Remark on writing down an SDE

If we are asked to write down an SDE for a process X_t, answer should be presented in form of

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t.$$

The drift and diffusion coefficients should always be expressed in X_t.

Question from the week 10 class test: Suppose X satisfies $dX_t = \frac{\alpha}{X_t}dt + dB_t$. Write down an SDE for Q_t where $Q_t = X_t^2$.

Applying Ito's lemma will give

$$\begin{aligned} dQ_t = d(X_t^2) &= 2X_t dX_t + (dX_t)^2 \\ &= (2\alpha + 1)dt + 2X_t dB_t. \end{aligned}$$

But $dQ_t = (2\alpha + 1)dt + 2X_t dB_t$ should not be presented as the final solution since the coefficient of dB_t is in X but not in Q. Instead we should write X in terms of Q and the answer should be written as

$$dQ_t = (2\alpha + 1)dt + 2\sqrt{Q_t}dB_t.$$

(Local) martingale property of a stochastic integral

Let X be a martingale, and C be an adapted process.

- In discrete time, we know that $(C \cdot X)_n := \sum_{k=1}^{n} C_{k-1}(X_k - X_{k-1})$ is a martingale.
- As a continuous time analogue, $(C \cdot X)_t := \int_0^t C_s dX_s$ is a local martingale.

In the continuous time if we further know that $\int_0^t C_s dX_s$ is a true (i.e not just local) martingale, then by the martingale property of $\mathbb{E}(M_t) = M_0$, we can conclude $\mathbb{E}(\int_0^t C_s dX_s) = 0$.

Warning: In general a stochastic integral is not always a true martingale, and hence it may not have zero expectation. In the ST909 module in term 2, you will see some tools which help check whether a local martingale is a true martingale.

Stochastic integral (against a Brownian motion) with deterministic integrand

In the last part of Q4(B) of the Jan2015 exam paper, it asks about identifying the distribution of Y_t where

$$Y_t = Y_0 e^{-\kappa t} + m(1 - e^{-\kappa t}) + \int_0^t \sigma e^{-\kappa(t-s)} dB_s.$$

The first two terms are just constants. Thus it is all about asking what the distribution of $\int_0^t \sigma e^{-\kappa(t-s)} dB_s$ is. Indeed, this is having a normal distribution of $N(0, \int_0^t \sigma^2 e^{-2\kappa(t-s)} ds)$ (see the lemma below), and hence Y_t is distributed as

$$\begin{aligned} Y_t &\sim N(Y_0 e^{-\kappa t} + m(1 - e^{-\kappa t}), \int_0^t \sigma^2 e^{-2\kappa(t-s)} ds)) \\ &= N\left(Y_0 e^{-\kappa t} + m(1 - e^{-\kappa t}), \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t}) \right). \end{aligned}$$

Lemma

If C is a deterministic process, then $\int_0^t C_s dB_s \sim N(0, \int_0^t C_s^2 ds)$.

Sketch of proof for the lemma

Proof.

Let $f(t) := \int_0^t C_s^2 ds$, which is a deterministic and increasing function. Let $f^{-1}(t)$ be the (left-continuous) inverse function of f, which is guaranteed to be well-defined since f is increasing.

Also define $I_t := \int_0^{f^{-1}(t)} C_s dB_s$. We first claim that the process $(I_t)_{t \geqslant 0}$ is a Brownian motion, since

1. I_t, as a stochastic integral, is a continuous local martingale;
2. The quadratic variation of I is given by $[I]_t = \int_0^{f^{-1}(t)} C_s^2 ds = f(f^{-1}(t)) = t$.

I is then a Brownian motion by Levy's theorem. Then $I_t \sim N(0,t)$ and equivalently $\int_0^{f^{-1}(t)} C_s dB_s \sim N(0,t)$. On replacing t by $f(t)$, we get

$$\int_0^t C_s dB_s \sim N(0, f(t)) = N(0, \int_0^t C_s^2 ds).$$

□