Sketch of solutions to Sheet 7

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- Try not to consult these before you have tried the questions thoroughly.
- Very likely the solutions outlined below only represent a tiny subset of all possible ways of solving the problems. You are highly encouraged to explore alternative approaches!

1. Since X_1 and X_2 are exponential random variables which are positive by definition. $Y := X_1/X_2$ is a positive random variable as well. For $y > 0$, the cumulative distribution function of Y is given by

$$\begin{aligned} F_Y(y) = \mathbb{P}(Y \leqslant y) = \mathbb{P}(X_1/X_2 \leqslant y) = \mathbb{P}(X_1 \leqslant yX_2) \\ &= \iint\limits_{\{(x_1,x_2)\in(0,\infty)^2 : x_1 \leqslant yx_2\}} \exp(-x_1)\exp(-x_2)dx_1dx_2 \\ &= \int_{x_1=0}^{x_1=\infty}\int_{x_2=x_1/y}^{x_2=\infty} \exp(-x_1)\exp(-x_2)dx_2dx_1 \\ &= \int_{x_1=0}^{x_1=\infty} \exp(-(1+1/y)x_1)dx_1 \\ &= 1 - \frac{1}{y+1}. \end{aligned}$$

The density of Y is given by $\frac{d}{dy}F_Y(y) = (y+1)^{-2}$ for $y \geqslant 0$ (and the density is 0 on $y < 0$).

$\mathbb{P}(X_1 < X_2) = \mathbb{P}(Y < 1) = F_Y(1) = 1/2$.

2. (X, Y) is supported on the domain R_1 where $R_r := \{(x, y) : x^2 + y^2 \leqslant r^2\}$ (which represents a circle centering at the origin with radius r). Using the property that a density function integrates to 1, we have $1 = \iint_{R_1} f(x, y)dxdy = c\iint_{R_1} dxdy = c\pi$ which gives $c = 1/\pi$.

We integrate the joint density to obtain the marginal density of X and Y. In particular, for

$$f_X(x) = \int_{y\in\mathbb{R}} f(x,y)dy = c\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy = \frac{2}{\pi}\sqrt{1-x^2},$$

for $-1 \leqslant x \leqslant 1$, and similarly we have $f_Y(y) = \frac{2}{\pi}\sqrt{1-y^2}$ for $-1 \leqslant y \leqslant 1$. We see that in general $f(x, y) \neq f_X(x)f_Y(y)$, thus X and Y are not independent.

D is supported on $[0, 1]$. We first work out the cumulative distribution function of D as follow: for $0 \leqslant d \leqslant 1$,

$$F_D(d) = \mathbb{P}(D \leqslant d) = \mathbb{P}(\sqrt{X^2+Y^2} \leqslant d) = \mathbb{P}(X^2+Y^2 \leqslant d^2) = \iint\limits_{R_d} f(x,y)dxdy = c\pi d^2 = d^2.$$

The density function of D is then given by $f_D(d) = F'_D(d) = 2d$ on $0 \leqslant d \leqslant 1$ (and is zero elsewhere).

3. $\mathbb{P}(N \text{ is even}) = \mathbb{P}(N \in \{0,2,4,...\}) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^{2k}}{(2k)!} = e^{-\lambda}\left(1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots\right)$.

   Recall that
$$e^{\lambda} = 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots$$
   and
$$e^{-\lambda} = 1 - \frac{\lambda}{1!} + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \cdots .$$
   Hence
$$\frac{1}{2}\left(e^{\lambda} + e^{-\lambda}\right) = 1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots$$
   and thus $\mathbb{P}(N \text{ is even}) = \frac{e^{-\lambda}}{2}\left(e^{\lambda} + e^{-\lambda}\right)$.

   On the other hand, $\mathbb{P}(N = n, N \text{ is even}) = \frac{e^{-\lambda}\lambda^n}{n!}$ if n is even, or otherwise is 0 when n is odd. Therefore
$$\begin{aligned}\mathbb{P}(N = n | N \text{ is even}) &= \frac{\mathbb{P}(N = n, N \text{ is even})}{\mathbb{P}(N \text{ is even})} \\ &= \begin{cases} \frac{2\lambda^n}{(e^{\lambda}+e^{-\lambda})n!}, & n = 0,2,4,... \\ 0, & n = 1,3,5,... \end{cases}\end{aligned}$$

$$\begin{aligned}\mathbb{E}(N | N \text{ is even}) &= \sum_{n=0}^{\infty} n\mathbb{P}(N = n | N \text{ is even}) \\ &= \frac{2}{e^{\lambda} + e^{-\lambda}} \sum_{n=0}^{\infty} (2n)\frac{\lambda^{2n}}{(2n)!} \\ &= \frac{2\lambda}{e^{\lambda} + e^{-\lambda}} \sum_{n=1}^{\infty} \frac{\lambda^{2n-1}}{(2n-1)!} \\ &= \frac{2\lambda}{e^{\lambda} + e^{-\lambda}} \left(\frac{\lambda}{1!} + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \cdots\right) \\ &= \frac{\lambda}{e^{\lambda} + e^{-\lambda}} \left(e^{\lambda} - e^{-\lambda}\right) \\ &= \lambda \tanh(\lambda).\end{aligned}$$

4. $1 = \sum_{k=0}^{\infty} \mathbb{P}(Z = k) = \frac{\theta}{C_\theta}\sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} + \frac{1}{C_\theta}\sum_{k=1}^{\infty} \frac{\lambda^{2k-1}}{(2k-1)!} = \frac{\theta}{2C_\theta}(e^{\lambda} + e^{-\lambda}) + \frac{1}{2C_\theta}(e^{\lambda} - e^{-\lambda})$, which gives $C_\theta = \frac{1}{2}\left((\theta+1)e^{\lambda} + (\theta-1)e^{-\lambda}\right)$. Check that $C_\theta \to \infty$ and $\frac{\theta}{C_\theta} \to \frac{2}{e^{\lambda}+e^{-\lambda}}$ as $\theta \to \infty$. Hence we have
$$\mathbb{P}_\theta(Z = z) \to \begin{cases} \frac{2}{e^{\lambda}+e^{-\lambda}}\frac{\lambda^z}{z!}, & z = 0,2,4,\cdots; \\ 0, & z = 1,3,5,\cdots \end{cases}$$
   as $\theta \to \infty$. This expression is equivalent to the one computed in question 3.

5. (a) The two events "$X = 0$" and "$Y \neq 0$" are mutually exclusive and cannot happen at the same time, thus $\mathbb{P}(X = 0, Y \neq 0) = 0$. On the other hand, $\mathbb{P}(X = 0) = 1/3$, $\mathbb{P}(Y \neq 0) = \mathbb{P}(X = 1) + \mathbb{P}(X = -1) = 2/3$. In particular, $\mathbb{P}(X = 0)\mathbb{P}(Y \neq 0) = 2/9 \neq 0 = \mathbb{P}(X = 0, Y \neq 0)$. Thus X and Y are not independent.

   (b) The joint probability mass function $p_{XY}(x,y)$ and marginal probability mass function $p_X(x)$ and $p_Y(y)$ can be represented below:
   X and Y are not independent since one can check that $p_{XY}(x,y) \neq p_X(x)p_Y(y)$ (for example, $p_{XY}(-1,1) = 1/3$ but $p_X(-1)p_Y(1) = (1/3)(2/3) = 2/9 \neq p_{XY}(-1,1)$.)

			x		$p_Y(y)$
		-1	0	1	
y	1	1/3	0	1/3	2/3
	0	0	1/3	0	1/3
$p_X(x)$		1/3	1/3	1/3	

(c) From the joint pmf, we can work out $\mathbb{E}(XY) = \sum_{x,y} xy\mathbb{P}(X = x, Y = y) = (1)(-1)(1/3) + (0)(0)(1/3) + (1)(1)(1/3) = 0$. From the marginal pmf's of X and Y it is also easy to check $\mathbb{E}(X) = 0$ and $\mathbb{E}(Y) = 2/3$. Hence $\mathbb{E}(XY) = 0 = \mathbb{E}(X)\mathbb{E}(Y)$. Here the covariance/correlation between X and Y is zero, although they are not independent.

6. For $X \sim Bin(n, p)$, its pgf is

$$G_X(t) = \mathbb{E}(t^X) = \sum_{k=0}^{n} t^k C_k^n p^k (1-p)^{n-k} = \sum_{k=0}^{n} C_k^n (pt)^k (1-p)^{n-k} = (pt + 1 - p)^n.$$

For $Y \sim Bin(m, p)$ which is independent of X, the pgf of $X + Y$ is given by $G_X(t)G_Y(t) = (pt + 1 - p)^n (pt + 1 - p)^m = (pt + 1 - p)^{n+m}$, which is identical to the pgf of $Bin(n + m, p)$. Hence $X + Y \sim Bin(n + m, p)$ by the unique correspondence between distribution and pgf.

7. With a given pgf $g_X(\cdot)$, $\mathbb{P}(X = k) = \frac{g_X^{(k)}(0)}{k!}$. In the case of $g_X(t) = e^{\theta(t-1)}$ we have $g_X^{(k)}(t) = \theta^k e^{\theta(t-1)}$. Hence $\mathbb{P}(X = k) = \frac{\theta^k e^{-\theta}}{k!}$ (for $k = 0, 1, 2, ...$), i.e. X has a Poisson(θ) distribution.

8. For $X \sim \exp(\lambda)$, its mgf is given by $m_X(t) = \mathbb{E}(e^{tX}) = \int_0^\infty e^{tu}\lambda e^{-\lambda u} du = \lambda \int_0^\infty e^{-(\lambda-t)u} du = \frac{\lambda}{\lambda - t}$. (Need $\lambda > t$ for the mgf to be well-defined, otherwise the indefinite integral diverges.)

We can obtain $m_X'(t) = \frac{\lambda}{(\lambda-t)^2}$ and $m_X''(t) = \frac{2\lambda}{(\lambda-t)^3}$. Hence $\mathbb{E}(X) = m_X'(0) = 1/\lambda$ and $\mathbb{E}(X^2) = m_X''(0) = 2/\lambda^2$. Then $\text{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 1/\lambda^2$.

9. We first obtain the density function f by differentiating the CDF:

$$f(x) = \frac{d}{dx}(1 - (1 + \lambda x)e^{-\lambda x}) = \lambda^2 x e^{-\lambda x}$$

for $x > 0$. Then the mgf is computed via

$$\begin{aligned} m(t) = \mathbb{E}(e^{tX}) &= \int_0^\infty e^{tx}\lambda^2 x e^{-\lambda x} dx \\ &= \lambda^2 \int_0^\infty x e^{-(\lambda-t)x} dx \\ &= \frac{\lambda^2}{\lambda - t} x e^{-(\lambda-t)x}|_\infty^0 + \frac{\lambda^2}{\lambda - t}\int_0^\infty e^{-(\lambda-t)x} dx \\ &= 0 + \frac{\lambda^2}{(\lambda-t)^2} e^{-(\lambda-t)x}|_\infty^0 \\ &= \left(\frac{\lambda}{\lambda - t}\right)^2. \end{aligned}$$

$\mathbb{E}(X)$ can be computed via $m'(0)$. Here $m'(t) = 2\lambda^2(\lambda - t)^{-3}$ and hence $\mathbb{E}(X) = m'(0) = 2/\lambda$.