Sketch of solutions to Sheet 2

September 25, 2018

- Try not to consult these before you have tried the questions thoroughly.
- Very likely the solutions outlined below only represent a tiny subset of all possible ways of solving the problems. You are highly encouraged to explore alternative approaches!

1. X is NOT measurable w.r.t $\mathcal{F}$ because, for example, $X^{-1}(1) = \{1\} \notin \mathcal{F}$. Y is measurable w.r.t $\mathcal{F}$.

2. The range of X consists of three possible values 1, -1 or 0. Check that $X^{-1}(1) = \{HH\}$, $X^{-1}(-1) = \{TT\}$ and $X^{-1}(0) = \{HT, TH\}$. Thus $\sigma(X)$ must at least contain the three events $\{HH\}$, $\{TT\}$ and $\{HT, TH\}$. All we need to do is to add in a few suitable extra elements (unions, compliments, the empty set, etc...) to turn it into a σ-algebra. We then obtain

$$\sigma(X) = \{\emptyset, \Omega, \{HH\}, \{TT\}, \{HT, TH\}, \{HH, TT\}, \{HT, TH, TT\}, \{HH, HT, TH\}\}.$$

3. A probability mass function (pmf) $p_X(k)$ has to satisfy two conditions: i) $p_X(k) > 0$ for each k belonging to the support of X; ii) $\sum_k p_X(k) = 1$. Here the first condition is clearly satisfied.

   On the second condition, $\sum_k p_X(k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1$ (recall the Taylor's expansion of the function e^{λ}). Hence $p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}$ is a well defined pmf.

   $\mathbb{E}(X) = \sum_k k p_X(k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{k\lambda^k}{k!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$, and

$$\begin{aligned}
\mathbb{E}(X^2) = \sum_k k^2 p_X(k) &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{k^2 \lambda^k}{k!} \\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(k(k-1)+k)\lambda^k}{k!} \\
&= e^{-\lambda} \left(\lambda^2 \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} + \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \right) \\
&= e^{-\lambda} \left(\lambda^2 e^{\lambda} + \lambda e^{\lambda} \right) \\
&= \lambda^2 + \lambda.
\end{aligned}$$

   Thus $\mathrm{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \lambda$.

4.

$$\begin{aligned}
\mathbb{E}(Y) &= \sum_{k=0}^{\infty} \frac{1}{(1+k)} e^{-\lambda} \frac{\lambda^k}{k!} \\
&= \frac{e^{-\lambda}}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{(k+1)!} \\
&= \frac{e^{-\lambda}}{\lambda} \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} - 1 \right) \\
&= \frac{e^{-\lambda}}{\lambda} (e^{\lambda} - 1) \\
&= \frac{1}{\lambda} (1 - e^{-\lambda}).
\end{aligned}$$

5. We have a sequence of "head" and "tail" after flipping the coin n times. What is the number of such sequence with "head" appearing k times? It is given by C_k^n (think of it as selecting k flips from the total n flips such that we assign "head" to each selected flip). The probability of getting any one of these C_k^n outcomes is $p^k(1-p)^{n-k}$. Hence we have $P(H = k) = C_k^n p^k (1-p)^{n-k}$ for $k = 0, 1, 2..., n$. This is a binomial distribution with parameters (n, p).

   H can be interpreted as a sum of n independent and identically distributed Bernoulli random variables with rate of success p, where each of these Bernoulli random variables has identical mean p and variance $p(1-p)$. By linearity of expectation and variance (where the latter relies on independence), we have $\mathbb{E}(H) = \sum_{k=1}^{n} \mathbb{E}(\mathbb{1}_k^H) = \sum_{k=1}^{n} p = np$ and $\text{var}(H) = \sum_{k=1}^{n} \text{var}(\mathbb{1}_k^H) = \sum_{k=1}^{n} p(1-p) = np(1-p)$.

6. If we need $N = k$, then the light bulb does not fail in the first $k-1$ days which has probability $(1-p)^{k-1}$, and fails on day k which has probability p. Then $\mathbb{P}(N = k) = (1-p)^{k-1}p$ where $k = 1, 2, 3...$ Here N has a geometric distribution with parameter p.

   We first work out $\mathbb{P}(N > k) = 1 - \mathbb{P}(N \leqslant k) = 1 - \sum_{i=1}^{k}(1-p)^{i-1}p = (1-p)^k$. Then

$$\begin{aligned}
\mathbb{P}(\text{survives for extra 5 days}|\text{has survived previous 10 days}) &= \mathbb{P}(N > 10 + 5 | N > 10) \\
&= \frac{\mathbb{P}(N > 15, N > 10)}{\mathbb{P}(N > 10)} \\
&= \frac{\mathbb{P}(N > 15)}{\mathbb{P}(N > 10)} \\
&= (1-p)^{15}/(1-p)^{10} \\
&= (1-p)^5.
\end{aligned}$$

   Similarly,

$$\begin{aligned}
\mathbb{P}(\text{survives for extra 5 days}|\text{has survived previous 100 days}) &= \mathbb{P}(N > 100 + 5 | N > 100) \\
&= \frac{\mathbb{P}(N > 105)}{\mathbb{P}(N > 100)} \\
&= (1-p)^5,
\end{aligned}$$

   where probability is the same.

7. A probability density function (pdf) $f(x)$ has to satisfy two conditions: i) $f(x) \geqslant 0$ for all x, and ii) $\int_{\mathbb{R}} f(x)dx = 1$.

   The first condition is clearly satisfied. On the second condition, $\int_{\mathbb{R}} f(x)dx = \int_0^{\infty} \lambda e^{-\lambda x} dx = e^{-\lambda x}|_{\infty}^{0} = 1$.

$\mathbb{E}(X) = \int_{\mathbb{R}} xf(x)dx = \int_0^\infty \lambda x e^{-\lambda x}dx = xe^{-\lambda x}|_\infty^0 + \int_0^\infty e^{-\lambda x}dx = 0 + \frac{1}{\lambda}e^{-\lambda x}|_\infty^0 = \frac{1}{\lambda}$, and

$$\begin{aligned}\mathbb{E}(X^2) = \int_{\mathbb{R}} x^2 f(x)dx &= \int_0^\infty \lambda x^2 e^{-\lambda x}dx \\ &= x^2 e^{-\lambda x}|_\infty^0 + \int_0^\infty 2xe^{-\lambda x}dx \\ &= 0 + \frac{2}{\lambda}\int_0^\infty \lambda x e^{-\lambda x}dx \\ &= \frac{2}{\lambda}\mathbb{E}(X) \\ &= \frac{2}{\lambda^2}.\end{aligned}$$

Thus $\mathrm{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{1}{\lambda^2}$.

8. We find c by using the property of a pdf that $\int_{\mathbb{R}} f(x)dx = 1$. Then $1 = \int_0^\infty c(1+x)^{-3}dx = \frac{c}{2}(1+x)^{-2}|_\infty^0 = \frac{c}{2}$ which gives $c = 2$.
$\mathbb{E}(X) = \int_0^\infty 2x(1+x)^{-3}dx = x(1+x)^{-2}|_\infty^0 + \int_0^\infty (1+x)^{-2}dx = 0 + (1+x)^{-1}|_\infty^0 = 1$.
The cumulative distribution function of X is given by $F(x) = \mathbb{P}(X \leqslant x)$. Since X only has non-zero density for $x > 0$, X is a positive random variable and thus $F(x) = \mathbb{P}(X \leqslant x) = 0$ for $x < 0$. Otherwise for $x \geqslant 0$, $F(x) = \mathbb{P}(X \leqslant x) = \int_0^x f(u)du = \int_0^x 2(1+u)^{-3}du = (1+u)^{-2}|_x^0 = 1 - (1+x)^{-2}$. In summary,
$$F(x) = \begin{cases} 0, & x < 0; \\ 1 - (1+x)^{-2}, & x \geqslant 0. \end{cases}$$

9. Let f be the density function of Y. Then
$$\int_0^\infty \mathbb{P}(Y > y)dy = \int_{y=0}^{y=\infty}\int_{u=y}^{u=\infty} f(u)dudy = \int_{u=0}^{u=\infty}\int_{y=0}^{y=u} f(u)dydu = \int_{u=0}^{u=\infty} uf(u)du = \mathbb{E}(Y).$$
(Indeed, the relationship $\mathbb{E}(Y) = \int_0^\infty \mathbb{P}(Y > y)dy$ holds for any positive random variable Y, not just those with absolutely continuous density. The proof of this general result is more subtle since Y may not have a density function.)

10. Denote the cumulative distribution function of Y by $F_Y(x)$ which we try to work out as follows:
$$F_Y(x) = \mathbb{P}(Y \leqslant x) = \mathbb{P}(e^X \leqslant x) = \mathbb{P}(X \leqslant \ln x) = \mathbb{P}\left(\frac{X-\mu}{\sigma} \leqslant \frac{\ln x - \mu}{\sigma}\right) = \mathbb{P}\left(Z \leqslant \frac{\ln x - \mu}{\sigma}\right),$$
where Z is a standard normal random variable. Hence $F_Y(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$, where Φ is the cumulative distribution of a standard normal distribution. The density of Y is then obtained by differentiation
$$f_Y(x) = \frac{d}{dx}F_Y(x) = \frac{d}{dx}\Phi\left(\frac{\ln x - \mu}{\sigma}\right) = \frac{1}{\sigma x}\Phi'\left(\frac{\ln x - \mu}{\sigma}\right) = \frac{1}{\sigma x\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right).$$
for $x > 0$, and the density is zero elsewhere.
(Recall that $\Phi'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ which is the density function of a standard normal random variable.)

The mean of Y can be computed as

$$\begin{aligned}
\mathbb{E}(Y) &= \int_0^\infty x f_Y(x) dx \\
&= \int_0^\infty \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right) dx \\
&= \int_{-\infty}^\infty \frac{\exp(\mu + \sigma u)}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \quad \text{(By change of variable } u = \frac{\ln x - \mu}{\sigma}\text{)} \\
&= e^{\mu + \sigma^2/2} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(u-\sigma)^2\right) du \\
&= e^{\mu+\sigma^2/2}.
\end{aligned}$$

Notice that we have performed a completing square trick from the third to the forth line. In the last line we have used the fact that the density of a standard normal variable integrates to 1, i.e. $\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du = 1$.

11. From the question setup, $T = \min(X, C)$ where $X \sim Exp(\lambda)$. Now we would like to find the CDF of T. Clearly T is non-negative. For $t \geqslant 0$.

$$\begin{aligned}
F_T(t) &= \mathbb{P}(T \leqslant t) \\
&= \mathbb{P}(\min(X, C) \leqslant t) \\
&= \mathbb{P}(X \leqslant t \text{ or } C \leqslant t) \\
&= \mathbb{P}(X \leqslant t) + \mathbb{P}(C \leqslant t) - \mathbb{P}(X \leqslant t \text{ and } C \leqslant t).
\end{aligned}$$

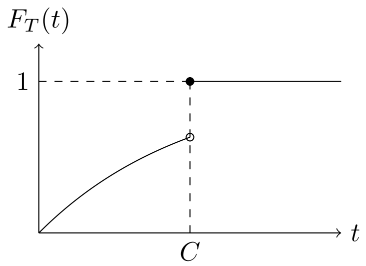
Notice that in the above expression C and t are just some non-random numbers. If $t \geqslant C$, then the event "$t \geqslant C$" always happens with probability one. Otherwise if $t < C$, then "$t \geqslant C$" will be an impossible event. This gives:

$$F_T(t) = \begin{cases} \mathbb{P}(X \leqslant t) + 0 - 0, & t < C; \\ \mathbb{P}(X \leqslant t) + 1 - \mathbb{P}(X \leqslant t), & t \geqslant C. \end{cases}$$

Since X follows $Exp(\lambda)$, $\mathbb{P}(X \leqslant t) = \int_0^t \lambda e^{-\lambda u} du = 1 - e^{-\lambda t}$. Substitution into the above gives

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t}, & t < C; \\ 1, & t \geqslant C. \end{cases}$$

A sketch of CDF is given below. There is a discontinuity in the function at $t = C$.



T is not a continuous random variable. It is because $\mathbb{P}(T = C) = \mathbb{P}(T \leqslant C) - \mathbb{P}(T < C) = 1 - (1 - e^{-\lambda C}) = e^{-\lambda C} > 0$. But a continuous random variable always has zero probability of assuming any particular value, i.e $\mathbb{P}(T = x) = 0$ must hold for any x if T was continuous. So T cannot be continuous.

T is also not discrete because the possible range of T is $[0, C]$, which is not a countable set.