On a zero-sum game of stopping and control

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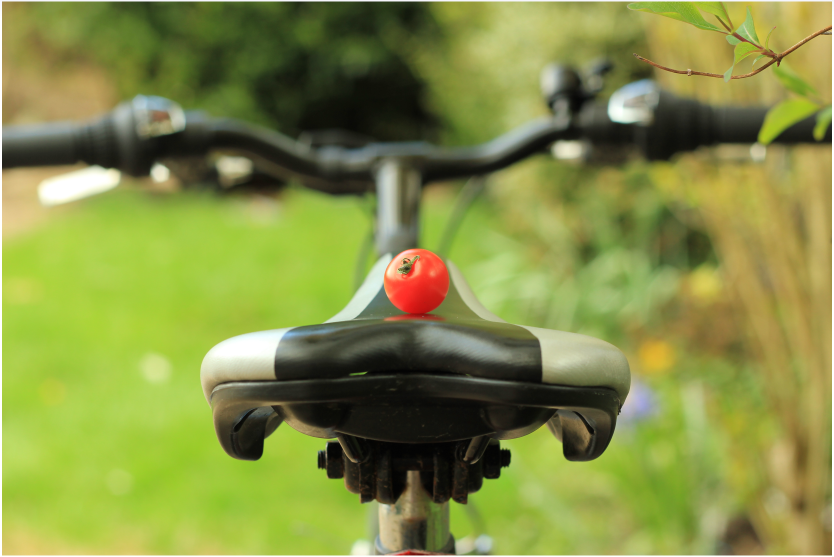

Optimal stopping with regime switching volatility

- Dynamics:

$$dX_t = X_t(Y_t\, dB_t + \mu\, dt),$$

with $\mu \in \mathbb{R}$; Y is a continuous time MC with finite state space $\mathcal{S} = \{1, 2, \dots, d\}$.

- Consider the value function

$$v(x, y) = \sup_{\tau} \mathbb{E}_{x,y}\, e^{-\alpha\tau} g(X_\tau),$$

where $\alpha > 0$ and g is measurable.

- Typical examples in American option pricing:
  - Put option, $g(x) = \max\{K - x, 0\}$
  - Call option, $g(x) = \max\{x - K, 0\}$

Guo and Zhang (2004), Jobert and Rogers (2006): Perpetual American put option.

- The optimal strategy is of the form

$$\tau^* = \inf\{t \geq 0 : X_t \leq b[Y_t]\}$$

- The *thresholds* $b[i]$, $1 \leq i \leq d$, are unknown in principle.
- An algebraic algorithm is given to compute $v(x, y)$, assuming that the $b[i]$'s are given and in a specific order. But there are d! possible orderings!
- Numerical examples in these papers suggest that the thresholds *may be monotone*.

Question 1: can we show that the thresholds are monotone?

YES!... To the extent that Y is skip-free.

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YES!... To the extent that Y is skip-free.

Intuitive fact:

The faster X moves (due to larger Y), the sooner X reaches the high values of g.

Theorem

Suppose Y is a skip-free MC and $g \geq 0$ and measurable. If either $\mu = 0$ or g is non-increasing, then $v(x, \cdot)$ is non-decreasing.

This in turn yields the unique order

$$b[1] \geq b[2] \geq \ldots \geq b[d].$$

Otherwise, $\exists\, x : \; b[i] < x \leq b[i+1]$ for some i, leading to the contradiction:

$$v(x, i) > g(x) = v(x, i+1).$$

Question 2: if Y is a diffusion, is $y \mapsto v(x, y)$ monotone?

YES!... under suitable assumptions, but same intuition.

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THE UNIVERSITY OF WARWICK

Parameter uncertainty

- Dynamics:

$$dX_t = \sigma(X_t)\, Y_t^{\pi}\, dB_t,$$
$$dY_t^{\pi} = \eta(Y_t^{\pi})dB_t^Y + \pi_t\, dt.$$

- Parameter π is only known to be in the class $\mathcal{A}$ of predictable processes $\pi = (\pi_t)_{t\geq 0}$ such that

$$0 \leq a(Y_t) \leq \pi_t \leq b(Y_t).$$

σ, η, a, b are real-valued continuous functions and $a(\cdot) \leq b(\cdot)$.

- Description of the game:

$$\underbrace{g(X_\tau^{\pi})}$$

minimizer ↙ ↘ maximizer

chooses π chooses τ

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minimizer chooses π — maximizer chooses τ

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Question 3: does the game have a value? That is, is it true that

$$\sup_\tau \inf_\pi E_{x,y} e^{-\alpha\tau} g(X^\pi_\tau) = \inf_\pi \sup_\tau E_{x,y} e^{-\alpha\tau} g(X^\pi_\tau)\,?$$

- We show the existence of a saddle point $(\hat\tau, \hat\pi)$, that is,

$$\mathbb{E}_{x,y}[e^{-\alpha\tau} g(X^{\hat\pi}_\tau)] \quad \le \quad \underbrace{\mathbb{E}_{x,y}[e^{-\alpha\hat\tau} g(X^{\hat\pi}_{\hat\tau})]}_{\text{value}} \quad \le \quad \mathbb{E}_{x,y}[e^{-\alpha\hat\tau} g(X^{\pi}_{\hat\tau})]$$

  for all stopping rules τ and controls π.
- Intuitive guess:
  - $\hat\pi_t = a(Y_t)$,
  - $\hat\tau$ the optimal stopping rule for the problem $\sup_\tau \mathbb{E}_{x,y} e^{-\alpha\tau} g(X^{\hat\pi}_\tau)$.
- The value of the game may be interpreted as the price of an American-type option in the
  - worst-case scenario for the writer (seller), or
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Monotonicity: diffusion case

- Let (X, Y) have dynamics (weakly unique solution)

$$X_t = X_0 + \int_0^t \sigma(X_s)\, Y_s\, dB_s, \qquad Y_t = Y_0 + \int_0^t \eta(Y_s) dB_s^Y + \int_0^t \theta(Y_s) ds.$$

  with values in $\mathbb{R} \times \mathcal{S}$, where $\mathcal{S} \subseteq (0, \infty)$.

- Consider $v(x, y) = \sup_\tau E_{x,y}\, e^{-\alpha\tau} g(X_\tau)$,

  g is only assumed to be non-negative and measurable.

- We apply time-change and coupling techniques to show that

$$y \mapsto v(x, y) \quad \text{is monotone.}$$

- Method of proof inspired by (amongst others)
  - E. Ekström (2004),
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Time-change heuristics

- We can rewrite

$$X_t = x + \int_0^t \sigma(X_s)\, dM_s, \qquad \text{where } M_t = \int_0^t Y_s dB_s.$$

- M defines a time-change $\Gamma = \langle M \rangle^{-1}$.
- Assuming $\langle M \rangle_\infty = \infty$, $\quad G = X \circ \Gamma, \xi = Y \circ \Gamma \quad$ satisfy

$$G_t = x + \int_0^t \sigma(G_s)\, dW_s, \qquad \xi_t = y + \int_0^t \eta(\xi_s)\, \xi_s^{-1} dW_s^\xi + \int_0^t \pi(\xi_s)\, \xi_s^{-2} ds.$$

- Heuristically, set $A = \Gamma^{-1}$ and $\rho = A_\tau$ so that

$$E_{x,y}\, e^{-\alpha\tau} g(X_\tau) \longleftrightarrow E_{x,y}\, e^{-\alpha\Gamma_{A_\tau}} g(G_{A_\tau}) \longleftrightarrow E_{x,y}\, e^{-\alpha\Gamma_\rho} g(G_\rho)$$

- Advantage: G does NOT depend on Y.

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Time-change

- Assume that there is a unique non-exploding strong solution in $\mathbb{R} \times \mathcal{S}$ for:

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on some $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ carrying (W, W^{ξ}) with natural, augmented filtration $(\tilde{\mathcal{F}}_t)$.

- Simple path-comparison: for $0 < y \leq y'$,

$$0 < \xi_t \leq \xi'_t, \quad t \geq 0 \text{ a.s.}$$

by strong uniqueness.

- Define $\Gamma_t = \int_0^t \xi_s^{-2} ds$, $\Gamma'_t = \int_0^t (\xi'_s)^{-2} ds$ so that $\Gamma_t \geq \Gamma'_t$ and also define

$$\tilde{X} = G \circ A,\ \tilde{Y} = \xi \circ A; \qquad \tilde{X}' = G \circ A',\ \tilde{Y}' = \xi' \circ A'.$$

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Coupling

- We constructed $(\tilde{X}, \tilde{Y})$ and $(\tilde{X}', \tilde{Y}')$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ such that

$$(\tilde{X}, \tilde{Y}) \overset{law}{=} \underbrace{(X, Y)}_{\text{under } P_{x,y}} \qquad \text{and} \qquad (\tilde{X}', \tilde{Y}') \overset{law}{=} \underbrace{(X', Y')}_{\text{under } P_{x,y'}}$$

- It remains to establish

$$v(x, y) = \sup_{\rho \in \mathcal{M}} \tilde{E}\, e^{-\alpha \Gamma_\rho} g(G_\rho)$$

where $\mathcal{M}$ is the class of stopping times ρ w.r.t. $(\tilde{\mathcal{F}}_t)_{t \geq 0}$. Similarly for $v(x, y')$.

- Since $\Gamma_t \geq \Gamma'_t$, for all $t \geq 0$,

$$\tilde{E}\, e^{-\alpha \Gamma_\rho} g(G_\rho) \leq \tilde{E}\, e^{-\alpha \Gamma'_\rho} g(G_\rho)$$

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"Back to the game"

We show the existence of a saddle point $(\hat{\tau}, \hat{\pi})$, that is,

$$\mathbb{E}_{x,y}[e^{-\alpha\tau} g(X^{\hat{\pi}}_{\tau})] \quad \leq \quad \underbrace{\mathbb{E}_{x,y}[e^{-\alpha\hat{\tau}} g(X^{\hat{\pi}}_{\hat{\tau}})]}_{\text{value}} \quad \leq \quad \mathbb{E}_{x,y}[e^{-\alpha\hat{\tau}} g(X^{\pi}_{\hat{\tau}})]$$

for all stopping rules τ and controls π.

Assume

- g is continuous, non-negative and bounded (for convenience).
- $(X^{\hat{\pi}}, Y^{\hat{\pi}})$ is a strong solution of

$$dX_t = \sigma(X_t)\, Y_t\, dB_s, \qquad dY_t = \eta(Y_t)\, dB^Y_t + \hat{\pi}(Y_t)\, dt.$$

Recall. Intuitive guess:

- $\hat{\pi}_t = a(Y_t)$,
- $\hat{\tau}$ the optimal stopping rule for $\hat{v}(x, y) = \sup_{\tau} \mathbb{E}_{x,y} e^{-\alpha\tau} g(X^{\hat{\pi}}_{\tau})$:

$$\hat{\tau} = \hat{\tau}^{x,y,\pi} = \inf\{t \geq 0 : (X^{\pi}_t, Y^{\pi}_t) \notin \mathcal{C}\}$$

where $\mathcal{C} = \{(x, y) \in \mathbb{R} \times \mathcal{S} : \hat{v}(x, y) > g(x)\}$ is fixed.

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- $\hat\tau$ the optimal stopping rule for $\hat v(x,y) = \sup_\tau \mathbb{E}_{x,y} e^{-\alpha\tau} g(X_\tau^{\hat\pi})$:

$$\hat\tau = \hat\tau^{x,y,\pi} = \inf\{t \geq 0 : (X_t^\pi, Y_t^\pi) \notin \mathcal{C}\}$$

where $\mathcal{C} = \{(x,y) \in \mathbb{R} \times \mathcal{S} : \hat v(x,y) > g(x)\}$ is fixed.

"Back to the game"

We show the existence of a saddle point $(\hat\tau, \hat\pi)$, that is,

$$\underbrace{\mathbb{E}_{x,y}[e^{-\alpha\tau} g(X^{\hat\pi}_{\tau})] \quad \le \quad \mathbb{E}_{x,y}[e^{-\alpha\hat\tau} g(X^{\hat\pi}_{\hat\tau})]}_{\text{By optimality of } \hat\tau} \quad \le \quad \mathbb{E}_{x,y}[e^{-\alpha\hat\tau} g(X^{\pi}_{\hat\tau})]$$

for all stopping rules τ and controls π.

Assume

- g is continuous, non-negative and bounded (for convenience).
- $(X^{\hat\pi}, Y^{\hat\pi})$ is a strong solution of

$$dX_t = \sigma(X_t)\, Y_t\, dB_s, \qquad dY_t = \eta(Y_t)\, dB^Y_t + \hat\pi(Y_t)\, dt.$$

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Bellman's principle

Define

$$u(x, y) := \inf_{\pi} E_{x,y} e^{-\alpha\hat{\tau}} g(X^{\pi}_{\hat{\tau}}).$$

Bellman's principle:

$e^{-\alpha\hat{\tau}\wedge t} u(X^{\pi}_{\hat{\tau}\wedge t}, Y^{\pi}_{\hat{\tau}\wedge t})$ is a submartingale for arbitrary π and a martingale for the optimal.

Candidate value function is

$$w(x, y) := E_{x,y}\, e^{-\alpha\hat{\tau}} g(X^{\hat{\pi}}_{\hat{\tau}}).$$

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Optimality of $\hat{\pi} = a$

Recall that $\pi \in \mathcal{A}$ satisfies $0 \le a(Y_t) \le \pi_t \le b(Y_t)$.

- Note that $w(x,y) \equiv \sup_\tau E_{x,y} e^{-\alpha\tau} g(X_\tau^{\hat{\pi}})$ and so
  - $w(x,\cdot)$ is non-decreasing.
  - $(\mathbb{L}^{\hat{\pi}} w - \alpha w)(x,y) = 0$ for $(x,y) \in \mathcal{C}$ where

$$\mathbb{L}^{\hat{\pi}} w(x,y) = \frac{1}{2} w_{xx}(x,y)\sigma^2(x)y^2 + \frac{1}{2} w_{yy}(x,y)\eta(y)^2 + w_y(x,y)\hat{\pi}(y).$$

- Since $\pi_t \ge \hat{\pi}_t$ and $w_y \ge 0$ we have

$$(L_t^\pi w - \alpha w) \ge (\mathbb{L}^{\hat{\pi}} w - \alpha w) = 0 \qquad \text{in } \mathcal{C}.$$

- Using a localization argument, the process

$$N_t(\pi) := e^{-\alpha \hat{\tau} \wedge t} w(X^\pi_{\hat{\tau}\wedge t}, Y^\pi_{\hat{\tau}\wedge t})$$

  stopped at $\tau_R =$ first exit from the open ball of radius R, is a submartingale.
- Dominated convergence (g is bounded) yields, as $R \to \infty$,

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Conclusion

1. Value function $v(x, y)$ of the optimal stopping problem associated to $\hat{\pi}$ is monotone in y.
2. By time-change we transfer

$$\text{dependance of } X \text{ on } y \quad \longrightarrow \quad \text{dependance of } \Gamma \text{ on } y$$

3. By coupling we compare the time-changes Γ pathwise.
4. Bellman's principle establishes the existence of a saddle point.

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THANK YOU!

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