Monotonicity of the value function for a two-dimensional optimal stopping problem

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Theorem 1 (Monotonicity)
Assumptions: Let $(X, Y)$ be as in (1)-(2) and assume that

a) $Y$ takes values in $S = (0, \infty)$ and for $(x, y) \in R \times S$, $P_x \left( \int_0^y \rho^2 ds < \infty, \forall t \geq 0 \right) = 1$.  $P_x \left( \lim_{t \to \infty} \int_0^y \rho^2 ds = \infty \right) = 1$. (4)

b) The measurable functions $a, \theta, \gamma$ are such that the system

\[ a(x,y)\frac{dX}{dY} = a(x) dW + \theta(x, y) d\xi + \gamma(x, y) dt \]

has a unique non-exploiting strong solution with $\xi \in S, t \geq 0$.

Result: for every $x \in R$ and $y, y' \in S$ with $y \leq y'$, $v(x, y) \leq v(x, y')$.

idea of proof: we sketch the proof in three steps (see middle column):

1. Construct a weak solution $\bar{X}, \bar{Y}$ of (1)-(2) by time-changing $(\ast)$.

2. Formulate an equivalent optimal stopping problem in the time-changed setting.

3. Use coupling to make path comparison of processes.

Theorem 2 (Continuity)
Assumptions: Let the assumptions of Theorem 1 hold and let the $\xi$ component of $(\ast)$ result in a Feller process with state space $S = (0, \infty)$.

Result:

i) If the horizon $T = \infty$ then $v(x, y)$ is continuous (although for right-continuity one has to assume a stronger integrability condition).

ii) If the horizon $T < \infty$ and $\xi$ is continuous then $v(x, y)$ is continuous.

Idea of proof:

i) Fix $[0, \infty)$ such that $y < y_0, y_0 \in (0, \infty)$.

ii) Let $(\xi_t)^{y_0}$ be the solution of $(\ast)$ starting from $(x, y_0), n = 1, 2, \ldots$ and construct $f_n, A^n, (\tilde{X}_t^y, \tilde{Y}_t^y)$ as in Step 1.

iii) If the sequence $(\xi_t)^{y_0}$ is monotone, then the coupling argument and the Feller property of $\xi$ imply $\gamma_n \to \gamma, A^n \to A$ if $n \to \infty$ for all $t \geq 0$, a.s.

Also, the strong monotonicity of $v(x, y)$ is the key tools to show left-continuity and right-continuity of $v(x, y)$.

Example: Hull and White model

The SV model proposed by Hull and White (after a change of variable) is of the form

\[ dx = XY dB, \quad dy = Y dB + \theta Y dt \]

where $\gamma, \theta$ are positive constants.

The time-changed volatility takes the form $d\xi = \gamma dW + \theta(t)^{-1} dt$.

The process $Z(t) = \xi(t)$ is a Bessel process of dimension $\phi = 1 + 2\theta/\gamma^2$.

Thus the condition $b)$ is satisfied if and only if $\phi \geq 2$.

Also, $\phi \geq 2$ implies that $\gamma_n \to \gamma, \xi_n \to \xi$ and $\xi_n \to \xi$ satisfies $\xi$.

That is, if a Feller process with state space $S$ is a known property of Bessel processes.

Some extensions

- All the previous results can be stated for a general measurable gain function $g$ which may take negative values. In the case $T = \infty$.

- We assume that $g(0) = 0$ and $P_{x,y}(\inf \{t \geq 0 : g(X_t) > 0\} < \infty)$, which is sufficient for $v(x,y) = \sup_{t \geq 0} E_x e^{-\gamma t} g(X_t)$ where $X_t$ consists of all the finite stopping times such that $g(X_t) > 0$.

- In the context of American option pricing, the payoff takes the form $g(e^{\gamma t} X_t)$ if exercised at $t$. The setup becomes

\[ P(X_t) = \sup_{t \geq 0} E_x e^{-\gamma t} g(e^{\gamma t} X_t) \]

Monotonicity of $P(x, y)$ will follow if $g$ is a non-increasing function, while continuity carries over if we further assume that $g$ is continuous.

Our results would not change in principle if $S \subseteq (\infty, 0)$ instead. Instead, we can change only that increasing would turn into decreasing.