Monotonicity of the value function for a two-dimensional optimal stopping problem

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Introduction

Setting: consider a two-dimensional strong Markov process (X,Y) on $(\Omega,\mathcal{F},P_{x,y},(x,y)\in \boldsymbol{R}\times\boldsymbol{S})$ with $\boldsymbol{S}\subseteq(0,\infty)$, such that

$$dX = a(X)YdB \qquad (1)$$

and Y is any of the two classes:
Regime-switching: Y is a continuous-time MC, independent of B.
Diffusion: Y solves a stochastic differential equation of the type

$$dY = \eta(Y)dB^Y + \theta(Y)dt \qquad (2)$$

where $B^Y=(B_t^Y)_{t\geq 0}$ is a standard Brownian motion such that $\langle B,B^Y\rangle_t=\delta t$, $t\geq 0$, for some real parameter $\delta\in[-1,1]$.

Aim: our main focus is on proving the monotonicity of $y\mapsto v(x,y)$,

$$v(x,y)=\sup_{0\leq\tau\leq T} E_{x,y}[e^{-q\tau}g(X_\tau)], \quad (x,y)\in\boldsymbol{R}\times\boldsymbol{S}, \qquad (3)$$

where $q>0$, $T\in[0,\infty]$, the gain function $g:\boldsymbol{R}\to\boldsymbol{R}$ is a general measurable function and the supremum is taken over all finite stopping times with respect to the filtration generated by (X,Y).

As a consequence of the monotonicity, we are also able to show continuity of $y\mapsto v(x,y)$, in both finite and infinite horizon.

Application: pairs (X,Y) as above are common in mathematical finance:
- $X\rightsquigarrow$ discounted price of an asset
- $Y\rightsquigarrow$ stochastic volatility.

The results can be applied to prices of American options in popular models such as Heston or Hull-White.

Some background: the monotonicity property was investigated for
- European options with convex gain function by Hobson [2] with (1)-(2),
- American options by Ekström [1] in the case that $Y=1$ and $a=a(x,t)$.

Note: in this presentation we concentrate on the diffusion case and $g\geq 0$.

[1] Ekström, E., Properties of American options prices, *Stoch. Proc. Appl.* **114**, 265-278 (2004).
[2] Hobson, D., Comparison results for stochastic volatility models via coupling, *Finance and Stochastics* **14**, 129-152 (2010).

Theorem 1 (Monotonicity)

Assumptions: let (X,Y) be as in (1)-(2) and assume that

a) Y takes values in $\boldsymbol{S}=(0,\infty)$ and for $(x,y)\in\boldsymbol{R}\times\boldsymbol{S}$,

$$P_{x,y}\left(\int_0^t Y_s^2 ds<\infty,\ \forall t\geq 0\right)=1,\quad P_{x,y}\left(\lim_{t\uparrow\infty}\int_0^t Y_s^2 ds=\infty\right)=1. \qquad (4)$$

b) The measurable functions a,η,θ are such that the system

$$(\star)\begin{cases} dG = a(G)dW \\ d\xi=\eta(\xi)\xi^{-1}dW^\xi+\theta(\xi)\xi^{-2}dt\end{cases}$$

has a unique non-exploding strong solution with $\xi_t\in\boldsymbol{S}$, $t\geq 0$.

Result: for every $x\in\boldsymbol{R}$ and $y,y'\in\boldsymbol{S}$ with $y\leq y'$, $v(x,y)\leq v(x,y')$.

Idea of proof: we sketch the proof in three steps (see middle column):
1. Construct a weak solution $(\tilde X,\tilde Y)$ of (1)-(2) by time-changing $(\star)$.
2. Formulate an equivalent optimal stopping problem in the time-changed setting.
3. Use coupling to make path comparison of processes.

Step 1: constructing a weak solution $(\tilde X,\tilde Y)$.

- Choose $(\tilde\Omega,\tilde{\mathcal F},\tilde P)$, complete, big enough to carry a pair of BM's (W,W^ξ) with $\langle W,W^\xi\rangle_t=\delta t$, and denote by $\tilde{\mathcal F}_t$ the augmentation of the filtration generated by (W,W^ξ).
- Let (G,ξ) be the unique solution of $(\star)$ starting from (x,y).
- Setting $M_t:=\int_0^t Y_u\,dB_u$, we verify that $Y\circ\langle M\rangle^{-1}$ solves weakly the second component of $(\star)$, then ξ has the same law as $Y\circ\langle M\rangle^{-1}$ under $P_{x,y}$.
- Define $\Gamma_t:=\int_0^t\xi_u^{-2}du$. Since $\int_0^t(Y\circ\langle M\rangle_u^{-1})^{-2}du$ is the inverse of $\int_0^t Y_u^2du$, the previous step and (4) imply

$$\tilde P(\Gamma_t<\infty,\forall t\geq 0)=1 \quad\text{and}\quad \tilde P\left(\lim_{t\uparrow\infty}\Gamma_t=\infty\right)=1. \qquad (5)$$

- As Γ_t is continuous and strictly increasing, so is $A_t=\Gamma_t^{-1}$, and

(P1) $\Gamma_{A_t}=A_{\Gamma_t}=t$ for all $t\geq 0$ $\tilde P$–a.s. and
(P2) $s<\Gamma_t$ if and only if $A_s<t$ for all $0\leq s,t<\infty$ $\tilde P$–a.s.

- As ξ is $\tilde{\mathcal F}_t$-adapted, we have that $\tilde M_t=\int_0^t\xi_u^{-1}dW_u$ and $\tilde M_t^\xi=\int_0^t\xi_u^{-1}dW_u^\xi$ are $\tilde{\mathcal F}_t$-local martingales (which do exist by (5)), and then the $\tilde{\mathcal F}_{A_t}$-adapted processes

$$\tilde X=G\circ A,\qquad \tilde Y=\xi\circ A \qquad (6)$$

constitute a non-exploding weak solution of (1)-(2), with driving BM's

$$\tilde B=\tilde M\circ A,\qquad \tilde B^Y=\tilde M^\xi\circ A.$$

Step 2: formulating an equivalent optimal stopping problem.

- The solution $(\tilde X,\tilde Y)$ constructed in Step 1, must have the same law as (X,Y) under $P_{x,y}$ by the assumption b). Then we can write

$$v(x,y)=\sup_{\tau\in\mathcal T_T}\tilde E[e^{-q\tau}g(\tilde X_\tau)]$$

where $\mathcal T_T=\{$ finite stopping times τ w.r.t. $(\tilde{\mathcal F}_{A_t})_{t\geq 0}$ such that $0\leq\tau\leq T\}$.

Remark that even though $\mathcal F_t^{\tilde X,\tilde Y}\subseteq\tilde{\mathcal F}_{A_t}$, $(\tilde X,\tilde Y)$ is a strong Markov process w.r.t. $\tilde{\mathcal F}_{A_t}$ and so we may take stopping times w.r.t this filtration.

- Using the properties (P1)-(P2) above is not difficult to verify that

$$v(x,y)=\sup_{\rho\in\mathcal M_T}\tilde E[e^{-q\rho}g(G_\rho)], \qquad (7)$$

where $\mathcal M_T=\{$ finite stopping times ρ w.r.t. $(\tilde{\mathcal F})_{t\geq 0}$ such that $0\leq\tau\leq A_T\}$.

Step 3: coupling of stochastic processes.

- Fix $x\in\boldsymbol{R}$ and $y,y'\in\boldsymbol{S}$ such that $y\leq y'$. Let (G,ξ) and (G,ξ') be the solutions of $(\star)$ starting from (x,y) and (x,y') respectively.
- Define $C=\inf\{t\geq 0:\xi_t>\xi_t'\}$ and set $\bar\xi_t=\xi_t I(t<C)+\xi_t' I(t\geq C)$. Then $(G,\bar\xi)$ solves $(\star)$ with $(G_0,\bar\xi_0)=(x,y)$ and hence $\xi_t=\bar\xi_t\leq\xi_t'$, $t\geq 0$, a.s. by strong uniqueness.
- Construct $\tilde X=G\circ A$, $\tilde X'=G\circ A'$ on $(\tilde\Omega,\tilde{\mathcal F},\tilde P)$ as in Step 1. It follows from $0\leq\xi_t\leq\xi_t'$ that $\Gamma_t\geq\Gamma_t'$, $t\geq 0$, a.s.
- Using that $A_t\leq A_t'$, we have $\mathcal M_T\subseteq\mathcal M_T'$. Thus by (7) one obtains $v(x,y)\leq v(x,y')$, Q.E.D.

Theorem 2 (Continuity)

Assumptions: let the assumptions of Theorem 1 hold and let the ξ component of $(\star)$ result in a Feller process with state space $\boldsymbol{S}=(0,\infty)$.

Result:

i) If the horizon $T=\infty$ then $v(x,\cdot)$ is continuous (although for right-continuity one has to assume a stronger integrability condition).

ii) If the horizon $T<\infty$ AND g is continuous then $v(x,\cdot)$ is continuous.

Idea of proof:
- Fix $(y_n)_{n=1}^\infty\subseteq(0,\infty)$ such that $y_n\to y_0$, $y_0\in(0,\infty)$.
- Let (G,ξ^n) be the solution of $(\star)$ starting from (x,y_n), $n=1,2,\ldots$, and construct Γ^n, A^n and $(\tilde X^n,\tilde Y^n)$ as in Step 1.
- If the sequence $(y_n)_{n=1}^\infty$ is monotone, then the coupling argument and the Feller property of ξ imply

$$\Gamma_t^n\to\Gamma_t^0\quad\text{and}\quad A_t^n\to A_t^0\quad\text{if}\quad n\to\infty$$

for all $t\geq 0$, a.s.

- The previous fact and the monotonicity of $v(x,\cdot)$ are the key tools to show left-continuity and right-continuity of $v(x,\cdot)$.

Example: Hull and White model

- The SV model proposed by Hull and White (after a change of variable) is of the form

$$dX = XYdB \qquad X_0=x>0,$$
$$dY=\eta YdB^Y+\theta Ydt \qquad Y_0=y>0,$$

where η,θ are positive constants.

- The time-changed volatility takes the form $d\xi_t=\eta\,dW_t^\xi+\theta\xi^{-1}dt$. The process $Z_t=\xi_t/\eta$ is a Bessel process of dimension $\phi=1+2\theta/\eta^2$. Thus the condition b) is satisfied if and only if $\phi\geq 2$.
- Also, $\phi\geq 2$ implies that $\Gamma_t=\int_0^t(\xi_u)^{-2}du=\eta^2\int_0^t(Z_u)^{-2}du$ satisfies (5).
- That ξ is a Feller process with state space $\boldsymbol{S}$ is a known property of Bessel processes.

Some extensions

- All the previous results can be stated for a general measurable gain function g which may take negative values, in the case $T=\infty$.

We assume that $\{g\geq 0\}\neq\emptyset$ and $P_{x,y}(\inf\{t\geq 0:g(X_t)\geq 0\}<\infty)=1$, which is sufficient for $v(x,y)=\sup_{\tau\in\mathcal K_\infty^{g+}}E_{x,y}e^{-q\tau}g(X_\tau)$, where $\mathcal K_\infty^{g+}$ consists of all the finite stopping times such that $g(X_\tau)\geq 0$.

- In the context of American option pricing, the payoff takes the form $g(e^{r\tau}X_\tau)$ if exercised at τ. The setup becomes

$$P(x,y)=\sup_{0\leq\tau\leq T}E_{x,y}[e^{-r\tau}g(e^{r\tau}X_\tau)].$$

Monotonicity of $P(x,\cdot)$ will follow if g is a non-increasing function, while continuity carries over if we further assume that g is continuous.

- Our results would not change in principle if $\boldsymbol{S}\subseteq(-\infty,0)$ instead. What would change is only that increasing would turn into decreasing.