Fundamental Tools Sheet 5: Probability spaces, conditional probabilities and counting

1. Write down the defining properties for a σ-algebra on some sample space Ω. Is $\mathcal{F} = \{\{1\}, \{2\}, \{3\}\}$ a σ-algebra on $\Omega = \{1, 2, 3\}$? Explain your answer.

2. Suppose that B_1, B_2, B_3 is a partition of a sample space Ω (so $B_i \cap B_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{3} B_i = \Omega$). Write down the smallest σ-algebra $\mathcal{F}$ of Ω such that $B_i \in \mathcal{F}$ for every i.

3. A coin will be tossed two times, and we are interested in whether the tosses give the same outcome. Write down the minimal σ-algebra which contains such information.

4. Suggest a probability space suitable for modelling the outcome of tossing 2 fair coins (so you must specify a sample space Ω, a σ-algebra $\mathcal{F}$ and a probability measure $\mathbb{P}$). Let E_H denote the event that at least one head occurs and E_T the event that at least one tail occurs. Write out E_H, E_T and $E_H \cap E_T$ explicitly in terms of the outcomes in your sample space. Show that E_H and E_T are not independent.

5. Write down the definition for a probability measure $\mathbb{P}$ defined on a σ-algebra $\mathcal{F}$. Using the fundamental properties of $\mathbb{P}$, show that

(a) $\mathbb{P}(A^C) = 1 - \mathbb{P}(A)$.

(b) $\mathbb{P}(F_1) \leqslant \mathbb{P}(F_2)$ if $F_1 \subseteq F_2$.

6. Two dice are rolled. What is the conditional probability that at least one shows a six given that the dice show different numbers?

7. The proportion of Jaguar cars manufactured in Coventry is 0.7, and the proportion of these with some fault is 0.2. All other Jaguars are made in Birmingham and the proportion of faulty Birmingham cars is 0.1. What is the probability that a randomly selected Jaguar car:

(a) is both faulty and manufactured in Coventry?

(b) is faulty?

(c) is manufactured in Coventry given that it is faulty?

8. Given an example of a probability space and three events which are pairwise independent but not independent!

9. There are 52 cards in a pack. In a game of poker you receive a hand of 5 cards. How many different hands are there? How many hands have exactly 4 spades?

10. You are dealt 5 cards from a well shuffled pack. What is the probability that you receive 3 spades and 2 hearts?

11. You now have n distinguishable boxes, and r indistinguishable balls. How many ways of distributing the balls among the n boxes are there? (Here, some boxes can be empty).

12. We have r red balls, and b blue balls, $r > b - 1$. Other than color, the balls are indistinguishable. How many orderings of the balls are there so that no two blue balls are next to each other?

13. A pack of cards is well shuffled. What is the probability that there are no two hearts next to each other?