required probability is $\frac{C^{40}_{13}}{C^{52}_{13}}$.