Ch4 Q13: first approach

- $u = 3/2$, $d = 2/3$, interest rate factor$= \beta = 13/12$, $P_0 = 1$
- The question doesn't specify the real-world probabilities of the up- and down-move. But for now let's suppose they are both 1/2
- Idea: find an adapted process L_n such that $L_n P_n$ and $L_n R_n$ are martingales, where P_n and R_n are the processes of the stock and bond price respectively. Note that $R_n = \beta^n R_0$
- Try $L_n = \left(\frac{P_n}{P_0}\right)^{\epsilon} \left(\frac{R_n}{R_0}\right)^{\gamma} = \left(\frac{P_n}{P_0}\right)^{\epsilon} \beta^{n\gamma}$, with $L_0 = 1$
- Exercise: establish the below equations from the martingale properties $\mathbb{E}(L_{n+1}R_{n+1}|\mathcal{F}_n) = L_n R_n$ and $\mathbb{E}(L_{n+1}P_{n+1}|\mathcal{F}_n) = L_n P_n$

$$\left(\frac{13}{12}\right)^{\gamma+1} \left(\frac{1}{2}\left(\frac{3}{2}\right)^{\epsilon} + \frac{1}{2}\left(\frac{2}{3}\right)^{\epsilon}\right) = 1 \tag{1}$$

$$\left(\frac{13}{12}\right)^{\gamma} \left(\frac{1}{2}\left(\frac{3}{2}\right)^{\epsilon+1} + \frac{1}{2}\left(\frac{2}{3}\right)^{\epsilon+1}\right) = 1 \tag{2}$$

Ch4 Q13: first approach (cont)

- Solving the equations would give $\epsilon = 0$ and $\gamma = -1$. Hence $L_n = \left(\frac{12}{13}\right)^n$
- The price of an ATM call option maturing at $n = 2$ with payoff $V_2 = (P_2 - 1)^+$ is given by

$$\begin{aligned}\frac{1}{L_0}\mathbb{E}(L_2 V_2) &= \left(\frac{12}{13}\right)^2 \mathbb{E}((P_2 - 1)^+) \\ &= \frac{144}{169} \times \frac{1}{4} \times \left(\frac{9}{4} - 1\right) \\ &= \frac{45}{169}\end{aligned}$$

Ch4 Q13: first approach (cont)

- What if the real-world probability of the up-move now becomes 1/3??
- The new set of equations derived from the martingale properties of $L_n R_n$ and $L_n P_n$ now become (check!)

$$\left(\frac{13}{12}\right)^{\gamma+1}\left(\frac{1}{3}\left(\frac{3}{2}\right)^{\epsilon}+\frac{2}{3}\left(\frac{2}{3}\right)^{\epsilon}\right)=1 \tag{3}$$

$$\left(\frac{13}{12}\right)^{\gamma}\left(\frac{1}{3}\left(\frac{3}{2}\right)^{\epsilon+1}+\frac{2}{3}\left(\frac{2}{3}\right)^{\epsilon+1}\right)=1 \tag{4}$$

- Exercise: from these equations, verify that ϵ and γ will satisfy $\left(\frac{3}{2}\right)^{\epsilon}=\sqrt{2}$ and $\left(\frac{13}{12}\right)^{\gamma}=\frac{9\sqrt{2}}{13}$
- The price of the same call option is given by

$$\begin{aligned}\frac{1}{L_0}\mathbb{E}(L_2V_2) &= \mathbb{E}\left(\beta^{2\gamma}\left(\frac{P_2}{P_0}\right)^{\epsilon}(P_2-1)^{+}\right)\\ &= \left(\frac{13}{12}\right)^{2\gamma}\times\frac{1}{9}\times\left(\frac{9}{4}\right)^{\epsilon}\times\left(\frac{9}{4}-1\right)\\ &= \left(\frac{9\sqrt{2}}{13}\right)^{2}\times\frac{1}{9}\times\left(\sqrt{2}\right)^{2}\times\left(\frac{9}{4}-1\right)=\frac{45}{169}\end{aligned}$$

Ch4 Q13: second approach

- Find the probability q of the up-move under which the discounted stock price process is a martingale:

$$q \times \frac{3}{2} \times \frac{1}{\beta} + (1-q) \times \frac{2}{3} \times \frac{1}{\beta} = 1$$

$$q = \frac{\beta - 2/3}{3/2 - 2/3} = \frac{13/12 - 2/3}{3/2 - 2/3} = 1/2$$

- Option price is given by the expected value of the discounted payoff under this q-probability

$$\mathbb{E}_{\mathbb{Q}}\left(\frac{V_2}{\beta^2}\right) = \frac{1}{\beta^2} \times q^2 \times \left(\frac{9}{4} - 1\right)$$

$$= \left(\frac{12}{13}\right)^2 \times \left(\frac{1}{2}\right)^2 \times \left(\frac{9}{4} - 1\right) = \frac{45}{169}$$

Some comments

- The first approach is more tedious computationally
- Moreover, it has an undesirable feature that one needs to specify the real-world probabilities of the stock price evolution in order to work out L_n, which is difficult in practice as well as irrelevant to pricing of options
- By working with the second approach (the risk-neutral probability q), one doesn't even need to worry about the chance of the stock moving up or down (mean return of the stock is not important). It is a feature of typical option pricing models