Regularity of the value function of optimal stopping problems

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What is an optimal stopping problem? In general...

Given a stochastic process $X = (X_t)_{t \geq 0}$, an optimal stopping problem is to find two things: (1) The value of

$$v = \sup_{\tau} \mathbb{E} \left[ \int_0^\tau f(X_t) \, dt + g(X_\tau) \right].$$

where the supremum is taken over a specified class of stopping times, and

(2) A stopping rule $\tau^*$ such that

$$v = \mathbb{E} \left[ \int_0^{\tau^*} f(X_t) \, dt + g(X_{\tau^*}) \right].$$

We will concentrate in the special case where $f \equiv 0$, i.e., in the problem

$$\sup_{\tau} \mathbb{E} g(X_\tau).$$
What is an optimal stopping problem? In general...

Given a stochastic process \( X = (X_t)_{t \geq 0} \), an optimal stopping problem is to find two things: (1) The value of

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\nu = \sup_{\tau} \mathbb{E} \left[ \int_0^\tau f(X_t) \, dt + g(X_\tau) \right].
\]

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(2) A stopping rule \( \tau^* \) such that

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(2)

We will concentrate in the special case where \( f \equiv 0 \), i.e., in the problem

\[
\sup_{\tau} \mathbb{E} g(X_\tau).
\]
... in particular

If $X$ is a Markov process, one can formulate the problem as

$$v(x) = \sup_{\tau} \mathbb{E}_x g(X_\tau).$$

(3)

where the expectation is with respect to the measure $P_x(X_0 = x) = 1$.

This approach is convenient:

- When the initial state of the process is relevant.
- At time $t$, the decision to stop or to continue only depends on the present state of $X_t$.
- The problem becomes a problem of optimal stopping for a random path in the state space $E$ (instead of the probability space $\Omega$), which in general is $E = \mathbb{R}^n$. 
Stochastic Analysis: Doob inequalities

It is well-known that if $B$ is a standard Brownian motion and $\tau$ is any stopping time for $B$ with $\mathbb{E}\tau < \infty$ then Doob’s $L^2$-maximal inequality holds:

$$\mathbb{E}\left(\max_{0 \leq t \leq \tau} |B_t|^2\right) \leq 4\mathbb{E}|B_\tau|^2.$$  

(Actually it holds for any càdàg martingale or positive submartingale $X$)

If the Brownian motion starts at any given point $x \geq 0$ under $P_x$, i.e., $P_x(B_0 = x) = 1$, one can show that

$$\mathbb{E}_x\left(\max_{0 \leq t \leq \tau} |B_t|^2\right) \leq 4\mathbb{E}_x|B_\tau|^2 - 2x^2,$$

by solving the optimal stopping problem

$$V(y, s) = \sup_{\tau} \mathbb{E}_{y,s}(S_\tau - c\tau)$$

with the Markov process $(Y, S)$, where, for $0 \leq y \leq s$,

$$Y_t = |B_t|^2, \quad S_t = \left(\max_{0 \leq r \leq t} |B_r|^2\right) \vee s, \quad Y_0 = y = x^2, \quad S_0 = s.$$  

The idea is to realize that, setting $\nu(c) = \sup_{\tau} \mathbb{E}_{y,y}(S_\tau - c\tau)$ for $c > 0$,

$$\mathbb{E}_x\left(\max_{0 \leq t \leq \tau} |B_t|^2\right) \leq \inf_{c > 0} \{\nu(c) + c\mathbb{E}_x\tau\}.$$
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The idea is to realize that, setting $v(c) = \sup_{\tau} E_{y,y}(S_\tau - c\tau)$ for $c > 0$,

$$E_x\left(\max_{0 \leq t \leq \tau} |B_t|^2\right) \leq \inf_{c > 0} \{v(c) + cE_x \tau\}.$$
Statistics: sequential testing

At time $t = 0$ we begin to observe a Poisson process $X = (X_t)_{t \geq 0}$ with intensity $\lambda > 0$, which is either $\lambda_0$ or $\lambda_1$. The true value of $\lambda$ is not known to us.

Problem: to decide ASAP and with a minimal error probability the true value of $\lambda$. We now formalize this mathematically:

1. On a probability space $(\Omega, \mathcal{F}; P_\pi, \pi \in [0, 1])$ where

   $$P_\pi = \pi P_1 + (1 - \pi) P_0$$

   we assume that the r.v. $\lambda = \lambda(\omega)$ takes two values $\lambda_1$ and $\lambda_0$ according to the a priori distribution $P_\pi$, i.e.

   $$P_\pi(\lambda = \lambda_1) = \pi \quad \text{and} \quad P_\pi(\lambda = \lambda_0) = 1 - \pi.$$

2. Concerning the observable process $X$,

   $$P_\pi(X \in \cdot | \lambda = \lambda_i) = P_i(X \in \cdot),$$

   where $P_i(X \in \cdot)$ coincides with the distribution of a Poisson process with intensity $\lambda_i$. 
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  where $P_i(X \in \cdot)$ coincides with the distribution of a Poisson process with intensity $\lambda_i$. 
To test sequentially the hypothesis $H_1 : \lambda = \lambda_1$ and $H_0 : \lambda = \lambda_0$, assume we can decide

- A stopping time $\tau = \tau(\omega)$ to stop the observation of $X$, and
- A terminal decision function $d = d(\omega)$ which indicates that either $H_0$ or $H_1$ should be accepted.

Each decision rule $(\tau, d)$ implies losses:

- $c\mathbb{E}_\pi \tau$, $c > 0$ - due to a cost of observation.
- $aP_\pi(d = 0, \lambda = \lambda_1) + bP_\pi(d = 1, \lambda = \lambda_0)$, $a, b > 0$ - due to a wrong terminal decision.

The total average loss of the decision rule $(\tau, d)$ is

$$L_\pi(\tau, d) = \mathbb{E}_\pi \left( c \tau + a I_{(d=0, \lambda=\lambda_1)} + b I_{(d=1, \lambda=\lambda_0)} \right)$$

and the problem is then to compute

$$V(\pi) = \inf_{(\tau, d)} L_\pi(\tau, d).$$

The optimal $(\tau^*, d^*)$ is called $\pi$-Bayes decision rule.
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Mathematical Finance: American options

Let $X$ be an asset price and assume that it follows the SDE

$$dX_t = \sigma(X_t)dB_t + b(X)dt, \quad X_0 = x > 0.$$ 

An American-type option is a contract which gives the holder (or writer) the right to sell (or buy) the asset at any time before some fixed time $T$, and if she decides to exercise the option at time $t$ then she receives a payoff $g(X_t)$.

If the holder chooses the stopping time $\tau \leq T$, the payoff has present value (after lots of considerations)

$$\mathbb{E}_x e^{-r\tau} g(X_\tau), \quad r > 0 \text{ is the interest rate.}$$

Since the holder wants the highest reward but she doesn’t know what the price will be in the future, she should take the stopping time $\tau^*$ such that

$$\mathbb{E}_x e^{-r\tau^*} g(X_\tau) = \sup_{\tau \leq T} \mathbb{E}_x e^{-r\tau} g(X_\tau),$$

and this again is an optimal stopping problem.
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and this again is an optimal stopping problem.
Consider a two-dimensional strong Markov process 
\((X, Y) = (X_t, Y_t, t \geq 0)\) with state space \(\mathbb{R} \times S, S \subseteq \mathbb{R}\), where

\[
dX = a(X)Y dB
\]  

(4)

and \(Y\) is any of two classes:

1) **Regime-switching**: \(Y\) is an irreducible continuous-time MC independent of \(B\).

2) **Diffusion**: \(Y\) solves an SDE of the type

\[
dY = \eta(Y)dB^Y + \theta(Y)dt
\]  

(5)

where \(B\) and \(B^Y\) might be correlated and \(a, \eta, \theta\) are measurable functions.

*Note that \(Y\) does NOT depend on \(X\).*
The problem is...

...the regularity of the value function $v(x, y)$. Specifically, the monotonicity and continuity of $v(x, y)$ with respect to $y \in S$, where

$$v(x, y) = \sup_{0 \leq \tau \leq T} \mathbb{E}_{x,y} [e^{-q \tau} g(X_\tau)], \quad (x, y) \in \mathbb{R} \times S, \quad (6)$$

with

- $q > 0, \ T \in [0, \infty]$,
- the gain function $g : \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function,
- and the supremum is taken over all stopping times with respect to the filtration generated by $(X, Y)$.

For ease of presentation we will assume $T = \infty$. 
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Object of study
Optimal stopping problems
Some examples

Aim
The setting
The problem

Regime-switching model
Difficulties
Dealing with difficulties: time-change and coupling technique

Diffusion model
What else?
Game against nature

Regime-switching case

Assume that $Y$ is a continuous-time MC with $Q$-matrix $(q_{ij})$ taking values on $S = \{y_i : i = 1, 2, \ldots\} \subset (0, \infty)$ and,

$$X_t = x + \int_0^t a(X_s) Y_s \, dB_s,$$

The goal is to show that for fixed $x \in \mathbb{R}$ and $y, y' \in S$, it holds that

$$\text{if } y \leq y' \text{ then } v(x, y) \leq v(x, y'). \quad (7)$$

Recall that

$$v(x, y) = \sup_{\tau} \mathbb{E}_{x, y} [e^{-q\tau} g(X_\tau)]. \quad (x, y) \in \mathbb{R} \times S,$$

Immediate difficulties:

- $\mathbb{E}_{x, y}$ and $\mathbb{E}_{x, y'}$ are defined for different measures.
- $X_t$ depends on the value of $Y_t$, so path comparison is not effective.
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if $y \leq y'$ then $v(x, y) \leq v(x, y').$ \hspace{1cm} (7)

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Working on only ONE probability space

One difficulty is that $\mathbb{E}_{x,y}$ and $\mathbb{E}_{x,y'}$ are defined for different measures.

To overcome this we choose an arbitrary $(\Omega, \mathcal{F}, P)$ BIG enough to carry three processes $Y$, $Y'$, $W$ such that

- $Y$ has the same law as the original MC under $P_{x,y}$,
- $Y'$ has the same law as the original MC under $P_{x,y'}$, and
- $W$ is a Brownian motion independent of $(Y, Y')$.

Technical consideration: $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_\infty^Y \vee \mathcal{F}_\infty^{Y'}$.

We will construct two processes $X$ and $X'$ on $(\Omega, \mathcal{F}, P)$ such that they solve (weakly) the original SDE.

Then we can write

$$v(x, y) = \sup_{\tau} \mathbb{E} [e^{-q\tau} g(X_\tau)]$$

$$v(x, y') = \sup_{\tau'} \mathbb{E} [e^{-q\tau'} g(X'_{\tau'})]$$

where $\tau$ and $\tau'$ are stopping times with respect to the filtration generated by $(X, Y)$ and $(X', Y')$, respectively.
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where $\tau$ and $\tau'$ are stopping times with respect to the filtration generated by $(X, Y)$ and $(X', Y')$, respectively.
Another difficulty is that $X_t$ depends on the value of $Y_t$. To overcome this, we use a time-change method: given the MC $Y$, define

$$A_t = \int_0^t Y_u^2 du, \quad \text{and} \quad \Gamma_t = \inf\{s \geq 0 : A_s > t\},$$

so that $A = \Gamma^{-1}$. Consider $G = (G_t)_{t \geq 0}$ as a unique strong solution to the SDE

$$G_t = G_0 + \int_0^t a(G_s) dW_s, \quad G_0 = x.$$ 

Note that $G$ does not depend on $Y$.

Also define the local martingale $M_t = \int_0^t dW_u / Y_{\Gamma_u}$. The process $B = M \circ A = (M_{A_t})_{t \geq 0}$ is a Brownian motion.

For $X = G \circ A$ and $B = M \circ A$ we have

$$X_t = x + \int_0^t a(X_s) Y_s dB_s, \quad t \geq 0, \text{ a.s.},$$

Similarly with $Y'$ to get $X' = G \circ A'$ and $B' = M' \circ A'$.
Time-change method

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Similarly with $Y'$ to get $X' = G \circ A'$ and $B' = M' \circ A'$.
Putting all together

Observe that for $X = G \circ A$ and $\tau$ stopping time w.r.t. $(\mathcal{F}_{A_t})_{t \geq 0}$,

$$v(x, y) = \sup_{\tau} \mathbb{E} e^{-q \tau} g(X_{\tau}) = \sup_{\tau} \mathbb{E} e^{-q \Gamma A_{\tau}} g(G_{A_{\tau}})$$

$$= \sup_{\rho} \mathbb{E} e^{-q \Gamma \rho} g(G_{\rho}).$$

where we set $\rho = A_{\tau}$.

Similarly for $X' = G \circ A'$ and $\tau'$ stopping time w.r.t. $(\mathcal{F}_{A'_t})_{t \geq 0}$,

$$v(x, y') = \sup_{\tau'} \mathbb{E} e^{-q \tau} g(X'_{\tau'}) = \sup_{\rho} \mathbb{E} e^{-q \Gamma' \rho} g(G_{\rho})$$

where we set $\rho = A'_{\tau'}$.

Remark two important facts:

- In both cases, $\rho$ is a stopping time w.r.t. SAME $(\mathcal{F}_t)_{t \geq 0}$
- The dependence on the Markov chain is ONLY on the discount factors $\Gamma$ and $\Gamma'$ which are the inverses of

$$A_t = \int_0^t Y_u^2 du, \quad \text{and} \quad A'_t = \int_0^t (Y'_u)^2 du$$

Therefore, we WANT to obtain $A_t \leq A'_t$ for all $t \geq 0$ a.s.
Putting all together

Observe that for $X = G \circ A$ and $\tau$ stopping time w.r.t. $(\mathcal{F}_t^A)_{t \geq 0}$,

$$v(x, y) = \sup_{\tau} \mathbb{E} e^{-q \tau} g(X_\tau) = \sup_{\tau} \mathbb{E} e^{-q A_\tau} g(G_{A_\tau})$$

$$= \sup_{\rho} \mathbb{E} e^{-q \Gamma_{\rho}} g(G_{\rho}).$$

where we set $\rho = A_\tau$.

Similarly for $X' = G \circ A'$ and $\tau'$ stopping time w.r.t. $(\mathcal{F}_t^{A'})_{t \geq 0}$,

$$v(x, y') = \sup_{\tau'} \mathbb{E} e^{-q \tau} g(X_{\tau'}) = \sup_{\rho} \mathbb{E} e^{-q \Gamma'_{\rho}} g(G_{\rho})$$

where we set $\rho = A'_{\tau'}$.

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Coupling of Markov chains

Suppose that $\tilde{Y}$ and $Y'$ are independent MC’s with the same intensity matrix $Q$ but $\tilde{Y}_0 = y$ and $Y'_0 = y'$, with $y \leq y'$. Assume that $Q$ is tridiagonal.

Define the coupling time

$$C = \inf\{t \geq 0 : \tilde{Y}_t = Y'_t\}$$

and the Markov chain

$$Y_t = \begin{cases} 
\tilde{Y}_t & \text{if } t < C \\
Y'_t & \text{if } t \geq C.
\end{cases}$$

We have that $Y$ has the same law as $\tilde{Y}$ and $Y_0 = \tilde{Y}_0 = y$.

Moreover, the MC $\tilde{Y}$ does NOT overtake $Y'$ before $C$, then

$$Y_t \leq Y'_t \quad \forall t \geq 0.$$ 

Therefore, we can ALWAYS obtain

$$A_t \leq A'_t \quad \forall t \geq 0 \quad \text{a.s.}$$

as we required for showing $v(x, y) \leq v(x, y')$. 

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"
Diffusion case

Now the two-dimensional strong Markov process \((X, Y)\) is given by

\[
\begin{align*}
dX &= a(X)YdB \\
dY &= \eta(Y)dY + \theta(Y)dt.
\end{align*}
\]

Again, it holds true that for fixed \(x, y, y' \in \mathbb{R}\),

\[
\text{if } y \leq y' \text{ then } v(x, y) \leq v(x, y').
\]

The technique is adapted from the MC case, i.e., it is based on time-change and coupling but with more technicalities.

From the monotonicity result we can also show continuity of \(v(x, y)\) in the parameter \(y\), at least in the finite-horizon \((T < \infty)\) case.
Diffusion case

Now the two-dimensional strong Markov process \((X, Y)\) is given by

\[
\begin{align*}
    dX &= a(X)YdB \\
    dY &= \eta(Y)dB^Y + \theta(Y)dt.
\end{align*}
\]

Again, it holds true that for fixed \(x, y, y' \in \mathbb{R}\),

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Game against nature

The time-change/coupling method proved to be very powerful in comparison of processes.

Consider the following game against nature:

The payoff of the game is given by

\[ f_{x,y}(\tau, Q) = \mathbb{E}_{x,y} e^{-q\tau} g(X^Q_T) \]

where \( X^Q \) stands for the solution to \( dX = a(X)YdB \) if \( Y \) has intensity matrix \( Q \).

Suppose that

- **nature** chooses \( Q \) so as to minimize \( f \), while
- **the player** chooses the stopping rule \( \tau \) so as to maximize \( f \).

Then we could say that

- Nature plays the worst strategy against the player \( \rightarrow \) If Woody Allen is the player he would say "I am at two with nature."
- or, the player plays the best strategy assuming the worst situation \( \rightarrow \) pessimistic view.
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- Nature plays the worst strategy against the player \( \rightarrow \) If Woody Allen is the player he would say "I am at two with nature."
- or, the player plays the best strategy assuming the worst situation \( \rightarrow \) pessimistic view.
The value functions of the player and nature are given by

\[ v^P(x, y) = \sup_{\tau} \inf_Q f_{x,y}(\tau, Q), \]

\[ v^N(x, y) = \inf_Q \sup_{\tau} f_{x,y}(\tau, Q). \]

One can check that \( v^P(x, y) \leq v^N(x, y) \) and so the player wants to achieve the equality so as to maximize its value function.

Suppose there exist strategies \( \tau^* \) and \( Q^* \) such that

\[ \tau^* = \arg \max f_{x,y}(\cdot, Q^*), \]

\[ Q^* = \arg \min f_{x,y}(\tau^*, \cdot). \]

It follows that

\[ f_{x,y}(\tau, Q^*) \leq f_{x,y}(\tau^*, Q^*) \leq f_{x,y}(\tau^*, Q) \]

for all strategies \( \tau \) and \( Q \). If the above inequalities hold then \( v^P(x, y) = v^N(x, y) \).

Hence we look for the \textit{saddle point} \((\tau^*, Q^*)\)!
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Solution

One can find a saddle point \((\tau^*, Q^*)\), i.e,

\[ f_{x,y}(\tau, Q^*) \leq f_{x,y}(\tau^*, Q^*) \leq f_{x,y}(\tau^*, Q). \]

The left-hand side inequality is simply the solution to the optimal stopping problem

\[ f(\tau^*, Q^*) = \sup_{\tau} f_{x,y}(\cdot, Q^*), \]

which is known to exist.

The right-hand side inequality can be solved using time-change and coupling!

Work in progress is to study the problem in the case where \(Y\) is a controlled diffusion process.
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Thank you!