Follow-up questions on Ch4 Q7

An asymmetric random walk is defined by $S_0 = x$ and $S_n = x + \sum_{k=1}^{n} X_k$ for $n \geqslant 1$, where

$$X_n = \begin{cases} +1, & \text{with probability } p \\ -1, & \text{with probability } q \end{cases}$$

with $p \neq q$. Fix $\theta = q/p$, we have shown that $M_n := \theta^{S_n}$ is a martingale and $\mathbb{P}(H_a < H_b) = \frac{\theta^x - \theta^b}{\theta^a - \theta^b}$, where $H_a = \inf(n \geqslant 0 : S_n = a)$ and $H_b = \inf(n \geqslant 0 : S_n = b)$.

Some follow-up questions:

1. Does $\lim_{n \to} M_n$ exist? What can be said about this limit?
2. Let $T = H_a \wedge H_b$. Find $\mathbb{E}(S_T)$.
3. Show that $H_n = S_n - n(p - q)$ is a martingale. Hence find $\mathbb{E}(T)$.

Part 1

Note that $M_n = (q/p)^{S_n}$ is always non-negative. Hence by Martingale Convergence Theorem (MCT), $\lim_{n\to} M_n$ exists.

Let μ be the common mean of X_n. By definition of X_n we have $\mu = p - q$. Since X_n's are i.i.d random variables with finite variance, strong law of large numbers gives $S_n/n \to \mu = p - q$ almost surely as $n \to \infty$.

Now consider two cases:
Case 1: Suppose $p > q$. We have $S_n \approx n(p-q) \to +\infty$ almost surely as $n \to \infty$. Moreover, under $p > q$ we have $\theta < 1$. Hence $M_n = \theta^{S_n} \to 0$.
Case 2: Suppose $q > p$. We have $S_n \approx n(p-q) \to -\infty$ almost surely as $n \to \infty$. Moreover under $q > p$ we have $\theta > 1$. Hence $M_n = \theta^{S_n} \to 0$.

In summary, M_n converges to 0 almost surely.

Remark: the argument via $S_n \approx n(p-q)$ for large n is quite loose. But it can be formally proved that $S_n \to \pm\infty$ when $p \gtrless q$ by tracing the formal definitions from analysis. See also Ch3 Q6 where a similar idea is used.

Part 2

We know that S_T is a binary random variable, which takes value of a or b with probability $\mathbb{P}(H_a < H_b)$ and $\mathbb{P}(H_b < H_a)$ respectively. Then

$$\begin{aligned}\mathbb{E}(S_T) &= a\mathbb{P}(H_a < H_b) + b\mathbb{P}(H_b < H_a)\\ &= \frac{a(\theta^x - \theta^b) + b(\theta^a - \theta^x)}{\theta^a - \theta^b}.\end{aligned}$$

Part 3

Verify that $H_n = S_n - n(p-q)$ satisfies the three required properties of a martingale:

1. H_n is clearly adapted.
2. $\mathbb{E}|H_n| \leqslant \mathbb{E}|S_n| + n|p-q| \leqslant x + n + n|p-q| < \infty$.
3. Check that

$$\begin{aligned}
\mathbb{E}(H_{n+1}|\mathcal{F}_n) &= \mathbb{E}(S_{n+1}|\mathcal{F}_n) - (n+1)(p-q) \\
&= \mathbb{E}(S_n + X_{n+1}|\mathcal{F}_n) - (n+1)(p-q) \\
&= S_n + \mathbb{E}(X_{n+1}) - (n+1)(p-q) \\
&= S_n + (p-q) - (n+1)(p-q) \\
&= S_n - n(p-q) = H_n.
\end{aligned}$$

We have a martingale H. Now it's tempting to directly use optimal stopping theorem (OST) on H_T to conclude $\mathbb{E}(H_T) = H_0$. But it can't be done in this way since H_n is not bounded for $n \leqslant T$ so the condition for OST fails.

Part 3 (cont')

Remember our trick on Ch4 Q6. If we don't have a bounded martingale which allows direct use of OST, the second best attempt is to consider a capped (bounded) stopping time, and see if we can take limit to get the results we want.

Let $N > 0$ be some fixed constant and set $T_N = \min(T, N)$. Then T_N is a bounded stopping time. OST can now be used which gives $\mathbb{E}(H_{T_N}) = H_0 = S_0 = x$. Then $x = \mathbb{E}(S_{T_N} - T_N(p - q))$ and in turn

$$\mathbb{E}(T_N) = \frac{1}{p-q}(\mathbb{E}(S_{T_N}) - x).$$

Part 3 (cont')

Take limit $N \to \infty$ on both side.

- The left-hand-side becomes

$$\lim_{N\to\infty} \mathbb{E}(T_N) = \mathbb{E}\left(\lim_{N\to\infty} T_N\right) = \mathbb{E}(T).$$

  The swap of $\mathbb{E}$ and lim is valid due to monotone convergence theorem (MON) as T_N is positive and increasing to T as $N \to \infty$.

- The right-hand-side becomes

$$\lim_{N\to\infty} \mathbb{E}(S_{T_N}) = \mathbb{E}\left(\lim_{N\to\infty} S_{T_N}\right) = \mathbb{E}(S_T).$$

  The swap of $\mathbb{E}$ and lim is valid due to bounded convergence theorem (BDD) as S_{T_N} is bounded between a and b for all N.

We thus obtain:

$$\mathbb{E}(T) = \frac{1}{p-q}\mathbb{E}(S_T - x) = \frac{1}{p-q}\left(\frac{a(\theta^x - \theta^b) + b(\theta^a - \theta^x)}{\theta^a - \theta^b} - x\right)$$
$$= \frac{(a-x)((q/p)^x - (q/p)^b) + (b-x)((q/p)^a - (q/p)^x)}{(p-q)((q/p)^a - (q/p)^b)}.$$