THE UNIVERSITY OF WARWICK

MSc EXAMINATION: September 27th 2018

FUNDAMENTAL TOOLS

Time Allowed: **3 hours**

Candidates should attempt **all 4** questions. However more credit will be given for complete answers than for a number of fragments.

Read carefully the instructions on the answer book and make sure that the numbers required are entered on each answer book.

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1. a) (i) Give the definition of a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. **[3]**

(ii) Write down the definition of the distribution function for a random variable. Why is this well-defined? **[2]**

b) (i) Suppose X is a continuous random variable; i.e. $\mathbb{P}(X = x) = 0$ for each $x \in \mathbb{R}$. Let F be its distribution function. Show that $F \circ X \sim \text{Unif}([0, 1])$. **[4]**

(ii) Suppose X is not a continuous random variable. Show that $F \circ X$ does *not* have the Unif$([0, 1])$ distribution. **[4]**

(iii) Show that if $U \sim \text{Unif}([0, 1])$, and $F^{-1}(y) := \min\{x \in \mathbb{R} : F(x) \geq y\}$, then $F^{-1} \circ U$ has distribution function F. **[2]**

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2. a) Write down the definition for $\mathcal{F}$ to be a σ-algebra on a given space Ω. **[4]**

b) Let $\mathcal{P}$ be a partition of the space Ω.

(i) In the case that $\mathcal{P}$ is countable, describe the smallest σ-algebra on Ω which contains $\mathcal{P}$, denoted $\sigma(\mathcal{P})$. Be precise about checking all the defining properties of σ-algebras in your arguments. **[3]**

(ii) How does your answer to (c)(i) change if $\mathcal{P}$ is not countable? **[2]**

(iii) If $|\mathcal{P}| = n \in \mathbb{N}$, what is $|\sigma(\mathcal{P})|$? **[3]**

c) A security password contains three or four characters. Each character can be one of 26 letters or 10 digits. Any password must contain at least two digits. How many such passwords are there? **[4]**

d) Suppose that in the previous question we pick a password consisting only of digits. A hacker tries to guess our password by selecting each character randomly with uniform probability. In a single try:

(i) What are the chances of him guessing the password without any prior knowledge? **[2]**

(ii) What is the posterior probability of him guessing the password if he is given the knowledge that our password consists only of digits? **[3]**

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3. a) Say what it means for $\mathcal{F}$ to be a σ-algebra on Ω. **[4]**

b) Of the following:

(i) $\mathcal{F}_1 := \{\emptyset, 1, 2, \{1,2\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$,

(ii) $\mathcal{F}_2 := \{\{1\}, \{2\}\}$,

(iii) $\mathcal{F}_3 := \{\emptyset, \{1,2,3,4\}\}$,

which, if any, is a σ-algebra on $\{1,2,3,4\}$? **[3]**

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4. a) Write down the definition of what it means for two random variables to be independent. **[3]**

b) When are two random variables said to be uncorrelated? **[3]**

c) Let X and Y be random variables (with finite expectation). Of the two statements:

(i) X and Y are independent.

(ii) X and Y are uncorrelated.

which implies the other? Give a counter-example showing the reverse implication fails to be true in general. **[6]**

d) Let now $X = (X_1, X_2, \ldots)$ be a sequence of iid, $\mathrm{Ber}(p)$, $p \in (0,1)$, random variables (so $\mathbb{P}(X_1 = 1) = 1 - \mathbb{P}(X_1 = 0) = p$). Let further $Y = (Y_1, Y_2, \ldots)$ be a sequence of iid $\mathrm{Ber}(q)$, $q \in (0,1)$, random variables, independent of the sequence X.

(i) What (known!) distribution does X_1Y_1 have? What (known!) distribution does $S_n := X_1Y_1 + \cdots + X_nY_n$ have ($n \geq 1$)? **Remark**: You need not prove that $X_1Y_1, X_2Y_2, \ldots$ is a sequence of independent random variables (as it is), nor do you need to argue that for each $i \in \mathbb{N}$, X_i is independent of Y_i (as it is). **[5]**

(ii) State the weak and strong law of large numbers for the sequence $XY := (X_1Y_1, X_2Y_2, \ldots)$. In particular, give the limit explicitly, and don't forget to explain the different modes of convergence. **[5]**

 Question 4 continued overleaf

(iii) How do your answers to question (di) change, if instead of Bernoulli random variables, we have $\mathbb{P}(X_1 = 1) = 1 - \mathbb{P}(X_1 = -1) = p$ and $\mathbb{P}(Y_1 = 1) = 1 - \mathbb{P}(Y_1 = -1) = q$, but X and Y remain independent sequences of iid random variables? Again you can express your answers (albeit indirectly) in terms of known distributions (more precisely, as affine transformations of known distributions)! **[3]**

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5. a) Define what it means for a random variable X to be discrete, explaining the terms probability mass function (pmf) of X; support of X; atom of the law of X. **[5]**

b) Let Y be the random variable showing a 1 if we see a head and a 0 if we see a tail, in a single throw of a fair coin. Is Y discrete (why)? If so, what is its support? **[2]**

c) Say what it means for the random variables in an infinite sequence, $X_1, X_2, \ldots$, of random variables, to be independent. **[3]**

d) In the following, let $p_n^{(k)}$ denote the probability of seeing a run of $k \in \mathbb{N}$ or more heads (i.e. k or more *consecutive* heads) in a sequence of $n \in \mathbb{N}$ independent tosses of a fair coin. (For example, for $n = 4$, $HHTH$ has a run of 1, *and* a run of 2, heads, but no run of 3 heads; $THHH$ has a run of 3 heads, as does $HHHH$; the latter in addition has a run of 4 heads, and so on.)

(i) Why is $p_n^{(k)} = 0$ for $n < k$? **[2]**

(ii) Determine $p_k^{(k)}$ (for $k \in \mathbb{N}$). **[2]**

(iii) Show that, for $n \geq k$:

$$p_{n+1}^{(k)} = p_n^{(k)} + \frac{1}{2^{k+1}}(1 - p_{n-k}^{(k)}).$$

[5]

(iv) Conclude that $\lim_{n\to\infty} p_n^{(k)} = 1$. **[2]**

(v) What is the probability of seeing a run of k or more heads in an *infinite* sequence of throws (i.e. eventually)? Why? **[4]**

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6. a) Explain what it means for $\mathcal{F}$ to be a σ-algebra on Ω. **[4]**

b) If $\mathcal{A}$ is a subset of the power set of Ω (i.e. a collection of subsets of Ω), define $\sigma_\Omega(\mathcal{A})$, the σ-algebra generated by $\mathcal{A}$ on Ω. **[3]**

c) Let X be a random variable defined on some sample space Ω. Consider the collection $\mathcal{B}_X := \{X^{-1}(A) : A \in \mathcal{B}(\mathbb{R})\}$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$.

(i) Why is $\emptyset \in \mathcal{B}_X$? Why is $\Omega \in \mathcal{B}_X$? **[3]**

(ii) If $B \in \mathcal{B}_X$, is it necessarily the case that $\Omega \backslash B \in \mathcal{B}_X$ also? Why/why not? **[3]**

(iii) Let $B_1, B_2, \ldots$ be a sequence of members of $\mathcal{B}_X$. Show that $\cup_{n \geq 1} B_n \in \mathcal{B}_X$! Is $\mathcal{B}_X$ a σ-algebra on Ω? **[5]**

Let Y be another random variable, defined on the same sample space Ω, $\mathcal{B}_Y := \{Y^{-1}(A) : A \in \mathcal{B}(\mathbb{R})\}$. Suppose (i) $\mathcal{B}_X = \mathcal{B}_Y$ and (ii) there is a Borel set Z such that $\mathbb{P}(X^{-1}(Z)) = \mathbb{P}(X \in Z) \notin \{0, 1\}$.

(iv) Can X and Y be independent? **[4]**

(v) Can X and Y be uncorrelated? **[3]**

In both parts (civ) and (cv), either provide an example if the answer is to the affirmative (including specifying the whole probability space-triplet $(\Omega, \mathcal{F}, \mathbb{P})$, and of course X and Y); or else give an argument as to why the answer is to the negative.

MATHEMATICS DEPARTMENT
MSC EXAMS

Course Title: FUNDAMENTAL TOOLS

Model Solution No: 1

a) A random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a mapping $X : \Omega \to \mathbb{R}$ which is measurable, i.e. $\{X \in B\} \in \mathcal{F}$ for all Borel sets B. **[3]**

b) $F_X(x) = \mathbb{P}(X \leq x)$; well-defined, since X is measurable and $(-\infty, x]$ is a Borel set $(x \in \mathbb{R})$. **[2]**

c) (i) $\mathbb{P}(X > x) = \int_x^\infty \lambda e^{-\lambda y} dy = e^{-\lambda x}$. **[1]**

(ii) Since $\mathbb{P}(X \wedge Y > x) = \mathbb{P}(X > x, Y > x) = \mathbb{P}(X > x)\mathbb{P}(Y > x) = e^{-\lambda x} e^{-\mu y} = e^{-(\lambda+\mu)x}$ for all $x \in (0, \infty)$, it follows that $F_{X \wedge Y}(x) = (1 - e^{-(\lambda+\mu)x})\mathbb{1}_{(0,\infty)}(x)$. **[2]**

(iii) $\mathbb{P}(X > x + s \mid X > x) = \mathbb{P}(X > x + s)/\mathbb{P}(X > x) = e^{\lambda x} e^{-\lambda(x+s)} = e^{-\lambda s} = \mathbb{P}(X > s)$ by (ci). **[2]**

d) (i) Fix $x \in (0, 1)$. Let $x^* := \max\{y \in \mathbb{R} : F(y) \leq x\}$. This maximum is indeed attained, since F has no jumps by the continuity of X. Moreover, $F(x^*) = x$. Next note that for each ω from the sample space, $F(X(\omega)) \leq x \Leftrightarrow X(\omega) \leq x^*$. It follows that $\mathbb{P}(F \circ X \leq x) = \mathbb{P}(X \leq x^*) = F(x^*) = x$. This implies that $F \circ X \sim \text{Unif}([0, 1])$. **[4]**

(ii) Since X is not a continuous random variable, there is a real x with $\mathbb{P}(X = x) > 0$, and F has a jump of this size at x. Let $x_1 = \lim_{y \uparrow x} F(y)$ and $x_2 = F(x)$. Let $a = (x_1 + x_2)/2$. Then $\mathbb{P}(F \circ X \leq a) = \mathbb{P}(X < x) = x_1 < a$. Hence $F \circ X$ cannot be $\text{Unif}([0, 1])$. **[4]**

(iii) $\mathbb{P}(F^{-1} \circ U \leq x) = \mathbb{P}(U \leq F(x)) = F(x)$. **[2]**

e) (i) Since $\int_0^\infty x^2 e^{-x} dx = 2!$, it follows that $c = 1/2$. **[1]**

(ii) $\mathbb{E}[X] = \frac{1}{2} \int_0^\infty x^3 e^{-x} dx = 3!/2 = 3$. **[1]**

(iii) Since $\mathbb{E}[X^2] = \frac{1}{2} \int_0^\infty x^4 e^{-x} dx = 12$, it follows that $\text{var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 12 - 9 = 3$. **[1]**

(iv) Finally, $\phi_X(t) = \mathbb{E}[e^{tX}] = \int_0^\infty e^{tx} x^2 e^{-x} dx/2 = 1/(1 - t)^3$, for $t < 1$, by a change of variables. **[2]**

Breakdown: Seen: (a), (b); Bookwork: (c)(i)+(iii), (e); Unseen: (c)(ii), (d).

MA 901

MATHEMATICS DEPARTMENT
MSC EXAMS

Course Title: FUNDAMENTAL TOOLS

Model Solution No: 2

a) $\mathcal{F}$ is a σ-algebra on Ω if: (i) $\Omega \in \mathcal{F}$, (ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ and (iii) for every $A_1, A_2, \ldots$ countable sequence of sets in $\mathcal{F}$, $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$. **[4]**

b) No. Here is a counter-example. Let $\Omega := \{0, 1, 2\}$, $\mathcal{A} := \{\emptyset, \{0\}, \{1, 2\}, \Omega\}$ and $\mathcal{B} := \{\emptyset, \{2\}, \{0, 1\}, \Omega\}$. Then $\{1\} \notin \mathcal{A} \cup \mathcal{B}$ and yet $\{1\} = \{0, 1\} \cap \{1, 2\}$ must belong to every σ-algebra, which contains $\mathcal{A}$ and $\mathcal{B}$. **[4]**

c) (i) $\sigma(\mathcal{P}) = \{\text{countable unions of elements of } \mathcal{P}\} =: \mathcal{C}$. Since $\mathcal{P}$ is a countable set, whose union is Ω, then $\Omega \in \mathcal{C}$. Closure under complements follows from the countability of $\mathcal{P}$ and the fact that it is a partition. Finally a countable union of countably many sets from $\mathcal{P}$ is a countable union of sets of $\mathcal{P}$ which gives closure under countable unions for $\mathcal{C}$. It follows that $\mathcal{C}$ is a σ-algebra, it contains $\mathcal{P}$ and every σ-algebra containing $\mathcal{P}$, must contain $\mathcal{C}$ as well, due to the countable union closure property. **[3]**

(ii) If $\mathcal{P}$ is not countable, then $\sigma(\mathcal{P}) = \{\text{countable unions of elements of } \mathcal{P}\} \cup \{\text{complements of countable unions of elements of } \mathcal{P}\} =: \mathcal{D}$. One checks that in fact $\sigma(\mathcal{P}) = \mathcal{D}$ in a similar manner. **[2]**

(iii) By the first part (ci) the elements of $\sigma(\mathcal{P})$ are determined uniquely, in a one-to-one and onto fashion, by specifying whether or not any one given element of $\mathcal{P}$ is their subset. Thus, by the multiplication and bijection rules, it follows that $|\sigma(\mathcal{P})| = 2^n$. **[3]**

d) If the password contains 3 characters in total, then we can have 2 or 3 digits in total. For each of these cases, we must specify where the digits are placed in the password, and then which digits and letters we put in their allotted places. Similarly when there are 4 signs in total to the password. It follows by an application of the sum and multiplication rules, and the formula for combinations, that the total number of passwords is:

$$\binom{3}{2} \cdot 10^2 \cdot 26^1 + \binom{3}{3} \cdot 10^3 \cdot 26^0 + \binom{4}{2} \cdot 10^2 \cdot 26^2 + \binom{4}{3} \cdot 10^3 \cdot 26^1 + \binom{4}{4} \cdot 10^4 \cdot 26^0.$$

This gives 528400 possible passwords. **[4]**

e) (i) 1/528400. **[2]**

(ii) There are $10^3 + 10^4 = 11000$ different passwords containing only digits, all of which are equally likely. Hence the posterior probability is 1/11000. **[3]**

Breakdown: Seen: (a); Bookwork: (b); Unseen: (c), (d), (e).

MA 901

MATHEMATICS DEPARTMENT
MSC EXAMS

Course Title: FUNDAMENTAL TOOLS

Model Solution No: 3

a) A collection of subsets of Ω is a σ-algebra on Ω, if (i) $\Omega \in \mathcal{F}$, (ii) $A \in \mathcal{F} \Rightarrow \Omega \backslash A \in \mathcal{F}$ and (iii) $(A_i)_{i=1}^{\infty} \subset \mathcal{F} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

b) Let $[4] := \{1, 2, 3, 4\}$. $\mathcal{F}_1$ is not a σ-algebra on $[4]$, since, in fact, $\mathcal{F}_1 \not\subset 2^{[4]}$. $\mathcal{F}_2$ has $\{1\} \in \mathcal{F}_2$ and $\{2\} \in \mathcal{F}_2$, but $\{1\} \cup \{2\} = \{1, 2\} \notin \mathcal{F}_2$, so is not a σ-algebra either. Finally, $\mathcal{F}_3$ is, trivially, a σ-algebra on $[4]$.

MA 901

MATHEMATICS DEPARTMENT
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Course Title: FUNDAMENTAL TOOLS

Model Solution No: 4

a) Random variables X and Y are independent, if $\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$ for any Borel subsets A and B of $\mathbb{R}$.

b) Random variables X and Y are uncorrelated, if $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ (all the expectations needing to be well-defined and finite).

c) Independent random variables (of finite mean) are uncorrelated. The converse implication need not hold. For example, take:

$$X \sim \begin{pmatrix} -2 & -1 & 1 & 2 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$$

i.e. $X \sim \mathrm{Unif}(\{-2,-1,1,2\})$, discrete and uniformly distributed on the set $\{-2,-1,1,2\}$. Then:

$$Y := X^2 \sim \begin{pmatrix} 1 & 4 \\ 1/2 & 1/2 \end{pmatrix}.$$

Moreover, clearly $\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[X^3] - \mathbb{E}[X]\mathbb{E}[X^2] = 0$. To show X and Y are dependent take $B := \{4\}$ and $A := \{2\}$. Then:

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A) = 1/4 \neq 1/8 = (1/4)\cdot(1/2) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B).$$

d) (i) Clearly, with probability 1, $X_1Y_1 \in \{0,1\}$. Moreover, by independence, $\mathbb{P}(X_1Y_1 = 1) = \mathbb{P}(X_1 = 1, Y_1 = 1) = \mathbb{P}(X_1 = 1)\mathbb{P}(Y_1 = 1) = pq$. Thus $X_1Y_1 \sim \mathrm{Ber}(pq)$. Then $S_n \sim \mathrm{Bin}(n, pq)$, since $X_1Y_1, \ldots, X_nY_n$ are independent.

(ii) We remark that XY is, in fact, a sequence of iid random variables with finite mean. The weak law of large numbers then states that the sample means S_n/n converge in probability to $\mathbb{E}[X_1Y_1] = pq$, i.e. for each $\epsilon > 0$:

$$\mathbb{P}\left(\left|\frac{S_n - npq}{n}\right| \geq \epsilon\right) \to 0, \text{ as } n \to \infty.$$

The strong law of large numbers states that the sample means S_n/n converge to pq with probability 1, i.e.:

$$\mathbb{P}(\lim_{n\to\infty} S_n/n = pq) = 1.$$

(iii) As in (di), we find that $\mathbb{P}(X_1Y_1 = 1) = 1 - \mathbb{P}(X_1Y_1 = -1) = pq + (1-p)(1-q)$. Then $(X_1Y_1 + 1)/2 \sim \mathrm{Ber}(pq + (1-p)(1-q))$ and hence:

$$S_n = 2\frac{1}{2}\left((X_1Y_1 + 1) + \cdots + (X_nY_n + 1)\right) - n \sim 2\mathrm{Bin}(n, pq + (1-p)(1-q)) - n,$$

by an abuse of notation.

Breakdown: Seen: (a); (b); (c) Bookwork: (c); Unseen: (d).

MATHEMATICS DEPARTMENT
MSC EXAMS

Course Title: FUNDAMENTAL TOOLS

Model Solution No: 5

a) A random variable X is discrete if its law, μ_X, puts all its mass on a countable subset of $\mathbb{R}$, i.e. if there is a $C \subset \mathbb{R}$, denumerable, and such that $\mu_X(C) = \mathbb{P}(X \in C) = 1$. Given a discrete random variable X, $D := \{x \in \mathbb{R} : \mathbb{P}(X = x) > 0\}$ is its support. The probability mass function of X is then the mapping $p_X : D \to (0, 1]$, with $p_X(d) = \mathbb{P}(X = d)$ for $d \in D$. Finally, an atom of its law is an element of D.

b) Y is discrete, since it takes on only finitely many values (0 and 1). Its support is the set $\{0, 1\}$.

c) A sequence of random variables $X_1, X_2, \ldots$ is an independency, if any finite subsequence thereof is so. A finite sequence $X_1, \ldots, X_n$ ($n \in \mathbb{N}$) of random variables is an independency if for all Borel sets $A_1, \ldots, A_n$, $\mathbb{P}(X_1 \in A_1, \ldots, X_n \in A_n) = \mathbb{P}(X_1 \in A_1) \cdots \mathbb{P}(X_n \in A_n)$.

d) Let Y_i show a 1 or a 0, according as to whether on the i-th throw a head or a tail was seen, $i \in \mathbb{N}$. Remark the random variables Y_1, Y_2, ... are independent.

(i) There cannot have been more heads than there have been actual throws, so in this instance $p_n^{(k)}$ is the probability of the impossible event $\emptyset$, which is then zero.

(ii) $p_k^{(k)}$ is the probability of the event that in k consecutive throws of the coin, all were heads, i.e. $p_k^{(k)} = \mathbb{P}(Y_1 = 1, \ldots, Y_k = 1) = \mathbb{P}(Y_1 = 1) \cdots \mathbb{P}(Y_n = 1) = \underbrace{2^{-1} \cdots 2^{-1}}_{k-\text{times}} = 2^{-k}$ (owing to independence).

(iii) Note that:

$$\mathbb{P}(E_{n+1}) = \mathbb{P}(E_n) + \mathbb{P}(E_{n+1} \backslash E_n),$$

where E_n is the event of seeing a run of k or more heads in the first n tosses and E_{n+1} is the event of seeing a run of k or more heads in the first $n+1$ tosses. Clearly $\mathbb{P}(E_n) = p_n^{(k)}$. On the other hand $E_{n+1} \backslash E_n$ corresponds to the event that a head was seen last, $k-1$ heads before that, a tail before that, and then no run of k or more heads in the first $n-k$ tosses. Then, again owing to independence, $\mathbb{P}(E_{n+1} \backslash E_n) = (1 - p_{n-k}^{(k)})/2^{k+1}$.

(iv) Remark $p_n^{(k)}$ is clearly nondecreasing in n. Then take limits as $n \to \infty$ in the relation of (diii) to obtain (letting $p := \lim_{n\to\infty} p_{n+1}^{(k)} = \lim_{n\to\infty} p_n^{(k)} = \lim_{n\to\infty} p_{n-k}^{(k)}$) $p = p + (1-p)/2^k$, i.e. $p = 1$.

(v) Let A be the event of seeing a run of k heads or more, eventually. Then, for each n, A^c is included in the event A_n^c that a run of k or more heads is not seen in the first n tosses, whence by monotonicity of probability measures $1 - \mathbb{P}(A) = \mathbb{P}(A^c) \leq \mathbb{P}(A_n^c) = 1 - p_n^{(k)}$. Letting n tend to infinity, we conclude $\mathbb{P}(A) \geq 1$, so $\mathbb{P}(A) = 1$.

Breakdown: Seen: (a), (c); Bookwork: (b); Unseen: (d).

MA 901

MATHEMATICS DEPARTMENT
MSC EXAMS

Course Title: FUNDAMENTAL TOOLS

Model Solution No: 6

a) $\mathcal{F}$ is a σ-algebra on Ω, if (o) $\mathcal{F} \subset 2^\Omega$; (i) $\Omega \in \mathcal{F}$; (ii) $A \in \mathcal{F} \Rightarrow \Omega \backslash A \in \mathcal{F}$; and (iii) $(A_i)_{i\geq 1} \subset \mathcal{F} \Rightarrow \cup_{i\geq 1} A_i \in \mathcal{F}$.

b) $\sigma_\Omega(\mathcal{A}) := \cap_{\mathcal{F}\supset\mathcal{A},\ \mathcal{F} \text{ a } \sigma\text{-algebra on } \Omega}\mathcal{F}$. Remark 2^Ω is always a σ-algebra on Ω, containing $\mathcal{A}$.

c) Remark $\mathcal{B}(\mathbb{R})$ is a σ-algebra on Ω, hence has $\mathbb{R}$ for a member and is closed under complements (wrt $\mathbb{R}$) and denumerable unions.

(i) $\emptyset = X^{-1}(\emptyset)$ and $\emptyset \in \mathcal{B}(\mathbb{R})$. $\Omega = X^{-1}(\mathbb{R})$ and $\mathbb{R} \in \mathcal{B}(\mathbb{R})$.

(ii) Yes. If $B \in \mathcal{B}_X$, then there is an $A \in \mathcal{B}(\mathbb{R})$ such that $B = X^{-1}(A)$. Consequently $\mathbb{R}\backslash A \in \mathcal{B}(\mathbb{R})$ and since $\Omega\backslash B = X^{-1}(\mathbb{R}\backslash A)$, so $\Omega\backslash B \in \mathcal{B}_X$.

(iii) For each $i \in \mathbb{N}$, there is A_i Borel such that $B_i = X^{-1}(A_i)$. Then $\cup_{i\geq 1}A_i$ is Borel in turn, and hence $\cup_{i\geq 1}B_i = \cup_{i\geq 1}X^{-1}(A_i) = X^{-1}(\cup_{i\geq 1}A_i) \in \mathcal{B}_X$. It follows from the above that $\mathcal{B}_X$ is a σ-algebra on Ω (clearly $\mathcal{B}_X \subset 2^\Omega$).

(iv) No. Suppose X and Y were independent. Then, since $X^{-1}(Z) \in \mathcal{B}_X = \mathcal{B}_Y$, there is a W, Borel subset of $\mathbb{R}$, with $X^{-1}(Z) = Y^{-1}(W)$. By independence, it follows that

$$\mathbb{P}(X^{-1}(Z) \cap Y^{-1}(W)) = \mathbb{P}(X^{-1}(Z))\mathbb{P}(Y^{-1}(W)).$$

But $X^{-1}(Z) \cap Y^{-1}(W) = X^{-1}(Z) \cap X^{-1}(Z) = X^{-1}(Z)$, so $\mathbb{P}(X^{-1}(Z) \cap Y^{-1}(W)) = \mathbb{P}(X^{-1}(Z))$, whilst, since $X^{-1}(Z) = Y^{-1}(W)$, $\mathbb{P}(Y^{-1}(W)) = \mathbb{P}(X^{-1}(Z))$. We obtain from the displayed formula that:

$$\mathbb{P}(X^{-1}(Z)) = (\mathbb{P}(X^{-1}(Z)))^2,$$

whence $\mathbb{P}(X^{-1}(Z)) \in \{0, 1\}$, a contradiction.

(v) Yes. Let $\Omega = \{-2, -1, 1, 2\}$, $\mathcal{F} = 2^\Omega$, $\mathbb{P}$ be the discrete uniform probability measure on Ω, $X = \mathrm{id}_\Omega$, $Y(-2) = -1$, $Y(-1) = 2$, $Y(1) = -2$ and $Y(2) = 1$. Then $\mathcal{B}_X = \mathcal{F} = \mathcal{B}_Y$, whilst $\mathbb{E}[XY] = \mathbb{E}[X] = \mathbb{E}[Y] = 0$, so X and Y are uncorrelated. Also, e.g. $Z := \{2\}$, satisfies $\mathbb{P}(X \in Z) = 1/4 \notin \{0, 1\}$.

Breakdown: Seen: (a), (b); Bookwork: (ci), (cii), (ciii); Unseen: (civ), (cv).