Fundamental Tools Sheet 8: Some inequalities and limit theorems

1. Write down Markov's inequality. Now let X be a random variable with mean zero and finite variance σ^2, and $a, b > 0$. Prove that

$$\mathbb{P}(X \geq a) \leq \frac{\sigma^2 + b^2}{(a+b)^2}.$$

Find b to minimise the right hand side and hence conclude $\mathbb{P}(X \geq a) \leq \sigma^2/(\sigma^2 + a^2)$.

2. You are told that the waiting time for bus has mean 10 minutes and variance 4 minutes. Use the inequality in the previous part to bound the probability that you wait for 12 minutes or more. Compare this to the bound you get from Markov's inequality.

3. **Chernoff bound.** Suppose X is a random variable with moment generating function $\phi_X(t) = \mathbb{E}[\exp(tX)]$. Show that, for $t \geq 0$, $\mathbb{P}(X \geq a) \leq \exp(-ta)\phi_X(t)$. If $X \sim \text{Pois}(\lambda)$, what value of t minimises the right hand side? Show that, for $k > \lambda$, your minimiser gives $\mathbb{P}(X \geq k) \leq \exp(-\lambda)\lambda^k \exp(k)/k^k$.

4. Use Chebyshev's inequality to prove the weak law of large numbers.

5. A series of 10000 tosses of a fair coin is to be performed. Let X represent the number of Heads. Use the Central Limit Theorem to approximate the following probabilities:

(a) $\mathbb{P}(X > 6000)$, (b) $\mathbb{P}(X \leq 5100)$, (c) $\mathbb{P}(|X - 5000| < 50)$.

Write your answers in terms of Φ, the cumulative distribution function of the standard normal random variable.

6. **How can normal approximation go wrong.** The skewness measure of a random variable is defined as

$$skew(X) := \mathbb{E}\left(\left(\frac{X-\mu}{\sigma}\right)^3\right) = \frac{\mathbb{E}(X^3) - 3\mu\sigma^2 - \mu^3}{\sigma^3},$$

where $\mu = \mathbb{E}(X)$ and $\sigma = \sqrt{\text{var}(X)}$. It measures the degree of asymmetry of the distribution. For random variables with symmetric distributions (uniform and normal distributions), their skewness measures are zero.

(a) For $X \sim Bin(n,p)$, show that $skew(X) = \frac{1-2p}{\sqrt{np(1-p)}}$. You can use the facts that $\mu = np$, $\sigma^2 = np(1-p)$, and also

$$m^{(3)}(0) = np\left(3np - 3np^2 + n^2p^2 - 3p + 2p^2 + 1\right)$$

where $m(t) := (1 - p + pe^t)^n$ is the mgf of a $Bin(n,p)$ random variable.

(b) For fixed n, what will happen to $skew(X)$ when $p \to 0$, and $p \to 1$? Do you think normal distribution can well approximate a binomial distribution when p is close to either 0 or 1?

7. State central limit theorem. Using this to show that $\lim_{n\to\infty} \exp(-n)\sum_{k=1}^{n} n^k/k! = 1/2$.

(Hint: write the left hand side as a statement about some Poisson random variables.)